Some uniqueness results for Bernoulli interior free-boundary problems in convex domains *

Pierre Cardaliaguet & Rabah Tahraoui

Abstract

We establish the existence of a elliptic family of convex solutions for Bernoulli interior free-boundary problems in bounded convex domains. We also proved that there is a unique solution to the problem associated with the so-called Bernoulli constant, and give an estimate from above for this constant.

1 Introduction

Let $\Omega$ be an open bounded subset of $\mathbb{R}^N$ and let $\lambda > 0$ be fixed. For a subdomain $D \subset \Omega$, the capacity potential $u_D$ of $D$ in $\Omega$ is defined as the solution to

$$
\begin{align*}
-\Delta u &= 0 \quad \text{in } \Omega \setminus D \\
u &= 0 \quad \text{on } \partial \Omega \\
u &= 1 \quad \text{on } \partial D.
\end{align*}
$$

(1.1)

The interior Bernoulli free-boundary problem is stated as follows: Find a subdomain $D$ such that capacity potential $u_D$ satisfies

$$
\forall x \in \partial D, \quad \frac{\partial u_D(x)}{\partial n_x} = -\lambda,
$$

where $n_x$ is the outward normal to $D$ at $x$. In the sequel, such a domain is called a solution of Bernoulli problem of level $\lambda$. Bernoulli problem has been extensively studied and we refer the reader to the survey paper [14] for several motivations and references. It is known that this problem does not have a solution for any positive level $\lambda$. For instance, when $\Omega$ is convex, it is proved in [19] that there is some positive constant $\lambda_\Omega$ such that Bernoulli problem has a solution of level $\lambda$ if and only if $\lambda \geq \lambda_\Omega$. This constant $\lambda_\Omega$ is called Bernoulli constant. It is also known that even if there are solutions to the problem for some $\lambda$, there is no uniqueness in general. For instance, if $\Omega$ is a ball, there are exactly two solutions to the problem for any $\lambda > \lambda_\Omega$.

*Mathematics Subject Classifications: 35R35.
Key words: Bernoulli free-boundary problem, convex solutions, Borell’s inequality.
©2002 Southwest Texas State University.
Uniqueness results for Bernoulli problems EJDE–2002/102

while for $\lambda_\Omega$ the solution is unique. Let us now briefly describe the structure of the solutions when $\Omega$ is a ball. All the solutions are balls, with the same center as $\Omega$ (c.f. [22]). For any $\lambda > \lambda_\Omega$, let us denote by $D_\lambda$ the largest solution of level $\lambda$ and by $\overline{D}_\lambda$ the smallest one. Then the family of largest solutions $(\overline{D}_\lambda)_{\lambda > \lambda_\Omega}$ forms a continuous increasing family of balls, while the family of smallest solutions $(\tilde{D}_\lambda)_{\lambda > \lambda_\Omega}$ forms a continuous decreasing family of balls. In Beurling terminology [5], the increasing family is called an **elliptic family of solutions** while the decreasing family is called a **hyperbolic family of solutions**. The unique solution corresponding to the Bernoulli constant $\lambda_\Omega$ is called **parabolic**. It is the limit of the $(\overline{D}_\lambda)$ and of the $(\tilde{D}_\lambda)$ when $\lambda \to \lambda_\Omega$.

Finally, the limit of the elliptic family when $\lambda \to +\infty$ is equal to $\Omega$, while the limit of the hyperbolic family when $\lambda \to +\infty$ is reduced to the center of the ball. In particular, the boundary of the solutions of Bernoulli problem completely cover $\Omega$ but the center.

A very interesting - and open - question is whether for general convex bounded sets $\Omega$ the structure of the solutions of Bernoulli problem enjoys similar features. In this paper we try to provide some positive evidence towards this conjecture. Let us first recall that, for any bounded and convex domain $\Omega$, and for any fixed volume $\sigma > 0$, there is at least one convex solution to Bernoulli free boundary problem of volume $\sigma$. This has already been proved in [2], Theorem 3 and in [10], Theorem 5.1. In the first part of this paper, we give a new - and we hope enlighting - proof of this result. In [19], Henrot and Shahgholian proved the existence, for any $\lambda \geq \lambda_\Omega$, of a maximal convex solution $\tilde{D}_\lambda$ of level $\lambda$. Moreover the family $(\tilde{D}_\lambda)_{\lambda \geq \lambda_\Omega}$ turns out to be increasing with respect to the inclusion. In order to prove that this family is an elliptic family of solutions, it remains to show that it is continuous. This has partially been established by Acker in [1], Theorem 6.5., where it is proved the existence of some constant $\lambda_0$ (with $\lambda_0 \geq \lambda_\Omega$) sufficiently large such that the subfamily $(\tilde{D}_\lambda)_{\lambda \geq \lambda_0}$ is continuous ([1], Theorem 6.5 p. 1418). This implies the uniqueness of the solutions among the sets with a boundary close of the boundary of $\Omega$. One of the main contribution of this paper is the fact that the full family $(\tilde{D}_\lambda)_{\lambda \geq \lambda_\Omega}$ is continuous. More precisely we prove the following result:

**Theorem 1.1** The family $(\tilde{D}_\lambda)_{\lambda > \lambda_\Omega}$ is elliptic (i.e., increasing and continuous) and we have the following inclusion: For any $\lambda_0$ and $\lambda_1$ larger than $\lambda_\Omega$, for any $s \in [0, 1]$,

\[
(1 - s)\tilde{D}_{\lambda_0} + s\tilde{D}_{\lambda_1} \subset \tilde{D}_{\gamma_s},
\]

where $\gamma_s = 1/[(1 - s)/\lambda_0 + s/\lambda_1]$.

**Remark:** Since $\gamma_s \leq (1 - s)\lambda_0 + s\lambda_1$ and the family $(\tilde{D}_\lambda)_{\lambda \geq \lambda_\Omega}$ is increasing, Theorem 1.1 implies that

\[
(1 - s)\tilde{D}_{\lambda_0} + s\tilde{D}_{\lambda_1} \subset \tilde{D}_{(1 - s)\lambda_0 + s\lambda_1}.
\]

This property can be viewed as a “concavity property” of the family $(\tilde{D}_\lambda)_{\lambda \geq \lambda_\Omega}$. The continuity of the family is a simple consequence of the “inequality of con-
cavity” (1.2). This continuity is important to get uniqueness results. Indeed, by standard comparison principle, the ellipticity of the family implies that any classical solution of Bernoulli problem \( D \) containing \( \tilde{D}_{\lambda} \Omega \) belongs to the family \((\tilde{D}_{\lambda})_{\lambda \geq \lambda_\Omega}\). Namely there is some \( \lambda \geq \lambda_\Omega \) with \( D = \tilde{D}_{\lambda} \). This remark can be found in [1, Theorem 6.4]. Let us also point out that Theorem 1.1 (or, more precisely its proof) implies that \((\tilde{D}_{\lambda})_{\lambda \geq \lambda_\Omega}\) is the unique elliptic family of convex solution (cf. Corollary 3.4). Concerning the parabolic solution, our main contribution is the following result.

**Theorem 1.2** There is exactly one solution \( \tilde{D}_{\lambda_\Omega} \) of level \( \lambda_\Omega \).

More precisely, we show in Theorem 4.1 below that \( \tilde{D}_{\lambda_\Omega} \) is the unique subsolution of level \( \lambda_\Omega \) (the definition of a subsolution is given later). This result is much deeper than Theorem 1.1. Indeed, it cannot simply rely upon an “inequality of concavity” of the form (1.2). In fact the key point for the proof of Theorem 1.1 is an inequality due to Borell in [6], whereas for Theorem 1.2 it is necessary to investigate the cases of equality in Borell inequality. This later result has been obtained by the authors in [11]. Since the Bernoulli constant plays a crucial role in this study, we complete the paper by giving a new estimate from above for the Bernoulli constant. This inequality is optimal in the sense that it is exact for balls. In conclusion, this paper gives a fairly complete picture for the elliptic solutions and for a parabolic solution of the interior Bernoulli free boundary problem. However the question of the (conjectured) uniqueness of the hyperbolic family of solutions remains open. Let us just give a possible starting point in this direction: Following [14], it is known that, when the volume \( \sigma \to 0^+ \), the solutions of Bernoulli problem of volume \( \sigma \) become closer and closer to balls and concentrate at some points of \( \Omega \) called the harmonic centers of \( \Omega \). It is known that the harmonic center of a convex bounded domain is unique (this is proved in [9] for \( N = 2 \), and in [11] for \( N \geq 3 \)). Therefore the solutions of volume \( \sigma \) concentrate to this unique harmonic center when \( \sigma \to 0^+ \). Let us finally explain how this paper is organized. In section 2, we give a new proof for the existence, for any volume \( \sigma \), of a solution of Bernoulli problem of volume \( \sigma \).

Sections 3 and 4 are respectively devoted to the proof of Theorems 1.1 and 1.2. In the last part of the paper we give some estimate for the Bernoulli constant.

## 2 Existence of convex solutions

The aim of this section is to give a new proof of the following result.

**Theorem 2.1** ([9, 2]) If \( \Omega \) is an open bounded convex domain of \( \mathbb{R}^N \), then, for any volume \( \sigma \in (0, |\Omega|) \) there is at least one convex solution of Bernoulli problem of volume \( \sigma \). More precisely, there is a solution which is a minimizer of the problem:

\[
\inf \{ \text{cap}(D) \mid D \subseteq \Omega \text{ is convex and } |D| = \sigma \}.
\]
Remark: It would be interesting to know if, for any \( \sigma \geq |\widetilde{D}_\lambda| \), the set \( \widetilde{D}_\lambda \) of volume \( \sigma \) is a minimizer of the problem.

For proving Theorem 2.1, let us introduce the following function:

\[
F(\sigma) = \inf \{ \text{cap}(D) \mid D \Subset \Omega \text{ is convex and } |D| = \sigma \},
\]

where \( |D| \) stands for the volume of \( D \) and \( \text{cap}(D) \) is the capacity of \( D \) in \( \Omega \), i.e.,

\[
\text{cap}(D) = \inf \{ \int_\Omega |\nabla u|^2 \mid u \in H_0^1(\Omega), u \geq 1 \text{ in } D \} = \int_{\partial D} |\nabla u_D|.
\]

Standard arguments show that the infimum in the problem defining \( F \) is attained (see for instance [3, 7]).

**Lemma 2.2** The function \( F \) is monotonically increasing.

The proof follows from the fact that the capacity \( D \rightarrow \text{cap}(D) \) is monotonically increasing with respect to the inclusion.

**Lemma 2.3** Let \( \sigma \) be a point of derivability of \( F \). Then any convex domain realizing the minimum in (2.1) is a solution to Bernoulli problem of level \( \lambda = \sqrt{F'(\sigma)} \).

**Remark:** The behaviour of the function \( F \) seems to be extremely relevant for describing the general behaviour of the solutions of the Bernoulli free boundary problem. In particular the question of the derivability of \( F \), of its convexity (or its concavity) properties are of crucial importance. From Lemma 2.3 one could expect \( F \) to be convex on \([\bar{\sigma}, |\Omega|]\) and concave on \((0, \bar{\sigma})\), where \( \bar{\sigma} \) is the volume of the solution of level \( \lambda_\Omega \).

**Proof of Lemma 2.3:** We follow several arguments of Acker [1]. Let \( D \) realize the minimum in (2.1). From Poincaré’s variational formula for the capacity (which can be applied since the boundary of \( D \) is Lipschitz, see for instance [14]), we have

\[
\frac{d}{dh} \text{cap}(D - hB)_{|h=0} = -\int_{\partial D} |\nabla u_D|^2
\]

where we have set

\[D - hB = \{ x \in D \mid d_{\partial D}(x) > h \},\]

where \( d_{\partial D}(x) \) denotes the distance of the point \( x \) to the set \( \partial D \). Since, for \( h \) sufficiently small, we have

\[|D - hB| = |D| - h|\partial D| + o(h) = \sigma - h|\partial D| + o(h),\]

we can deduce that

\[F(\sigma - h|\partial D| + o(h)) \leq \text{cap}(D - hB).\]
Then, using (2.3) and equality $F(\sigma) = \text{cap}(D)$, we get:

\[ F'(\sigma)|\partial D| \geq \int_{\partial D} |\nabla u_D|^2. \quad (2.4) \]

Let us now consider $D_h = \{u_D > 1 - h\}$. We know that $\text{cap}(D_h) = \text{cap}(D)/(1 - h)$. Moreover,

\[ |D_h| = |D| + h \int_{\partial D} \frac{1}{|\nabla u_D|} + o(h). \]

Let us recall that $\nabla u \neq 0$ in $\Omega \backslash D$ (see [21]). Hence, since $|D| = \sigma$, this gives

\[ F(\sigma + h \int_{\partial D} \frac{1}{|\nabla u_D|} + o(h)) \leq \text{cap}(D_h) = \frac{\text{cap}(D)}{1 - h}. \]

Therefore,

\[ F'(\sigma) \left( \int_{\partial D} \frac{1}{|\nabla u_D|} \right) \leq \text{cap}(D) = \int_{\partial D} |\nabla u_D|. \quad (2.5) \]

Putting (2.4) and (2.5) together gives:

\[
\left( \int_{\partial D} \frac{1}{|\nabla u_D|} \right) \left( \int_{\partial D} |\nabla u_D|^2 \right) \leq |\partial D| \left( \int_{\partial D} |\nabla u_D| \right) = \left( \int_{\partial D} 1 \right) \left( \int_{\partial D} |\nabla u_D| \right),
\]

where we have used equality (2.2). Let us set for simplicity $S = \partial D$ and $a(x) = |\nabla u_D(x)|$. Then the previous inequality can be rewritten as

\[
\int_{S \times S} \left( \frac{a(y)^2}{a(x)} - a(x) \right) \leq 0.
\]

Since this expression is symmetric with respect to $x$ and $y$, it implies

\[
\int_{S \times S} \left( \frac{a(y)^2}{a(x)} + \frac{a(x)^2}{a(y)} - a(x) - a(y) \right) \leq 0,
\]

i.e.,

\[
\int_{S \times S} a(x) \left[ 1 + \frac{a(x)^3}{a(y)^3} - \frac{a(x)^2}{a(y)^2} - \frac{a(x)}{a(y)} \right] \leq 0.
\]

Since the polynomial $t \rightarrow 1 + t^3 - t^2 - t$ is positive for $t \geq 0$ unless $t = 1$, the previous inequality implies that

\[ a(x) = a(y) \quad \text{for almost every } (x, y) \in S \times S. \]

Therefore $a = |\nabla u_D|$ is constant on $S = \partial D$. This means that $D$ is a solution of Bernoulli problem. Using (2.4) and (2.5) shows easily that it is a solution of level $\lambda = \sqrt{F'(\sigma)}$. \qed
Proof of Theorem 2.1: Let $\sigma > 0$ be fixed. Since $F$ is almost everywhere derivable, there is a sequence $(\sigma_n)$ converging to $\sigma$ such that $F'(\sigma_n)$ exists. Let $D_n$ be a minimizer for $F(\sigma_n)$. Then, since the $D_n$ are convex and bounded, they converge, up to a subsequence again denoted $(D_n)$ to some convex set $D$ with $|D| = \sigma$. Moreover, from standard arguments in convex analysis, we also have that $|\partial D_n|$ converges to $|\partial D| > 0$. Let us now prove that the sequence $F'(\sigma_n)$ is bounded. Indeed, we have
\[ \text{cap}(D_n) = \int_{\partial D_n} |\nabla u_{D_n}| = |\partial D_n| \sqrt{F'(\sigma_n)}, \]
because, from Lemma 2.3, $D_n$ is a solution of Bernoulli problem of level $\sqrt{F'(\sigma_n)}$. Since $\text{cap}(D_n) = F(\sigma_n)$ and $1/|\partial D_n|$ are bounded, we have proved that $F'(\sigma_n)$ is bounded. Thus $(F'(\sigma_n))$ converges (up to a subsequence again denoted $(F'(\sigma_n))$) to some $\lambda \geq 0$. Then standard arguments show that $D$, as a limit of convex solutions of Bernoulli problem of level $F'(\sigma_n)$, is also a convex solution of Bernoulli problem of level $\lambda$. Since $|D| = \sigma$, this completes the proof of Theorem 2.1.

3 The elliptic family of solutions

The aim of this section is to prove Theorem 1.1. Let us first recall the main results of [19] concerning the construction of the maximal solution $\tilde{D}_\lambda$ of Bernoulli problem of level $\lambda$. A subsolution of the Bernoulli problem of level $\lambda$ is a set $D \supseteq \Omega$ such that $u_\sigma$ is Lipschitz continuous and $\frac{\partial u_\sigma}{\partial \nu_x} \geq -\lambda$ on $\partial D$.

Let us introduce for any $\lambda$ the family of subsolutions:
\[ \mathcal{F}_\lambda = \{ D \in \Omega \mid u_\sigma \text{ is Lipschitz continuous and } \frac{\partial u_\sigma}{\partial \nu_x} \geq -\lambda \text{ on } \partial D \}, \]
where $\nu_x$ denotes the outward normal to $D$ at $x$, and $u_\sigma$ is the capacity potential of $D$, i.e., the solution of (1.1). Let us point out that, if a domain $D$ is a solution of Bernoulli problem of level $\lambda$, then $D$ belongs to $\mathcal{F}_\lambda$. Let us set
\[ \lambda_\Omega = \inf \{ \lambda \mid \mathcal{F}_\lambda \neq \emptyset \}. \]
Then it is proved in [19] that $\lambda_\Omega > 0$ and that $\forall \lambda \geq \lambda_\Omega$, the set $\mathcal{F}_\lambda$ is not empty. The main result of [19] states that, for any $\lambda \geq \lambda_\Omega$, the set
\[ \tilde{D}_\lambda = \overline{\bigcup_{D \in \mathcal{F}_\lambda} D} \]
is the maximal solution of Bernoulli problem of level $\lambda$, where $\overline{A}$ denotes the closed convex hull of a set $A$. 
To prove Theorem 1.1, we need some preliminary results about an inequality due to Borell [6] that we describe now. Let $D_0$ and $D_1$ be two convex, open subdomains of $\Omega$. For $s \in [0,1]$, we denote by $D_s$ the following set:

$$D_s = (1-s)D_0 + sD_1$$

$$= \{ x \in \mathbb{R}^N, \exists x_0 \in D_0, \exists x_1 \in D_1 \text{ with } x = (1-s)x_0 + sx_1 \}.$$

Let us recall that $D_s$ is a convex, open subdomain of $\Omega$. Following Borell, we denote by $\tilde{u}_s$ the function

$$\forall x \in \Omega \setminus D_s, \quad \tilde{u}_s(x) = \sup_{x_0,x_1} \min\{ u_{D_0}(x_0), u_{D_1}(x_1) \} \quad (3.1)$$

where the supremum is taken over the $x_0 \in \Omega \setminus D_0$ and $x_1 \in \Omega \setminus D_1$ such that $x = (1-s)x_0 + sx_1$. Borell’s inequality states that

$$u_{D_s}(\cdot) \geq \tilde{u}_s(\cdot) \quad \text{in } \Omega \setminus D_s. \quad (3.2)$$

Moreover, $\tilde{u}_s$ is continuous on $\Omega \setminus D_s$, $\tilde{u}_s = 0$ on $\partial\Omega$, and $\tilde{u}_s = 1$ on $\partial D$.

In [11], we have refined Borell’s inequality in establishing that the map $\tilde{u}_s$ is in fact a subsolution of Laplace equation in the viscosity sense (for the definition of this notion, see [12]). Namely:

**Lemma 3.1** In the viscosity sense,

$$-\Delta \tilde{u}_s \leq 0 \quad \text{in } \Omega \setminus D_s.$$

We use this fact in the next section together with the following sharp estimate of the case of equality in Borell’s inequality, that we have established in [11].

**Theorem 3.2** Assume that for some $s \in (0,1)$ the function $\tilde{u}_s$ is harmonic. Then $D_0 = D_1$.

**Remark:** We shall mainly use this result combined with Lemma 3.1 and Borell’s inequality (3.2) in the following way: If $D_0 \neq D_1$ and $s \in (0,1)$, then $u_{D_s} - \tilde{u}_s$ is a non-negative, non-zero, superharmonic function. The key point of the proof of Theorem 1.1 is the following lemma.

**Lemma 3.3** Let $\lambda_0$ and $\lambda_1$ be not smaller than $\lambda_{\Omega}$. Then, for any convex sets $D_0$ and $D_1$, such that $D_0 \in \mathcal{F}_{\lambda_0}$ and $D_1 \in \mathcal{F}_{\lambda_1}$, for any $s \in [0,1]$, the set $D_s = (1-s)D_0 + sD_1$ belongs to $\mathcal{F}_{\gamma_s}$, where

$$\gamma_s = \frac{1}{\frac{1-s}{\lambda_0} + \frac{s}{\lambda_1}}.$$
Remark: Note that $\gamma_s \leq (1-s)\lambda_0 + s\lambda_1$ and thus

$$D_s = (1-s)D_0 + sD_1 \in F_{(1-s)\lambda_0 + s\lambda_1}.$$ 

Proof of Lemma 3.3: For simplicity, we set $u_0 = u_{D_0}$ and $u_1 = u_{D_1}$. Following Gabriel [15, 16, 17] and Lewis [21], the level sets of the functions $u_i$ are smooth and strictly convex, and $\nabla u_i \neq 0$ in $\Omega \setminus D_i$. From Lemma 2.2 in [19], $D_s$ belongs to $F_{\gamma_s}$ if and only if

$$\frac{\partial u_{D_s}(x)}{\partial \nu_x} \geq -\gamma_s$$

for almost all $x \in \partial D_s$, where $\nu_x$ denotes the outward normal to $D_s$ at $x$. The partial derivative has to be understood in the sense

$$\frac{\partial u_{D_s}(x)}{\partial \nu_x} = \lim_{h \to 0^+} \frac{u_{D_s}(x + h\nu_x) - 1}{h}$$

and exists almost everywhere on $\partial D_s$ (see [13]). Let us now fix some $x \in \partial D_s$ point where the previous limit exists and where $D_s$ has a unique unit normal $\nu_x$. From Borell’s inequality (3.2), we have

$$\lim_{h \to 0^+} \frac{u_{D_s}(x + h\nu_x) - 1}{h} \geq \limsup_{h \to 0^+} \frac{\tilde{u}_s(x + h\nu_x) - 1}{h}.$$ 

Let us now recall some results of [6] (see also [11]): First it is proved that $\tilde{u}_s$ is $C^1$ in $\Omega \setminus D_s$. Second, it is also proved that for any $x \in \Omega \setminus D_s$, there exists a unique pair $(x_0, x_1)$ belonging to $(\Omega \setminus D_0) \times (\Omega \setminus D_1)$, such that

$$\tilde{u}_s(x) = u_0(x_0) = u_1(x_1) \text{ and } x = (1-s)x_0 + sx_1.$$ 

Moreover, $\nabla \tilde{u}_s(x)$ is given by

$$\frac{\nabla \tilde{u}_s(x)}{||\nabla \tilde{u}_s(x)||} = \frac{\nabla u_0(x_0)}{||\nabla u_0(x_0)||} = \frac{\nabla u_1(x_1)}{||\nabla u_1(x_1)||}$$

and

$$\frac{1}{||\nabla \tilde{u}_s(x)||} = \frac{1-s}{||\nabla u_0(x_0)||} + \frac{s}{||\nabla u_1(x_1)||}.$$ 

Let us now consider $\xi_h \in [x, x + h\nu_x]$ such that

$$\tilde{u}_s(x + h\nu_x) - 1 \frac{h}{h} = \langle \nabla \tilde{u}_s(\xi_h), \nu_x \rangle. \quad (3.3)$$

Note that $\xi_h \to x$ when $h \to 0^+$. We now apply the results of [6] recalled above to the point $\xi_h$: There are $\xi_h^1 \in \Omega \setminus D_s$ such that

$$\tilde{u}_s(\xi_h) = u_0(\xi_h^1) = u_1(\xi_h^1) \text{ and } \xi_h = (1-s)\xi_h^0 + s\xi_h^1.$$
\[
\frac{\nabla u_s(\xi_h)}{|\nabla u_s(\xi_h)|} = \frac{\nabla u_0(\xi_h^0)}{|\nabla u_0(\xi_h^0)|} = \frac{\nabla u_1(\xi_h^1)}{|\nabla u_1(\xi_h^1)|}
\]

and, finally,
\[
\frac{1}{|\nabla u_s(\xi_h)|} = \frac{1 - s}{|\nabla u_0(\xi_h^0)|} + \frac{s}{|\nabla u_1(\xi_h^1)|}.
\] (3.4)

Since \( D_i \) belongs to \( F_{\lambda_i} \), we have
\[
|\nabla u_0(\xi_0^0)| \leq \lambda_0 \quad \text{and} \quad |\nabla u_1(\xi_1^1)| \leq \lambda_1.
\]

Hence, from (3.4) and the previous inequalities, we obtain
\[
|\nabla \bar{u}_s(\xi_h)| \leq 1/[(1 - s)/\lambda_0 + s/\lambda_1] = \gamma_s.
\] (3.5)

Let us now consider a sequence \( h_n \to 0^+ \) such that
\[
\limsup_{h \to 0^+} \frac{\bar{u}_s(x + h\nu_x) - 1}{h} = \lim_n \frac{\bar{u}_s(x + h_n\nu_x) - 1}{h_n}.
\] (3.6)

Let us set \( \xi_n = \xi_{h_n} \). Since \( u_n = -\nabla \bar{u}_s(\xi_n)/|\nabla \bar{u}_s(\xi_n)| \) is an outward normal to the convex set \( \{ \bar{u}_s \geq \bar{u}_s(\xi_n) \} \) at \( \xi_n \), a standard passage to the limit shows that \( a = \lim_n a_n \) is an outward normal to the set \( D_s \) at \( x \) because the \( \xi_n \) converge to \( x \), \( x \) belongs to \( \partial D_s \), and the convex set \( \{ \bar{u}_s \geq \bar{u}_s(\xi_n) \} \) converges to the convex set \( D_s \). Since, from our assumption, \( D_s \) has a unique outward normal at \( x \), namely \( \nu_x \), we have \( a = \nu_x \). Hence, from (3.3) and (3.6), we get
\[
\limsup_{h \to 0^+} \frac{\bar{u}_s(x + h\nu_x) - 1}{h} = \lim_n (\nabla \bar{u}_s(\xi_n), \nu_x) = -\lim_n |\nabla \bar{u}_s(\xi_n)|.
\]

Using (3.5), we prove the desired result:
\[
\lim_{h \to 0^+} \frac{u_{D_s}(x + h\nu_x) - 1}{h} \geq -\gamma_s.
\]

\( \square \)

**Proof of Theorem 1.1:** We first prove (1.2). Let \( \lambda_0 \) and \( \lambda_1 \) be fixed. From [19], the convex sets \( D_{\lambda_0} \) and \( D_{\lambda_1} \) belong respectively to \( F_{\lambda_0} \) and to \( F_{\lambda_1} \). Lemma 3.3 then states that, for any \( s \in [0, 1] \), the convex set
\[
D_s = (1 - s)\overline{D_{\lambda_0}} + s\overline{D_{\lambda_1}}
\]

belongs to \( F_{\gamma_s} \), where \( \gamma_s \) is defined as in Lemma 3.3. Accordingly, from the construction of the solution \( D_{\lambda} \), we have
\[
D_s \subset \overline{D_{\gamma_s}}
\]
This proves (1.2). We now prove the continuity of the family $(\tilde{D}_\lambda)_{\lambda > \lambda_\Omega}$. Let $\lambda > \lambda_\Omega$ be fixed. From the construction of the solution $\tilde{D}_\lambda$ and standard stability results, we have easily that

$$\tilde{D}_\lambda = \text{Int} \left( \bigcap_{\lambda' > \lambda} \tilde{D}_{\lambda'} \right).$$

This proves the continuity on the right. For proving the continuity on the left, let us set

$$D = \bigcup_{\lambda' < \lambda} \tilde{D}_{\lambda'}.$$

We already know that $D \subset \tilde{D}_\lambda$ and we want to prove the equality. We argue by contradiction, by assuming that $D \neq \tilde{D}_\lambda$. Let us first notice that $D$ is a convex solution of Bernoulli problem of level $\lambda$, as limit of convex solutions of Bernoulli problem of level $\lambda'$, with $\lambda' \to \lambda$. Thus its boundary is smooth (see for instance [19]). Since $\tilde{D}_\lambda$ is also a solution of the Bernoulli problem of level $\lambda$, the strong maximum principle implies that the boundary of the sets $D$ and $\tilde{D}_\lambda$ are disjoint. Hence $D \not\subset \tilde{D}_\lambda$. We now choose some $\lambda_0 \in (\lambda_\Omega, \lambda)$. Let $s$ be the largest real in $(0,1)$ such that

$$(1-s)\tilde{D}_{\lambda_0} + s\tilde{D}_{\lambda} \subset D.$$

Let us notice that $s$ belongs to $(0,1)$ and that, if we set $D_s = (1-s)\tilde{D}_{\lambda_0} + s\tilde{D}_{\lambda}$, the boundaries of $D_s$ and $D$ have a non empty intersection. Let $x$ belong to this intersection. From Lemma 3.3, we know that $x_s = 1/[(1-s)/\lambda_0 + s/\lambda]$. Let us notice that $\gamma_s$ is smaller than $\lambda$. Since $D_s \subset D$, we have $u_{D_s} \leq u_D$. Let $\nu_x$ be the normal to $D_s$ and $D$ at $x$. We have

$$\lambda = \lim_{h \to 0+} \frac{1 - u_{D}(x + h\nu_x)}{h} \leq \lim_{h \to 0+} \frac{1 - u_{D_s}(x + h\nu_x)}{h} \leq \gamma_s.$$ 

Hence there is a contradiction, since we have in fact $\gamma_s < \lambda$. This completes the proof.

The same proof shows that the family $(\tilde{D}_\lambda)_{\lambda > \lambda_\Omega}$ is the unique elliptic family of convex solutions. Namely:

**Corollary 3.4** Let $D_0$ and $D_1$ be two convex solutions of Bernoulli problem respectively of level $\lambda_0$ and $\lambda_1$. Assume that $D_0 \subset D_1$ and $\lambda_0 < \lambda_1$. Then $D_1 = \tilde{D}_{\lambda_1}$.

**Proof:** Replace in the proof of Theorem 1.1 the set $D_{\lambda_1}$ by $D_0$, $D$ by $D_1$ and $\tilde{D}_{\lambda}$ by $\tilde{D}_{\lambda_1}$. \[\square\]

## 4 Uniqueness of the parabolic solution

We finally investigate the special case of the parabolic solution, i.e., the solution of level $\lambda_\Omega$. Our aim is to establish the uniqueness of the solution. We are in fact going to prove a stronger result. Namely:
Theorem 4.1 With the notations of the previous section, we have
\[ \mathcal{F}_{\lambda_0} = \{ \hat{D}_{\lambda_0}\}. \]

Note that Theorem 4.1 implies Theorem 1.2, since a solution \( D \) of the Bernoulli problem of level \( \lambda_0 \) always belongs to \( \mathcal{F}_{\lambda_0} \).

Proof of Theorem 4.1: We argue by contradiction. Let us assume that there is an open set \( D \) belonging to \( \mathcal{F}_{\lambda_0} \), with \( D \neq \hat{D}_{\lambda_0} \). The definition of \( \hat{D}_{\lambda_0} \) implies that \( D \subset \hat{D}_{\lambda_0} \). From Lemma 2.4 of [19], we know that the convex hull of \( D \), denoted by \( D_0 \), also belongs to \( \mathcal{F}_{\lambda_1} \). We claim that \( D_0 \subset D_{\lambda_0} \).

Indeed, otherwise, there should exist some \( x \) belonging to the intersection of the boundary of \( D \) and the boundary of \( D_{\lambda_0} \). Since \( D \subset D_{\lambda_0} \) and \( D \neq D_{\lambda_0} \), Hopf maximum principle then would imply that \( |\nabla u_D(x)| > |\nabla u_{D_{\lambda_0}}(x)| \) at this point \( x \). But this is impossible since \( |\nabla u_D(x)| \leq \lambda_0 \) because \( D \) is a sub-solution and \( |\nabla u_{D_{\lambda_0}}(x)| = \lambda_0 \) because \( D_{\lambda_0} \) is a solution of Bernoulli problem of level \( \lambda_0 \). Hence we have proved that \( D_0 \subset D_{\lambda_0} \). We now consider the convex combination \( D_s = (1 - s)D_0 + sD_{\lambda_0} \) for some \( s \in (0, 1) \). To achieve the proof of our Theorem, it suffices to prove that \( D_s \) belongs in fact to \( \mathcal{F}_{\lambda_1 - \epsilon} \), for some \( \epsilon > 0 \). Indeed this leads to a contradiction because \( \mathcal{F}_{\lambda_1 - \epsilon} \) is empty from the definition of \( \lambda_1 \). We now prove that \( D_s \) belongs to \( \mathcal{F}_{\lambda_1 - \epsilon} \) for some \( \epsilon > 0 \). At this step, we have to underline that Borell’s inequality is no longer enough for proving that \( D_s \) belongs to some \( \mathcal{F}_{\lambda_1 - \epsilon} \). Indeed, Borell’s inequality only gives that \( D_s \) belongs to \( \mathcal{F}_{\lambda_0} \) (see Lemma 3.3). Therefore we have to use a stronger argument: This argument is Theorem 3.2, which states that, since \( D_0 \neq D_{\lambda_0} \), the map \( \tilde{u}_s \) defined by (3.1) cannot be a solution of Laplace equation. Since \( \tilde{u}_s \) is a subsolution of this equation (cf. Lemma 3.1), this shows that the map \( u_{D_s} - \tilde{u}_s \) is a non-negative viscosity supersolution of Laplace equation, vanishing at the boundary \( \partial \Omega \cup \partial D_s \). Using the fact that \( D_s \) is convex and bounded, Hopf maximum principle states that there is a neighborhood \( U \) of \( \partial D_s \) and some positive constant \( \epsilon \) such that
\[ \forall x \in U \setminus D_s, \quad u_{D_s}(x) - \tilde{u}_s(x) \geq \epsilon d_{D_s}(x), \]
where \( d_{D_s}(x) \) denotes the distance from the point \( x \) to the set \( D_s \). We now argue as in the proof of Lemma 3.3: We have, for almost every \( x \in \partial D_s \), where there is a unique outward normal \( \nu_s \) to \( D_s \) at \( x \),
\[ \lim_{h \to 0^+} \frac{u_{D_s}(x + h\nu_s) - 1}{h} \geq \lim_{h \to 0^+} \frac{\tilde{u}_s(x + h\nu_s) - 1}{h} + \epsilon, \]
since \( d_{D_s}(x + h\nu_s) = h \). We can estimate the term in \( \limsup \) as in the proof of Lemma 3.3 (with now \( \lambda_0 = \lambda_1 = \lambda_1 \)): This gives
\[ \limsup_{h \to 0^+} \frac{\tilde{u}_s(x + h\nu_s) - 1}{h} \geq -\lambda_1. \]
Hence we have
\[
\lim_{h \to 0^+} \frac{u_{D_s}(x + h\nu_x) - 1}{h} \geq -\lambda_\Omega + \epsilon.
\]
Using Lemma 2.2 of [19], this proves that \( D_s \) belongs to \( \mathcal{F}_{\lambda_\Omega - \epsilon} \) and gives the desired contradiction. \( \square \)

5 Estimate for the Bernoulli constant

Our aim is to estimate from above the Bernoulli constant \( \lambda_\Omega \). For an estimate from below, let us recall that the following question is still open (see [14]): Let \( \Omega \) be an open, bounded, convex subset of \( \mathbb{R}^N \) and let \( \tilde{\Omega} \) be the ball centered at 0 with \( |\Omega| = |\tilde{\Omega}| \). Do we always have \( \lambda_{\tilde{\Omega}} \leq \lambda_\Omega \)?

To explain our result, we have to introduce some definitions and notation. Let \( \Omega \) be an open, bounded convex subset of \( \mathbb{R}^N \). Let us denote by \( F = F(|\cdot|) \) the fundamental solution of Laplace equation in \( \mathbb{R}^n \) and, for any \( x \in \Omega \), let \( H_x(\cdot) \) be the regular part of the Green function of Laplace equation with Dirichlet boundary condition, i.e., the solution of
\[
-\Delta H_x(\cdot) = 0 \quad \text{in } \Omega \\
H_x(\cdot) = F(|\cdot - x|) \quad \text{on } \partial \Omega
\]
The Robin function \( t : \Omega \to \mathbb{R} \) and the harmonic radius \( r : \Omega \to \mathbb{R} \) are respectively defined by
\[
\forall x \in \Omega, \quad t(x) = H_x(x) \quad \text{and} \quad t(x) = F^{-1}(t(x)).
\]
We also denote by \( \bar{r}_\Omega \) the maximum of the harmonic radius in \( \Omega \) (which exists since \( \Omega \) is convex and bounded) and by \( \bar{x}_\Omega \) the harmonic center of \( \Omega \) (i.e., the point of maximum of the strictly concave function \( r(\cdot) \), see [11] for instance). Let us recall that in dimension \( N = 2 \), the maximum of the harmonic radius is usually called the conformal radius.

In [19], the following estimate from above of the Bernoulli constant is given: If \( \Omega \) is an open convex bounded subset of \( \mathbb{R}^2 \), we have
\[
\lambda_\Omega \leq 6.252/\bar{r}_\Omega.
\]
We improve this result as follows:

**Theorem 5.1** For any dimension \( N \geq 2 \), we have
\[
\lambda_\Omega \leq \lambda_{B_{\bar{r}_\Omega}(0)} = \begin{cases} 
\frac{N-2}{N-2 - (N-1)\frac{N-2}{N-2 - (N-1)\frac{1}{\bar{r}_\Omega}}} & \text{if } N \geq 3 \\
\frac{1}{\epsilon/\bar{r}_\Omega} & \text{if } N = 2
\end{cases}
\]
Remarks:

1. Let us recall that the following inequality holds true for open, bounded and convex sets: If
   \[ \Omega_1 \subset \Omega_2, \quad \lambda_{\Omega_1} \geq \lambda_{\Omega_2}. \]  
   (5.1)
   This is a straightforward consequence of the construction of [19]. This inequality gives an easy estimate from below of the Bernoulli constant
   \[ \lambda_{\Omega} \geq \lambda_{B_R(0)} \]
   where \( R \) is the radius of the smallest ball containing \( \Omega \).

2. Let us point out that the estimate from above given in the Theorem does not derive from inequality (5.1) because the ball \( B(\bar{x}_\Omega, \bar{r}_\Omega) \) is not contained in \( \Omega \), unless \( \Omega \) is a ball.

3. Let us finally notice that the estimate of the Theorem is optimal for balls, because in this case the maximum of the harmonic radius \( \bar{r}_\Omega \) is equal to the radius of the ball.

Proof of Theorem 5.1:  Let us set \( B = B_{\bar{r}_\Omega}(0) \) and \( \lambda = \lambda_B \) the Bernoulli constant of the ball \( B \). There is a unique radius \( r \in (0, \bar{r}_\Omega) \) such that \( D = B_r(0) \) is the solution of Bernoulli problem of level \( \lambda \) in \( B \) (indeed, in the case of balls, it is known that any solution is radial, cf. [22]). Let \( G_x(\cdot) \) and \( G_x(\cdot) \) be the Green functions of the sets \( B \) and \( \Omega \) respectively, for the Dirichlet problem, i.e., the solutions of

\[
-\Delta \bar{G}_x(\cdot) = \delta_x \quad \text{in } B \quad \text{and} \quad -\Delta G_x(\cdot) = \delta_x \quad \text{in } \Omega
\]

\[
\bar{G}_x(\cdot) = 0 \quad \text{on } \partial B \quad \text{and} \quad G_x(\cdot) = 0 \quad \text{on } \partial \Omega
\]

where \( \delta_x \) is the Dirac measure at \( x \). Then, for \( x = 0 \), the solution \( \bar{G}_0(\cdot) \) is radial and we denote by \( \bar{t} \) its value on \( \partial D \). Let us set

\[
\forall s \in [0, +\infty), \quad \phi(s) = \begin{cases} 
  s/\bar{t} & \text{if } s \leq \bar{t} \\
  1 & \text{otherwise}
\end{cases}
\]

Then \( \bar{u}(\cdot) = \phi \circ G_0(\cdot) \) is nothing but the capacity potential of \( D \) in \( B \).

Let us now consider the harmonic transplantation \( u_0 = \phi \circ G_x \) (see [20], [4]). Then \( u_0 \) is clearly the capacity potential of the set \( D_0 = \{ G_x > \bar{t} \} \). Following [4], Theorem 18, we have

\[
\text{cap}_B(D) = \int_B |\nabla \bar{u}|^2 = \int_\Omega |\nabla u_0|^2 = \text{cap}_\Omega(D_0)
\]

and

\[
|D| \leq |D_0|
\]
(from Theorem 18 of [4], part 2, with \( f(t) = 1 \) if \( t \geq 1 \), \( f(t) = 0 \) otherwise). Accordingly,

\[
\text{cap}_B(D) = \text{cap}_\Omega(D_0) \\
\geq \inf\{\text{cap}_\Omega(C) \mid C \subseteq \Omega, |C| = |D|, C \text{ convex}\} = F(|D|),
\]

where \( F \) is defined in section 2, because the capacity is non-decreasing with respect to the inclusion. From Theorem 2.1, we know that there is a open convex set \( C \) of volume \( |D| \), which is solution of Bernoulli problem for some level \( \bar{\lambda} > 0 \). Moreover, from the proof of Theorem 2.1 (see section 2), we can choose \( C \) as a minimizer of \( F(|D|) \). Accordingly, we have

\[
\text{cap}_B(D) = \lambda|\partial D| \geq \text{cap}_\Omega(C) = \bar{\lambda}|\partial C|.
\]

Since \( |D| = |C| \), the isoperimetric inequality states that \( |\partial C| \geq |\partial D| \). Hence we have proved that

\[
\lambda = \lambda_B \geq \bar{\lambda} \geq \lambda_\Omega.
\]

Since \( B = B_{\eta_\Omega}(0) \), the proof of Theorem 5.1 is complete. \( \Box \)

References


PIERRE CARDALIAGUET
Université de Bretagne Occidentale, Département de Mathématiques, 6, avenue Victor-le-Gorgeu, B.P. 809, 29285 Brest Cedex, France
e-mail: Pierre.Cardaliaguet@univ-brest.fr

RABAH TAHRAOUI
C.E.R.E.M.A.D.E., Université Paris IX Dauphine, Place du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16 France
and
I.U.F.M. de Rouen, 2, rue du Tronquet, B.P. 18, Mont Saint-Aignan 76131 France
e-mail: tahraoui@ceremade.dauphine.fr