Electronic Journal of Differential Equations, Vol. 2002(2002), No. 100, pp. 1–30.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

ON A PARABOLIC-HYPERBOLIC PENROSE-FIFE PHASE-FIELD SYSTEM

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Abstract. The initial and boundary value problem is studied for a non-conserved phase-field system derived from the Penrose-Fife model for the kinetics of phase transitions. Here the evolution of the order parameter is governed by a nonlinear hyperbolic equation which is characterized by the presence of an inertial term with small positive coefficient. This feature is a consequence of the assumption that the response of the phase variable to the generalized force which drives the system toward equilibrium states is not instantaneous but delayed. The resulting model consists of a nonlinear parabolic equation for the absolute temperature coupled with the hyperbolic equation for the phase. Existence of a weak solution is obtained as well as the convergence of any family of weak solutions of the parabolic-hyperbolic model to the weak solution of the standard Penrose-Fife phase-field model as the inertial coefficient goes to zero. In addition, continuous dependence estimates are proved for the parabolic-hyperbolic system as well as for the standard model.

1. Introduction

Penrose and Fife [23, 24, 7] proposed a thermodynamically consistent model to describe the kinetics of phase transitions. In this framework, one is led to formulate a system of nonlinear partial differential equations that governs the evolution of the absolute temperature $\theta : Q_T := \Omega \times (0,T) \to \mathbb{R}$ and of the order parameter $\chi : Q_T \to \mathbb{R}$. Here $T > 0$ is a reference time and $\Omega \subset \mathbb{R}^N$, $N \leq 3$, is a bounded domain with a smooth boundary Γ. When χ is non-conserved, in absence of mechanical stresses and/or convective motions, the Penrose-Fife system has the following form [23]

$$(\theta + \lambda(\chi))_t - \Delta(-\theta^{-1}) = f \tag{1.1}$$

$$\omega\chi_t - \nu\Delta\chi + g(\chi) + \lambda'(\chi)\theta^{-1} = 0 \tag{1.2}$$

in Q_T. Here λ is a smooth function which may have quadratic growth so that second-order phase transitions can be taken into account (see, e.g., [2, Sec. 4.4]). In addition, the datum $f : Q_T \to \mathbb{R}$ represents the heat supply and function g is a third-degree polynomial function with positive leading coefficient: a well-known

2000 *Mathematics Subject Classification.* 35G25, 35Q99, 80A22.
Key words and phrases. Phase-field, Penrose-Fife model, existence of solutions, nonlinear partial differential equations, continuous dependence on the data.
©2002 Southwest Texas State University.
Submitted October 11, 2002. Published November 26, 2002.
Pages 30-32 contain an erratum submitted on March 31, 2003.

example of g comes from the derivative of an oriented double-well potential and reads $g(r) = r^3 - r - \theta_c^{-1}$, $r \in \mathbb{R}$, where $\theta_c > 0$ is the critical temperature around which the phase transition occurs. Moreover, $\omega > 0$ is a time relaxation parameter and $\nu > 0$ is a correlation length.

A typical initial and boundary value problem that can be associated with (1.1)-(1.2) consists of the usual initial conditions

$$\theta(0) = \theta_0, \quad \chi(0) = \chi_0 \tag{1.3}$$

in Ω, along with the boundary conditions

$$(-\theta^{-1})_{\mathbf{n}} - \gamma\theta^{-1} = h, \tag{1.4}$$

$$\chi_{\mathbf{n}} = 0 \tag{1.5}$$

on $\Gamma_T := \Gamma \times (0, T)$ (see [16] and particularly [6, Introduction and Remark 4.8] for an in-depth discussion on condition (1.4)). Here the subscript $\mathbf{n}$ stands for the derivative with respect to the outward normal $\mathbf{n}$ to Γ, γ is a positive constant, and $h : \Gamma_T \to (-\infty, 0)$ is a known function. More precisely, h has the form $\gamma(-\theta_\Gamma^{-1})$, θ_Γ being the outside temperature at the boundary.

We thus obtain an initial and boundary value problem, namely (1.1)-(1.5), which has been widely investigated in the last decade (see, among others, [5, 6, 14, 15, 16, 17, 18, 19, 28, 29]).

System (1.1)-(1.2) reflects the balance equations of energy and momentum in terms of thermodynamic state variables and it is derived from a free energy functional $\mathcal{F}(\theta, \chi)$ in compliance with the basic laws of Thermodynamics. In particular, the phase-field equation (1.2) originates from the phenomenological assumption

$$\chi_t = -\frac{1}{\omega}\frac{\delta\mathcal{F}}{\delta\chi} \tag{1.6}$$

which is consistent with the second principle. Here, $\delta\mathcal{F}/\delta\chi$ denotes the functional derivative of $\mathcal{F}$ with respect to χ and has the form

$$\frac{\delta\mathcal{F}}{\delta\chi} = -\nu\Delta\chi + g(\chi) + \lambda'(\chi)\theta^{-1}. \tag{1.7}$$

This quantity may be considered as a generalized force which arises as a consequence of the tendency of the free energy to decay toward a minimum. Relationship (1.6) amounts to say that the response of χ to the generalized force is instantaneous. However, it has been recently supposed that in some situations the response of χ to the generalized force is subject to a delay expressed by a suitable time dependent relaxation kernel k (see [25, 26], cf. also [8, 10, 13, 21]). This means that (1.6) can be replaced by

$$\chi_t = -\int_{-\infty}^{t} k(t-s)\frac{\delta\mathcal{F}}{\delta\chi}(s)\,ds. \tag{1.8}$$

The simplest natural choice for the relaxation kernel is

$$k(t) = \frac{1}{\omega\mu}e^{-t/\mu} \quad t \geq 0$$

for some $\mu > 0$ sufficiently small. Notice that as $\mu \to 0$, then $k(t) \to \delta(t)/\omega$, where δ is the Dirac mass at zero, so that (1.8) formally reduces to (1.6). Differentiating equation (1.8) with respect to time, with k as above, we deduce

$$\mu\chi_{tt} + \chi_t + \frac{1}{\omega}\frac{\delta\mathcal{F}}{\delta\chi} = 0.$$

Hence, setting for simplicity $\omega = \nu = 1$, and recalling (1.7), we deduce the *hyperbolic* version of (1.2)

$$\mu \chi_{tt} + \chi_t - \Delta \chi + g(\chi) + \lambda'(\chi)\theta^{-1} = 0 \quad \text{in } Q_T. \tag{1.9}$$

It is interesting to point out that the presence of the inertial term $\mu \partial_{tt} \chi$ is also discussed in the analysis of dynamical phenomena around the critical region of the phase transition (see [22, Ch. 7]). In fact, even though one forgets about the interpretation of (1.9) as a special case of law (1.8), we underline that (1.9) may be actually considered as a direct time relaxation of (1.2), and thus worth to be investigated.

On account of our previous considerations, we can formally introduce the initial and boundary value problem

Problem $\mathbf{P}_\mu$. *Find a solution (θ, χ) to the system*

$$(\theta + \lambda(\chi))_t - \Delta(-\theta^{-1}) = f \quad \text{in } Q_T$$
$$\mu \chi_{tt} + \chi_t - \Delta \chi + g(\chi) + \lambda'(\chi)\theta^{-1} = 0 \quad \text{in } Q_T$$

that satisfies the initial and boundary conditions

$$\theta(0) = \theta_0, \quad \chi(0) = \chi_0, \quad \chi_t(0) = \chi_1 \quad \text{in } \Omega;$$
$$(-\theta^{-1})_{\mathbf{n}} - \gamma \theta^{-1} = h, \quad \chi_{\mathbf{n}} = 0 \quad \text{on } \Gamma_T.$$

The mathematical analysis of $\mathbf{P}_\mu$ is the main goal of this paper.

Note that, by linearizing the term θ^{-1} around the critical value θ_c^{-1}, we obtain a simplest version of $\mathbf{P}_\mu$ which has already been analyzed in some detail [9, 11, 12]. Here, our goal is to study problem $\mathbf{P}_\mu$, which look considerably more difficult due to the presence of the nonlinearity θ^{-1}. The main result is the existence of a weak solution in the case when λ has a quadratic growth. Then, we show any family of solutions (θ_μ, χ_μ) to $\mathbf{P}_\mu$ converges, as $\mu \downarrow 0$, to the weak solution to the corresponding initial and boundary value problem $\mathbf{P}_0$, associated with the standard Penrose-Fife system (1.1)-(1.2).

Our further results are concerned with continuous dependence estimates and uniqueness. We first obtain a (conditional) continuous dependence estimate which entails uniqueness for $N = 1$. Then, we show that $\mathbf{P}_\mu$ has a unique solution which continuously depends on the data provided that λ is linear. Finally, we report a continuous dependence estimate for $\mathbf{P}_0$ whose proof is inspired by some contracting arguments developed in [15].

2. Weak formulation and statements of the results

Before introducing the assumptions on λ, g, and on the data along with the weak formulation of $\mathbf{P}_\mu$, we need some notation.

We set

$$H := L^2(\Omega), \quad V := H^1(\Omega).$$

Consequently, we let V' be the dual space of V. As usual, we identify H with its dual space H' and we recall the continuous and dense embeddings

$$V \hookrightarrow H \equiv H' \hookrightarrow V'.$$

Moreover, we indicate by $(\cdot,\cdot)$ and $(\cdot,\cdot)_\Gamma$ the usual scalar products in H or H^N and in $L^2(\Gamma)$, respectively. Accordingly, the related norms will be indicated by $\|\cdot\|$ and $\|\cdot\|_\Gamma$. We will use the following scalar product in V

$$(\!(v_1,v_2)\!) := (\nabla v_1,\nabla v_2) + \gamma(v_1,v_2)_\Gamma \quad \forall\, v_1,v_2\in V.$$

However, just for the sake of convenience, the norms $\|\cdot\|_V$ and $\|\cdot\|_{V'}$ will be the usual ones instead of those associated with the scalar product defined above. The duality pairing between V' and V will be denoted by $\langle\cdot,\cdot\rangle$. We shall also use the notation $(1*a)(t)=\int_0^t a(s)\,ds$ for vector-valued functions a summable in $(0,T)$.

Our structural assumptions on λ and g read

(H1) $\lambda\in C^2(\mathbb{R})$
(H2) $\lambda''\in L^\infty(\mathbb{R})$
(H3) $g\in C^1(\mathbb{R})$
(H4) There exist $\tau_1,\tau_2>0$ such that $|g(r)|\le\tau_1|r|^3+\tau_2$ for all $r\in\mathbb{R}$
(H5) $\lim_{r\to\pm\infty} g(r)=\pm\infty$.

Remark 2.1. On account of (H3)-(H5), it turns out that g is allowed to be the derivative of a *multiple-well* potential.

Note that, by virtue of (H3)-(H5), there exists a primitive $\hat g$ of g such that

$$0\le\hat g(r)\le\tau_3|r|^4+\tau_4 \quad \forall\, r\in\mathbb{R} \tag{2.1}$$

for some positive constants τ_3 and τ_4. For instance, one can take

$$\hat g(r)=C_g+\int_{\alpha_0}^r g(s)ds \quad \forall\, r\in\mathbb{R} \tag{2.2}$$

where $\alpha_0\in\mathbb{R}$ is a fixed zero of g and $C_g\in\mathbb{R}$ is chosen accordingly.

As far as the data are concerned, we assume

(H6) $f\in L^2(0,T;L^p(\Omega))$
(H7) $h\in L^2(\Gamma_T)$
(H8) $h\le 0$ a.e. on Γ_T
(H9) $\theta_0\in L^p(\Omega)$, $\theta_0>0$ a.e. in Ω
(H10) $\ln\theta_0\in L^1(\Omega)$
(H11) $\chi_0\in V$
(H12) $\chi_1\in H$.

Here $p\in(\frac{6}{5},\frac{3}{2}]$. We observe that if $f\in L^2(0,T;L^q(\Omega))$ and $\theta_0\in L^q(\Omega)$ for some $q>3/2$, then (H6) and (H9) still hold.

We can now introduce the weak formulation of $\mathbf{P}_\mu$.

Problem $\mathbf{P}_\mu$. *Find $\theta\in H^1(0,T;V')\cap L^\infty(0,T;L^p(\Omega))$ and $\chi\in C^1([0,T];H)\cap C^0([0,T];V)$ such that*

$$\theta>0 \quad \text{a.e. in } Q \tag{2.3}$$

$$\theta^{-1}\in L^2(0,T;V) \tag{2.4}$$

$$\langle(\theta+\lambda(\chi))_t,v\rangle+(\!(-\theta^{-1},v)\!)=\langle f,v\rangle+(h,v)_\Gamma \quad \forall\, v\in V,\ \text{a.e. in } (0,T) \tag{2.5}$$

$$\langle\mu\chi_{tt},v\rangle+(\chi_t,v)+(\nabla\chi,\nabla v)+(g(\chi)+\lambda'(\chi)\theta^{-1},v)=0 \quad \forall\, v\in V,\ \text{a.e. in } (0,T) \tag{2.6}$$

$$\theta(0)=\theta_0,\quad \chi(0)=\chi_0,\quad \chi_t(0)=\chi_1 \quad \text{a.e. in } \Omega. \tag{2.7}$$

Remark 2.2. As $L^p(\Omega) \hookrightarrow V'$ because $N \le 3$, the right hand side of (2.5) makes sense and the first initial condition in (2.7) holds almost everywhere in Ω, due to the weak continuity of $t \mapsto \theta(t)$ from $[0,T]$ to $L^p(\Omega)$. Moreover, we point out that the regularity property $\chi \in C^0([0,T];L^6(\Omega))$ entails (cf. (H4) and (H2), (2.4)) $g(\chi) \in L^\infty(0,T;H)$ and $\lambda'(\chi)\theta^{-1} \in L^2(0,T;L^3(\Omega))$, whence, by comparison in (2.6), it follows that $\chi_{tt} \in C^0([0,T];V') + L^2(0,T;H)$.

The main result is as follows.

Theorem 2.3. *Let* (H1)-(H12) *hold. Then, for any $\mu > 0$, problem $\mathbf{P}_\mu$ has a solution (θ^μ, χ^μ).*

Consider now the formal limit problem, which corresponds to the standard Penrose-Fife model (1.1)-(1.2). Note that the regularity prescription on χ is different from the above, and it refers instead to the usual requirement for parabolic phase field models [6, 15].

Problem $\mathbf{P}_0$. *Find θ and χ satisfying*

$$\theta \in H^1(0,T;V') \cap L^\infty(0,T;L^p(\Omega)) \tag{2.8}$$

$$\theta > 0 \quad \text{a.e. in } Q_T \tag{2.9}$$

$$\theta^{-1} \in L^2(0,T;V) \tag{2.10}$$

$$\chi \in H^1(0,T;H) \cap L^2(0,T;H^2(\Omega)) \hookrightarrow C^0([0,T];V) \tag{2.11}$$

$$\langle(\theta + \lambda(\chi))_t, v\rangle + (\!(-\theta^{-1}, v)\!) = \langle f, v\rangle + (h,v)_\Gamma \quad \forall\, v \in V, \text{ a.e. in } (0,T) \tag{2.12}$$

$$\chi_t - \Delta\chi + g(\chi) + \lambda'(\chi)\theta^{-1} = 0 \quad \text{a.e. in } Q_T \tag{2.13}$$

$$\chi_{\mathbf{n}} = 0 \quad \text{a.e. on } \Gamma_T \tag{2.14}$$

$$\theta(0) = \theta_0, \quad \chi(0) = \chi_0 \quad \text{a.e. in } \Omega. \tag{2.15}$$

In the body of our arguments, we will check, in particular, existence and uniqueness of the solution to $\mathbf{P}_0$. In a first step, we prove the following theorem.

Theorem 2.4. *Let* (H1)-(H12) *hold and let $\mu \in (0, \mu_0]$, $\mu_0 > 0$ being fixed. Then there exists a positive constant K, independent of μ, such that, for any solution (θ^μ, χ^μ) to $\mathbf{P}_\mu$, there holds*

$$\begin{aligned} &\|\theta^\mu\|_{H^1(0,T;V')\cap L^\infty(0,T;L^p(\Omega))} + \|1/\theta^\mu\|_{L^2(0,T;V)} \\ &\quad + \sqrt{\mu}\,\|\chi^\mu_t\|_{L^\infty(0,T;H)} + \|\chi^\mu_t\|_{L^2(0,T;H)} + \|\chi^\mu\|_{L^\infty(0,T;V)} \le K. \end{aligned} \tag{2.16}$$

Consider now a sequence $\{(\theta^\mu, \chi^\mu)\}_{\mu \in (0,\mu_0]}$, where (θ^μ, χ^μ) denotes an arbitrary solution to $\mathbf{P}_\mu$. Then, the whole sequence $\{(\theta^\mu, \chi^\mu)\}$ weakly converges to the pair (θ, χ), which solves problem $\mathbf{P}_0$, in the sense that as $\mu \searrow 0$ the following holds:

$$\theta^\mu \to \theta \quad \text{weakly star in } L^\infty(0,T;L^p(\Omega)) \text{ and weakly in } H^1(0,T;V')$$

$$\theta^\mu \to \theta \quad \text{strongly in } C^0([0,T];V')$$

$$\frac{1}{\theta^\mu} \to \frac{1}{\theta} \quad \text{weakly in } L^2(0,T;V)$$

$$\mu\chi^\mu_t \to 0 \quad \text{strongly in } L^\infty(0,T;H)$$

$$\chi^\mu \to \chi \quad \text{weakly star in } L^\infty(0,T;V) \text{ and weakly in } H^1(0,T;H)$$

$$\chi^\mu \to \chi \quad \text{strongly in } C^0([0,T];L^4(\Omega)).$$

Remark 2.5. Note that Theorem 2.4 yields, as a by-product, an existence result for Problem $\mathbf{P}_0$, in which the solution is found as the asymptotic limit of the sequence $\{(\theta^\mu, \chi^\mu)\}$. The uniqueness of (θ, χ) follows from Theorem 2.10 below (cf. Remark 2.12 for a comparison with existing results).

A conditional continuous dependence estimate is given by the following theorem.

Theorem 2.6. *Let* (H1)-(H5) *hold. Suppose moreover that*

(H13) $\lambda' \in L^\infty(\mathbb{R})$ *if* $N = 2, 3$
(H14) *for some positive constant* c_1 *and all* $r \in \mathbb{R}$, $|g'(r)| \le c_1(1 + |r|^2)$.

Consider two sets of data $\{\theta_{0j}, \chi_{0j}, \chi_{1j}, f_j, h_j\}$, $j = 1, 2$, *satisfying assumptions* (H6)-(H12) *and denote by* (θ_j, χ_j) *a corresponding solution to problem* $\mathbf{P}_\mu$. *Assume that*

$$u_j := -\theta_j^{-1} \in L^2(0, T; L^\infty(\Omega)), \quad j = 1, 2 \tag{2.17}$$

and let M_1 *be a positive constant such that*

$$\max\left\{\|\chi_1\|_{L^\infty(0,T;V)}, \|\chi_2\|_{L^\infty(0,T;V)}, \|u_1\|_{L^2(0,T;L^\infty(\Omega))}, \|u_2\|_{L^2(0,T;L^\infty(\Omega))}\right\} \le M_1.$$

Then

$$\begin{aligned}
&\Big(\int_0^T\int_\Omega \frac{|\theta_1-\theta_2|^2}{1+|\theta_1|^2+|\theta_2|^2}\,dx\,ds\Big)^{1/2} + \Big(\int_0^T\int_\Omega \frac{|u_1-u_2|^2}{1+|u_1|^2+|u_2|}\,dx\,ds\Big)^{1/2} \\
&+\|1 * (u_1 - u_2)\|_{L^\infty(0,T;V)} + \|(\chi_1-\chi_2)_t\|_{L^\infty(0,T;H)} + \|\chi_1-\chi_2\|_{L^\infty(0,T;V)} \\
&\le C_1\Big(\|\theta_{01}-\theta_{02}\|_{V'} + \|f_1-f_2\|_{L^2(0,T;V')} + \|h_1-h_2\|_{L^2(\Gamma_T)} \\
&\quad + \|\chi_{01}-\chi_{02}\|_V + \|\chi_{11}-\chi_{12}\|\Big)
\end{aligned} \tag{2.18}$$

for some positive constant $C_1 = C_1(M_1)$ *also depending on* T, Ω, γ, μ, λ, *and* c_1. *In particular, if* $N = 1$, *then problem* $\mathbf{P}_\mu$ *has a unique solution.*

Remark 2.7. Note that the first integral on the left hand side makes sense though it is not clear whether $\theta_j \notin L^2(Q_T)$, $j = 1, 2$. Indeed, as $\theta_j \in L^\infty(0, T; L^p(\Omega))$ it turns out that (cf. also (2.4))

$$0 \le \frac{|\theta_1-\theta_2|^2}{1+|\theta_1|^2+|\theta_2|^2} \le (\theta_1-\theta_2)(u_1-u_2)$$

which is in $L^1(Q_T)$. Referring now to (2.17), we underline that the existence result in Theorem 2.3 ensures the regularity $u_j \in L^2(0, T; L^\infty(\Omega))$ if $N = 1$ only. Moreover, in the one-dimensional case χ_1, $\chi_2 \in L^\infty(Q_T)$ (see the condition on M_1) so that if $N = 1$ (H2) (and (H13)) are no longer needed in Theorem 2.6.

In the case of a special class of λ, we have the following statement.

Theorem 2.8. *Let* (H3)-(H5) *and* (H14) *hold. In addition, suppose that*

(H15) $\lambda(r) = r$ *for all* $r \in \mathbb{R}$.

Consider two sets of data $\{\theta_{0j}, \chi_{0j}, \chi_{1j}, f_j, h_j\}$, $j = 1, 2$, *satisfying assumptions* (H6)-(H12) *and denote by* (θ_j, χ_j) *a corresponding solution to problem* $\mathbf{P}_\mu$. *Set* $u_j = -\theta_j^{-1}$ *and let* M_2 *be a positive constant such that*

$$\max\left\{\|\chi_1\|_{L^\infty(0,T;V)}, \|\chi_2\|_{L^\infty(0,T;V)}\right\} \le M_2.$$

Then there exists a positive constant $C_2 = C_2(M_2)$, also depending on T, Ω, γ, μ, and c_1, such that

$$\begin{aligned}
&\Big(\int_0^T\int_\Omega \frac{|\theta_1-\theta_2|^2}{1+|\theta_1|^2+|\theta_2|^2}\,dx\,ds\Big)^{1/2} + \Big(\int_0^T\int_\Omega \frac{|u_1-u_2|^2}{1+|u_1|^2+|u_2|}\,dx\,ds\Big)^{1/2}\\
&+\|1*(u_1-u_2)\|_{L^\infty(0,T;V)} + \|(\chi_1-\chi_2)_t\|_{L^\infty(0,T;V')}\\
&+\|\chi_1-\chi_2\|_{L^\infty(0,T;H)} + \|1*(\chi_1-\chi_2)\|_{L^\infty(0,T;V)} \qquad (2.19)\\
&\le C_2\Big(\|\theta_{01}-\theta_{02}\|_{V'} + \|f_1-f_2\|_{L^2(0,T;V')} + \|h_1-h_2\|_{L^2(\Gamma_T)}\\
&\quad + \|\chi_{01}-\chi_{02}\| + \|\chi_{11}-\chi_{12}\|_{V'}\Big).
\end{aligned}$$

Remark 2.9. Observe that (H15) is basically equivalent to assuming that λ is affine.

Finally, the following theorem implies uniqueness for the solution to $\mathbf{P}_0$.

Theorem 2.10. *Let* (H1)-(H5), (H14) *hold and let* $\{\theta_{0j}, \chi_{0j}, f_j, h_j\}$, $j = 1, 2$, *be two sets of data satisfying assumptions* (H6)-(H11). *Denote by* (θ_j, χ_j) *a pair fulfilling* (2.8)-(2.15) *and set* $u_j := -\theta_j^{-1}$. *Let* M_2 *be as in Theorem 2.8 and let* M_3 *specify a positive constant such that*

$$\max\big\{\|u_1\|_{L^2(0,T;V)},\ \|u_2\|_{L^2(0,T;V)}\big\} \le M_3.$$

Then there exists a positive constant $C_3 = C_3(M_2, M_3)$, also depending on T, Ω, γ, λ, and c_1, such that

$$\begin{aligned}
&\Big(\int_0^T\int_\Omega \frac{|\theta_1-\theta_2|^2}{1+|\theta_1|^2+|\theta_2|^2}\,dx\,ds\Big)^{1/2} + \Big(\int_0^T\int_\Omega \frac{|u_1-u_2|^2}{1+|u_1|^2+|u_2|}\,dx\,ds\Big)^{1/2}\\
&+\|1*(u_1-u_2)\|_{L^\infty(0,T;V)} + \|\chi_1-\chi_2\|_{L^\infty(0,T;H)} + \|\chi_1-\chi_2\|_{L^2(0,T;V)}\\
&\le C_3\Big(\|e_{01}-e_{02}\|_{V'} + \|\chi_{01}-\chi_{02}\| + \|f_1-f_2\|_{L^2(0,T;V')} + \|h_1-h_2\|_{L^2(\Gamma_T)}\Big)
\end{aligned} \qquad (2.20)$$

where $e_{0j} := \theta_{0j} + \lambda(\chi_{0j})$, $j = 1, 2$. If in place of (H14) *the following condition holds*

(H16) $(g(r_1)-g(r_2))(r_1-r_2) \ge -c_2|r_1-r_2|^2$ *for all* $r_1, r_2 \in \mathbb{R}$

for some nonnegative constant c_2, then C_3 depends on M_3, T, Ω, γ, λ, c_2 only.

Remark 2.11. Note that the sample choice $g(r) = r^3 - r - \theta_c^{-1}$, $r \in \mathbb{R}$, corresponding to a double-well potential in the free energy, satisfies both (H14) and (H16).

Remark 2.12. If one uses the enthalpy variables $e_j = \theta_j + \lambda(\chi_j)$, $j = 1, 2$, in Remark 2.9 and Theorem 2.10, then the norm $\|e_1 - e_2\|_{L^\infty(0,T;V')}$ can be estimated in terms of the right hand side of (2.19) and (2.20), respectively. Indeed, it suffices to integrate the difference of equations (2.12) written for e_i, u_i, $i = 1, 2$, with respect to time and compare the resulting terms. However, we point out that an estimate like (2.20) referred to the enthalpy has been already proved in [15, Theorem 3.1], where a problem more general than $\mathbf{P}_0$ is considered. There, the enthalpy depends nonlinearly on θ and equation (2.13) also contains a maximal monotone graph with bounded domain: this constraint forces χ to be necessarily bounded, which is rather helpful in the mathematical analysis.

3. Proof of Theorem 2.3

The proof is split into several steps. First, we construct a suitable sequence of approximating problems $\mathbf{P}_\mu^n$, $n \in \mathbb{N}$, and the related sequence of solutions (θ^n, χ^n), the index μ being omitted for the sake of brevity. Then, a series of a priori estimates on the sequence (θ^n, χ^n) will allow us to get a solution to $\mathbf{P}_\mu$ by passing to the limit as $n \to +\infty$.

Approximating $\mathbf{P}_\mu$. Let us introduce approximations of λ and g first. For $n \in \mathbb{N}$, we set

$$\lambda_n(r) := \begin{cases} \lambda(-n) + \lambda'(-n)(r+n) & \text{if } r < -n \\ \lambda(r) & \text{if } -n \le r \le n \\ \lambda(n) + \lambda'(n)(r-n) & \text{if } r > n \end{cases} \tag{3.1}$$

and observe that

$$\lambda_n \in C^{1,1}(\mathbb{R}), \quad \lambda_n', \lambda_n'' \in L^\infty(\mathbb{R}), \quad \lambda_n \to \lambda \quad \text{a.e. in } \mathbb{R}. \tag{3.2}$$

Also, by (H1)-(H2) and the mean value theorem we easily infer

$$|\lambda_n'(r)| \le c_\lambda\,(1 + |r|) \quad \forall\, r \in \mathbb{R} \tag{3.3}$$

where c_λ is a positive constant only depending on λ. Then, for any integer $n \in \mathbb{N}$, we consider an approximation $\hat{g}_n$ of $\hat{g}$ (cf. (2.1)) such that

$$0 \le \hat{g}_n(r) \le \hat{g}(r) \quad \forall\, r \in \mathbb{R} \tag{3.4}$$

and, letting $g_n = \hat{g}_n'$,

$$g_n \in C^{0,1}(\mathbb{R}), \quad g_n \to g \quad \text{a.e. in } \mathbb{R}. \tag{3.5}$$

For instance, recalling (2.2), we can find two sequences $\{s_n\}$ and $\{r_n\}$ such that

$$s_n < \alpha_0 < r_n \tag{3.6}$$

$$g(s_n) = -n, \quad g(r_n) = n \tag{3.7}$$

$$g(r) \le -n \quad \forall\, r < s_n, \quad g(r) \ge n \quad \forall\, r > r_n \tag{3.8}$$

and set

$$g_n(r) := \begin{cases} -n & \text{if } r \le s_n \\ g(r) & \text{if } s_n < r < r_n \\ n & \text{if } r \ge r_n. \end{cases} \tag{3.9}$$

Note that

$$g(r) \le g_n(r) \quad \text{if } r \le s_n, \quad g(r) \ge g_n(r) \quad \text{if } r \ge r_n; \tag{3.10}$$

$$\hat{g}_n(r) = C_g + \int_{\alpha_0}^{r} g_n(s)\,ds, \quad r \in \mathbb{R}, \tag{3.11}$$

satisfies (3.4).

To introduce a suitable approximation of the remaining nonlinearity, we set

$$\rho(u) := (-u)^{-1} \quad \forall\, u < 0 \tag{3.12}$$

$$a_n := -(n+1), \quad b_n := -\frac{1}{n+1} \quad \forall\, n \in \mathbb{N} \tag{3.13}$$

and define, for any $n \in \mathbb{N}$,

$$\rho_n(u) := \begin{cases} \rho(b_n) & \text{if } u > b_n \\ \rho(u) & \text{if } a_n \le u \le b_n \\ \rho(a_n) & \text{if } u < a_n. \end{cases} \tag{3.14}$$

We approximate the initial datum θ_0 as well. Define the measurable and negative function (cf. (H9))

$$u_0 := -(\theta_0)^{-1} \tag{3.15}$$

and, consequently, for any $n \in \mathbb{N}$, the approximating data

$$\theta_{0n} := \rho_n(u_0), \quad u_{0n} := -(\theta_{0n})^{-1}. \tag{3.16}$$

Note that u_{0n}, $\theta_{0n} \in L^\infty(\Omega)$. Moreover, it can be proved

$$\theta_{0n} \le \theta_0 + 1 \quad \text{a.e. in } \Omega \tag{3.17}$$

$$a_n \le u_{0n} \le b_n \quad \text{a.e. in } \Omega \tag{3.18}$$

$$\theta_{0n} \to \theta_0 \quad \text{a.e. in } \Omega \quad \text{and in } L^p(\Omega), \quad \text{as } n \to +\infty \tag{3.19}$$

by virtue of (H9) and the Lebesgue dominated convergence theorem. Also, setting

$$\nu_n := (1 + n^2)^{-1} \tag{3.20}$$

for any $n \in \mathbb{N}$, we can infer

$$\nu_n \|u_{0n}\|^2 \le C \tag{3.21}$$

where henceforth C denotes a positive constant independent of n and μ, but depending on T, Ω, Γ, γ, p, λ, and g, at most. Observe, in particular, that (3.20)-(3.21) entail

$$\nu_n u_{0n} \to 0 \quad \text{in } H, \quad \text{as } n \to +\infty. \tag{3.22}$$

For the sake of simplicity, we also approximate the source term f with a sequence $\{f_n\} \subset L^2(0,T;H)$ such that

$$f_n \to f \quad \text{in } L^2(0,T;L^p(\Omega)), \quad \text{as } n \to +\infty. \tag{3.23}$$

We can now formulate the approximating problem for any $n \in \mathbb{N}$.

Problem $\mathbf{P}^n_\mu$. *Find $u^n \in C^0([0,T];H) \cap L^2(0,T;V)$ and $\chi^n \in W^{2,\infty}(0,T;V') \cap C^1([0,T];H) \cap C^0([0,T];V)$ such that*

$$\langle(\nu_n u^n + \rho_n(u^n) + \lambda_n(\chi^n))_t, v\rangle + ((u^n, v)) = (f_n, v) + (h, v)_\Gamma \qquad \forall\, v \in V, \text{ a.e. in } (0,T) \tag{3.24}$$

$$\langle \mu\chi^n_{tt}, v\rangle + (\chi^n_t, v) + (\nabla\chi^n, \nabla v) + (g_n(\chi^n) + \lambda'_n(\chi^n)(\rho_n(u^n))^{-1}, v) = 0 \qquad \forall\, v \in V, \text{ a.e. in } (0,T) \tag{3.25}$$

$$u^n(0) = u_{0n}, \quad \chi^n(0) = \chi_0, \quad \chi^n_t(0) = \chi_1 \quad \text{a.e. in } \Omega. \tag{3.26}$$

Existence and uniqueness for $\mathbf{P}^n_\mu$. We can apply a fixed-point argument based on the Contraction Principle. Define the Banach space

$$X_T = L^2(0,T;H) \times C^0([0,T];H)$$

and let $(\widetilde{u}^n, \widetilde{\chi}^n) \in X_T$. Then, consider the Cauchy problem

$$\begin{gathered}\langle \mu \chi^n_{tt}, v\rangle + (\chi^n_t, v) + (\nabla \chi^n, \nabla v) + (\chi^n, v) = (\mathcal{G}(\widetilde{u}^n, \widetilde{\chi}^n), v) \quad \forall\, v \in V, \text{ a.e. in } (0,T) \\ \chi^n(0) = \chi_0, \quad \chi^n_t(0) = \chi_1 \quad \text{a.e. in } \Omega\end{gathered} \tag{3.27}$$

where

$$\mathcal{G}(\widetilde{u}^n, \widetilde{\chi}^n) = \widetilde{\chi}^n - g_n(\widetilde{\chi}^n) - \lambda'_n(\widetilde{\chi}^n)(\rho_n(\widetilde{u}^n))^{-1} \in L^\infty(0,T;H).$$

As (3.27) is a linear hyperbolic problem, it turns out that (cf. [1, Theorem 3.3]) there is a unique solution

$$\chi^n \in W^{2,\infty}(0,T;V') \cap C^1([0,T];H) \cap C^0([0,T];V)$$

to (3.27) (one may also see [30, pp. 74–79]). Moreover, the usual energy estimate

$$\begin{aligned}&\mu \|\chi^n_t\|^2_{C([0,t];H)} + \|\chi^n_t\|^2_{L^2(0,t;H)} + \|\chi^n\|^2_{C^0([0,t];V)} \\ &\leq C\Big(\mu\|\chi_1\|^2 + \|\chi_0\|^2_V + \int_0^t \|\mathcal{G}(\widetilde{u}^n(s), \widetilde{\chi}^n(s))\|^2\, ds\Big)\end{aligned}$$

holds for any $t \in [0,T]$. Next, it is not difficult to realize that (see, for instance, [5, Lemma 3.4]) there exists a unique solution $u^n \in C^0([0,T];H) \cap L^2(0,T;V)$ to

$$\langle (\nu_n u^n + \rho_n(u^n))_t, v\rangle + (\!(u^n, v)\!) = -((\lambda_n(\chi^n))_t - f_n, v) + (h,v)_\Gamma$$
$$\forall\, v \in V, \text{ a.e. in } (0,T) \tag{3.28}$$
$$u^n(0) = u_{0n} \quad \text{a.e. in } \Omega. \tag{3.29}$$

We have thus constructed a mapping S from X_T into itself by setting $S(\widetilde{u}^n, \widetilde{\chi}^n) := (u^n, \chi^n)$, with the property that

$$(u^n, \chi^n) \in \big[C^0([0,T];H) \cap L^2(0,T;V)\big] \times \big[C^1([0,T];H) \cap C^0([0,T];V)\big]\,.$$

Consider now $(\widetilde{u}^n_j, \widetilde{\chi}^n_j) \in X_T$, $j = 1,2$, and the corresponding (u^n_j, χ^n_j). Observe that, integrating with respect to time the equation (3.28) written for the difference $u^n_1 - u^n_2$, we obtain (cf. also (3.29))

$$\begin{aligned}&\langle \nu_n(u^n_1 - u^n_2) + \rho_n(u^n_1) - \rho_n(u^n_2), v\rangle + (\!(1 * (u^n_1 - u^n_2), v)\!) \\ &= -(\lambda_n(\chi^n_1) - \lambda_n(\chi^n_2), v) \quad \forall\, v \in V, \text{ in } (0,T).\end{aligned}$$

Then, taking $v = u^n_1 - u^n_2$ and recalling (3.1)-(3.2), (3.14), it is not difficult to deduce the estimate

$$\|u^n_1 - u^n_2\|^2_{L^2(0,t;H)} \leq \Lambda_n \|\chi^n_1 - \chi^n_2\|^2_{L^2(0,t;H)} \quad \forall\, t \in [0,T]$$

where Λ_n denotes a positive constant blowing up as n goes to $+\infty$.

On the other hand, the energy estimate related to the difference $\chi^n_1 - \chi^n_2$ (of solutions to the respective problems (3.27)) yields

$$\|\chi^n_1 - \chi^n_2\|^2_{C^0([0,t];H)} \leq C \int_0^t \|\mathcal{G}(\widetilde{u}^n_1(s), \widetilde{\chi}^n_1(s)) - \mathcal{G}(\widetilde{u}^n_2(s), \widetilde{\chi}^n_2(s))\|^2\, ds.$$

Combining the last two estimates and recalling the definition of $\mathcal{G}$ along with (H1)-(H3), (3.1)-(3.2), (3.9), and (3.14), we eventually deduce

$$\begin{aligned}&\|u^n_1 - u^n_2\|^2_{L^2(0,t;H)} + \|\chi^n_1 - \chi^n_2\|^2_{C^0([0,t];H)} \\ &\leq \Lambda_n \int_0^t \Big(\|\widetilde{u}^n_1 - \widetilde{u}^n_2\|^2_{L^2(0,s;H)} + \|\widetilde{\chi}^n_1 - \widetilde{\chi}^n_2\|^2_{C^0([0,s];H)}\Big)\, ds\end{aligned}$$

for any $t \in (0,T]$. Thus, for any fixed $n \in \mathbb{N}$, we can find an integer $m = m(n)$ such that S^m is a contraction of X_T into itself. Therefore S has a unique fixed-point in X_T; that is, $\mathbf{P}^n_\mu$ has a unique solution.

A priori estimates. Suppose, for the sake of simplicity, $\mu \in (0,1]$. Let us set

$$\rho_n^*(r) := \int_{-1}^{r} (1 - (\rho_n(s))^{-1})\, ds$$

for all $r \in \mathbb{R}$. Note that $\rho_n^* \geq 0$ in $\mathbb{R}$. Moreover, a straightforward computation gives (cf. (3.12)-(3.14))

$$\rho_n^*(r) = \frac{r^2}{2} + r + \frac{1}{2} \quad \forall\, r \in [a_n, b_n]. \tag{3.30}$$

Then, define

$$\theta^n := \rho_n(u^n), \quad w^n := (\rho_n(u^n))^{-1} \tag{3.31}$$

and observe that θ^n and w^n both belong to $C^0([0,T];H) \cap L^2(0,T;V)$, due to the Lipschitz continuity of ρ_n and $1/\rho_n$.

Let us point out first that the estimates we are performing on equation (3.24) are formal since we only know that $\nu_n u^n + \theta^n \in H^1(0,T;V')$, but we would need to know that both u^n and θ^n belong to $H^1(0,T;V')$ at least, separately. In order to make the estimates rigorous, we should better approximate f, h, and u_0 by smoother functions $f_n \in H^1(0,T;H)$, $h_n \in H^1(0,T;L^2(\Gamma))$, and $u_{0n} \in V$, arguing then on the regularized version (see also [5, remarks at p. 321] and references therein).

Consider therefore (3.24) with $v = 1 - w^n$ and note that

$$\langle (\nu_n u^n(t) + \theta^n(t))_t, 1 - w^n(t)\rangle = \frac{d}{dt}\int_\Omega (\nu_n \rho_n^*(u^n(t)) + \theta^n(t) - \ln \theta^n(t))\, dx. \tag{3.32}$$

Recalling again (3.12)-(3.14) and the definition of the scalar product in V (cf. Sec. 2), one can easily check that

$$(\!(u^n(t), 1 - w^n(t))\!) = \|\nabla w^n(t)\|^2 + \gamma (u^n(t), 1 - w^n(t))_\Gamma. \tag{3.33}$$

On the other hand, we have that

$$(u^n(t), 1 - w^n(t))_\Gamma \geq (-w^n(t), 1 - w^n(t))_\Gamma. \tag{3.34}$$

Hence, integrating (3.24) with $v = 1 - w^n(t)$ with respect to t and using (3.32)-(3.34), we deduce the estimate

$$\begin{aligned}
&\int_\Omega (\nu_n \rho_n^*(u^n(t)) + \theta^n(t) - \ln \theta^n(t)) dx \\
&+ \int_0^t \|\nabla w^n(s)\|^2\, ds + \gamma \int_0^t \|w^n(s)\|^2_{L^2(\Gamma)}\, ds - \gamma \int_0^t \|w^n(s)\|_{L^1(\Gamma)}\, ds \\
&\leq \int_\Omega (\nu_n \rho_n^*(u_{0n}) + \rho_n(u_{0n}) - \ln \rho_n(u_{0n})) dx \\
&\quad + \int_0^t (f_n(s) - (\lambda_n(\chi^n(s))_s, 1 - w^n(s))\, ds + \int_0^t (h(s), 1 - w^n(s))_\Gamma\, ds.
\end{aligned} \tag{3.35}$$

On account of (3.14) and (3.16), we have

$$\rho_n(u_{0n}) = \theta_{0n}. \tag{3.36}$$

Recalling (3.18), (3.20) and (3.30), we infer

$$\int_\Omega \nu_n \rho_n^*(u_{0n})\,dx \le C. \tag{3.37}$$

Moreover, recalling (H9)-(H10) and using (3.17) and (3.36), we obtain

$$\int_\Omega (\rho_n(u_{0n}) - \ln \rho_n(u_{0n}))\,dx \le C + \|\theta_0\|_{L^1(\Omega)} + \|\ln\theta_0\|_{L^1(\Omega)}. \tag{3.38}$$

Therefore, on account of the elementary inequality

$$r - \ln r \ge \frac{1}{3}(r + |\ln r|) \quad \forall\, r > 0$$

one sees that (3.35) and (3.37)-(3.38) yield

$$\begin{aligned}
&\frac{1}{3}\big(\|\theta^n(t)\|_{L^1(\Omega)} + \|\ln\theta^n(t)\|_{L^1(\Omega)}\big)\\
&+ \int_0^t \|\nabla w^n(s)\|^2\,ds + \gamma\int_0^t \|w^n(s)\|^2_{L^2(\Gamma)}\,ds - \gamma\int_0^t \|w^n(s)\|_{L^1(\Gamma)}\,ds\\
\le{}& C + \|\theta_0\|_{L^1(\Omega)} + \|\ln\theta_0\|_{L^1(\Omega)}\\
&\quad + \int_0^t (f_n(s) - (\lambda_n(\chi^n(s))_s, 1 - w^n(s))\,ds + \int_0^t (h(s), 1 - w^n(s))_\Gamma\,ds.
\end{aligned} \tag{3.39}$$

Define now

$$\tilde\rho_n(r) := \int_{-1}^r ((\rho_n(s))^{p-1} - 1)\,ds$$

for all $r \in \mathbb{R}$. Note that, on account of (3.12)-(3.14), we have

$$\tilde\rho_n(r) = \frac{(-r)^{2-p}}{p-2} - \frac{1}{p-2} - (r+1) \quad \forall\, r \in [a_n, b_n]. \tag{3.40}$$

Moreover, it is easy to check that $\tilde\rho_n \ge 0$ in $\mathbb{R}$. Consider then the identity

$$\langle(\nu_n u^n(t) + \theta^n(t))_t, (\theta^n(t))^{p-1} - 1\rangle = \frac{d}{dt}\int_\Omega \big(\nu_n\tilde\rho_n(u^n(t)) + \frac{1}{p}|\theta^n(t)|^p - \theta^n(t)\big)\,dx. \tag{3.41}$$

Arguing as for (3.33), we deduce

$$(\!(u^n(t), (\theta^n(t))^{p-1} - 1)\!) = (-\nabla w^n(t), \nabla((\theta^n(t))^{p-1} - 1)) + \gamma(u^n(t), (\theta^n(t))^{p-1} - 1)_\Gamma \tag{3.42}$$

as well as (cf. (3.34))

$$(u^n(t), (\theta^n(t))^{p-1} - 1)_\Gamma \ge (-w^n(t), (\theta^n(t))^{p-1} - 1)_\Gamma. \tag{3.43}$$

Therefore, (3.42) and (3.43) give

$$\begin{aligned}
&(\!(u^n(t), (\theta^n(t))^{p-1} - 1)\!)\\
&\qquad \ge (p-1)\|(\theta^n(t))^{p/2-1}\nabla\ln\theta^n(t)\|^2 + \gamma(-w^n(t), (\theta^n(t))^{p-1} - 1)_\Gamma.
\end{aligned} \tag{3.44}$$

Set now $v = (\theta^n(t))^{p-1} - 1$ in (3.24) and integrate the resulting identity with respect to time. On account of (3.41) and (3.44), we can obtain the inequality

$$\begin{aligned}
&\int_\Omega \Big(\nu_n \tilde\rho_n(u^n(t)) + \frac{1}{p}|\theta^n(t)|^p - \theta^n(t)\Big)\,dx \\
&+(p-1)\int_0^t \|(\theta^n(s))^{p/2-1}\nabla \ln\theta^n(s)\|^2\,ds + \gamma\int_0^t (-w^n(s),(\theta^n(s))^{p-1}-1)_\Gamma\,ds \\
&\le \int_\Omega (\nu_n\tilde\rho_n(u_{0n}) + \frac{1}{p}|\rho_n(u_{0n})|^p - \rho_n(u_{0n}))\,dx \\
&\quad + \int_0^t (f_n(s) - (\lambda_n(\chi^n(s))_s, (\theta^n(s))^{p-1}-1)\,ds + \int_0^t (h(s),(\theta^n(s))^{p-1}-1)_\Gamma\,ds.
\end{aligned} \tag{3.45}$$

Recalling (3.36) and using (3.18), (3.20), and (3.40), we find

$$\int_\Omega \nu_n\tilde\rho_n(u_{0n})\,dx \le C.$$

Also, owing to (3.17) and (3.36), we get

$$\int_\Omega \Big(\frac{1}{p}|\rho_n(u_{0n})|^p - \rho_n(u_{0n})\Big)\,dx \le C\big(\|\theta_0\|^p_{L^p(\Omega)} + \|\theta_0\|_{L^1(\Omega)} + 1\big). \tag{3.46}$$

Consequently, from (3.45)-(3.46) we derive

$$\begin{aligned}
&\frac{1}{p}\|\theta^n(t)\|^p_{L^p(\Omega)} - \|\theta^n(t)\|_{L^1(\Omega)} + (p-1)\int_0^t \|(\theta^n(s))^{p/2-1}\nabla\ln\theta^n(s)\|^2\,ds \\
&+\gamma\int_0^t (-w^n(s),(\theta^n(s))^{p-1}-1)_\Gamma\,ds \\
&\le C\big(\|\theta_0\|^p_{L^p(\Omega)} + \|\theta_0\|_{L^1(\Omega)} + 1\big) + \int_0^t (f_n(s) - (\lambda_n(\chi^n(s))_s,(\theta^n(s))^{p-1}-1)\,ds \\
&\quad + \int_0^t (h(s),(\theta^n(s))^{p-1}-1)_\Gamma\,ds.
\end{aligned} \tag{3.47}$$

We now set $v = u^n(t)$ in (3.24); that is,

$$\begin{aligned}
&\frac{\nu_n}{2}\frac{d}{dt}\|u^n(t)\|^2 + (\theta^n_t(t),u^n(t)) + (\!(u^n(t),u^n(t))\!) \\
&\qquad = (f_n(t) - (\lambda_n(\chi^n(t))_t, u^n(t)) + (h(t),u^n(t))_\Gamma.
\end{aligned} \tag{3.48}$$

Observe that, on account of (3.12)-(3.14), we have

$$\int_1^r \rho_n^{\rm inv}(s)\,ds = -\ln r \quad \forall\, r \in [-b_n,-a_n]$$

where $\rho_n^{\rm inv}$ denotes the inverse function of the restriction of ρ_n to $[a_n,b_n]$. Hence, since $\theta^n(t) \in [-b_n,-a_n]$, by (3.31) we deduce

$$(\theta^n_t(t),u^n(t)) = -\frac{d}{dt}\int_\Omega \ln\theta^n(t)\,dx. \tag{3.49}$$

Then, owing to (3.49), an integration of (3.48) with respect to time yields

$$\frac{\nu_n}{2}\|u^n(t)\|^2 - \int_\Omega \ln\theta^n(t)\,dx + \int_0^t (\!(u^n(s),u^n(s))\!)\,ds$$

$$
\begin{aligned}
&=\frac{\nu_n}{2}\|u_{0n}\|^2 - \int_\Omega \ln\theta_{0n}\,dx + \int_0^t (f_n(s) - (\lambda_n(\chi^n(s))_s, u^n(s))\,ds \\
&\quad + \int_0^t (h(s), u^n(s))_\Gamma\,ds
\end{aligned}
$$

and, recalling (3.21) and (3.38), we obtain the inequality

$$
\begin{aligned}
&\frac{\nu_n}{2}\|u^n(t)\|^2 - \int_\Omega \ln\theta^n(t)\,dx + \int_0^t ((u^n(s), u^n(s)))\,ds \\
&\le C + \|\ln\theta_0\|_{L^1(\Omega)} + \int_0^t (f_n(s) - (\lambda_n(\chi^n(s)))_s, u^n(s))\,ds \\
&\quad + \int_0^t (h(s), u^n(s))_\Gamma\,ds.
\end{aligned}
\tag{3.50}
$$

Consider now equation (3.25) and pick formally $v = \chi_t^n$ (see Appendix in [4] to make this argument rigorous). Integrating the resulting identity with respect to time, we get the estimate

$$
\begin{aligned}
&\frac{\mu}{2}\|\chi_t^n(t)\|^2 + \frac{1}{2}\|\nabla\chi^n(t)\|^2 + \int_\Omega \hat{g}_n(\chi_n(t))dx + \int_0^t \|\chi_s^n(s)\|^2\,ds \\
&\quad \le \frac{\mu}{2}\|\chi_1\|^2 + \frac{1}{2}\|\nabla\chi_0\|^2 + \int_\Omega \hat{g}_n(\chi_0)dx - \int_0^t ((\lambda_n(\chi^n(s)))_s, w^n(s))\,ds.
\end{aligned}
\tag{3.51}
$$

Let us add (3.39) and (3.51). This gives

$$
\begin{aligned}
&\frac{1}{3}\big(\|\theta^n(t)\|_{L^1(\Omega)} + \|\ln\theta^n(t)\|_{L^1(\Omega)}\big) \\
&+ \int_0^t \|\nabla w^n(s)\|^2\,ds + \gamma\int_0^t \|w^n(s)\|^2_{L^2(\Gamma)}\,ds - \gamma\int_0^t \|w^n(s)\|_{L^1(\Gamma)}\,ds \\
&+\frac{\mu}{2}\|\chi_t^n(t)\|^2 + \frac{1}{2}\|\nabla\chi^n(t)\|^2 + \int_\Omega \hat{g}_n(\chi_n(t))dx + \int_0^t \|\chi_s^n(s)\|^2\,ds \\
&\le C + \|\theta_0\|_{L^1(\Omega)} + \|\ln\theta_0\|_{L^1(\Omega)} + \frac{\mu}{2}\|\chi_1\|^2 + \frac{1}{2}\|\nabla\chi_0\|^2 \\
&\quad + \int_0^t (f_n(s), 1 - w^n(s))\,ds - \int_0^t\int_\Omega \lambda_n'(\chi^n)\chi_s^n\,dx\,ds \\
&\quad + \int_0^t (h(s), 1 - w^n(s))_\Gamma\,ds + \int_\Omega \hat{g}_n(\chi_0)dx.
\end{aligned}
\tag{3.52}
$$

In view of (3.3) and (3.26), by Young and Hölder inequalities we have that

$$
\begin{aligned}
&-\int_0^t\int_\Omega \lambda_n'(\chi^n)\chi_s^n\,dx\,ds \\
&\le C\Big(1 + \int_0^t \|\chi^n(s)\|^2\,ds\Big) + \frac{1}{2}\int_0^t \|\chi_s^n(s)\|^2\,ds \\
&\le C\Big(1 + \|\chi_0\|^2 + \int_0^t \|\chi_s^n\|^2_{L^2(0,s;H)}\,ds\Big) + \frac{1}{2}\int_0^t \|\chi_s^n(s)\|^2\,ds.
\end{aligned}
$$

Then, recalling (H1)-(H2), (H11), (2.1), and (3.4), from (3.52) we deduce

$$
\frac{1}{3}\big(\|\theta^n(t)\|_{L^1(\Omega)} + \|\ln\theta^n(t)\|_{L^1(\Omega)}\big)
$$

$$
\begin{aligned}
&+\int_0^t \|\nabla w^n(s)\|^2\,ds+\gamma\int_0^t \|w^n(s)\|_\Gamma^2\,ds\\
&+\frac{\mu}{2}\|\chi_t^n(t)\|^2+\frac12\|\nabla\chi^n(t)\|^2+\frac12\int_0^t \|\chi_s^n(s)\|^2\,ds\\
&\le C\Big(1+\|\theta_0\|_{L^1(\Omega)}+\|\ln\theta_0\|_{L^1(\Omega)}+\|\chi_1\|^2+\|\chi_0\|_V^4+\int_0^t \|\chi_s^n\|^2_{L^2(0,s;H)}\,ds\Big)\\
&\quad+\|f_n\|_{L^2(0,T;L^p(\Omega))}\big(1+\|w_n\|_{L^2(0,t;L^{p'}(\Omega))}\big)+\|h\|_{L^2(\Gamma_T)}\big(1+\|w^n(s)\|_{L^2(\Gamma_t)}\big)
\end{aligned}
$$

where we have assumed $\mu\in(0,1]$ for the sake of simplicity. Hence, the injection $V\hookrightarrow L^{p'}(\Omega)$ and Young and Gronwall inequalities allow us to obtain the bound

$$
\begin{aligned}
&\|\theta^n(t)\|_{L^1(\Omega)}+\|\ln\theta^n(t)\|_{L^1(\Omega)}+\int_0^t \|\nabla w^n(s)\|^2\,ds+\gamma\int_0^t \|w^n(s)\|_\Gamma^2\,ds\\
&+\mu\|\chi_t^n(t)\|^2+\|\nabla\chi^n(t)\|^2+\int_0^t \|\chi_s^n(s)\|^2\,ds\\
&\le C\big(1+\|\theta_0\|_{L^1(\Omega)}+\|\ln\theta_0\|_{L^1(\Omega)}+\|\chi_1\|^2+\|\chi_0\|_V^4\\
&\quad+\|f_n\|^2_{L^2(0,T;L^p(\Omega))}+\|h\|^2_{L^2(\Gamma_T)}\big)
\end{aligned}
\tag{3.53}
$$

for any $t\in[0,T]$. Going back to (3.50), we easily see that

$$
\begin{aligned}
&\frac{\nu_n}{2}\|u^n(t)\|^2+\int_0^t ((u^n(s),u^n(s)))\,ds\\
&\le C+\|\ln\theta_0\|_{L^1(\Omega)}+\|\ln\theta^n(t)\|_{L^1(\Omega)}+\int_0^t (f_n(s),u^n(s))\,ds\\
&\quad+\int_0^t (h(s),u^n(s))_\Gamma\,ds-\int_0^t (\lambda_n'(\chi^n(s))\chi_s^n(s),u^n(s))\,ds
\end{aligned}
\tag{3.54}
$$

from which, on account of (3.3) and (H1)-(H2), using Young and Hölder inequalities, we derive

$$
\begin{aligned}
&\nu_n\|u^n(t)\|^2+\|u^n\|^2_{L^2(0,t;V)}\\
&\le C\Big(1+\|\ln\theta_0\|_{L^1(\Omega)}+\|\ln\theta^n(t)\|_{L^1(\Omega)}+\|f_n\|^2_{L^2(0,T;L^p(\Omega))}+\|h\|^2_{L^2(\Gamma_T)}\\
&\quad+\int_0^t \big(1+\|\chi^n(s)\|_{L^4(\Omega)}\big)\,\|\chi_s^n(s)\|\,\|u^n(s)\|_{L^4(\Omega)}\,ds\Big).
\end{aligned}
$$

Then, using the injection $V\hookrightarrow L^4(\Omega)$ and the bound (3.53), an application of Gronwall lemma yields

$$
\begin{aligned}
\nu_n\|u^n(t)\|^2+\|u^n\|^2_{L^2(0,t;V)}\le{}& C\Big(1+\|\theta_0\|_{L^1(\Omega)}+\|\ln\theta_0\|_{L^1(\Omega)}+\|\chi_1\|^2\\
&+\|\chi_0\|_V^4+\|f_n\|^2_{L^2(0,T;L^p(\Omega))}+\|h\|^2_{L^2(\Gamma_T)}\Big)^2
\end{aligned}
\tag{3.55}
$$

for any $t \in [0,T]$. At this point, in the light of (3.30) and (H8), from (3.47) we can infer

$$
\begin{aligned}
&\frac{1}{p}\|\theta^n(t)\|^p_{L^p(\Omega)} \\
&\le C\Big(1 + \|\theta_0\|^p_{L^p(\Omega)} + \|f_n\|_{L^1(Q_T)} + \|h\|_{L^1(\Gamma_T)} + \|(w^n)^{2-p}\|_{L^1(\Gamma_t)}\Big) \\
&\quad + \int_0^t (f_n(s), (\theta^n(s))^{p-1})\, ds - \int_0^t (\lambda_n'(\chi^n(s))\chi_s^n(s), (\theta^n(s))^{p-1} - 1)\, ds.
\end{aligned}
\tag{3.56}
$$

Recalling (H1)-(H2), (3.3), (3.53) and using Young and Hölder inequalities, we have

$$
\begin{aligned}
&\int_0^t (\lambda_n'(\chi^n(s))\chi_s^n(s), (\theta^n(s))^{p-1} - 1)\, ds \\
&\le C\int_0^t \Big(\int_\Omega |\lambda_n'(\chi^n(s))\chi_s^n(s)|^p\, dx + \int_\Omega |(\theta^n(s))^{p-1} - 1|^{p'}\, dx\Big)\, ds \\
&\le C\int_0^t \Big(\big(1 + \|\chi^n(s)\|^p_{L^{2p/(2-p)}(\Omega)}\big)\|\chi_s^n(s)\|^p + \|\theta^n(s)\|^p_{L^p(\Omega)} + 1\Big)\, ds \\
&\le \widetilde{C}\int_0^t \Big(\|\chi_s^n(s)\|^2 + \|\theta^n(s)\|^p_{L^p(\Omega)} + 1\Big)\, ds \\
&\le \widetilde{C}\int_0^t \big(\|\theta^n(s)\|^p_{L^p(\Omega)} + 1\big)\, ds
\end{aligned}
\tag{3.57}
$$

where $\widetilde{C}$ denotes a positive constant having the same dependencies as C and additionally depending on the quantities on the right hand side of (3.53). We have also used the fact that $p \in (\frac{6}{5}, \frac{3}{2}]$ and the continuous embedding $V \hookrightarrow L^{2p/(2-p)}(\Omega)$.

Combining (3.56) with (3.57), taking (3.53) into account, and making use of Hölder inequality and Gronwall lemma, we obtain that for any $t \in [0,T]$,

$$
\|\theta^n(t)\|^p_{L^p(\Omega)} \le \widetilde{C}\big(1 + \|\theta_0\|^p_{L^p(\Omega)} + \|f_n\|^2_{L^p(\Omega_T)}\big). \tag{3.58}
$$

Collecting (3.53), (3.55), and (3.58), owing to (3.23), we eventually deduce the a priori bounds

$$
\begin{aligned}
&\sqrt{\nu_n}\,\|u^n\|_{L^\infty(0,T;H)} + \|u^n\|_{L^2(0,T;V)} \\
&+\|\theta^n\|_{L^\infty(0,T;L^p(\Omega))} + \|\ln\theta^n\|_{L^\infty(0,T;L^1(\Omega))} + \|w^n\|_{L^2(0,T;V)} \\
&+\sqrt{\mu}\,\|\chi_t^n\|_{L^\infty(0,T;H)} + \|\chi_t^n\|_{L^2(0,T;H)} + \|\chi^n\|_{L^\infty(0,T;V)} \le K
\end{aligned}
\tag{3.59}
$$

with the constant K depending only on data and being independent of μ (cf. Theorem 2.4). In addition, recalling (H1)-(H2), (H4), (3.1), (3.3)-(3.5), and (3.31), on account of (3.59) and by comparison in (3.24) and (3.25), we can infer the further bounds

$$
\|\nu_n u^n + \theta^n\|_{H^1(0,T;V')} + \mu\|\chi_{tt}^n\|_{L^2(0,T;V')} \le K. \tag{3.60}
$$

Passage to the limit as $n \to +\infty$. In this subsection, all the convergences have to be understood for suitable subsequences. From (3.59) we deduce the existence of a pair (θ, χ) such that as $n \to +\infty$,

$$
\theta^n \to \theta \quad \text{weakly star in } \ L^\infty(0,T;L^p(\Omega)) \tag{3.61}
$$

$$
w^n \to w \quad \text{weakly in } \ L^2(0,T;V) \tag{3.62}
$$

$$
u^n \to u \quad \text{weakly in } \ L^2(0,T;V) \tag{3.63}
$$

$$\nu_n u^n \to 0 \quad \text{strongly in } \ C^0([0,T];H) \tag{3.64}$$

$$\chi^n \to \chi \quad \text{weakly star in } \ W^{1,\infty}(0,T;H)\cap L^\infty(0,T;V) \tag{3.65}$$

$$\chi^n_{tt} \to \chi_{tt} \quad \text{weakly in } \ L^2(0,T;V')\,. \tag{3.66}$$

Note that (3.59) and (3.60) entail

$$\|\nu_n u^n + \theta^n\|_{H^1(0,T;V')\cap L^\infty(0,T;L^p(\Omega))} \le K \tag{3.67}$$

and, due to the compact injection $L^p(\Omega)\hookrightarrow V'$ and (3.64), we have (see, e.g., [27, Corollary 8])

$$\theta^n \to \theta \quad \text{strongly in } \ C^0([0,T];V') \tag{3.68}$$

as $n\to+\infty$. Also, by (3.65) we infer that, as $n\to+\infty$,

$$\chi^n \to \chi \quad \text{strongly in } \ C^0([0,T];L^4(\Omega)). \tag{3.69}$$

We now have all the ingredients to pass to the limit as n goes to $+\infty$ in $\mathbf{P}^n_\mu$. First of all, let us analyze the nonlinearities. Observe that, for any $v\in L^2(0,T;H)$ such that $\rho(v)\in L^2(0,T;H)$, it turns out that

$$\rho_n(v)\to\rho(v) \quad \text{strongly in } \ L^2(0,T;H)$$

as $n\to+\infty$. Hence, recalling (3.31), (3.63) and (3.68), in view of the monotonicity of ρ_n and the maximal monotonicity of the graph induced by ρ on $\mathbb{R}\times\mathbb{R}$ and $L^2(Q_T)\times L^2(Q_T)$, taking the limit in

$$\int_0^T\int_\Omega (\theta_n-\rho_n(v))(u_n-v)\,dxdt$$

we obtain (cf., e.g., [3, Definition 2.2, p. 22])

$$u<0, \quad \theta=\rho(u) \tag{3.70}$$

almost everywhere in Q_T.

On the other hand, on account of (H1)-(H2), (3.1), (3.2), (3.59) and (3.69), one easily proves that, as n goes to $+\infty$,

$$\lambda_n'(\chi^n)\to\lambda'(\chi) \quad \text{strongly in } \ C^0([0,T];L^4(\Omega)). \tag{3.71}$$

Therefore, combining (3.65) with (3.71), we get

$$\lambda_n'(\chi^n)\chi^n_t \to \lambda'(\chi)\chi_t \quad \text{weakly in } \ L^2(0,T;L^{4/3}(\Omega)). \tag{3.72}$$

Since g_n uniformly converges to g on compact subsets of $\mathbb{R}$ and (cf. (3.69))

$$\chi_n\to\chi \quad \text{a.e. in } \ Q_T$$

at least for a subsequence, as n goes to $+\infty$ we have that

$$g_n(\chi^n)\to g(\chi) \quad \text{a.e. in } \ Q_T. \tag{3.73}$$

Also, using the injection $V\hookrightarrow L^6(\Omega)$, we infer

$$\begin{aligned}\int_0^T\int_\Omega |g_n(\chi^n(x,t))|^2dxdt &\le \int_0^T\int_\Omega |g(\chi^n(x,t))|^2dxdt\\ &\le \int_0^T\int_\Omega \big(\tau_3|\chi^n(x,t)|^3+\tau_4\big)^2dx\,dt\\ &\le C\left(\|\chi^n\|^6_{L^\infty(0,T;H^1(\Omega))}+1\right).\end{aligned}$$

Thus we get

$$\{g_n(\chi^n)\} \quad \text{is bounded in } L^2(0,T;L^2(\Omega)). \tag{3.74}$$

From (3.73) and (3.74) we deduce (see, e.g., [20, p. 13])

$$g_n(\chi^n) \to g(\chi) \quad \text{weakly in } L^2(0,T;H). \tag{3.75}$$

Recalling (3.31) and owing to the maximal monotonicity of the inverse graph of ρ, by (3.62), (3.68), and [3, Prop. 2.5, p. 27] we infer that $-w = -\theta^{-1}$ and consequently

$$\frac{1}{\theta^n} \to \frac{1}{\theta} \quad \text{weakly in } L^2(0,T;V). \tag{3.76}$$

Thus, (3.71) and (3.76) give

$$\frac{\lambda_n'(\chi^n)}{\theta^n} \to \frac{\lambda'(\chi)}{\theta} \quad \text{weakly in } L^2(0,T;H). \tag{3.77}$$

Summing up, the convergences (3.19), (3.23), (3.61)-(3.66), (3.72) (3.75), (3.77), along with (3.16), (3.70), allow us to pass to the limit in (3.24)-(3.26). Therefore, (θ,χ) happens to be a solution to $\mathbf{P}_\mu$. We recall that the regularity $\chi \in C^1([0,T];H)\cap C^0([0,T];V)$ follows from a standard argument for linear hyperbolic equations (see, for instance, [30, Lemma 4.1, p. 76]).

4. Proof of Theorem 2.4

We know that the solution (θ^μ,χ^μ) to $\mathbf{P}_\mu$ we have obtained from the limit procedure in our approximation scheme certainly satisfies the a priori bound (2.16), due to (3.59) and (3.60). Indeed, any bounding constant which appears in the previous proof does not depend on μ.

As a matter of fact, we now prove that any solution (θ^μ,χ^μ) to Problem $\mathbf{P}_\mu$ necessarily satisfies estimate (2.16). Indeed, on account of (H1)-(H4) and (2.1)-(2.2) we observe that

$$\hat g(\chi^\mu) \in H^1(0,T;L^1(\Omega)) \ \text{ and } \ (\hat g(\chi^\mu))_t = g(\chi^\mu)\chi^\mu_t \ \text{ a.e. in } Q \tag{4.1}$$

$$\lambda'(\chi^\mu)(\theta^\mu)^{-1} \in L^2(0,T;H) \tag{4.2}$$

$$F_\mu := (\lambda'(\chi^\mu))_t = \lambda'(\chi^\mu)\chi^\mu_t \in L^2(0,T;L^{3/2}(\Omega)). \tag{4.3}$$

Referring to the previous proof, we take a sequence $\{F_n\} \subset L^2(0,T;H)$ such that

$$F_n \to F_\mu \quad \text{in } L^2(0,T;L^{3/2}(\Omega)) \tag{4.4}$$

and we consider the Cauchy problem (cf. (3.24) and (3.26))

$$\langle(\nu_n u^n + \rho_n(u^n))_t, v\rangle + (\!(u^n,v)\!) = (f_n - F_n, v) + (h,v)_\Gamma \quad \forall\, v\in V, \text{ a.e. in } (0,T) \tag{4.5}$$

$$u^n(0) = u_{0n} \quad \text{a.e. in } \Omega. \tag{4.6}$$

Then, it is not difficult to realize that there exists a unique $u^n \in C^0([0,T];H) \cap L^2(0,T;V)$ which solves (4.5)-(4.6). Moreover, with the same positions as in (3.31), the estimates (3.39), (3.47), and (3.50) still hold with F_n in place of $(\lambda_n(\chi^n))_t$.

Therefore, multiplying (3.39) by 6 and adding it to (3.47) and (3.50), we deduce

$$\begin{aligned}
&\frac{1}{p}\|\theta^n(t)\|^p_{L^p(\Omega)} + \|\theta^n(t)\|_{L^1(\Omega)} + \|\ln\theta^n(t)\|_{L^1(\Omega)} \\
&+ 6\int_0^t (\!(w^n(s), w^n(s))\!)\,ds - 6\gamma\int_0^t \|w^n(s)\|_{L^1(\Gamma)}\,ds \\
&- \gamma\int_0^t (w^n(s), (\theta^n(s))^{p-1})_\Gamma\,ds + \frac{\nu_n}{2}\|u^n(t)\|^2 + \int_0^t (\!(u^n(s), u^n(s))\!)\,ds \\
&\le C\big(1 + \|\theta_0\|_{L^1(\Omega)} + \|\ln\theta_0\|_{L^1(\Omega)} + \|\theta_0\|^p_{L^p(\Omega)}\big) \\
&\quad + 6\int_0^t (f_n(s) - F_n(s), 1 - w^n(s))\,ds + 6\int_0^t (h(s), 1 - w^n(s))_\Gamma\,ds \\
&\quad + \int_0^t (f_n(s) - F_n(s), (\theta^n(s))^{p-1} - 1)\,ds + \int_0^t (h(s), (\theta^n(s))^{p-1} - 1)_\Gamma\,ds \\
&\quad + \int_0^t (f_n(s) - F_n(s), u^n(s))\,ds + \int_0^t (h(s), u^n(s))_\Gamma\,ds.
\end{aligned} \tag{4.7}$$

Then, thanks to (H8) and (3.31) we infer

$$\begin{aligned}
&\frac{1}{p}\|\theta^n(t)\|^p_{L^p(\Omega)} + \|\theta^n(t)\|_{L^1(\Omega)} + \|\ln\theta^n(t)\|_{L^1(\Omega)} \\
&+6\int_0^t (\!(w^n(s), w^n(s))\!)\,ds + \frac{\nu_n}{2}\|u^n(t)\|^2 + \int_0^t (\!(u^n(s), u^n(s))\!)\,ds \\
&\le 6\gamma\int_0^t \|w^n(s)\|_{L^1(\Gamma)}\,ds + \gamma\int_0^t \|(w^n(s))^{2-p}\|_{L^1(\Gamma)}\,ds \\
&\quad + C\big(1 + \|\theta_0\|_{L^1(\Omega)} + \|\ln\theta_0\|_{L^1(\Omega)} + \|\theta_0\|^p_{L^p(\Omega)}\big) \\
&\quad + 6\int_0^t (f_n(s) - F_n(s), 1 - w^n(s))\,ds - 6\int_0^t (h(s), w^n(s))_\Gamma\,ds \\
&\quad + \int_0^t (f_n(s) - F_n(s), (\theta^n(s))^{p-1} - 1)\,ds + 5\int_0^t \|h(s)\|_{L^1(\Gamma)}\,ds \\
&\quad + \int_0^t (f_n(s) - F_n(s), u^n(s))\,ds + \int_0^t (h(s), u^n(s))_\Gamma\,ds.
\end{aligned}$$

Using now Young inequality, from the above inequality we get (cf. also (3.31))

$$\begin{aligned}
&\frac{1}{p}\|\theta^n(t)\|^p_{L^p(\Omega)} + \|\theta^n(t)\|_{L^1(\Omega)} + \|\ln\theta^n(t)\|_{L^1(\Omega)} \\
&+ \int_0^t (\!((\theta^n)^{-1}(s), (\theta^n)^{-1}(s))\!)\,ds + \frac{\nu_n}{2}\|u^n(t)\|^2 + \frac{1}{2}\int_0^t (\!(u^n(s), u^n(s))\!)\,ds \\
&\le C\left(1 + \|\theta_0\|^p_{L^p(\Omega)} + \|\ln\theta_0\|_{L^1(\Omega)} + \|h\|^2_{L^2(\Gamma_T)}\right) \\
&\quad + 5\int_0^t \|f_n(s) - F_n(s)\|_{L^1(\Omega)}\,ds - 6\int_0^t (f_n(s) - F_n(s), (\theta^n(s))^{-1}(s))\,ds \\
&\quad + \int_0^t (f_n(s) - F_n(s), (\theta^n(s))^{p-1})\,ds + \int_0^t (f_n(s) - F_n(s), u^n(s))\,ds.
\end{aligned} \tag{4.8}$$

Applying Young inequality once more, we have

$$\int_0^t (f_n(s) - F_n(s), (\theta^n(s))^{p-1})\, ds \\ \le C \int_0^t \Big(\|f_n(s) - F_n(s)\|^p_{L^p(\Omega)} + \|\theta^n(s)\|^p_{L^p(\Omega)} \Big)\, ds. \quad (4.9)$$

Then, on account of (4.9), from (4.8) we easily deduce

$$\begin{aligned} &\|\theta^n(t)\|^p_{L^p(\Omega)} + \|\theta^n(t)\|_{L^1(\Omega)} + \|\ln\theta^n(t)\|_{L^1(\Omega)} \\ &+ \int_0^t \|(\theta^n)^{-1}(s)\|^2_V\, ds + \nu_n \|u^n(t)\|^2 + \int_0^t \|u^n(s)\|^2_V\, ds \\ &\le C\Big(1 + \|\theta_0\|^p_{L^p(\Omega)} + \|\ln\theta_0\|_{L^1(\Omega)} + \|h\|^2_{L^2(\Gamma_T)} \\ &\quad + \int_0^t \|f_n(s) - F_n(s)\|^p_{L^p(\Omega)}\, ds + \int_0^t \|\theta^n(s)\|^p_{L^p(\Omega)}\, ds \\ &\quad + \int_0^t \|f_n(s) - F_n(s)\|_{V'} \Big(\|(\theta^n)^{-1}(s)\|_V + \|u^n(s)\|_V \Big) ds \Big) \end{aligned}$$

and, recalling the embedding $L^p(\Omega) \hookrightarrow V'$ and using Young inequality, we obtain

$$\begin{aligned} &\|\theta^n(t)\|^p_{L^p(\Omega)} + \|\theta^n(t)\|_{L^1(\Omega)} + \|\ln\theta^n(t)\|_{L^1(\Omega)} \\ &+ \int_0^t \|(\theta^n)^{-1}(s)\|^2_V\, ds + \nu_n \|u^n(t)\|^2 + \int_0^t \|u^n(s)\|^2_V\, ds \\ &\le C\Big(1 + \|\theta_0\|^p_{L^p(\Omega)} + \|\ln\theta_0\|_{L^1(\Omega)} + \|h\|^2_{L^2(\Gamma_T)} \\ &\quad + \int_0^t \|f_n(s) - F_n(s)\|^2_{L^p(\Omega)}\, ds + \int_0^t \|\theta^n(s)\|^p_{L^p(\Omega)}\, ds \Big). \end{aligned} \quad (4.10)$$

An application of Gronwall lemma to (4.10) yields the bound (cf. also (3.23) and (4.4))

$$\sqrt{\nu_n}\, \|u^n\|_{L^\infty(0,T;H)} + \|u^n\|_{L^2(0,T;V)} + \|\theta^n\|_{L^\infty(0,T;L^p(\Omega))} \\ + \|\ln\theta^n\|_{L^\infty(0,T;L^1(\Omega))} + \|(\theta^n)^{-1}\|_{L^2(0,T;V)} \le K_\mu. \quad (4.11)$$

Here, K_μ denotes a generic positive constant which does depend on the quantity $\|F_\mu\|_{L^2(0,T;L^{3/2}(\Omega))}$, but it is independent of n. Consequently, by comparison in (4.5), we also have

$$\|\nu_n u^n + \theta^n\|_{H^1(0,T;V')} \le K_\mu. \quad (4.12)$$

Then, arguing as in the last subsection of the existence proof, we obtain

$$\theta^n \to \theta^\mu \quad \text{weakly star in } L^\infty(0,T;L^p(\Omega)) \quad (4.13)$$

$$\theta^n \to \theta^\mu \quad \text{strongly in } C^0([0,T];V') \quad (4.14)$$

$$u^n \to -(\theta^\mu)^{-1} \quad \text{weakly in } L^2(0,T;V) \quad (4.15)$$

$$(\theta^n)^{-1} \to (\theta^\mu)^{-1} \quad \text{weakly in } L^2(0,T;V) \quad (4.16)$$

$$\nu_n u^n \to 0 \quad \text{strongly in } C^0([0,T];H) \quad (4.17)$$

where the limit θ^μ should solve the Cauchy problem (cf. (4.5) and (4.6))

$$\langle \theta^\mu_t, v\rangle + (\!(-(\theta^\mu)^{-1}, v)\!) = \langle f - F_\mu, v\rangle + (h, v)_\Gamma \quad \forall\, v \in V, \text{ a.e. in } (0,T)$$

$$\theta^\mu(0) = \theta_0 \quad \text{a.e. in } \Omega$$

and therefore (cf. (4.3)) coincide with the first component of the solution (θ^μ, χ^μ) to $\mathbf{P}_\mu$ fixed at the beginning of this section. Indeed, the above Cauchy problem admits a unique (positive) solution. Uniqueness can be checked by a contradiction argument (see, e.g., the proof of Theorem 2.6 below).

Observe now that, owing to (4.14), we have, for any $t \in [0,T]$,

$$\theta^n(t) \to \theta^\mu(t) \quad \text{in } V'. \tag{4.18}$$

On the other hand, since $t \to \theta^n(t)$ is weakly continuous from $[0,T]$ to $L^p(\Omega)$ and $\{\theta^n\}$ is bounded in $L^\infty(0,T;L^p(\Omega))$ (cf. (4.11)), it follows that, for any $t \in [0,T]$, there exists a subsequence $\{\theta^{n_k}(t)\}$ and some element $\eta^t \in L^p(\Omega)$ such that

$$\theta^{n_k}(t) \to \eta^t \quad \text{weakly in } L^p(\Omega). \tag{4.19}$$

Hence, combining (4.18) with (4.19) and exploiting the uniqueness of the first limit, we deduce

$$\theta^n(t) \to \theta^\mu(t) \quad \text{weakly in } L^p(\Omega). \tag{4.20}$$

Then, since (3.39) holds with w^n, $(\lambda_n(\chi^n))_t$ replaced by $(\theta^n)^{-1}$, F_n, respectively, by (H8) and Hölder and Young inequalities we have

$$\begin{aligned}
&\frac{1}{3}\|\theta^n(t)\|_{L^1(\Omega)} + \int_0^t \|\nabla(\theta^n)^{-1}\|^2\,ds + \frac{\gamma}{2}\int_0^t \|(\theta^n)^{-1}(s)\|^2_{L^2(\Gamma)}\,ds \\
&\le C\big(1 + \|\theta_0\|_{L^1(\Omega)} + \|\ln\theta_0\|_{L^1(\Omega)} + \|h\|^2_{L^2(\Gamma_T)}\big) \\
&\quad + \int_0^t \|f_n(s) - F_n(s)\|_{L^1(\Omega)}\,ds - \int_0^t (f_n(s) - F_n(s), (\theta^n)^{-1}(s))\,ds.
\end{aligned} \tag{4.21}$$

On account of (3.23) and (4.4), using (4.16), (4.20) and the (weak) lower semicontinuity of the norm, we deduce from (4.21) the following inequality

$$\begin{aligned}
&\frac{1}{3}\|\theta^\mu(t)\|_{L^1(\Omega)} + \int_0^t \|\nabla(\theta^\mu)^{-1}\|^2\,ds + \frac{\gamma}{2}\int_0^t \|(\theta^\mu)^{-1}(s)\|^2_{L^2(\Gamma)}\,ds \\
&\le C\Big(1 + \|\theta_0\|_{L^1(\Omega)} + \|\ln\theta_0\|_{L^1(\Omega)} + \|h\|^2_{L^2(\Gamma_T)}\Big) \\
&\quad + \int_0^t \|f(s) - F_\mu(s)\|_{L^1(\Omega)}\,ds - \int_0^t (f(s) - F_\mu(s), (\theta^\mu)^{-1}(s))\,ds.
\end{aligned} \tag{4.22}$$

Observe now that our fixed χ_μ satisfies equation (2.6) and related initial conditions in (2.7). Hence, we can formally take $v = \chi^\mu_t(t)$ in (2.6) and integrate with respect to time over $(0,t)$. In view of (4.1) and (4.3), we get the energy identity

$$\begin{aligned}
&\frac{\mu}{2}\|\chi^\mu_t(t)\|^2 + \frac{1}{2}\|\nabla\chi^\mu(t)\|^2 + \int_\Omega \hat{g}(\chi_\mu(t))dx + \int_0^t \|\chi^\mu_s(s)\|^2\,ds \\
&\quad = \frac{\mu}{2}\|\chi_1\|^2 + \frac{1}{2}\|\nabla\chi_0\|^2 + \int_\Omega \hat{g}(\chi_0)dx - \int_0^t (F_\mu(s), (\theta^\mu)^{-1}(s)(s))\,ds.
\end{aligned} \tag{4.23}$$

We recall that the above argument can be made rigorous by using a suitable regularization of $\chi^\mu_t(t)$ (see [4, Appendix]).

Adding (4.22) and (4.23) and arguing as we did in the previous proof to obtain (3.53), we deduce the estimate

$$\begin{aligned}
&\|\theta^\mu(t)\|_{L^1(\Omega)} + \int_0^t \|\nabla(\theta^\mu)^{-1}(s)\|^2\,ds + \gamma\int_0^t \|(\theta^\mu)^{-1}(s)\|_\Gamma^2\,ds \\
&+ \mu\|\chi_t^n(t)\|^2 + \|\nabla\chi^n(t)\|^2 + \int_0^t \|\chi_s^n(s)\|^2\,ds \\
&\le C\Big(1 + \|\theta_0\|_{L^1(\Omega)} + \|\ln\theta_0\|_{L^1(\Omega)} + \|\chi_1\|^2 + \|\chi_0\|_V^4 \\
&\quad + \|f\|^2_{L^2(0,T;L^p(\Omega))} + \|h\|^2_{L^2(\Gamma_T)}\Big)
\end{aligned} \tag{4.24}$$

for any $t \in [0,T]$. In the light of the definition (4.3) of F_μ, it turns out that (4.24) yields a fortiori a bound for $\|F_\mu\|_{L^2(0,T;L^{3/2}(\Omega))}$ independent of μ. Hence, from (4.11) and (4.13) we conclude that

$$\|\theta^\mu\|_{L^\infty(0,T;L^p(\Omega))} \le K. \tag{4.25}$$

Moreover, a comparison in (2.5) entails

$$\|\theta_t^\mu\|_{L^2(0,T;V')} \le K. \tag{4.26}$$

Finally, (4.24)-(4.26) enable us to deduce (2.16).

Thanks to (2.16), there exist a sequence $\{\mu_n\}$ that converges to 0 and a pair (θ,χ) such that

$$\theta^{\mu_n} \to \theta \quad \text{weakly star in } L^\infty(0,T;L^p(\Omega)) \tag{4.27}$$

$$\theta^{\mu_n} \to \theta \quad \text{weakly in } H^1(0,T;V') \tag{4.28}$$

$$\theta^{\mu_n} \to \theta \quad \text{strongly in } C^0([0,T];V') \tag{4.29}$$

$$\frac{1}{\theta^{\mu_n}} \to \frac{1}{\theta} \quad \text{weakly in } L^2(0,T;V) \tag{4.30}$$

$$\mu_n\chi_t^{\mu_n} \to 0 \quad \text{strongly in } C^0([0,T];H) \tag{4.31}$$

$$\chi^{\mu_n} \to \chi \quad \text{weakly star in } L^\infty(0,T;V) \tag{4.32}$$

$$\chi^{\mu_n} \to \chi \quad \text{weakly in } H^1(0,T;H) \tag{4.33}$$

$$\chi^{\mu_n} \to \chi \quad \text{strongly in } C^0([0,T];L^4(\Omega)) \tag{4.34}$$

as $n \to +\infty$ (cf. also the last subsection of the existence proof).

Integrating equation (2.6) with respect to time over $(0,t)$ and taking initial conditions into account, we obtain

$$\begin{aligned}
&(\mu_n\chi_t^{\mu_n} + \chi^{\mu_n}, v) + (\nabla(1*\chi^{\mu_n}),\nabla v) + \big(1*(g(\chi^{\mu_n}) + \lambda'(\chi^{\mu_n})(\theta^{\mu_n})^{-1})\,, v\big) \\
&\qquad = (\mu_n\chi_1 + \chi_0, v) \quad \forall\, v \in V, \text{ a.e. in } (0,T).
\end{aligned} \tag{4.35}$$

Then, thanks to (4.27)-(4.34), we can pass to the limit as $n \to +\infty$ in (2.5) and (4.35), arguing as above for the nonlinearities, and we can deduce that (θ,χ) satisfies the equations

$$\langle(\theta + \lambda(\chi))_t, v\rangle + (\!(-\theta^{-1}, v)\!) = \langle f, v\rangle + (h,v)_\Gamma \quad \forall\, v \in V, \text{ a.e. in } (0,T) \tag{4.36}$$

$$\begin{aligned}
&(\chi, v) + (\nabla(1*\chi),\nabla v) + \big(1*(g(\chi) + \lambda'(\chi)\theta^{-1}), v\big) = (\chi_0, v) \\
&\qquad \forall\, v \in V, \text{ a.e. in } (0,T)
\end{aligned} \tag{4.37}$$

$$\theta(0) = \theta_0 \quad \text{in } V'. \tag{4.38}$$

On the other hand, owing to (H1)-(H4), (H11) and (4.30), (4.32)-(4.34), from (4.37) we can infer

$$(\chi_t, v) + (\nabla\chi, \nabla v) + \big(g(\chi) + \lambda'(\chi)\theta^{-1}, v\big) = 0 \quad \forall\, v \in V, \text{ a.e. in } (0,T) \tag{4.39}$$

$$\chi(0) = \chi_0 \quad \text{a.e. in } \Omega. \tag{4.40}$$

Moreover, since

$$g(\chi) + \lambda'(\chi)\theta^{-1} - \chi_t \in L^2(0,T;H)$$

we deduce from (4.39) and the standard elliptic regularity theory that χ satisfies (2.11) and (2.14). Hence, the pair (θ,χ) fulfills (2.8)-(2.15) and solves problem $\mathbf{P}_0$. Consequently, the uniqueness of solutions to $\mathbf{P}_0$ (cf. Theorem 2.10) implies that the whole family $\{(\theta^\mu, \chi^\mu)\}$ converges to (θ,χ) according to (4.27)-(4.34) as $\mu \searrow 0$, and Theorem 2.4 is completely proved.

5. Proof of Theorem 2.6

Observe that, for $j = 1, 2$, (θ_j, χ_j) solves the problem $\mathbf{P}_\mu$ with θ_{0j}, f_j, h_j, χ_{0j}, χ_{1j} in place of θ_0, f, h, χ_0, χ_1, respectively, and $u_j = -\theta_j^{-1}$. Then, setting

$$\theta = \theta_1 - \theta_2, \quad u = u_1 - u_2, \quad \chi = \chi_1 - \chi_2$$

$$\theta^0 = \theta_{01} - \theta_{02}, \quad f = f_1 - f_2, \quad h = h_1 - h_2$$

$$\chi^0 = \chi_{01} - \chi_{02}, \quad \chi^1 = \chi_{11} - \chi_{12}$$

we have

$$\langle(\theta + \lambda(\chi_1) - \lambda(\chi_2))_t, v\rangle + (\!(u,v)\!) = \langle f, v\rangle + (h,v)_\Gamma \quad \forall\, v \in V, \text{ a.e. in } (0,T) \tag{5.1}$$

$$\langle \mu\chi_{tt}, v\rangle + (\chi_t, v) + (\nabla\chi, \nabla v) + (g(\chi_1) - g(\chi_2) - \lambda'(\chi_1)u_1 + \lambda'(\chi_2)u_2, v) = 0 \quad \forall\, v \in V, \text{ a.e. in } (0,T) \tag{5.2}$$

$$\theta(0) = \theta^0, \quad \chi(0) = \chi^0, \quad \chi_t(0) = \chi^1 \quad \text{a.e. in } \Omega. \tag{5.3}$$

Let us integrate (5.1) with respect to time over $(0,t)$, take $v = u(t)$ and integrate in time once more. We obtain

$$\begin{aligned}
&\int_0^t (\theta(s), u(s))\, ds + \frac{1}{2}(\!((1*u)(t), (1*u)(t))\!) \\
&= -\int_0^t (\lambda(\chi_1(s)) - \lambda(\chi_2(s)), u(s))\, ds \\
&\quad + \int_0^t \langle \theta^0 + \lambda(\chi_{01}) - \lambda(\chi_{02}) + (1*f)(s), u(s)\rangle\, ds + \int_0^t ((1*h)(s), u(s))_\Gamma\, ds.
\end{aligned} \tag{5.4}$$

Observe now that

$$(\theta(s), u(s)) \ge \frac{1}{2}\int_\Omega \frac{|\theta(s)|^2}{1 + |\theta_1(s)|^2 + |\theta_2(s)|^2}\, dx + \frac{1}{2}\int_\Omega \frac{|u(s)|^2}{1 + |u_1(s)|^2 + |u_2(s)|^2}\, dx. \tag{5.5}$$

Integrations by parts lead to

$$\begin{aligned}
&\int_0^t \langle \theta^0 + \lambda(\chi_{01}) - \lambda(\chi_{02}) + (1*f)(s), u(s)\rangle\, ds \\
&= \langle \theta^0 + \lambda(\chi_{01}) - \lambda(\chi_{02}) + (1*f)(t), (1*u)(t)\rangle - \int_0^t \langle f(s), (1*u)(s)\rangle\, ds
\end{aligned} \tag{5.6}$$

and

$$\int_0^t ((1*h)(s), u(s))_\Gamma\, ds = ((1*h)(t), (1*u)(t))_\Gamma - \int_0^t (h(s), (1*u)(s))_\Gamma\, ds. \tag{5.7}$$

Then, recalling (H1) and (H13) and using (5.5)-(5.7) and Young inequality, from (5.4) we infer

$$\begin{aligned}
&\int_0^t\int_\Omega \frac{|\theta|^2}{1+|\theta_1|^2+|\theta_2|^2}\,dx\,ds + \int_0^t\int_\Omega \frac{|u|^2}{1+|u_1|^2+|u_2|}\,dx\,ds + \|(1*u)(t)\|_V^2 \\
&\le C\Big(\int_0^t\int_\Omega |\chi|\,|u|\,dx\,ds + \int_0^t \|(1*u)(s)\|_V^2\,ds + \|\theta^0\|_{V'} + \|\chi^0\| \\
&\quad + \|(1*f)(t)\|_{V'}^2 + \|f\|_{L^2(0,T;V')}^2 + \|(1*h)(t)\|_{L^2(\Gamma)}^2 + \|h\|_{L^2(\Gamma_T)}^2\Big).
\end{aligned} \tag{5.8}$$

Here and in the sequel of the proof, C denotes a positive constant depending on T, Ω, γ, λ, and g, at most. Note that if $N=1$ there is no need of (H13) since $C^0([0,T];V) \hookrightarrow C^0(\overline{Q}_T)$. In this case the constant C depends on M_1 as well. We now have

$$\begin{aligned}
&\int_\Omega |\chi(s)|\,|u(s)|\,dx \\
&= \int_\Omega \frac{|u(s)|}{\sqrt{1+|u_1(s)|^2+|u_2(s)|^2}}\sqrt{1+|u_1(s)|^2+|u_2(s)|^2}|\chi(s)|\,dx \\
&\le \frac{1}{2}\int_\Omega \frac{|u(s)|^2}{1+|u_1(s)|^2+|u_2(s)|^2}\,dx + \frac{1}{2}\int_\Omega \big(1+|u_1(s)|^2+|u_2(s)|^2\big)\,|\chi(s)|^2\,dx
\end{aligned} \tag{5.9}$$

so that we deduce

$$\int_\Omega |\chi(s)|\,|u(s)|\,dx \le \frac{1}{2}\int_\Omega \frac{|u(s)|^2}{1+|u_1(s)|^2+|u_2(s)|^2}\,dx + C\Lambda(s)\|\chi(s)\|^2 \tag{5.10}$$

where

$$\Lambda(t) = 1 + \|u_1(t)\|_{L^\infty(\Omega)}^2 + \|u_2(t)\|_{L^\infty(\Omega)}^2 \quad \text{for a.a. } t \in (0,T). \tag{5.11}$$

Note that $\Lambda \in L^1(0,T)$, due to (2.17). Then, a combination of (5.8) with (5.10) gives

$$\begin{aligned}
&\int_0^t\int_\Omega \frac{|\theta|^2}{1+|\theta_1|^2+|\theta_2|^2}\,dx\,ds + \int_0^t\int_\Omega \frac{|u|^2}{1+|u_1|^2+|u_2|}\,dx\,ds + \|(1*u)(t)\|_V^2 \\
&\le C\Big(\|\theta^0\|_{V'}^2 + \|\chi^0\|_H^2 + \|f\|_{L^2(0,T;V')}^2 + \|h\|_{L^2(\Gamma_T)}^2 \\
&\quad + \int_0^t \|(1*u)(s)\|_V^2\,ds + \int_0^t \Lambda(s)\|\chi(s)\|^2\,ds\Big).
\end{aligned} \tag{5.12}$$

Take now $v = \chi_t$ in (5.2) and integrate over $(0,t)$. Thus, we obtain

$$\begin{aligned}
&\frac{\mu}{2}\|\chi_t(t)\|^2 + \int_0^t \|\chi_s(s)\|^2\,ds + \frac{1}{2}\|\nabla\chi(t)\|^2 \\
&= \frac{\mu}{2}\|\chi^1\|^2 + \frac{1}{2}\|\nabla\chi^0\|^2 - \int_0^t (g(\chi_1(s)) - g(\chi_2(s)), \chi_s(s))\,ds \\
&\quad + \int_0^t ((\lambda'(\chi_1(s)) - \lambda'(\chi_2(s)))u_1(s) + \lambda'(\chi_2(s))u(s), \chi_s(s))\,ds.
\end{aligned} \tag{5.13}$$

We notice once more that this argument is formal since $\chi_t(t)$ does not belong to V; however, it can be made rigorous using, for instance, [4, Appendix].

Observe that, thanks to (H14), Hölder inequality and the injection $V \hookrightarrow L^6(\Omega)$, we have

$$
\begin{aligned}
&\|g(\chi_1(s)) - g(\chi_2(s))\|^2 \\
&\le c_1^2 \int_\Omega (1 + |\chi_1(s)|^2 + |\chi_2(s)|^2)^2 |\chi(s)|^2 \, dx \\
&\le C \Big(1 + \|\chi_1(s)\|^4_{L^6(\Omega)} + \|\chi_2(s)\|^4_{L^6(\Omega)}\Big) \|\chi(s)\|^2_{L^6(\Omega)} \\
&\le C \Big(1 + \|\chi_1\|^4_{L^\infty(0,T;V)} + \|\chi_2\|^4_{L^\infty(0,T;V)}\Big) \|\chi(s)\|^2_V .
\end{aligned}
\tag{5.14}
$$

On the other hand, due to (H1)-(H2), (2.17), and (5.11) we deduce that

$$
\int_\Omega |\lambda'(\chi_1(s)) - \lambda'(\chi_2(s))| \, |u_1(s)| |\chi_t(s)| \, dx \le C\Lambda(s) \int_\Omega |\chi(s)| \, |\chi_t(s)| \, dx. \tag{5.15}
$$

Moreover, thanks to (H13), we have (cf. (5.10))

$$
\begin{aligned}
\int_\Omega |\lambda'(\chi_2(s)) u(s) \chi_t(s)| \, dx \le & C \int_\Omega |u(s)\chi_t(s)| \, dx \\
\le & \frac{1}{2} \int_\Omega \frac{|u(s)|^2}{1 + |u_1(s)|^2 + |u_2(s)|^2} \, dx + C\Lambda(s) \|\chi_t(s)\|^2 .
\end{aligned}
\tag{5.16}
$$

Collecting (5.13)-(5.16) and using Hölder inequality, we obtain

$$
\begin{aligned}
&\frac{\mu}{2} \|\chi_t(t)\|^2 + \int_0^t \|\chi_s(s)\|^2 \, ds + \frac{1}{2} \|\nabla \chi(t)\|^2 \\
&\le \frac{\mu}{2} \|\chi^1\|^2 + \frac{1}{2} \|\nabla \chi^0\|^2 + C \int_0^t \Lambda(s) \left(\|\chi(s)\|^2 + \|\chi_s(s)\|^2\right) ds \\
&\quad + C \Big(1 + \|\chi_1\|^4_{L^\infty(0,T;V)} + \|\chi_2\|^4_{L^\infty(0,T;V)}\Big) \int_0^t \|\chi(s)\|^2_V \, ds \\
&\quad + \frac{1}{2} \int_0^t \int_\Omega \frac{|u|^2}{1 + |u_1|^2 + |u_2|^2} \, dx \, ds.
\end{aligned}
\tag{5.17}
$$

Hence, a combination of (5.12) with (5.17) gives

$$
\begin{aligned}
&\int_0^t \int_\Omega \frac{|\theta|^2}{1 + |\theta_1|^2 + |\theta_2|^2} \, dx \, ds + \frac{1}{2} \int_0^t \int_\Omega \frac{|u|^2}{1 + |u_1|^2 + |u_2|} \, dx \, ds \\
&+ \|(1 * u)(t)\|^2_V + \frac{\mu}{2} \|\chi_t(t)\|^2 + \int_0^t \|\chi_s(s)\|^2 \, ds + \frac{1}{2} \|\nabla \chi(t)\|^2 \\
&\le C \Big(\|\theta^0\|^2_{V'} + \|\chi^0\|^2_V + \mu \|\chi^1\|^2 + \|f\|^2_{L^2(0,T;V')} + \|h\|^2_{L^2(\Gamma_T)} \\
&\quad + \int_0^t \|(1 * u)(s)\|^2_V \, ds + \int_0^t \Lambda(s) \left(\|\chi(s)\|^2 + \|\chi_s(s)\|^2\right) ds \\
&\quad + \Big(1 + \|\chi_1\|^4_{L^\infty(0,T;V)} + \|\chi_2\|^4_{L^\infty(0,T;V)}\Big) \int_0^t \|\chi(s)\|^2_V \, ds \Big)
\end{aligned}
$$

and eventually an application of Gronwall lemma yields (2.18).

6. Proof of Theorem 2.8

Referring to the previous proof, observe that, owing to (H15), (5.4) becomes

$$\begin{aligned}&\int_0^t (\theta(s), u(s))\, ds + \frac{1}{2}(((1*u)(t), (1*u)(t)))\\&= -\int_0^t (\chi(s), u(s))\, ds - \int_0^t \langle \theta^0 + \chi^0 + (1*f)(s), u(s)\rangle\, ds \\&\quad + \int_0^t ((1*h)(s), u(s))_\Gamma\, ds.\end{aligned} \tag{6.1}$$

On the other hand, integrating by parts with respect to time yields

$$\int_0^t (\chi(s), u(s))\, ds = (\chi(s), (1*u)(s)) - \int_0^t (\chi_s(s), (1*u)(s))\, ds. \tag{6.2}$$

Therefore, recalling (5.5)-(5.7), the analog of (5.8) reads

$$\begin{aligned}&\int_0^t\int_\Omega \frac{|\theta|^2}{1+|\theta_1|^2+|\theta_2|^2}\, dx\, ds + \int_0^t\int_\Omega \frac{|u|^2}{1+|u_1|^2+|u_2|}\, dx\, ds + \|(1*u)(t)\|_V^2\\&\le C\Big(\|\chi(s)\|_{V'}\|(1*u)(s)\|_V + \int_0^t \|\chi_s(s)\|_{V'}\|(1*u)(s)\|_V\, ds + \int_0^t \|(1*u)(s)\|_V^2\, ds\\&\quad + \|\theta^0+\chi^0+(1*f)(t)\|_{V'}^2 + \|f\|^2_{L^2(0,T;V')} + \|(1*h)(t)\|^2_{L^2(\Gamma)} + \|h\|^2_{L^2(\Gamma_T)}\Big).\end{aligned}$$

Here and in the sequel of this proof, C stands for a positive constant that depends on T, Ω, γ, μ, and c_1, at most. Other possible dependencies will be pointed out explicitly.

Using then Young and Hölder inequalities, we easily deduce

$$\begin{aligned}&\int_0^t\int_\Omega \frac{|\theta|^2}{1+|\theta_1|^2+|\theta_2|^2}\, dx\, ds + \int_0^t\int_\Omega \frac{|u|^2}{1+|u_1|^2+|u_2|}\, dx\, ds + \|(1*u)(t)\|_V^2\\&\le C\Big(\|\chi(s)\|_{V'}^2 + \int_0^t \|\chi_s(s)\|_{V'}^2\, ds + \int_0^t \|(1*u)(s)\|_V^2\, ds\\&\quad + \|\theta^0\|_{V'}^2 + \|\chi^0\|^2 + \|f\|^2_{L^2(0,T;V')} + \|h\|^2_{L^2(\Gamma_T)}\Big).\end{aligned} \tag{6.3}$$

Consider now equation (5.2) and integrate it with respect to time. Recalling (H15) and (5.3), we obtain

$$\begin{aligned}&\langle \mu\chi_t, v\rangle + (\chi, v) + (\nabla(1*\chi), \nabla v) + (1*(g(\chi_1)-g(\chi_2)) - 1*u, v)\\&\qquad = \langle \mu\chi^1, v\rangle + (\chi^0, v) \qquad \forall\, v\in V,\ \text{a.e. in } (0,T).\end{aligned} \tag{6.4}$$

Pick $v=\chi$ in (6.4) and integrate with respect to time once again. We get

$$\begin{aligned}&\frac{\mu}{2}\|\chi(t)\|^2 + \int_0^t \|\chi(s)\|^2\, ds + \frac{1}{2}\|\nabla(1*\chi)(t)\|^2\\&= -\int_0^t ([1*(g(\chi_1)-g(\chi_2))](s) - (1*u)(s), \chi(s))\, ds\\&\quad + \frac{\mu}{2}\|\chi^0\|^2 + (\mu\chi^1+\chi^0, (1*\chi)(s))\end{aligned} \tag{6.5}$$

for $t \in [0,T]$. An integration by parts yields

$$\int_0^t ([1*(g(\chi_1)-g(\chi_2))](s),\chi(s))\,ds = ([1*(g(\chi_1)-g(\chi_2))](t),(1*\chi)(t)) - \int_0^t (g(\chi_1(s))-g(\chi_2(s)),(1*\chi)(s))\,ds. \tag{6.6}$$

Observe now that, using (H14), Hölder inequality and the injection $V \hookrightarrow L^6(\Omega)$, we have

$$\begin{aligned} &\|g(\chi_1(s))-g(\chi_2(s))\|_{L^{6/5}(\Omega)} \\ &\le c_1\Big\{\int_\Omega (1+|\chi_1(s)|^2+|\chi_2(s)|^2)^{6/5}|\chi(s)|^{6/5}\,dx\Big\}^{5/6} \\ &\le C\Big\{\int_\Omega (1+|\chi_1(s)|^6+|\chi_2(s)|^6)\,dx\Big\}^{1/3}\|\chi(s)\| \\ &\le C\Big(1+\|\chi_1\|^2_{L^\infty(0,T;V)}+\|\chi_2\|^2_{L^\infty(0,T;V)}\Big)\|\chi(s)\|. \end{aligned} \tag{6.7}$$

Hence, on account of (6.7) and Young inequality, from (6.6) we deduce

$$\begin{aligned} &-\int_0^t ([1*(g(\chi_1)-g(\chi_2))](s),\chi(s))\,ds \\ &\le C\int_0^t \|g(\chi_1(s))-g(\chi_2(s))\|^2_{L^{6/5}(\Omega)}\,ds + \frac{1}{8}\|(1*\chi)(t)\|_V^2 + \int_0^t \|(1*\chi)(s)\|_V^2\,ds \\ &\le C(M_2)\int_0^t \|\chi(s)\|^2\,ds + \frac{1}{8}\|(1*\chi)(t)\|_V^2 + \int_0^t \|(1*\chi)(s)\|_V^2\,ds. \end{aligned} \tag{6.8}$$

Using (6.8) and Young inequality once more, we infer from (6.5)

$$\begin{aligned} &\frac{\mu}{2}\|\chi(t)\|^2 + \frac{1}{2}\int_0^t \|\chi(s)\|^2\,ds + \frac{1}{4}\|\nabla(1*\chi)(t)\|^2 \\ &\le C\Big(\|\chi^0\|^2 + \|\chi^1\|^2_{V'} + \int_0^t \|(1*u)(s)\|^2\,ds \\ &\quad + \int_0^t \|\nabla(1*\chi)(s)\|^2\,ds\Big) + C(M_2)\int_0^t \|\chi(s)\|^2\,ds. \end{aligned} \tag{6.9}$$

Thanks to (6.7) and (6.9), by comparison in equation (6.4) we also derive

$$\begin{aligned} \mu^2\|\chi_t(t)\|^2_{V'} &\le C(M_2)\Big(\|\chi_0\|^2 + \|\chi_1\|^2_{V'} + \int_0^t \|(1*u)(s)\|^2\,ds \\ &\quad + \int_0^t \|\chi(s)\|^2\,ds + \int_0^t \|\nabla(1*\chi)(s)\|^2\,ds\Big) + 2\|(1*u)(t)\|^2. \end{aligned} \tag{6.10}$$

Finally, multiplying (6.10) by $1/4$, then adding it to (6.3) and (6.9), a subsequent application of Gronwall lemma leads to (2.19).

7. Proof of Theorem 2.10

In this section, we also set

$$e^0 = \theta^0 + \lambda(\chi_{01}) - \lambda(\chi_{02})$$

and let the generic constant C depend on T, Ω, γ, $\|\lambda''\|_{L^\infty(\mathbb{R})}$, and c_1 or c_2, at most. We still have (5.4), due to (2.12). On the other hand, observe that (cf. (2.13))

$$\chi_t + \Delta\chi + g(\chi_1) - g(\chi_2) = \lambda'(\chi_1)u_1 - \lambda'(\chi_2)u_2 \quad \text{a.e. in } Q_T. \tag{7.1}$$

Therefore, multiplying equation (7.1) by χ and integrating over space and time, with the help of Green formula we get

$$\begin{aligned}\frac{1}{2}\|\chi(t)\|^2 + \int_0^t \|\nabla\chi(s)\|^2\,ds = -\int_0^t (g(\chi_1(s)) - g(\chi_2(s)), \chi(s))\,ds + \frac{1}{2}\|\chi^0\|^2 \\ + \int_0^t (\lambda'(\chi_1(s))u_1(s) - \lambda'(\chi_2(s))u_2(s), \chi(s))\,ds.\end{aligned} \tag{7.2}$$

Adding (5.4) and (7.2), in view of (5.5)-(5.7) we can infer

$$\begin{aligned}&\frac{1}{2}\int_0^t\int_\Omega \frac{|\theta|^2}{1+|\theta_1|^2+|\theta_2|^2}\,dx\,ds + \frac{1}{2}\int_0^t\int_\Omega \frac{|u|^2}{1+|u_1|^2+|u_2|}\,dx\,ds \\ &+ \frac{1}{2}\|(1*u)(t)\|_V^2 + \frac{1}{2}\|\chi(t)\|^2 + \int_0^t \|\nabla\chi(s)\|^2\,ds \\ &\le \langle e^0 + (1*f)(t), (1*u)(t)\rangle - \int_0^t \langle f(s), (1*u)(s)\rangle\,ds \\ &\quad + ((1*h)(t), (1*u)(t))_\Gamma - \int_0^t (h(s), (1*u)(s))_\Gamma\,ds + \frac{1}{2}\|\chi^0\|^2 \\ &\quad - \int_0^t (g(\chi_1(s)) - g(\chi_2(s)), \chi(s))\,ds - \int_0^t (\lambda(\chi_1(s)) - \lambda(\chi_2(s)), u(s))\,ds \\ &\quad + \int_0^t (\lambda'(\chi_1(s))u_1(s) - \lambda'(\chi_2(s))u_2(s), \chi(s))\,ds.\end{aligned} \tag{7.3}$$

Let us estimate the last three integrals on the right hand side. Assume that (H14) holds. Then, owing to (6.7) and Young inequality, we have

$$\begin{aligned}&-\int_0^t (g(\chi_1(s)) - g(\chi_2(s)), \chi(s))\,ds \\ &\le C\|g(\chi_1(s)) - g(\chi_2(s)\|_{L^{6/5}(\Omega)}^2 + \frac{1}{8}\int_0^t \|\chi(s)\|_V^2\,ds \\ &\le C(M_2)\int_0^t \|\chi(s)\|^2\,ds + \frac{1}{8}\int_0^t \|\nabla\chi(s)\|^2\,ds.\end{aligned} \tag{7.4}$$

Next, owing to (H1)-(H2), Taylor expansion, and Hölder inequality, we have

$$\begin{aligned}&-\int_0^t (\lambda(\chi_1(s)) - \lambda(\chi_2(s)), u(s))\,ds \\ &+ \int_0^t (\lambda'(\chi_1(s))u_1(s) - \lambda'(\chi_2(s))u_2(s), \chi(s))\,ds \\ &= \int_0^t\int_\Omega u_1(\lambda(\chi_2) - \lambda(\chi_1) - \lambda'(\chi_1)(\chi_2 - \chi_1))\,dx\,ds \\ &+ \int_0^t\int_\Omega (u_2(\lambda(\chi_1) - \lambda(\chi_2) - \lambda'(\chi_2)(\chi_1 - \chi_2))\,dx\,ds\end{aligned}$$

$$
\begin{aligned}
&\le \|\lambda''\|_{L^\infty(\mathbb{R})} \int_0^t\int_\Omega (|u_1|+|u_2|)|\chi|^2\,dx\,ds \\
&\le C\int_0^t \big(\|u_1(s)\|_{L^4(\Omega)} + \|u_2(s)\|_{L^4(\Omega)}\big)\,\|\chi(s)\|\|\chi(s)\|_V\,ds \\
&\le C\int_0^t \big(1+\|u_1(s)\|_V^2 + \|u_2(s)\|_V^2\big)\,\|\chi(s)\|^2\,ds + \frac{1}{8}\int_0^t \|\nabla\chi(s)\|^2\,ds \\
&\le C(M_3)\int_0^t \|\chi(s)\|^2\,ds + \frac{1}{8}\int_0^t \|\nabla\chi(s)\|^2\,ds.
\end{aligned}
$$

By virtue of (7.4), the above inequality, and Young inequality, from (7.3) it is straightforward to deduce (cf. (5.8))

$$
\begin{aligned}
&\int_0^t\int_\Omega \frac{|\theta|^2}{1+|\theta_1|^2+|\theta_2|^2}\,dx\,ds + \int_0^t\int_\Omega \frac{|u|^2}{1+|u_1|^2+|u_2|}\,dx\,ds \\
&+ \|(1*u)(t)\|_V^2 + \|\chi(t)\|^2 + \int_0^t \|\nabla\chi(s)\|^2\,ds \\
&\le C\Big(\|e^0\|_{V'}^2 + \|\chi^0\|^2 + \|f\|_{L^2(0,T;V')}^2 + \|h\|_{L^2(\Gamma_T)}^2\Big) \\
&\quad + \int_0^t \|(1*u)(s)\|_V^2\,ds + C(M_2,M_3)\int_0^t \|\chi(s)\|^2\,ds.
\end{aligned} \tag{7.5}
$$

Then, an application of Gronwall lemma yields (2.20).

Finally, suppose that (H16) holds instead of (H14). Then, in place of (7.4) we have

$$
-\int_0^t (g(\chi_1(s)) - g(\chi_2(s)),\chi(s))\,ds \le c_2\int_0^t \|\chi(s)\|^2\,ds.
$$

Therefore, the constant C_3 appearing in (2.20) does not depend on M_2 and Theorem 2.10 is proved.

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8. Erratum: submitted on March 31, 2003.

1. [p. 1, last line, and p. 2, first line] We should point out that the well-known example of g we give, i.e., $g(r) = r^3 - r - \theta_c^{-1}$, $r \in \mathbb{R}$, where $\theta_c > 0$ is the critical temperature around which the phase transition occurs, applies for solid-liquid phase transitions to the simplest case $\lambda'(r) = 1$ for all $r \in \mathbb{R}$ (cf. also Remark 2.11, p. 7). In the general case, g can still be a third-degree polinomial with the same leading term, but with more general first and possibly second-order terms.

2. [p. 15, line +4] This line must be converted into

$$\begin{aligned} &+ \|f_n\|_{L^2(0,T;L^p(\Omega))}\big(C + \|w_n\|_{L^2(0,t;L^{p'}(\Omega))}\big) \\ &+ \big(C + \|h\|_{L^2(\Gamma_T)}\big)\|w^n(s)\|_{L^2(\Gamma_t)} \end{aligned}$$

that yields the correct last two terms on the right hand side of the involved inequality.

3. [p. 16] At line +1, (3.31) should be recalled along with (3.30) and (H8). Moreover, in formula (3.58) $L^p(\Omega_T)$ must be replaced by $L^p(Q_T)$.

4. [p. 17, lines from +12 to +21] This part must be changed as follows.
First of all, let us analyze the nonlinearities. Observe that, for any $v \in L^2(0,T;L^{p'}(\Omega))$ such that $\rho(v) \in L^2(0,T;L^p(\Omega))$, it turns out that

$$\rho_n(v) \to \rho(v) \quad \text{strongly in} \;\; L^2(0,T;L^p(\Omega))$$

as $n \to +\infty$. On the other hand, it is known that ρ induces a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ and, by regarding ρ as the subdifferential of a proper convex lower semicontinuous function, one can adapt the arguments in Example 3, pp. 61-63, of [V. Barbu, Nonlinear semigroups and differential equations in Banach spaces, Noordhoff, Leyden (1976)] to show that the graph relation

$$z \in \rho(v) \quad \text{almost everywhere in} \;\; Q_T \tag{$*$}$$

between two functions $v \in L^2(0,T;L^{p'}(\Omega))$ and $z \in L^2(0,T;L^p(\Omega))$ yields a *maximal* monotone operator in the product space.

Hence, recalling (3.31), (3.61), (3.63) and (3.68), in view of the monotonicity of ρ_n we can take the limit in

$$\begin{aligned}
&\int_0^T\int_\Omega (\theta_n - \rho_n(v))(u_n - v)\,dxdt \\
&= \int_0^T \langle \theta_n(t), u_n(t)\rangle dt - \int_0^T\int_\Omega (\theta_n v + \rho_n(v)(u_n - v))\,dxdt
\end{aligned}$$

and obtain

$$\int_0^T\int_\Omega (\theta - z)(u - v)\,dxdt \ge 0$$

for all functions $v \in L^2(0,T;L^{p'}(\Omega))$ and $z \in L^2(0,T;L^p(\Omega))$ fulfilling (*). Now, this implies (cf., e.g., [3, Definition 2.2, p. 22])

$$u < 0, \quad \theta = \rho(u) \tag{3.70}$$

almost everywhere in Q_T, where ρ here denotes the function again.

5. [p. 29, line +4] *This line must be deleted* so that (7.5) becomes

$$\begin{aligned}
&\int_0^t\int_\Omega \frac{|\theta|^2}{1+|\theta_1|^2+|\theta_2|^2}\,dx\,ds + \int_0^t\int_\Omega \frac{|u|^2}{1+|u_1|^2+|u_2|}\,dx\,ds \\
&+ \|(1*u)(t)\|_V^2 + \|\chi(t)\|^2 + \int_0^t \|\nabla\chi(s)\|^2\,ds \\
&\le C\Big(\|e^0\|_{V'}^2 + \|\chi^0\|^2 + \|f\|_{L^2(0,T;V')}^2 + \|h\|_{L^2(\Gamma_T)}^2\Big) \\
&\quad + \int_0^t \|(1*u)(s)\|_V^2\,ds \\
&\quad + C(M_2)\int_0^t \big(1 + \|u_1(s)\|_V^2 + \|u_2(s)\|_V^2\big)\,\|\chi(s)\|^2\,ds.
\end{aligned} \tag{7.5}$$

and still one can conclude via Gronwall lemma, with $\exp(M_3)$ entering the constant C_3 in (2.20). $\square$

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