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Asymptotic behavior of solutions for some nonlinear partial differential equations on unbounded domains *

Jacqueline Fleckinger, Evans M. Harrell II, & François de Thélin

Abstract

We study the asymptotic behavior of positive solutions u of

$$-\Delta_p u(\mathbf{x}) = V(\mathbf{x})u(\mathbf{x})^{p-1}, \quad p > 1;\ \mathbf{x} \in \Omega,$$

and related partial differential inequalities, as well as conditions for existence of such solutions. Here, Ω contains the exterior of a ball in $\mathbb{R}^N$ $1 < p < N$, Δ_p is the p-Laplacian, and V is a nonnegative function. Our methods include generalized Riccati transformations, comparison theorems, and the uncertainty principle.

1 Introduction

In an earlier article [9] we studied the behavior of solutions of equations and inequalities containing the p-Laplacian near the boundary of a bounded domain. In this article we consider unbounded domains, and estimate the behavior of solutions as $|\mathbf{x}| \to \infty$. We restrict ourselves to positive solutions of

$$-\Delta_p u(\mathbf{x}) = \lambda V(\mathbf{x})u(\mathbf{x})^{p-1}, \tag{1.1}$$

which decrease at infinity. Here, $p > 1$, $\lambda > 0$, and $\mathbf{x}$ runs over a domain containing the exterior of a large ball in $\mathbb{R}^N$. The value of the constant λ is important for some questions of existence, but it is not essential for most of our purposes and will set to 1 in the following sections. The power $p-1$ on the right provides the same homogeneity in u as the p-Laplacian. The weight function V is always assumed nonnegative, and some further restrictions will be imposed.

The number of previous articles on the subject of asymptotics of solutions of equations like (1.1) does not appear to be large. Many of those of which we are aware use or adapt lemmas of Serrin [17] and of Ni and Serrin [14]; e.g., see [2]. In case $N > p$, a positive radial solution u of the partial differential inequality

$$-\Delta_p u(\mathbf{x}) \geq 0$$

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will satisfy bounds of the form [14],

$$\begin{aligned} u(r) &\geq C_1 r^{-\frac{N-p}{p-1}} \\ u'(r) &\geq C_2 r^{-\frac{N-1}{p-1}}. \end{aligned} \tag{1.2}$$

Other related estimates are to be found in [4] when u is a ground–state solution of (1.1). For some conditions guaranteeing the existence of solutions to (1.1) we refer to [18] and references therein. (In some circumstances our results on asymptotics will imply nonexistence of solutions.) For certain equations resembling (1.1), but where the degree of homogeneity on the right differs from that of the p-Laplacian, there is some work on asymptotic estimates and existence theory: See [6, 12, 19], and especially the books [4] and [5] for these and background material on equations like (1.1).

A *ground-state solution* is understood as a positive solution on $\mathbb{R}^N$ which tends to 0 as $|\mathbf{x}| \to \infty$. In case $N > p$, it is shown in [4], Theorem 4.1, that for some $\lambda =: \lambda_1$, there exists a ground-state solution, which is in L^q for any $q \in [p^*, \infty]$, where $p* := Np/(N-p)$. It is also remarked in [4] that the same bound applies when u is a positive, decaying solution on an exterior domain Ω, given Dirichlet boundary conditions on $\partial\Omega$.

Henceforth we absorb the eigenvalue into V, setting $\lambda = 1$.

In this article, we assume initially that $-\Delta_p u$ is bounded from below. We study the radial case with a generalized Riccati transformation, and establish a priori bounds on the logarithmic derivative of u. These in turn imply some lower bounds on u or, in some circumstances, its nonexistence.

Then we turn our attention to the non-radial case. We find it convenient to study averages of expressions containing u over suitable sets rather than attempting pointwise estimates. We modify the Riccati transformation for the non-radial case and use it to derive some bounds analogous to those of the radial case.

In the fourth section we make the complementary assumption, that $-\Delta_p u$ is bounded from above. Here we adapt the techniques of [9] to unbounded domains and establish upper bounds on u.

2 The Logarithmic Rate of Decrease of Radial Solutions

In this section we assume radial symmetry, and study the positive radial solutions of the inequality

$$-\Delta_p u(\mathbf{x}) \geq V(\mathbf{x}) u(\mathbf{x})^{p-1},\ p > 1. \tag{2.1}$$

Since this section concerns only radial solutions, (2.1) may be written as

$$-(r^{N-1}|u'|^{p-2}u')' \geq V r^{N-1} u^{p-1}. \tag{2.2}$$

In [14], it is shown that:

Proposition 2.1 ([14]) *Assume that $V(r) > 0$ is bounded and measurable on any finite subset of $\{r > r_0\}$, where r is the radial coordinate in $\mathbb{R}^N, N > p$. Suppose that $u(r)$ is a positive radial ground–state solution of (2.1) for $r > r_0$. Then there exist two positive constants c_1 and c_2 such that*

$$u(r) \geq c_1 r^{-\left(\frac{N-p}{p-1}\right)},$$
$$|u'(r)| \geq c_2 r^{-\left(\frac{N-1}{p-1}\right)}.$$

Under certain circumstances we shall improve Ni and Serrin's bound on u with a Riccati transformation adapted to the p-Laplacian. An interesting aspect of this is that our bound can be viewed as an oscillation theorem. We shall also comment on the consequences for the possible existence of positive solutions. Let

$$\rho := -\Big|\frac{u'}{u}\Big|^{p-2}\frac{u'}{u}. \tag{2.3}$$

The sign is reasonable because we shall show that $u' < 0$ for large r. By inserting (2.3) into (2.2), we derive

$$\rho' \geq V + (p-1)\,|\rho|^{\frac{p}{p-1}} - (N-1)\frac{\rho}{r}. \tag{2.4}$$

We note here that since ρ determines the logarithmic derivative of u, bounds on ρ as $r \to \infty$ correspond roughly to decay estimates for u. Moreover, at finite r a divergence of ρ may simply arise from a zero of u; it may be possible to continue through the zero in standard ways [10].

We consider here positive radial solutions of (2.1) such that

$$\limsup_{r\to+\infty} \rho(r) < +\infty. \tag{2.5}$$

The following proposition states that any positive radial solution must either satisfy an *a priori* bound on all $r > r_0$ or else blow up at some finite value of r.

Proposition 2.2 *Assume $V(r) > 0$ is bounded and measurable on any finite subset of $\{r > r_0\}$, where r is the radial coordinate in $\mathbb{R}^N, N \geq 2$. Let $u(r)$ be a positive radial solution of (2.1) for all $r > r_0$, and define ρ by (2.3). Assume further that ρ satisfies (2.5). Then a)*

$$\rho < \rho_{cr} := \left\{\frac{N-1}{(p-1)r}\right\}^{p-1}. \tag{2.6}$$

b) If $u(r) > 0$, $u'(r) < 0$ for $r > r_0$, and if (2.6) is violated at $r = r_0$, then $u(r_1) = 0$ for some r_1 which could be bounded above explicitly in terms of $\rho(r_0), N$, and r_0.

Proof: a) First we show by contradiction that if $\rho(r_0) > 0$, then $\rho(r) > 0$ for all $r > r_0$. Suppose otherwise and let R be the first zero of ρ larger than r_0: $\rho(R) = 0$ and $\rho'(R) \leq 0$. This contradicts (2.4) at the point R, by which $\rho'(R) > 0$.

Moreover, statement a) is trivial in the case where $\rho(r) < 0$ for any r. We conclude that we may assume that $\rho(r) > 0$ for any r.

Now consider the critical curve defined by (2.6) in the (r, ρ) plane, $\rho > 0$:

$$(p-1)\rho_{cr}^{\frac{p}{p-1}} - (N-1)\frac{\rho_{cr}}{r} = 0.$$

The function ρ_{cr} decreases as r increases, and by (2.4), if $\rho \geq \rho_{cr}$, then $\rho' \geq V > 0$.

Hence if (2.6) were false at $r = R$ for some finite R, then ρ would be an increasing function for all $r > R$. Consequently, it would either approach a finite positive limit or else diverge to $+\infty$. A finite positive limit, however, is incompatible with (2.4) for large r, and therefore ρ would become arbitrary large as $r \to \infty$. For large r and small positive ε, it then follows from (2.4) that $\rho' > (p-1-\varepsilon)\rho^{\frac{p}{p-1}}$, which implies by comparison that $\rho \geq \tilde{\rho}$ with $\tilde{\rho}$ a positive solution of

$$\tilde{\rho}' = (p-1-\varepsilon)\tilde{\rho}^{\frac{p}{p-1}}.$$

It is elementary to solve the comparison equation: We find

$$\tilde{\rho}(r) = \Big[\frac{p-1}{(p-1-\varepsilon)(r_2 - r)}\Big]^{p-1}, \quad \text{for some } r_2 > R.$$

Since any such solution is singular at the finite point $r = r_2$, any solution ρ violating (2.6) likewise blows up at some finite $r_1 \leq r_2$, which contradicts (2.5).

b) The proof above shows that ρ blows up at r_1 and therefore $u(r_1) = 0$. $\square$

Proposition 2.2 makes no use of the detailed nature of V, and therefore it can be sharpened, given more information about V :

Lemma 2.3 *For a given $b > 0$ and $x > b/(N-1)$, set*

$$\varphi_b(x) := \left(\frac{(N-1)x - b}{p-1}\right)^{\frac{p-1}{p}}.$$

The concave function φ_b is increasing and admits a fixed point if and only if $b \leq (\frac{N-1}{p})^p$. By concavity there are at most two fixed points, and we denote by $a_*(b)$, or a_* for short, the larger (or only) one. An explicit bound on the fixed point is:

$$a_* \leq \left(\frac{N-1}{p-1}\right)^{p-1} - b\left(\frac{p-1}{N-1}\right). \tag{2.7}$$

Proof: Since $\varphi_b\left(\frac{b}{N-1}\right) = 0$, we seek $x > \frac{b}{N-1}$ such that $\varphi_b(x) = x$, which is equivalent to $\psi(x) = b$ with $\psi(x) := (N-1)x - (p-1)x^{\frac{p}{p-1}}$. The extremum of ψ is obtained for $\tilde{x} = \left(\frac{N-1}{p}\right)^{p-1}$ and $\psi(\tilde{x}) = (\frac{N-1}{p})^p$. Hence φ_b admits at least one fixed point if and only if

$$b \le b_{\max}(N,p) := \left(\frac{N-1}{p}\right)^p. \tag{2.8}$$

The roots of ψ are $x = 0$ and $x = \hat{x} := (\frac{N-1}{p-1})^{p-1}$. Obviously $a_* \le \hat{x}$. In fact, since ψ is concave, the curve ψ lies below the tangent at point $\hat{x}$, which leads to the estimate (2.7). □

Proposition 2.4 *Suppose that for some $b > 0$, $V(r) \ge br^{-p}$ on the interval $[r_0;\infty)$.*

a) If $0 < b \le b_{\max}(N,p) := \frac{(N-1)^p}{p^p}$, then for any solution u of (2.1) *on $[r_0;\infty)$, such that its ρ satisfies* (2.5)*, we have $\rho \le a_*(b)r^{-(p-1)}$, where $a_*(b)$ is as defined in Lemma 2.3.*

b) If $b > b_{\max}(N,p)$, then there are no solutions ρ of (2.4) *which satisfy* (2.5)*, and thus there are no positive, decreasing radial solutions u of* (2.1).

Proof: a) We assume that $b \le \frac{(N-1)^p}{p^p}$ and extend Proposition 2.2 by a bootstrap argument. Suppose that it has been established that $\rho \le ar^{-(p-1)}$ for some a. Then from (2.4) it follows that $\rho' \ge 0$ provided that

$$-((N-1)a - b)r^{-p} + (p-1)\rho^{\frac{p}{p-1}} \ge 0,$$

which corresponds to the critical curve

$$\rho_{cr}(r;a) = \varphi_b(a)r^{-(p-1)}.$$

With the same argument as in Proposition 2.2, we conclude that ρ lies below the critical curve, $\rho(r) < \varphi_b(a)r^{-(p-1)}$. By iteration we improve the above estimate with a decreasing sequence of a_k, and as $k \to \infty$ we obtain

$$a_* = \left[\frac{(N-1)a_* - b}{p-1}\right]^{\frac{p-1}{p}}$$

as the largest fixed point of φ_b: $a_* = \varphi_b(a_*)$, which exists by Lemma 2.3.

b) Assume now that $b > \frac{(N-1)^p}{p^p}$. We have shown above that if $a > \frac{b}{N-1}$ and $\rho \le ar^{-(p-1)}$, then ρ also satisfies $\rho \le \varphi_b(a)r^{-(p-1)}$. Recalling Proposition II.2, there exists $a_0 > \frac{b}{N-1}$ such that $\rho \le a_0r^{-(p-1)}$. We define $a_{k+1} := \varphi_b(a_k)$. For large a, $\varphi_b(a) < a$, so the sequence a_k is bounded from above. Hence either

(i) there exists a subsequence $a_{k_j} \to a_* > \frac{b}{N-1}$ as $j \to \infty$; or

(*ii*) there exists k such that $a_{k+1} \le \frac{b}{N-1} < a_k$.
Case (i) is excluded by the previous lemma. In Case (ii), we may decrease b as necessary to $b_\epsilon := (N-1)a_{k+1} - (\epsilon(p-1))^{\frac{p}{p-1}}$ so that $a_{k+1} \ge \frac{b_\epsilon}{N-1}$. Then we define $\tilde{a}_{k+2} := \varphi_{b_\epsilon}(a_{k+1}) = \epsilon$. It follows that $\rho \le \epsilon r^{-(p-1)}$, and as $\epsilon \searrow 0$ we find $\rho \le 0$. Therefore u cannot be a positive decreasing solution. □

Corollary 2.5 *If $V(r) \ge br^{-p}$ for $0 < b < b_{\max}(N,p)$, then any positive solution u on $[r_0;\infty[$ satisfies*

$$u(r) \ge u(r_0)\left(\frac{r}{r_0}\right)^{-a_*^{\frac{1}{p-1}}},$$

where a_* is as in Lemma 2.3.

Proof: Because of (2.3),

$$\big(-\frac{u'}{u}\big) \le a_*^{\frac{1}{p-1}}\frac{1}{r},$$

and by integrating, for $r > r_0$,

$$-\ln\big|\frac{u(r)}{u(r_0)}\big| \le a_*^{\frac{1}{p-1}} \ln\big(\frac{r}{r_0}\big),$$

which for $u > 0$ implies the claim. □

Remark. Corollary 2.5 is weaker than Proposition 2.1 for small b, but improves it for some values of N, p, and b. More specifically, Corollary 2.5 is an improvement when $N > p$ and $a_* < (\frac{N-p}{p-1})^{p-1}$. From (2.7), for this it suffices to have

$$\big(\frac{N-1}{p-1}\big)^{p-1} - b\big(\frac{p-1}{N-1}\big) < \big(\frac{N-p}{p-1}\big)^{p-1}.$$

For $b = b_{\max}$, this condition becomes

$$1 - \big(\frac{p-1}{p}\big)^p < \big(1 - \frac{p-1}{N-1}\big)^{p-1},$$

which is clearly true for sufficiently large N.

Corollary 2.6 *Suppose that for some $C > 0$ and $m < p$, $V(r) \ge Cr^{-m}$ for all $r > r_0$. Then Equation* (2.1) *has no solutions which remain positive on* (r_0, ∞).

Remark. The proof of this corollary is merely an application of Propositon 2.4 b). This corollary is a special case of [18, Theorem 3.2].

Lemma 2.7 *Let $u(r)$ be a positive, radial, decreasing solution of* (2.1) *for $r \ge r_0$, with $V(r) > 0$, and define ρ as before. Then*

$$\rho(r) \ge \rho(r_0)\left(\frac{r}{r_0}\right)^{-(N-1)}$$

for all $r > r_0$.

Proof: Under these circumstances, it follows from (2.4) that

$$\rho' \geq -\frac{(N-1)\rho}{r},$$

so by the comparison principle, ρ is bounded from below by the solution of

$$\tilde{\rho}' = -\frac{(N-1)\tilde{\rho}}{r},$$

which agrees with ρ at $r = r_0$. This yields the claimed bound. □

Finally we observe that our bounds imply some simple and fairly standard nonexistence criteria.

Corollary 2.8 *Suppose:*

a) that $p > N$. Then there are no ground–state solutions of (2.1).

b) that $p = N$, and that $V(r) \geq \left(\frac{p-1}{p}\right)^p r^{-p}$ on the interval $[r_0; \infty)$. Then there are no ground–state solutions of (2.1).

Proof: a) If $p > N$, then the lower bound of Lemma 2.7 would exceed the upper bound of Proposition 2.2 for large r. Part b) is the same as Proposition 2.4 b) whith $p = N$. □

3 The Nonradial Case

We now turn our attention to (2.1) when V is not necessarily radial. We suppose throughout this section that $u > 0$ on the exterior of some ball, and consider (2.1) on this exterior domain. Without assumptions of symmetry some control is lost on the decrease of the solutions. Instead of estimating the solutions pointwise, we shall estimate certain integrals over large balls and spheres.

We frequently use the following standard notation:
B_R is the ball of radius R.
$\mathbf{n}$ is the unit radial vector.
ω_N is the surface area of ∂B_1,
and we adapt the definition of ρ to the nonradial case:

$$\rho := -\left|\frac{\nabla u}{u}\right|^{p-2} \frac{\nabla u}{u}. \tag{3.1}$$

Theorem 3.1 *Let*

$$W(R) := \int_{B_R} \left(\frac{|\nabla u|^{p-2} \, |\nabla u \cdot \mathbf{n}|}{|u|^{p-1}} \right)^{\frac{p}{p-1}}.$$

Then have the following estimates:

$$W(R) \leq (p-1)^{-p}(N-p)^{p-1} \omega_N R^{N-p},$$

$$\int_{B_R} \left| \frac{\partial \ln u}{\partial r} \right|^p \leq (p-1)^{-p}(N-p)^{p-1} \omega_N R^{N-p}.$$

Proof: The second estimate is merely a simplification and slight weakening of the first, since $\nabla u \cdot \mathbf{n} = \frac{\partial u}{\partial r}$. Inequality (2.1) may be rewritten as:

$$\nabla \cdot \rho \geq V + (p-1)\, |\rho|^{p/(p-1)} \,. \tag{3.2}$$

By integration and by Gauß's divergence theorem we have

$$\int_{\partial B_r} \rho \cdot \mathbf{n} \geq \int_{B_r} V + (p-1) \int_{B_r} |\rho|^{p/(p-1)} \,. \tag{3.3}$$

Set

$$g(r) = \int_{B_r} V \tag{3.4}$$

and

$$U(r) = \int_{\partial B_r} |\rho \cdot \mathbf{n}|^{p/(p-1)}. \tag{3.5}$$

It follows from (3.3) that

$$\int_{\partial B_r} \rho \cdot \mathbf{n} \geq g(r) + (p-1) \int_0^r \int_{\partial B_s} [|\rho \cdot \mathbf{n}|^2 + \rho_\tau^2]^{p/2(p-1)},$$

where ρ_τ designates the tangential component of ρ, i.e., $\rho_\tau = \rho - (\rho \cdot \mathbf{n})\mathbf{n}$.

$$\int_{\partial B_r} \rho \cdot \mathbf{n} \geq g(r) + (p-1) \int_0^r \int_{\partial B_s} |\rho \cdot \mathbf{n}|^{p/(p-1)}. \tag{3.6}$$

Set $U = W'$, so that

$$W(r) = \int_0^r U(s) ds. \tag{3.7}$$

By Hölder's inequality and (3.5),

$$\int_{\partial B_r} \rho \cdot \mathbf{n} \leq [U(r)]^{(p-1)/p} \omega_N^{1/p} r^{(N-1)/p}.$$

Inequality (3.6) may be rewritten as

$$g(r) + (p-1)W(r) \leq \omega_N^{1/p} \left(W'(r)\right)^{(p-1)/p} r^{(N-1)/p}. \tag{3.8}$$

Since $V > 0$, $g > 0$, and hence

$$(p-1)W(r) \leq \omega_N^{1/p} \left(W'(r)\right)^{(p-1)/p} r^{(N-1)/p}.$$

Therefore, by integration, for any r and r_0 satisfying $r > r_0 > 0$, we have:

$$W(r)^{-(\frac{1}{p-1})} \leq (W(r_0))^{-(\frac{1}{p-1})} + (p-1)^{(\frac{p}{p-1})} (N-p)^{-1} \omega_N^{-(\frac{1}{p-1})} \big[r^{\frac{p-N}{p-1}} - r_0^{\frac{p-N}{p-1}} \big]. \tag{3.9}$$

Now we claim that

$$W(r) \leq K r^{N-p} \text{ for } r \geq r_0. \tag{3.10}$$

Assume for the purpose of contradiction that $W(r_0) > Kr_0^{N-p}$. Let r_1 be defined by

$$(p-1)^{(\frac{p}{p-1})}(N-p)^{-1}\omega_N^{-(\frac{1}{p-1})}\big(r_1^{\frac{p-N}{p-1}} - r_0^{\frac{p-N}{p-1}}\big) = (Kr_0^{N-p})^{-(\frac{1}{p-1})}.$$

We deduce from this and from (3.9) that

$$W^{(\frac{1}{p-1})}(r) \geq \Big[(p-1)^{(\frac{p}{p-1})}(N-p)^{-1}\omega_N^{-(\frac{1}{p-1})} r_1^{\frac{p-N}{p-1}}\Big]^{-1} \times \Big[r^{\frac{p-N}{p-1}} - r_1^{\frac{p-N}{p-1}}\Big]^{-1}.$$

It follows that there is some $r^* \in (r_0; r_1]$ such that $W(r^*) = \infty$, which is impossible since $W(r)$ is finite for all r. Hence (3.10) is proved with $K = (p-1)^{-p}(N-p)^{p-1}\omega_N$. We also conclude from (3.8) that

$$g(r) \leq \omega_N^{1/p}\,(W'(r))^{(p-1)/p}\, r^{(N-1)/p}.$$

If V grows as above, then $g(r) \geq r^{N-p}$, and

$$W'(r) \geq kr^{\frac{Np-p^2-N+1}{p-1}},$$

so

$$W(r) \geq A + kr^{N-p}. \tag{3.11}$$

□

As in the previous proof, let ρ_τ designate the tangential component of ρ. It can be controlled as follows:

Corollary 3.2 *Under the same conditions as in Theorem 3.1,*

$$\int_0^R \int_{B_r} |\rho_\tau|^{\frac{p}{p-1}} \leq KR^{N-p+1}.$$

Proof: From (3.3), we have

$$\int_{\partial B_r} \rho\cdot\mathbf{n} \geq (p-1)\int_{B_r} |\rho_\tau|^{(p/(p-1))}.$$

The desired estimate follows by integrating this from 0 to R and applying Hölder's inequality. □

Remark. If u is radially symmetric, and if ρ satisfies the estimate $\rho(r) \leq a_* r^{-(p-1)}$ with $a_* \leq \left(\frac{N-p}{p-1}\right)^{p-1}$, then by integration on B_r, we obtain

$$W(r) \leq (a_*)^{\frac{p}{p-1}} \int_0^r s^{-p+N-1}\omega_N ds \leq Kr^{N-p},$$

where $K = (a_*)^{\frac{p}{p-1}} \frac{\omega_N}{N-p} \leq \left(\frac{N-p}{p-1}\right)^p \frac{\omega_N}{N-p}$, which implies the result of Theorem 3.1 for u radial.

4 Decay Estimates Using the Uncertainty Principle

Our purpose in this section is to find decay estimates for Equation (1.1) to complement those of the previous section, which essentially apply to the partial differential inequality (2.1). In this section we pose the opposite inequality. Specifically, we shall assume, in contrast to (2.1), that

$$-\Delta_p u(x) \leq V(x)u^{p-1}(x), \quad x \in \mathbb{R}^N. \tag{4.1}$$

An important tool will be an L^p uncertainty principle from [9], which is analogous to Hardy's inequality as used in [7, 3, 9]. Indeed, like some other Hardy–type inequalities, it can be derived from an inequality of Boggio [1], as generalized and discussed in [9]. For further information about Hardy–type inequalities, see [13, 15] and references therein.

In what follows, k will denote positive constants with various values that we do not compute precisely. Henceforth let

$$c_p := \left(\frac{p}{N-p}\right)^{\left(\frac{p-1}{p}\right)}.$$

In [9] we generalized the uncertainty–principle lemma, classical when $p = 2$ (cf. [8, 11, 16]), to arbitrary $p < N$. Here we recall [9, Corollary II.4], and extend it by closure to the Sobolev space $\mathcal{D}^{1,p}(\mathbb{R}^N)$:

Lemma 4.1 *Assume that $p < N$. For any $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$, we have*

$$\int_{\mathbb{R}^N} \left|\frac{u}{r}\right|^p \leq c_p^p \int_{\mathbb{R}^N} |\nabla u|^p. \tag{4.2}$$

Proof: : In [9, Corollary II.4], (4.2) was established for functions in $\mathcal{C}_0^\infty(\mathbb{R}^N)$. For any $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$, there exists $u_n \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, tending to u with respect to the $\mathcal{D}^{1,p}$ norm. We infer that

$$\int_{\mathbb{R}^N} |\nabla u_n|^p \to \int_{\mathbb{R}^N} |\nabla u|^p,$$

and u_n converges to u strongly in L^{p^*}, and therefore *a.e.* in $\mathbb{R}^N$ (for a subsequence still denoted by u_n). Then from Fatou's Lemma,

$$\begin{aligned}\int_{\mathbb{R}^N} \lim\left|\frac{u}{r}\right|^p &\leq \limsup \int_{\mathbb{R}^N} \left|\frac{u_n}{r}\right|^p \\ &\leq c_p^p \limsup \int_{\mathbb{R}^N} |\nabla u_n|^p = c_p^p \int_{\mathbb{R}^N} |\nabla u|^p. \qquad \square\end{aligned}$$

Consider $u > 0$ in the Sobolev space $\mathcal{D}^{1,p}(\mathbb{R}^N)$, and satisfying (4.1). We assume further:

Hypothesis (H):

(1) $V = V(|x|) = V(r) > 0$ and $V \in L^{N/p}(\mathbb{R}^N)$, with $N > p$.

(2) There exists $\mu > p$ such that

$$V(r) \le \frac{k}{(1+r^2)^{\frac{\mu}{2}+\frac{p}{2c}}},$$

where $c := \hat{m}c_p$, and the positive constant $\hat{m} = \hat{m}(p,N)$ is defined in [9].

Theorem 4.2 *Assume Hypothesis* **(H)**. *Then there exists $k > 0$ such that for any $\varepsilon > 0$, any positive solution u of* (4.1) *satisfies the estimate:*

$$\int_{\{r>\left(\frac{1}{\varepsilon}\right)^c\}} \left(\frac{u}{r}\right)^p \le k\varepsilon^p \|u\|^p_{\mathcal{D}^{1,p}}.$$

Proof: Let φ be piecewise $\mathcal{C}^1$; by Lemma 3.1 of [9], we have

$$\int_{\mathbb{R}^N} |\nabla(\varphi u)|^p \le \hat{m}^p \int_{\mathbb{R}^N} |u\nabla\varphi|^p + \hat{k}\int_{\mathbb{R}^N} u|\varphi|^p(-\Delta_p u).$$

Here $\hat{k} = 2^{\frac{p-2}{2}} p^{2-p}(p-1)^{p-1}$. By Lemma 4.1 combined with the definition of c, we get

$$\int_{\mathbb{R}^N} \left|\frac{\varphi u}{r}\right|^p \le c^p \int_{\mathbb{R}^N} |u\nabla\varphi|^p + \hat{k}c_p^p \int_{\mathbb{R}^N} V|u\varphi|^p,$$

which is equivalent to

$$\int_{\mathbb{R}^N} \left(\left|\frac{\varphi}{r}\right|^p - c^p|\nabla\varphi|^p\right) u^p \le \hat{k}c_p^p \int_{\mathbb{R}^N} V|\varphi|^p u^p. \tag{4.3}$$

We define

$$\varphi(\mathbf{x}) := \varphi(r) := \min\left(r^{\frac{1}{c}}, \frac{1}{\varepsilon}\right). \tag{4.4}$$

By a computation, if $r < \left(\frac{1}{\varepsilon}\right)^c$, then

$$|\nabla\varphi| = \varphi'(r) = \frac{1}{c} r^{\frac{1}{c}-1} = \frac{1}{c}\frac{\varphi}{r}.$$

From (4.4) it follows that

$$\int_{\{r>\left(\frac{1}{\varepsilon}\right)^c\}} \left|\frac{\varphi}{r}\right|^p u^p \le \hat{k}c_p^p \int_{\mathbb{R}^N} V|\varphi|^p u^p. \tag{4.5}$$

Since $\mu > p$, Hypothesis **(H)** implies that

$$\begin{aligned}
\int_{\mathbb{R}^N} V|\varphi|^p u^p \;&\le\; k\int_{\mathbb{R}^N} \frac{u^p}{(1+r^2)^{\frac{\mu}{2}}} \\
&\le k\Big(\int_{\mathbb{R}^N} u^{p^*}\Big)^{p/p^*} \Big(\int_{\mathbb{R}^N} \frac{1}{(1+r^2)^{\frac{N\mu}{2p}}}\Big)^{p/N} \\
&\le k\|u\|^p_{L^{p^*}} \le k\|u\|^p_{\mathcal{D}^{1,p}}.
\end{aligned} \tag{4.6}$$

From (4.4) and (4.6), we derive

$$\int_{\{r>\left(\frac{1}{\varepsilon}\right)^c\}} \left(\frac{u}{r}\right)^p \leq k\varepsilon^p \|u\|^p_{\mathcal{D}^{1,p}}. \tag{4.7}$$

$\square$

Remark. To illustrate these bounds, assume that asymptotically $u(r) \simeq r^{-\alpha}$ as $r \to \infty$. Then by Theorem 4.2,

$$\int_{\varepsilon^{-c}}^{+\infty} r^{N-1-p(\alpha+1)} dr \leq k\varepsilon^p. \tag{4.8}$$

This implies that $N < p(\alpha + 1)$ and $\varepsilon^{cp(\alpha+1)-cN} \leq k\varepsilon^p$ and therefore $\alpha \geq \frac{1}{c} + \frac{N-p}{p}$.

Examples

(i) Suppose that $p = 2$ and $N = 3$; in this case $c_p = \sqrt{2}$, $\hat{m} = 1$; $c = \sqrt{2}$, $\hat{k} = 1$. Hence, from (4.8), $\alpha \geq \frac{1+\sqrt{2}}{2}$.

(ii) Suppose that $2 < p < N < 2p$; in this case $c_p = \left(\frac{p}{N-p}\right)^{\left(\frac{p-1}{p}\right)}$; $\hat{m} = (p-1)2^{\frac{p-2}{2p}}$; $c = \hat{m}c_p$. Hence from (4.8), $\alpha \geq \frac{1}{c} + \frac{N-p}{p}$.

Finally suppose that $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ is a positive solution of (4.1) with $V = V_1 - V_2$, where V_1 satisfies **(H)** and $V_2(r) \geq 0$ for any $r \geq 0$. Since $k \geq 1$, Equation (4.5) implies that

$$\begin{aligned}\int_{\{r>\left(\frac{1}{\varepsilon}\right)^c\}} V_2 \left(\frac{u}{\varepsilon}\right)^p &\leq k \int_{\mathbb{R}^N} V_2|\varphi|^p u^p + \int_{\{r>\left(\frac{1}{\varepsilon}\right)^c\}} \left|\frac{\varphi}{r}\right|^p u^p \\ &\leq k \int_{\mathbb{R}^N} V_1|\varphi|^p u^p \leq k.\end{aligned}$$

Hence we obtain the following:

Corollary 4.3 *Assume that $V = V_1 - V_2$, where V_1 satisfies* **(H)** *and $V_2(r) \geq 0$ for any $r \geq 0$. Then there exists $k > 0$ such that for any $\varepsilon > 0$, any positive solution u of* (4.1) *satisfies:*

$$\int_{\{r>(1/\varepsilon)^c\}} V_2 u^p \leq k\varepsilon^p.$$

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In Section 4, before Lemma 4.1, the constant c_p defined as

$$c_p := \Big(\frac{p}{N-p}\Big)^{(\frac{p-1}{p})}.$$

should be replaced by

$$c_p := \Big(\frac{p}{N-p}\Big)$$

(no exponent). This error propagated to the examples at the end of the article. Example (i) should read:

(i) Suppose that $p = 2$ and $N = 3$; in this case $c_p = 2$, $\hat{m} = 1$; $c = 2$, $\hat{k} = 1$. Hence, from (4.8), $\alpha \geq 1$.

In Example (ii), the expression

$$c_p := \Big(\frac{p}{N-p}\Big)^{(\frac{p-1}{p})}.$$

should be replaced by

$$c_p := \Big(\frac{p}{N-p}\Big).$$

(no exponent)

Jacqueline Fleckinger
CEREMATH & UMR MIP, Université Toulouse-1
Université Toulouse-1
21 allées de Brienne, 31000 Toulouse, France
e-mail: jfleck@univ-tlse1.fr

Evans M. Harrell II
School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-0160, USA
e-mail: harrell@math.gatech.edu

François de Thélin
UMR MIP, Université Paul Sabatier
31062 Toulouse, France
e-mail: dethelin@mip.ups-tlse.fr