On periodic solutions of superquadratic Hamiltonian systems

Guihua Fei

Abstract

We study the existence of periodic solutions for some Hamiltonian systems \( \dot{z} = JH_z(t, z) \) under new superquadratic conditions which cover the case \( H(t, z) = |z|^2(\ln(1 + |z|^p))^{q/2} \) with \( p, q > 1 \). By using the linking theorem, we obtain some new results.

1 Introduction

We consider the superquadratic Hamiltonian system

\[
\dot{z} = JH_z(t, z)
\]  

(1.1)

where \( H \in C^1([0, 1] \times \mathbb{R}^{2N}, \mathbb{R}) \) is a 1-periodic function in \( t \), \( J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix} \) is the standard \( 2N \times 2N \) symplectic matrix, and

\[
\frac{H(t, z)}{|z|^2} \to +\infty \text{ as } |z| \to +\infty \text{ uniformly in } t.
\]  

(1.2)

We assume \( H \) satisfies the following conditions.

(H1) \( H(t, z) \geq 0 \), for all \( (t, z) \in [0, 1] \times \mathbb{R}^{2N} \).

(H2) \( H(t, z) = o(|z|^2) \) as \( |z| \to 0 \) uniformly in \( t \).

In [12], Rabinowitz established the existence of periodic solutions for (1.1) under the following superquadratic condition: there exist \( \mu > 0 \) and \( r_1 > 0 \) such that for all \( |z| \geq r_1 \) and \( t \in [0, 1] \)

\[
0 < \mu H(t, z) \leq z \cdot H_z(t, z).
\]  

(1.3)

Since then, the condition (1.3) has been used extensively in the literature; see [1-14] and the references therein.

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It is easy to see that (1.3) does not include some superquadratic nonlinearity like
\[ H(t, z) = |z|^2 \ln(1 + |z|^p), \quad p, q > 1. \] (1.4)

In this paper, we shall study the periodic solutions of (1.1) under some superquadratic conditions which cover the cases like (1.4). We assume \( H \) satisfies the following condition.

(H3) There exist constants \( \beta > 1, 1 < \lambda < 1 + \frac{\beta - 1}{\beta}, c_1, c_2 > 0 \) and \( L > 0 \) such that
\[
\begin{align*}
z \cdot H_z(t, z) - 2H(t, z) &\geq c_1 |z|^\beta, \quad \forall |z| \geq L, \forall t \in [0, 1]; \\
|H_z(t, z)| &\leq c_2 |z|^\lambda, \quad \forall |z| \geq L, \forall t \in [0, 1].
\end{align*}
\]

**Theorem 1.1** Suppose \( H \in C^1([0, 1] \times \mathbb{R}^{2N}, \mathbb{R}) \) is 1-periodic in \( t \) and satisfies (1.2), (H1)–(H3). Then (1.1) possesses a nonconstant 1-periodic solution.

A straightforward computation shows that if \( H \) satisfies (1.4), for any \( T > 0 \), the system (1.1) has a nonconstant \( T \)-periodic solution with minimal period \( T \).

One can see Remark 2.2 and Corollary 2.3 for more examples.

For the second order Hamiltonian system
\[
\ddot{u}(t) + V'(t, u(t)) = 0, \\
\dot{u}(0) - u(1) = \ddot{u}(0) - \dot{u}(1) = 0
\] (1.5)
we have a similar result.

**Theorem 1.2** Suppose \( V \in C^1([0, 1] \times \mathbb{R}^N, \mathbb{R}) \) is 1-periodic in \( t \) and satisfies

(V1) \( V(t, x) \geq 0, \) for all \((t, x) \in [0, 1] \times \mathbb{R}^N\)

(V2) \( V(t, x) = o(|x|^2) \) as \( |x| \to 0 \) uniformly in \( t \)

(V3) \( V(t, x)/|x|^2 \to +\infty \) as \( |x| \to +\infty \) uniformly in \( t \)

(V4) There exist constants \( 1 < \lambda \leq \beta, d_1, d_2 > 0 \) and \( L > 0 \) such that
\[
\begin{align*}
x \cdot V'(t, x) - 2V(t, x) &\geq d_1 |x|^\beta, \quad \forall |x| \geq L, \forall t \in [0, 1]; \\
|V'(t, x)| &\leq d_2 |x|^\lambda, \quad \forall |x| \geq L, \forall t \in [0, 1].
\end{align*}
\] (1.6)

( or \( V(t, x) \leq d_2 |x|^\lambda + 1, \forall |x| \geq L, \forall t \in [0, 1] \)). (1.7)

Then (1.5) possesses a nonconstant 1-periodic solution.

We shall use the linking theorem [13, Theorem 5.29] to prove our results. The idea comes from [11, 12, 13]. Theorem 1.1 is proved in Section 2 while the proof of Theorem 1.2 is carried out in Section 3.
2 First order Hamiltonian system

Let $S^1 = \mathbb{R}/(2\pi \mathbb{Z})$ and $E = W^{1/2,2}(S^1, \mathbb{R}^N)$. Then $E$ is a Hilbert space with norm $\| \cdot \|$ and inner product $\langle \cdot, \cdot \rangle$. We define

$$\langle Ax, y \rangle = \int_0^1 \langle -J \dot{x}, y \rangle dt, \quad \forall x, y \in E; \quad (2.1)$$

$$f(z) = \frac{1}{2} \langle Az, z \rangle - \int_0^1 H(t, z) dt, \quad \forall z \in E. \quad (2.2)$$

Then $A$ is a bounded selfadjoint operator and $\ker A = \mathbb{R}^{2N}$. (H1)–(H3) imply that

$$|H(t, z)| \leq a_1 + a_2 |z|^{\lambda + 1}, \quad \forall z \in \mathbb{R}^{2N}.$$  

This implies that $f \in C^1(E, \mathbb{R})$ and looking for the solutions of (1.1) is equivalent to looking for the critical points of $f$ [12, 13]. Let $E^0 = \ker(A)$, $E^+ = $ positive definite subspace of $A$, and $E^- = $ negative definite subspace of $A$. Then $E = E^0 \oplus E^- \oplus E^+$.  

Lemma 2.1 Under the conditions of Theorem 1.1, $f$ satisfies the (PS) condition.  

Proof. Let $\{ z_m \}$ be a (PS)-sequence, i.e.,

$$|f(z_m)| \leq M; \quad f'(z_m) \to 0 \quad \text{as } m \to \infty.$$  

We want to show that $\{ z_m \}$ is bounded. Then by a standard argument, $\{ z_m \}$ has a convergent subsequence [13]. Suppose $\{ z_m \}$ is not bounded, then passing to a subsequence if necessary, $\| z_m \| \to +\infty$ as $m \to +\infty$. By (H3), there exists $C_3 > 0$ such that for all $z \in \mathbb{R}^{2N}$, $t \in [0, 1]$

$$z \cdot H_z(t, z) - 2H(t, z) \geq C_1 |z|^\beta - C_3.$$  

Therefore, we have

$$2f(z_m) - \langle f'(z_m), z_m \rangle = \int_0^1 [z_m \cdot H_z(t, z_m) - 2H(t, z_m)]dt \geq \int_0^1 [C_1 |z_m|^\beta - C_3]dt = C_1 \int_0^1 |z_m|^\beta dt - C_3.$$ 

This implies

$$\int_0^1 |z_m|^\beta dt \to 0 \quad \text{as } m \to \infty. \quad (2.3)$$

Note that from (H3), $1 < \lambda < 1 + \frac{\beta - 1}{\beta}$. Let $\alpha = \frac{\beta - 1}{\beta(\lambda - 1)}$. Then

$$\alpha > 1, \quad \alpha \lambda - 1 = \alpha - \frac{1}{\beta}. \quad (2.4)$$
By (H3), there exists $C_4 > 0$ such that
\[ |H_z(t, z)|^\alpha \leq C_4^\alpha |z|^\lambda + C_4, \quad \forall (t, z) \in [0, 1] \times \mathbb{R}^N. \]

Denote $z_m = z_m^+ + z_m^- + z_m^0 \in E^+ \oplus E^- \oplus E^0$. We have
\[
\langle f'(z_m), z_m^+ \rangle = \langle Az_m^+, z_m^+ \rangle - \int_0^1 [H_z(t, z_m) \cdot z_m^+] dt \\
\geq \langle Az_m^+, z_m^+ \rangle - \int_0^1 |H_z(t, z_m)|z_m^+| dt \\
\geq \langle Az_m^+, z_m^+ \rangle - \int_0^1 |H_z(t, z_m)|^\alpha \frac{1}{\alpha} \cdot C_\alpha \|z_m^+\|,
\]
where $C_\alpha > 0$ is a constant independent of $m$. By (2.5),
\[
\int_0^1 |H_z(t, z_m)|^\alpha dt \leq \int_0^1 (C_2^\alpha |z_m|^\lambda + C_4) dt \\
\leq C_5 \int_0^1 |z_m|^\beta dt^{1/\beta} \left( \int_0^1 |z_m|^{(\lambda-1)} \frac{\alpha}{\lambda} dt \right)^{1-\frac{1}{\beta}} + C_4 \\
\leq C_6 \left( \int_0^1 |z_m|^\beta \right)^{1/\beta} \|z_m\|^{(\lambda-1)} + C_4.
\]
Combining this inequality with (2.3) and (2.4) yields that
\[
\frac{\alpha}{\lambda} \left( \frac{\int_0^1 |H_z(t, z_m)|^\alpha dt}{\|z_m\|} \right)^{\frac{1}{\alpha}} \leq \frac{C_6 \left( \int_0^1 |z_m|^\beta dt \right)^{1/\beta}}{\|z_m\|^{\frac{(\lambda-1)}{\lambda}}} \cdot \frac{\|z_m\|^{(\lambda-1)}}{\|z_m\|^{\frac{(\lambda-1)}{\lambda}}} + \frac{C_4 \|z_m^+\|^{\frac{1}{\alpha}}}{\|z_m^+\|^{\frac{1}{\alpha}}} \to 0
\]
as $m \to \infty$. By (2.6) we have
\[
\frac{\langle f'(z_m), z_m^+ \rangle}{\|z_m\| \|z_m^+\|} \leq \frac{\|f'(z_m)\|}{\|z_m\| \|z_m^+\|} + \left( \frac{\int_0^1 |H_z(t, z_m)|^\alpha dt}{\|z_m\|} \right)^{\frac{1}{\alpha}} \cdot \frac{C_\alpha \|z_m^+\|}{\|z_m^+\|} \to 0
\]
as $m \to \infty$. This implies
\[
\frac{\|z_m^+\|}{\|z_m\|} \to 0 \quad \text{as } m \to \infty. \quad (2.7)
\]
Similary, we have
\[
\frac{\|z_m^-\|}{\|z_m\|} \to 0 \quad \text{as } m \to \infty. \quad (2.8)
\]
By (H3) there exist $C_7, C_8 > 0$ such that
\[ z \cdot H_z(t, z_m) - 2H(t, z) \geq C_7 |z| - C_8, \quad \forall (t, z) \in [0, 1] \times \mathbb{R}^N. \]
This implies
\[
2f(z_m) - \langle f'(z_m), z_m \rangle = \int_0^1 [z_m \cdot H_z(t, z_m) - 2H(t, z_m)]dt \geq \int_0^1 |C_\gamma|z_m| - C_0|dt
\]
\[
\geq \int_0^1 |C_\gamma|z_m^0| - C_\gamma|z_m^+| - C_\gamma|z_m^-| - C_0|dt
\]
\[
\geq C_0\|z_m^0\| - C_10(\|z_m^+\| + \|z_m^-\| + 1).
\]
Therefore, by (2.7) and (2.8)
\[
\frac{\|z_m^0\|}{\|z_m\|} \to 0 \quad \text{as } m \to \infty.
\]
Combine this with (2.7) and (2.8), we get
\[
\frac{\|z_m^0\|}{\|z_m\|} \to 0 \quad \text{as } m \to \infty,
\]
a contradiction. Therefore, \{z_m\} must be bounded. □

**Proof of Theorem 1.1** We prove that \( f \) satisfies the conditions of Theorem 5.29 in [13].

**Step 1:** By (H1)–(H3), we have
\[
H(t, z) \leq a_1 + a_2|z|^\lambda + 1, \quad \forall (t, z) \in [0, 1] \times \mathbb{R}^{2N}.
\]
By (H2), for any \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that
\[
H(t, z) \leq \varepsilon|z|^2, \quad \forall t \in [0, 1], \ |z| \leq \delta.
\]
Therefore, there exists \( M = M(\varepsilon) > 0 \) such that
\[
H(t, z) \leq \varepsilon|z|^2 + M|z|^\lambda + 1, \quad \forall (t, z) \in [0, 1] \times \mathbb{R}^{2N}.
\]
Note that \( \lambda + 1 > 2 \). By the same arguments as in [13, Lemma 6.16], there exist \( \rho > 0 \) and \( \tilde{a} > 0 \), such that for \( z \in E_1 = E^+ \)
\[
f(z) \geq \tilde{a} \quad \text{if } \|z\| = \rho,
\]
i.e., \( f \) satisfies \((I_7)(i)\) in [13, Theorem 5.29] with \( S = \partial B_\rho \cap E_1 \).

**Step 2:** Let \( e \in E^+ \) with \( \|e\| = 1 \) and \( \tilde{E} = E^- \oplus E^0 \oplus \text{span}\{e\} \). We denote
\[
K = \{z \in \tilde{E} : \|z\| = 1\}, \quad \lambda^- = \inf_{z \in E^- : \|z\| = 1} |\langle Az^-, z^- \rangle|, \quad \gamma = (\frac{\|A\|}{\lambda^-})^{1/2}.
\]
For \( z \in K \), we write \( z = z^- + z^0 + z^+ \in \tilde{E} \).

i) If \( \|z^-\| > \gamma\|z^+ + z^0\| \), by (H1) we have, for any \( r > 0 \),
\[
f(rz) = \frac{1}{2} \langle Arz^-, rz^- \rangle + \frac{1}{2} \langle Arz^+, rz^+ \rangle - \int_0^1 H(t, z)dt
\]
\[
\leq -\frac{1}{2} \lambda^- r^2\|z^-\|^2 + \frac{1}{2} \|A\|r^2\|z^+\|^2 \leq 0.
\]
ii) If \( \|z^-\| \leq \gamma \|z^+ + z^0\| \), we have
\[
1 = \|z\|^2 = \|z^-\|^2 + \|z^+ + z^0\|^2 \leq (1 + \gamma^2) \|z^+ + z^0\|^2,
\]
i.e.,
\[
\|z^+ + z^0\|^2 \geq \frac{1}{1 + \gamma^2} > 0.
\] (2.9)

Denote \( \tilde{K} = \{ z \in K : \|z^-\| \leq \gamma \|z^+ + z^0\| \} \).

**Claim:** There exists \( \varepsilon_1 > 0 \) such that, \( \forall u \in \tilde{K} \),
\[
\text{meas}\{ t \in [0, 1] : |u(t)| \geq \varepsilon_1 \} \geq \varepsilon_1.
\] (2.10)

For otherwise, \( \forall k > 0 \), \( \exists u_k \in \tilde{K} \) such that
\[
\text{meas}\{ t \in [0, 1] : |u_k(t)| \geq \frac{1}{k} \} < \frac{1}{k}.
\] (2.11)

Write \( u_k = u_k^- + u_k^0 + u_k^+ \in \tilde{E} \). Notice that \( \dim(E^0 \oplus \text{span}\{e\}) < +\infty \) and \( \|u_k^0 + u_k^+\| \leq 1 \). In the sense of subsequence, we have
\[
u_k^0 + u_k^+ \rightarrow u_0^0 + u_0^+ \in E^0 \oplus \text{span}\{e\} \quad \text{as} \quad k \rightarrow +\infty.
\]

Then (2.9) implies that
\[
\|u_0^0 + u_0^+\|^2 \geq \frac{1}{\gamma^2 + 1} > 0.
\] (2.12)

Note that \( \|u_k^-\| \leq 1 \), in the sense of subsequence \( u_k^- \rightarrow u_0^- \in E^- \) as \( k \rightarrow +\infty \). Thus in the sense of subsequences,
\[
u_k \rightarrow u_0 = u_0^- + u_0^0 + u_0^+ \quad \text{as} \quad k \rightarrow +\infty.
\]

This means that \( u_k \rightarrow u_0 \) in \( L^2 \), i.e.,
\[
\int_0^1 |u_k - u_0|^2 dt \rightarrow 0 \quad \text{as} \quad k \rightarrow +\infty.
\] (2.13)

By (2.12) we know that \( \|u_0\| > 0 \). Therefore, \( \int_0^1 |u_0|^2 dt > 0 \). Then there exist \( \delta_1 > 0 \), \( \delta_2 > 0 \) such that
\[
\text{meas}\{ t \in [0, 1] : |u_0(t)| \geq \delta_1 \} \geq \delta_2.
\] (2.14)

Otherwise, for all \( n > 0 \), we must have
\[
\text{meas}\{ t \in [0, 1] : |u_0(t)| \geq \frac{1}{n} \} = 0, \quad \text{i.e.,} \quad \text{meas}\{ t \in [0, 1] : |u_0(t)| < \frac{1}{n} \} = 1;
\]
\[
0 < \int_0^1 |u_0|^2 dt < \frac{1}{n^2} \cdot 1 \rightarrow 0 \quad \text{as} \quad n \rightarrow +\infty.
\]
We get a contradiction. Thus (2.14) holds. Let \( \Omega_0 = \{ t \in [0, 1] : |u_0(t)| \geq \delta_1 \} \), 
\( \Omega_k = \{ t \in [0, 1] : |u_k(t)| < 1/k \} \), and \( \Omega_k^+ = [0, 1] \setminus \Omega_k \). By (2.11), we have

\[
\text{meas}(\Omega_k \cap \Omega_0) = \text{meas}(\Omega_0 - \Omega_0 \cap \Omega_k^+) \geq \text{meas}(\Omega_0) - \text{meas}(\Omega_0 \cap \Omega_k^+) \geq \delta_2 - \frac{1}{k}.
\]

Let \( k \) be large enough such that \( \delta_2 = \frac{1}{k} \geq \frac{1}{2} \delta_2 \) and \( \delta_1 = \frac{1}{k} \geq \frac{1}{2} \delta_1 \). Then we have

\[
|u_k(t) - u_0(t)|^2 \geq (\delta_1 - \frac{1}{k})^2 \geq (\frac{1}{2} \delta_1)^2, \quad \forall t \in \Omega_k \cap \Omega_0.
\]

This implies that

\[
\int_0^1 |u_k(t) - u_0(t)|^2 dt \geq \int_{\Omega_k \cap \Omega_0} |u_k - u_0|^2 dt \geq (\frac{1}{2} \delta_1)^2 \cdot \text{meas}(\Omega_k \cap \Omega_0)
\]
\[
\geq (\frac{1}{2} \delta_1)^2 \cdot (\delta_2 - \frac{1}{k}) \geq (\frac{1}{2} \delta_1)^2 \cdot (\frac{1}{2} \delta_2) > 0.
\]

This is a contradiction to (2.13). Therefore the claim is true and (2.10) holds.

For \( z = z^- + z^0 + z^+ \in \tilde{K} \), let \( \Omega_2 = \{ t \in [0, 1] : |z(t)| \geq \varepsilon_1 \} \). By (1.2), for 
\( M = \frac{\|A\|}{\varepsilon_1} > 0 \), there exists \( L_1 > 0 \) such that

\[
H(t, x) \geq M|x|^2, \quad \forall |x| \geq L_1, \text{ uniformly in } t.
\]

Choose \( r_1 \geq L_1/\varepsilon_1 \). For \( r \geq r_1 \),

\[
H(t, rz(t)) \geq M|rz(t)|^2 \geq Mr^2\varepsilon_1^2, \quad \forall t \in \Omega_2.
\]

By (H1), for \( r \geq r_1 \)

\[
f(rz) = \frac{1}{2} r^2 \langle Az^+ + z^+ \rangle + \frac{1}{2} r^2 \langle Az^- + z^- \rangle - \int_0^1 H(t, rz) dt
\]
\[
\leq \frac{1}{2} \|A\|^2 r^2 - \int_{\Omega_2} H(t, rz) dt \leq \frac{1}{2} \|A\|^2 r^2 - M r^2 \varepsilon_1^2 \cdot \text{meas}(\Omega_2)
\]
\[
\leq \frac{1}{2} \|A\|^2 r^2 - M \varepsilon_1^2 r^2 = -\frac{1}{2} \|A\|^2 r^2 < 0.
\]

Therefore, we have proved that

\[
f(rz) \leq 0, \quad \text{for any } z \in K \text{ and } r \geq r_1.
\]

Let \( E_2 = E^- \oplus E^0 \), \( Q = \{ re : 0 \leq r \leq 2r_1 \} \oplus \{ z \in E_2 : \|z\| \leq 2r_1 \} \). By (H1) and (2.16) we have \( f|_{\partial Q} \leq 0 \), i.e., \( f \) satisfies (I7)(ii) in [13, Theorem 5.29].

**Step 3:** By Lemma 2.1, \( f \) satisfies the (PS) condition. Similar to the proof of [13, Theorem 6.10], by the linking theorem [13, Theorem 5.29], there exists a critical point \( z^* \in E \) of \( f \) such that \( f(z^*) \geq \bar{a} > 0 \). Moreover, \( z^* \) is a classical solution of (1.1) and \( z^* \) is nonconstant by (H1). \( \square \)
Remark 2.2 i) Suppose \( H(t, z) = \frac{1}{2}(B(t)z, z) + \tilde{H}(t, z) \) with \( B(t) \) being a \( 2N \times 2N \) matrix, continuous and 1-periodic in \( t \) and \( \tilde{H}(t, z) \) satisfies (1.2) and (H1)-(H3). We have the same conclusion as Theorem 1.1. The proof is similar and we omit it.

ii) Suppose \( H(t, z) = H(z) \) is independent on \( t \), i.e., (1.1) is an autonomous Hamiltonian system. Then under similar conditions as (1.2) and (H1)-(H3), for any \( T > 0 \), the system (1.1) has a nonconstant \( T \)-periodic solution. Moreover, if \( H(z) \in C^2(\mathbb{R}^{2N}, \mathbb{R}) \) and satisfies some strictly convex conditions such as \( H''(x) \) is positive definite for \( x \neq 0 \), then for any \( T > 0 \), (1.1) has a nonconstant \( T \)-periodic solution with minimal period \( T \). We omit the proof which is similar to the one in [4, 5].

iii) Suppose (1.4) holds, i.e.,

\[
H(t, z) = H(z) = |z|^2(\ln(1 + |z|^p))^{q-1}, \quad \forall (t, z) \in [0, 1] \times \mathbb{R}^{2N},
\]

where \( p > 1 \) and \( q > 1 \). Obviously, (1.2), (H1) and (H2) hold. Note that

\[
z \cdot H_z(z) - 2H(z) = |z|^2q(\ln(1 + |z|^p))^{q-1} \frac{p|z|^p}{1 + |z|^p} \geq |z|^2q(\ln 2)^{q-1}, \quad \forall |z| \geq 1.
\]

\[
|H_z(z)| \leq 2(\ln(1 + |z|^p))|z| + \frac{p|z|^p}{1 + |z|^p}q(\ln(1 + |z|^p))^{q-1}|z| \leq 2|z|^\frac{q}{2}, \quad \forall |z| \geq L,
\]

for \( L \) being large enough. This implies (H3). By directly computation, \( H''(z) \) is positive definite for \( z \neq 0 \). Therefore, for any \( T > 0 \), (1.1) possesses a \( T \)-periodic solution with minimal period \( T \).

iv) There are many examples which satisfy (H1)-(H3) and (1.2) but do not satisfy (1.3). For example

\[
H(t, z) = |z|^2 \ln(1 + |z|^2) \ln(1 + 2|z|^3).
\]

Corollary 2.3 Suppose \( H(t, z) = |z|^2h(t, z) \) with \( h \in C^1([0, 1] \times \mathbb{R}^{2N}, \mathbb{R}) \) being 1-periodic in \( t \) and satisfies

\((H'_1)\) \( h(t, z) \geq 0 \), for all \((t, z) \in [0, 1] \times \mathbb{R}^{2N}\).

\((H'_2)\) \( h(t, z) \to 0 \) as \(|z| \to 0\); \( h(t, z) \to +\infty \) as \(|z| \to +\infty\).

\((H'_3)\) There exist \( 0 \leq \delta < 1 \), \( L > 0 \), \( \varepsilon_0 > 0 \) and \( M > 0 \) such that

\[
|z|^{\delta}h_z(t, z) \cdot z \geq \varepsilon_0, \quad |z|h_z(t, z) \leq Mh, \quad \forall |z| \geq L;
\]

\[
\frac{h(t, z)}{|z|^\gamma} \to 0 \quad \text{as} \quad |z| \to \infty \quad \text{for any} \quad \gamma > 0.
\]

Then system (1.1) possesses a nonconstant 1-periodic solution.

Proof Obviously, \((H'_1) - (H'_3)\) imply (1.2), (H1) and (H2).

\[
z \cdot H_z(t, z) - 2H(t, z) = |z|^2|h_z(t, z) \cdot z | \geq \varepsilon_0|z|^{2-\delta}, \quad \forall |z| \geq L;
\]

\[
|H_z(t, z)| \leq |2h(t, z)||z| + |z|^{\delta}|h_z(t, z)|
\]

\[
\leq (2 + M)|z|h(t, z) \leq (2 + M)|z|^{1+\gamma}, \quad \forall |z| \geq L'.
\]
Let $\beta = 2 - \delta$ and $\lambda = 1 + \gamma$ with $0 < \gamma < (1 - \delta)/(2 - \delta)$. Then (H3) holds. By Theorem 1.1 we get the conclusion. □

3 Second order Hamiltonian System

Let $E = W^{1,2}(S^1, \mathbb{R}^N)$ with the norm $\| \cdot \|$ and inner product $(\cdot, \cdot)$. Then $E \subset C(S^1, \mathbb{R}^N)$ and $\|u\|^2 = \int_0^1 (|\dot{u}|^2 + |u|^2)dt$. Define

$$(Kx, y) = \int_0^1 x \cdot ydt, \quad \forall x, y \in E;$$

$$f(z) = \frac{1}{2}((id - K)z, z) - \int_0^1 V(t, z)dt, \quad \forall z \in E.$$  

Then $K$ is compact, $\ker(id - K) = \mathbb{R}^N$, and the negative definite subspace of $id - K$, $M^-(id - K) = \{0\}$, i.e., $E = E^0 \oplus E^+$ where $E^0 = \ker(id - K)$ and $E^+$ is the positive definite subspace of $id - K$. Note that (V1)–(V4) imply

$$V(t, x) \leq d_2 |x|^\lambda + d_3. \quad (3.1)$$

This implies that $f \in C^1(E, \mathbb{R})$ and critical points of $f$ are 1-periodic solutions of (1.5) [11].

Lemma 3.1 Suppose (V1)–(V4) hold. Then $f$ satisfies the (PS) condition.

Proof Let $\{z_m\}$ be a (PS) sequence. Suppose $\{z_m\}$ is not bounded. Passing to a subsequence if necessary, $\|z_m\| \to +\infty$ as $m \to \infty$. Then by (V4)

$$2f(z_m) - \langle f'(z_m), z_m \rangle = \int_0^1 [z_m \cdot V'(t, z_m) - 2V(t, z_m)]dt \geq d_1 \int_0^1 |z_m|^{\beta}dt - d_4.$$  

This implies

$$\frac{\int_0^1 |z_m|^\beta dt}{\|z_m\|} \to 0 \quad \text{as } m \to +\infty.$$  

If (1.6) holds, we have

$$\langle f'(z_m), z_m^+ \rangle = \langle (id - K)z_m^+, z_m^+ \rangle - \int_0^1 V'(t, z_m) \cdot z_m^+ dt \geq \langle (id - K)z_m^+, z_m^+ \rangle - \|z_m\| \int_0^1 |V'(t, z_m)|dt \geq \langle (id - K)z_m^+, z_m^+ \rangle - d_5 \|z_m\| (\int_0^1 |z_m|^\lambda dt + d_6).$$  

Since $\lambda \leq \beta$, we have

$$\frac{\|z_m^+\|}{\|z_m\|} \to 0 \quad \text{as } m \to +\infty. \quad (3.2)$$
If (1.7) holds, we have

\[
f(z_m) = \frac{1}{2} \langle (id-K)z_m^+, z_m^- \rangle - \int_0^1 V(t, z_m)dt \\
\geq \frac{1}{2} \langle (id-K)z_m^+, z_m^- \rangle - d_5 \int_0^1 |z_m|^{1+\lambda} dt - d_7 \\
\geq \langle (id-K)z_m^+, z_m^- \rangle - d_8 \|z_m\| \int_0^1 |z_m|^{1+\lambda} dt - d_7.
\]

Since \(\lambda \leq \beta\), we obtain (3.2). On the other hand, (V1)–(V4) imply

\[
x \cdot V'(t, x) - 2V(t, x) \geq d_9 |x| - d_{10}, \quad \forall t \in S^1 \times \mathbb{R}^N.
\]

Choose \(e \in E^+\) with \(\|e\| = 1\). Let \(\tilde{E} = \text{span}\{e\} \oplus E^0\) and \(K = \{u \in \tilde{E} : \|u\| = 1\}\). Note that \(\dim \tilde{E} < +\infty\). By using similar arguments as in the proof of (2.10), there exists \(\varepsilon_1 > 0\) such that

\[
\text{meas}\{t \in [0, 1] : |u(t)| \geq \varepsilon_1\} \geq \varepsilon_1, \quad \forall u \in K.
\]

By (3.2) and (3.3), we get a contradiction. Therefore \(\{z_m\}\) is bounded. By a standard argument, \(\{z_m\}\) has a convergent subsequence [11].

**Proof of Theorem 1.2** As in Step 1 of the proof of Theorem 1.1, by (V2) and (3.1), there exist \(\tilde{a} > 0, \rho > 0\) such that

\[
f(z) \geq \tilde{a}, \quad \forall z \in E^+ \quad \text{with } \|z\| = \rho.
\]

Choose \(e \in E^+\) with \(\|e\| = 1\). Let \(\tilde{E} = \text{span}\{e\} \oplus E^0\) and \(K = \{u \in \tilde{E} : \|u\| = 1\}\). Note that \(\dim \tilde{E} < +\infty\). By using similar arguments as in the proof of (2.10), there exists \(\varepsilon_1 > 0\) such that

\[
\text{meas}\{t \in [0, 1] : |u(t)| \geq \varepsilon_1\} \geq \varepsilon_1, \quad \forall u \in K.
\]

By (V1), (V3) and similar arguments as in the proof of Theorem 1.1, there exists \(r_1 > 0\) such that

\[
f|_{\partial Q} \leq 0, \quad \text{where } \ Q = \{re : 0 \leq r \leq 2r_1\} \oplus \{z \in E^0 : \|z\| \leq 2r_1\}.
\]

Now by Lemma 3.1, [13, Theorem 5.29], and (V1), \(f\) has a nonconstant critical point \(z^*\) such that \(f(z^*) \geq \tilde{a} > 0\). \(z^*\) is 1-periodic solution of (1.5).
Remark 3.2  (i) Suppose $V(t,x) = V(x)$ is independent on $t$ and $V(x)$ satisfies (V1)–(V4). Then for any $T > 0$, (1.5) possesses a nonconstant $T$-periodic solution.

(ii) There are many examples which satisfy (V1)–(V4) but do not satisfy a condition similar to (1.3). For example,

\[ V(t,x) = [1 + (\sin 2\pi t)^2] \cdot |x|^2 \ln(1 + 2|x|^2); \quad \text{or} \]
\[ V(t,x) = |x|^2 \ln(1 + |x|^2) \ln(1 + 2|x|^4). \]

By using similar arguments as in the proof of Theorem 1.2, we can prove the following corollary. Details are ommited.

Corollary 3.3  Suppose $V(t,x) = |x|^2 h(t,x)$ with $h \in C^1(S^1 \times \mathbb{R}^N, \mathbb{R})$ satisfies (V1'), $h(t,x) \geq 0, \ \forall (t,x) \in S^1 \times \mathbb{R}^N$.

(V2') $h(t,x) \to 0$ as $|x| \to 0$; \quad $h(t,x) \to +\infty$ as $|x| \to +\infty$.

(V3') There exist $L > 0, \lambda > 0, C_1, C_2 > 0$ such that for $t \in S^1$

\[ C_1|x|(h'(t,x) \cdot x) \geq h(t,x), \quad h(t,x) \leq C_2|x|^\lambda, \quad \forall |x| \geq L. \]

Then (1.5) possesses a nonconstant 1-periodic solution.

References


Guihua Fei
Department of Mathematics and statistics
University of Minnesota
Duluth, MN 55812, USA
e-mail : gfei@d.umn.edu