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Positive and monotone solutions of an m-point boundary-value problem *

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Abstract

We study the second-order ordinary differential equation

$$y''(t) = -f(t, y(t), y'(t)), \quad 0 \le t \le 1,$$

subject to the multi-point boundary conditions

$$\alpha y(0) \pm \beta y'(0) = 0, \quad y(1) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i)\,.$$

We prove the existence of a positive solution (and monotone in some cases) under superlinear and/or sublinear growth rate in f. Our approach is based on an analysis of the corresponding vector field on the (y, y') face-plane and on Kneser's property for the solution's funnel.

1 Introduction

Recently an increasing interest has been observed in investigating the existence of positive solutions of boundary-value problems. This interest comes from situations involving nonlinear elliptic problems in annular regions. Erbe and Tang [5] noted that, if the boundary-value problem

$$-\Delta u = F(|x|, u) \quad \text{in } R < |x| < \hat{R}$$

with

$$u = 0 \quad \text{for } |x| = R, \quad u = 0 \quad \text{for } |x| = \hat{R}; \quad \text{or}$$

$$u = 0 \quad \text{for } |x| = R, \quad \frac{\partial u}{\partial |x|} = 0 \quad \text{for } |x| = \hat{R}; \quad \text{or}$$

$$\frac{\partial u}{\partial |x|} = 0 \quad \text{for } |x| = R, \quad u = 0 \quad \text{for } |x| = \hat{R}$$

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is radially symmetric, then the boundary-value problem can be transformed into the scalar Sturm-Liouville problem

$$x''(t) = -f(t, x(t)), \quad 0 \le t \le 1, \tag{1.1}$$

$$\alpha x(0) - \beta x'(0) = 0, \quad \gamma x(1) + \delta x'(1) = 0. \tag{1.2}$$

where α, β, γ, δ are positive constants.

By a positive solution of (1.1)-(1.2), we mean a function $x(t)$ which is positive for $0 < t < 1$ and satisfies the differential equation (1.1) with the boundary conditions (1.2).

Erbe and Wang [6] using Green's functions and the Krasnoselskii's fixed point theorem on cones proved the existence of a positive solution of (1.1)-(1.2), under the following assumptions:

(B1) The function f is continuous and positive on $[0, 1] \times [0, \infty)$ and

$$f_0 := \lim_{y\to 0+} \max_{0\le t\le 1} \frac{f(t, y)}{y} = 0, \quad f_\infty := \lim_{y\to +\infty} \min_{0\le t\le 1} \frac{f(t, y)}{y} = +\infty \tag{1.3}$$

i.e. f is *superlinear* at both ends points $x = 0$ and $x = \infty$; or

$$f_0 := \lim_{y\to 0+} \min_{0\le t\le 1} \frac{f(t, y)}{y} = +\infty, \quad f_\infty := \lim_{y\to +\infty} \max_{0\le t\le 1} \frac{f(t, y)}{y} = 0. \tag{1.4}$$

i.e. f is *sublinear* at both $x = 0$ and $x = \infty$.

(B2) $\rho := \beta\gamma + \alpha\gamma + \alpha\delta > 0$.

Also nonlinear boundary constraints have been studied, among others by Thompson [22] and by the author of this paper and Jackson [9]. There are common ingredients in these papers: an (assumed) Nagumo-type growth condition on the nonlinearity f or/and the presence of upper and lower solutions.

The multi-point boundary-value problem for second-order ordinary differential equations was initiated by Ilin and Moiseev [10, 11]. Gupta [14] studied the three-point boundary-value problems for nonlinear ordinary differential equations. Since then, more general nonlinear multi-point boundary-value problems have been studied by several authors. Most of them used the Leray-Schauder continuation theorem, nonlinear alternatives of Leray-Schauder, coincidence degree theory or a fixed-point theorem on cones. We refer the reader to [1, 8, 13, 20] for some recent results of nonlinear multipoint boundary-value problems.

Let $a_i \ge 0$ for $i = 1, \dots, m-2$ and let ξ_i satisfy $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$. Ma [21] applied a fixed-point theorem on cones to prove the existence of a positive solution of

$$u'' + a(t)f(u) = 0$$

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i)$$

under superlinearity or sublinearity assumptions on f. He also assumed the following

(Γ1) $a \in C([0,1],[0,\infty))$, $f \in C([0,\infty),[0,\infty))$, and there exists $t_0 \in [\xi_{m-2},1]$ such that $a(t_0) > 0$

(Γ2) For $i = 1,\dots,m-2$, $a_i \geq 0$ and $\sum_{i=1}^{m-2} a_i\xi_i < 1$.

Recently, Gupta [16] obtained existence results for the boundary-value problem

$$\begin{gathered} y''(t) = f(t,y(t),y'(t)) + e(t), \quad 0 \leq t \leq 1 \\ y(0) = 0, \quad y(1) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i), \end{gathered}$$

by using the Leray-Schauder continuation theorem, under smallness assumptions of the form

$$|f(t,y,y')| \leq p(t)|y| + q(t)|y'| + r(t) \quad \text{and} \quad C_1\|p(t)\| + C_2\|q(t)\| \leq 1,$$

with $p(t)$, $q(t)$, $r(t)$ and $e(t)$ in $L^1(0,1)$ and C_1 and C_2 constants.

In this paper, we consider the problem of existence of positive solutions for the m-point boundary-value problem

$$y''(t) = -f(t,y(t),y'(t)), \quad 0 \leq t \leq 1, \tag{1.5}$$

$$\alpha y(0) - \beta y'(0) = 0, \quad y(1) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i). \tag{1.6}$$

We assume $\alpha > 0$, $\beta > 0$, the function f is continuous, and

$$f(t,y,y') \geq 0, \quad \text{for all } t \in [0,1],\ y \geq 0\ \ y' \in \mathbb{R}. \tag{1.7}$$

The presence of the third variable y' in the function $f(t,y,y')$ causes some considerable difficulties, especially, in the case where an approach relies on a fixed point theorem on cones and the growth rate of $f(t,y,y')$ is sublinear or superlinear. We overcome this predicament, by extending below the concept-assumptions (1.3) and (1.4) as follows:

Suppose that for any $M > 0$,

$$\begin{gathered} f_{0,0} := \lim_{(y,y')\to(0,0)} \max_{0\leq t\leq 1} \frac{f(t,y,y')}{y} = 0 \\ f_{+\infty} := \lim_{y\to+\infty} \min_{0\leq t\leq 1} \frac{f(t,y,y')}{y} = +\infty, \quad \text{for } |y'| \leq M \end{gathered} \tag{1.8}$$

i.e. f is *jointly superlinear* at the end point $(0,0)$ and *uniformly superlinear* at $+\infty$. Similarly

$$\begin{gathered} f_0 := \lim_{y\to 0+} \min_{0\leq t\leq 1} \frac{f(t,y,y')}{y} = +\infty, \quad \text{for } |y'| \leq M. \\ f_{+\infty,+\infty} := \lim_{(y,y')\to(+\infty,+\infty)} \max_{0\leq t\leq 1} \frac{f(t,y,y')}{y} = 0, \end{gathered} \tag{1.9}$$

i.e. f is *jointly sublinear* at $(+\infty,+\infty)$ and *uniformly sublinear* at 0.

Furthermore there exist $\bar{l}\in(0,\infty]$, such that for every $\bar{M}>0$

$$\lim_{y'\to-\infty}\max_{0\le t\le 1}\frac{f(t,y,y')}{y'}=-\bar{l},\quad \text{for } y\in[0,\bar{M}] \tag{1.10}$$

i.e. $f(t,y,.)$ is *linear or superlinear* at $-\infty$ and for every $\bar{\eta}>0$

$$\lim_{y'\to 0}\min_{0\le t\le 1}\frac{f(t,y,y')}{y'}=0,\quad \text{for } y\in(0,\bar{\eta}). \tag{1.11}$$

i.e. $f(t,y,.)$ is *superlinear* at 0.

Remark 1.1 Note that the differential equation (1.5) defines a vector field whose properties will be crucial for our study. More specifically, we look at the (y,y') face semi-plane $(y>0)$. From the sign condition on f (see assumption (1.7)), we immediately see that $y''<0$. Thus any trajectory $(y(t),y'(t))$, $t\ge 0$, emanating from the semi-line

$$E_0:=\{(y,y'):\alpha y-\beta y'=0,\ y>0\}$$

"trends" in a natural way, (when $y'(t)>0$) toward the positive y-semi-axis and then (when $y'(t)<0$) trends toward the negative y'-semi-axis. Lastly, by setting a certain growth rate on f (say superlinearity) we can control the vector field, so that some trajectory satisfies the given boundary condition

$$y(1)=\sum_{i=1}^{m-2}\alpha_i y(\xi_i)$$

at the time $t=1$. These properties will be referred as "*The nature of the vector field*" throughout the rest of paper.

So the technique presented here is different to that given in the above mentioned papers [16, 6, 3, 13, 5], but it is closely related with those in [9, 21]. Actually, we rely on the above "nature of the vector field" and on the simple shooting method. Finally, for completeness we refer to the well-known Kneser's theorem (see for example Copel's text-book [2]).

Theorem 1.2 *Consider the system*

$$x''=f(t,x,x'),\quad (t,x,x')\in\Omega:=[\alpha,\beta]\times\mathbb{R}^{2n}, \tag{1.12}$$

*with the function f continuous. Let $\hat{E}_0$ be a continuum (compact and connected) set in $\Omega_0:=\{(t,x,x')\in\Omega:t=\alpha\}$ and let $\mathcal{X}(\hat{E}_0)$ be the family of all solutions of (1.12) emanating from $\hat{E}_0$. If any solution $x\in\mathcal{X}(\hat{E}_0)$ is defined on the interval $[\alpha,\tau]$, then the set (*cross-section *at the point τ)*

$$\mathcal{X}(\tau;\hat{E}_0):=\Big\{(x(\tau),x'(\tau)):x\in\mathcal{X}(\hat{E}_0)\Big\}$$

is a continuum in $\mathbb{R}^{2n}$.

Now consider (1.5)-(1.6) with the following notation.

$$\sigma := \sum_{i=1}^{m-2} \alpha_i \xi_i < 1, \quad \sigma^* := \sum_{i=1}^{m-2} \alpha_i \xi_i + \frac{\beta}{\alpha}\Big\{ \sum_{i=1}^{m-2} \alpha_i \frac{\xi_i}{\xi_1} - 1 \Big\} < 1,$$
$$K_0 := \max\Big\{ \frac{2\alpha}{\beta},\ 2\big[\frac{\alpha+\beta}{\beta} - \frac{\sigma}{\xi_{m-2}}\big] \Big\},$$
$$\mu_0 := \min\Big\{ (1-m^*)\frac{\varepsilon\alpha}{\beta},\ 2\big[\frac{\varepsilon(\alpha+\beta)}{\beta} - 1\big] \Big\}$$

where $\beta/(\alpha+\beta) < \varepsilon < 1$ and $\sigma^* < m^* < 1$.

So by (1.10), for any $\bar{K} \in (0, \bar{l})$ there exists $H > 0$ such that

$$\min_{0\le t\le 1} f(t,y,y') > -\bar{K}y', \ \ 0 \le y \le H\big(1+\frac{\alpha}{\beta}\big) \quad \text{and} \quad y' < -H. \tag{1.13}$$

By the superlinearity of $f(t,y,y')$ at $y = +\infty$ (see condition (1.8)), for any $K^* > K_0$ there exists $H^* > H$ such that

$$\min_{0\le t\le 1} f(t,y,y') > K^* y,\ \ y \ge H^* \quad \text{and} \quad -2H \le y' \le \frac{\alpha}{\beta}H. \tag{1.14}$$

Similarly by the superlinearity of $f(t,y,y')$ at $(0,0)$, for any $0 < \mu^* < \mu_0$ there is an $\eta^* > 0$ such that

$$0 < y \le \eta^* \text{ and } 0 < y' \le \frac{\alpha}{\alpha+\beta}\eta^* \Rightarrow \max_{0\le t\le 1} f(t,y,y') \le \mu^* y. \tag{1.15}$$

Also consider the rectangle

$$D := \big[0,\ \big(1+\frac{\alpha}{\beta}\big)H\big] \times \big[-2H,\ \frac{\alpha}{\beta}H\big]$$

and define a bounded continuous modification F of f such that

$$F(t,y,y') = f(t,y,y'),\ (t,y,y') \in [0,1] \times D.$$

2 An m-point boundary-value Problem

We consider now the boundary-value problem

$$\begin{gathered} y'' + F(t,y,y') = 0. \\ \alpha y(0) - \beta y'(0) = 0, \quad y(1) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i) \end{gathered} \tag{2.1}$$

Theorem 2.1 *Assume that (1.7) holds and*

$$\sigma^* = \sum_{i=1}^{m-2} \alpha_i \xi_i + \frac{\beta}{\alpha}\Big\{ \sum_{i=1}^{m-2} \alpha_i \frac{\xi_i}{\xi_1} - 1 \Big\} < 1. \tag{2.2}$$

Then the boundary-value problem (1.5)-(1.6) has a positive solution provided that:

- *The function f is superlinear (see (1.8)) along with (1.10), or*
- *The function f is sublinear (see (1.9)), (1.11) holds and in addition,*

$$\Big(\sum_{i=1}^{m-2}\alpha_i\xi_i\Big)\Big[\frac{1}{2\xi_{m-2}}+\frac{\alpha}{2\beta}\Big]>1. \tag{2.3}$$

Furthermore, there exists a positive number H such that

$$0<y(t)\le H \quad \text{and} \quad -2H\le y'(t)\le\frac{\alpha}{\beta}H,\quad 0\le t\le 1,$$

for any such solution.

Proof 1) *Superlinear case.* Since $f_\infty=+\infty$ and in view of (1.14), for any $K^*>K>K_0$ there exists $H^*\ge H>0$ such that

$$\min_{0\le t\le 1} f(t,y,y')>Ky,\quad y\ge H \quad \text{and} \quad \frac{\alpha}{\beta}H\ge y'\ge -2H. \tag{2.4}$$

Consider the function

$$W(P):=\sum_{i=1}^{m-2}\alpha_i y(\xi_i)-y(1),$$

where $y\in\mathcal{X}(P_1)$ is any solution of differential equation (2.1) starting at the point $P_1:=(y_1,y_1')\in E_0$ with $y_1=H$.

By the assumption (1.7) (i.e. the nature of the vector field, see Remark 1.1) it is obvious that $y(t)\ge y_1=H$ and $y'(t)\le y_1'=\frac{\alpha}{\beta}y_1=\frac{\alpha}{\beta}H$, for all t in a sufficiently small neighborhood of $t=0$.

Let's suppose that there is $t^*\in(0,1]$ such that

$$y(t)\ge H,\ -2H\le y'(t)\le\frac{\alpha}{\beta}H,\ 0\le t<t^* \text{ and } y(t^*)=H$$

or

$$y(t)\ge H,\ -2H\le y'(t)\le\frac{\alpha}{\beta}H,\ 0\le t<t^* \text{ and } y'(t^*)=-2H.$$

Consider first the case: $y(t^*)=H$. Then since $P_1\in E_0$, by the Taylor's formula we get $t\in[0,t^*]$ such that

$$H=y(t^*)\le H\big[1+\frac{\alpha}{\beta}\big]-\frac{1}{2}f(t,y(t),y'(t)) \tag{2.5}$$

and thus

$$H\frac{2\alpha}{\beta}\ge f(t,y(t),y'(t)).$$

But since $y(t)\ge H$ and $-2H\le y'(t)\le\frac{\alpha}{\beta}H,\ 0\le t<t^*$ by (2.4), we have

$$f(t,y(t),y'(t))\ge\min_{0\le t\le 1}f(t,y(t),y'(t))\ge Ky(t))\ge KH$$

and so we obtain $H2\alpha/\beta \geq KH$ contrary to the choice $K > \frac{2\alpha}{\beta}$. Furthermore, by (2.5),

$$H \leq y(t) < H\big[1 + \frac{\alpha}{\beta}\big], \quad 0 \leq t \leq 1. \tag{2.6}$$

We recall also (see (1.13)) that for any $\varepsilon^* \in \big(1,\ \min\big\{2, 1 + \bar{l}\big\}\big)$ there exists $\bar{K} \in \big(\varepsilon^* - 1,\ \bar{l}\big)$ such that

$$\min_{0 \leq t \leq 1} f(t, y, y') > -\bar{K}y', \quad 0 \leq y \leq H\big[1 + \frac{\alpha}{\beta}\big] \quad \text{and} \quad y' < -H. \tag{2.7}$$

We shall prove that

$$\frac{\alpha}{\beta}H \geq y'(t) \geq -\varepsilon^* H > -2H, \quad 0 \leq t \leq 1. \tag{2.8}$$

Indeed, since $y'(t)$ is decreasing on $[0, 1]$, let's assume that there exist $t_0, t_1 \in (0, 1)$ such that

$$y'(t_0) = -H\ ,\ -\varepsilon^* H < y'(t) < -H,\ t_0 \leq t \leq t_1 \quad \text{and} \quad y'(t_1) = -\varepsilon^* H$$

Then by (2.6)-(2.8), for some $\bar{t} \in (t_0, t_1)$, we get

$$\begin{aligned} -\varepsilon^* H &= y'(t_1) = y'(t_0) - f(\bar{t}, y(\bar{t}), y'(\bar{t})) \\ &\leq -H + \bar{K}y'(\bar{t}) \leq -H + \bar{K}y'(t_0) \\ &= -H - \bar{K}H, \end{aligned}$$

Thus we get another contradiction $\bar{K} \leq \varepsilon^* - 1$. On the other hand by the concavity of the solution $y \in \mathcal{X}(P_1)$ (due to the assumption (1.7)), we know that the function $y(\xi)/\xi$, $0 < \xi \leq 1$ is decreasing and so

$$\frac{y(\xi_i)}{\xi_i} \geq \frac{y(\xi_{m-2})}{\xi_{m-2}}, \quad i = 1, 2, \dots, m-2. \tag{2.9}$$

Thus in view of (2.6)

$$\begin{aligned} W(P_1) &= \sum_{i=1}^{m-2} \alpha_i y(\xi_i) - y(1) = \sum_{i=1}^{m-2} \alpha_i \xi_i \frac{y(\xi_i)}{\xi_i} - y(1) \\ &\geq \Big[\sum_{i=1}^{m-2} \alpha_i \xi_i\Big] \frac{H}{\xi_{m-2}} - y(1) = \sigma \frac{H}{\xi_{m-2}} - y(1). \end{aligned}$$

where we recall that $\sigma = \sum_{i=1}^{m-2} \alpha_i \xi_i < 1$. Consequently by Taylor's formula,

$$W(P_1) \geq \frac{\sigma}{\xi_{m-2}} H - \big(y_1 + \frac{\alpha}{\beta} y_1 - \frac{1}{2} f(t^*, y(t^*), y'(t^*))\big)$$

Thus by (2.4), (2.6) and (2.8), we get

$$\begin{aligned} W(P_1) &\geq \frac{\sigma}{\xi_{m-2}} H - \big[1 + \frac{\alpha}{\beta}\big]H + \frac{1}{2}Ky(t^*) \\ &\geq \frac{\sigma}{\xi_{m-2}} H - \big[1 + \frac{\alpha}{\beta}\big]H + \frac{1}{2}KH. \end{aligned}$$

In this way we get

$$W(P_1) \geq 0, \tag{2.10}$$

since by the choice of K at (2.4), we have

$$K > 2\big[1 + \frac{\alpha}{\beta} - \frac{\sigma}{\xi_{m-2}}\big].$$

Similarly by the superlinearity of $f(t, y, y')$ at $(0, 0)$, for any $\mu > 0$ there is an $\eta > 0$ such that

$$0 < y \leq \eta \text{ and } 0 \leq y' \leq \frac{2\alpha\varepsilon}{\beta}\eta \quad \text{imply} \quad \max_{0\leq t\leq 1} f(t, y, y') < \mu y, \tag{2.11}$$

where $\frac{\beta}{\alpha+\beta} < \varepsilon < 1$. We choose now (see (1.15))

$$\mu^* \leq \mu < \mu_0 = \min\big\{(1 - m^*)\frac{\varepsilon\alpha}{\beta},\ 2\big[\frac{\varepsilon(\alpha + \beta)}{\beta} - 1\big]\big\} \tag{2.12}$$

and then clearly $\eta \geq \eta^*$.

Let now $y \in \mathcal{X}(P_0)$ be a solution of differential equation (2.1) starting at the point $P_0 := (y_0, y_0') \in E_0$ with $y_0 = \varepsilon\eta$. We shall show that

$$\varepsilon\eta \leq y(t) \leq \eta \quad \text{and} \quad m^*\frac{\alpha\varepsilon}{\beta}\eta \leq y'(t) \leq \frac{\alpha\varepsilon}{\beta}\eta, \quad 0 \leq t \leq 1, \tag{2.13}$$

where we recall that

$$\sigma^* = \sum_{i=1}^{m-2} \alpha_i\xi_i + \frac{\beta}{\alpha}\Big\{\sum_{i=1}^{m-2} \alpha_i\frac{\xi_i}{\xi_1} - 1\Big\} < m^* < 1.$$

Indeed, if there is a least $t^* \in (0, 1]$ such that $m^*\frac{\alpha\varepsilon}{\beta}\eta = y'(t^*)$, and

$$\varepsilon\eta \leq y(t) \leq \eta \quad \text{and} \quad m^*\frac{\alpha\varepsilon}{\beta}\eta \leq y'(t) \leq \frac{\alpha\varepsilon}{\beta}\eta, \quad 0 \leq t < t^*,$$

then again by Taylor's formula,

$$m^*\frac{\alpha\varepsilon}{\beta}\eta = y'(t^*) = y_0\frac{\alpha}{\beta} - f(t, y(t), y'(t)) \geq y_0\frac{\alpha}{\beta} - \mu y(t) \geq \varepsilon\eta\frac{\alpha}{\beta} - \mu\eta,$$

and hence we obtain the contradiction $\mu \geq (1 - m^*)\,\frac{\varepsilon\alpha}{\beta}$, due to the choice of μ at (2.12). Similarly we may prove the first inequality of (2.13).

Consider once again the function

$$W(P) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i) - y(1)$$

and then by (2.9),

$$\begin{aligned} W(P_0) &= \sum_{i=1}^{m-2} \alpha_i y(\xi_i) - y(1) = \sum_{i=1}^{m-2} \alpha_i \xi_i \frac{y(\xi_i)}{\xi_i} - y(1) \\ &\leq \Big[\sum_{i=1}^{m-2} \alpha_i \xi_i \Big] \frac{y(\xi_1)}{\xi_1} - y(1). \end{aligned} \tag{2.14}$$

Now in view of (2.13),

$$\frac{y(\xi_1)}{\xi_1} = \frac{1}{\xi_1}\Big\{ y(0) + \int_0^{\xi_1} y'(s)ds \Big\} \leq \frac{\varepsilon\eta}{\xi_1} + \frac{\alpha\varepsilon}{\beta}\eta$$

and

$$y(1) = y(0) + \int_0^1 y'(s)ds \geq \varepsilon\eta + m^* \frac{\alpha\varepsilon}{\beta}\eta.$$

Consequently by (2.14),

$$\begin{aligned} W(P_0) &\leq \Big[\sum_{i=1}^{m-2} \alpha_i \xi_i \Big] \big(\frac{\alpha\varepsilon}{\beta}\eta + \frac{\varepsilon\eta}{\xi_1} \big) - m^* \frac{\alpha\varepsilon}{\beta}\eta - \varepsilon\eta \\ &= \Big(\sum_{i=1}^{m-2} \alpha_i \xi_i - m^* \Big) \frac{\alpha\varepsilon}{\beta}\eta + \Big\{ \sum_{i=1}^{m-2} \alpha_i \frac{\xi_i}{\xi_1} - 1 \Big\} \eta\varepsilon \leq 0 \end{aligned} \tag{2.15}$$

due to the choice of $m^* > \sum_{i=1}^{m-2} \alpha_i \xi_i + \frac{\beta}{\alpha}\big\{ \sum_{i=1}^{m-2} \alpha_i \frac{\xi_i}{\xi_1} - 1 \big\}$.

It is now clear that the function $W = W(P)$, $P \in [P_0, P_1]$ is continuous and thus by the Kneser's property (see Theorem 1.2), (2.10) and (2.15), we get a point $P \in [P_0, P_1]$ (we chose the last one to the "left" of P_1) such that $W(P) = 0$. This fact clearly means that there is a solution $y \in \mathcal{X}(P)$ of equation (2.1), such that

$$W(P) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i) - y(1) = 0.$$

It remains to be proved that the so obtaining solution $y = y(t)$ is actually a bounded function. Indeed, by the choice of P, the continuity of $y(t)$ with respect initial values, (2.10) and (2.15), it follows that

$$y(t) > 0, \quad 0 \leq t \leq 1,$$

because if

$$y(t) > 0,\ 0 \leq t < 1 \quad \text{and} \quad y(1) = 0,$$

then $W(P) > 0$. Moreover by the nature of the vector field (see Remark 1.1), there is $t_P \in (0,1)$ such that the so obtaining solution $y \in \mathcal{X}(P)$ is strictly increasing on $[0, t_p]$, strictly decreasing on $[t_p, 1]$ and further is strictly positive on $[0,1]$. Also it holds $y(t) \leq H$, $0 \leq t \leq 1$, i.e.

$$0 < y(t) \leq H, \quad 0 \leq t \leq 1. \tag{2.16}$$

Indeed, let's assume that there exist $t_0, t_1 \in [0,1]$ such that

$$y(t) \le H, \quad 0 \le t < t_0, \quad y(t_0) = H, \quad y(t) \ge H, \text{ and } y'(t) \ge 0, \quad t_0 \le t \le t_1.$$

Then we have $0 < y'(t_0) < \frac{\alpha}{\beta} y(t_0) \le \frac{\alpha}{\beta} H$ and further by (2.4), for some $\bar{t} \in (t_0, t_1)$

$$\begin{aligned} H \le y(t_1) =& y(t_0) + (t_1 - t_0) y'(t_0) - \frac{1}{2} f(\bar{t}, y(\bar{t}), y'(\bar{t})) \\ \le& H\big[1 + \frac{\alpha}{\beta}\big] - \frac{K}{2} y(\bar{t}) \\ \le& H\big[1 + \frac{\alpha}{\beta}\big] - \frac{K}{2} H. \end{aligned}$$

Thus we get the contradiction $K < 2\alpha/\beta$. Also by assumption (1.10), we may show (exactly as at (2.8)) that the above solution $y \in \mathcal{X}(P)$ implies further the inequalities

$$\frac{\alpha}{\beta} H \ge y'(t) \ge -\varepsilon^* H \ge 2H, \quad 0 \le t \le 1. \tag{2.17}$$

and hence by (2.16) and the definition of the modification F, the obtaining solution of (2.1) is actually a solution of the original equation (1.5).

2) *Sublinear case.* We choose $\varepsilon_0 > \frac{\alpha+\beta}{\beta}$ and recall that

$$\sum_{i=1}^{m-2} \alpha_i y(\xi_i) + \frac{\beta}{\alpha} \Big\{ \sum_{i=1}^{m-2} \alpha_i \frac{\xi_i}{\xi_1} - 1 \Big\} < m^* < 1.$$

Since $f_{+\infty,+\infty} = 0$, for $\mu < \min\big\{(1-m^*)\frac{\alpha}{\varepsilon_0 \beta},\ \frac{2}{\varepsilon_0}[\varepsilon_0 - \frac{\alpha+\beta}{\beta}]\big\}$, there exists $H > 0$ such that

$$\max_{0 \le t \le 1} f(t, y, y') < \mu y, \quad y \ge H, \quad \text{and} \quad \frac{\alpha}{\beta} H \ge y' \ge m^* \frac{\alpha}{\beta} H. \tag{2.18}$$

Let's consider a point $P_0 := (y_0, y_0') \in E_0$ with $y_0 = H$. We will prove first that for any solution $y \in \mathcal{X}(P_0)$,

$$H \le y(t) \le \varepsilon_0 H \quad \text{and} \quad \frac{m^* \alpha}{\beta} H \le y'(t) \le \frac{\alpha}{\beta} H, \quad 0 \le t \le 1. \tag{2.19}$$

Let us suppose that this is not the case. Then by the assumption (1.7), there is $t^* \in [0,1]$ such that

$$\begin{gathered} H \le y(t) \le \varepsilon_0 H, \quad \frac{m^* \alpha}{\beta} H \le y'(t) \le \frac{\alpha}{\beta} H, \quad 0 < t < t^*, \\ \text{and} \quad y(t^*) = \varepsilon_0 H \quad \text{or} \quad y'(t) = \frac{m^* \alpha}{\beta} H. \end{gathered} \tag{2.20}$$

Assume that $y(t^*) = \varepsilon_0 H$. Then by the Taylor's formula, (2.18) and (2.20) we obtain $t \in [0, t^*]$ such that

$$\begin{aligned}\varepsilon_0 H = y(t^*) =& y_0[1+\frac{\alpha}{\beta}] - \frac{1}{2} f(t, y(t), y'(t))\\ <& H[1+\frac{\alpha}{\beta}] + \frac{1}{2}\mu y(\bar{t}) \le H[1+\frac{\alpha}{\beta}] + \frac{1}{2}\mu\varepsilon_0 H\end{aligned}$$

and hence it contradicts

$$\mu < \frac{2}{\varepsilon_0}\big[\varepsilon_0 - \frac{\alpha+\beta}{\beta}\big].$$

Let's suppose now that $y'(t^*) = m^* \frac{\alpha}{\beta} H$. Then again by (2.18) and (2.20), we obtain

$$m^*\frac{\alpha}{\beta}H = y'(t^*) = y_0' - f(t, y(t), y'(t)) \ge \frac{\alpha}{\beta}H - \mu y(t) \ge \frac{\alpha}{\beta}H - \mu\varepsilon_0 H,$$

which contradicts $\mu < (1-m^*)\alpha/(\varepsilon_0\beta)$.

Consider the function $W(P)$. Then

$$\begin{aligned}W(P_0) =& \sum_{i=1}^{m-2}\alpha_i y(\xi_i) - y(1) = \sum_{i=1}^{m-2}\alpha_i\xi_i\frac{y(\xi_i)}{\xi_i} - y(1)\\ \le& \Big[\sum_{i=1}^{m-2}\alpha_i\xi_i\Big]\frac{y(\xi_1)}{\xi_1} - y(1)\end{aligned}$$

and so by the second inequality of (2.19) (see also (2.15)), we get

$$\begin{aligned}W(P_0) \le& \Big[\sum_{i=1}^{m-2}\alpha_i\xi_i\Big]\big(\frac{\alpha}{\beta}H + \frac{H}{\xi_1}\big) - m^*\frac{\alpha}{\beta}H - H\\ =& \big(\sum_{i=1}^{m-2}\alpha_i\xi_i - m^*\big)\frac{\alpha}{\beta}H + \big(\sum_{i=1}^{m-2}\alpha_i\frac{\xi_i}{\xi_1} - 1\big)H \le 0\end{aligned} \tag{2.21}$$

due to the fact that $m^* > \sum_{i=1}^{m-2}\alpha_i\xi_i + \frac{\beta}{\alpha}\big\{\sum_{i=1}^{m-2}\alpha_i\frac{\xi_i}{\xi_1} - 1\big\}$.

On the other hand, since $f_0 = +\infty$, for any $K > \max\{\frac{2(\alpha-\beta)}{\beta}, \frac{2\alpha}{\beta}\}$ there exist $\eta \in (0, H)$ such that

$$\min_{0\le t\le 1} f(t, y, y') > Ky,\ 0 < y \le \eta \quad\text{and}\quad -\eta \le y' \le \frac{\alpha}{\beta}\frac{\eta}{2}. \tag{2.22}$$

Consider a point $P_1 := (y_1, y_1') \in E_0$ with $y_1 = \frac{\eta}{2}$ and any $y \in \mathcal{X}(P_1)$. As above, by Taylor's formula, (2.22) and the choice $K > \max\{\frac{2(\alpha-\beta)}{\beta}, \frac{2\alpha}{\beta}\}$ we can easily prove that

$$\frac{\eta}{2} \le y(t) \le \eta, \quad 0 \le t \le 1. \tag{2.23}$$

We choose now $\varepsilon_0^* \in (1,2)$ and then by Assumption (1.11), there exist $\bar\eta_0 \in (0,\eta)$ and

$$0 < K^* < \min\Big\{\frac{\varepsilon_0^*-1}{\varepsilon_0^*},\ \min\{1, \frac{2\beta}{\alpha}\}[\frac{\sigma\xi_{m-2}}{2}]^{-1}[\frac{\sigma}{2\xi_{m-2}}+\frac{\sigma\alpha}{2\beta}-1]\Big\} \tag{2.24}$$

such that

$$\max_{0\le t\le 1} f(t,y,y') < K^*|y'|, \quad \frac{\eta}{2} \le y \le \eta, \quad \text{and} \quad -2\bar\eta_0 \le y' < \frac{\alpha\eta}{2\beta}. \tag{2.25}$$

Besides (2.23) we shall prove that

$$\frac{\alpha}{\beta}\frac{\eta}{2} \ge y'(t) \ge -\bar\eta_0 > -\eta, \quad 0 \le t \le 1. \tag{2.26}$$

Indeed since $y'(t)$ is decreasing on $[0,1]$ and $\varepsilon_0^* \in (1,2)$ is arbitrary, let's assume that there exist $t_0, t_1 \in [0,1]$ such that $y'(t_0) = -\bar\eta_0$,

$$-2\bar\eta_0 < -\varepsilon_0^*\bar\eta_0 \le y'(t) \le -\bar\eta_0, \quad t_0 \le t < t_1, \quad \text{and } y'(t_1) = -\varepsilon_0^*\bar\eta_0.$$

Thus by (2.23)-(2.25), we have for some $\bar t \in (t_0, t_1)$

$$-\varepsilon_0^*\bar\eta_0 = y'(t_1) = y'(t_0) - f(\bar t, y(\bar t), y'(\bar t)) \ge -\bar\eta_0 + K^* y'(\bar t) \ge -\bar\eta_0 - K^*\varepsilon_0^*\bar\eta_0,$$

and so, we get another contradiction $K^* \ge (\varepsilon_0^*-1)/\varepsilon_0^*$, due to (2.24).

Now as above (see (2.9) and (2.21)), we have

$$W(P_1) \ge \Big[\sum_{i=1}^{m-2}\alpha_i\xi_i\Big]\frac{y(\xi_{m-2})}{\xi_{m-2}} - y(1) = \sigma\frac{y(\xi_{m-2})}{\xi_{m-2}} - y(1).$$

Consequently by (2.23) and the Taylor's formula,

$$\begin{aligned} W(P_1) \ge& \frac{\sigma}{\xi_{m-2}}\Big(y_1 + \frac{\alpha}{\beta}y_1\xi_{m-2} - \frac{\xi_{m-2}^2}{2}f(\bar t, y(\bar t), y'(\bar t))\Big) - \eta \\ =& \frac{\sigma}{\xi_{m-2}}\frac{\eta}{2} + \sigma\frac{\alpha}{\beta}\frac{\eta}{2} - \frac{1}{2}\sigma\xi_{m-2}f(\bar t, y(\bar t), y'(\bar t)) - \eta \end{aligned}$$

Thus by (2.23) and (2.26), we get

$$\begin{aligned} W(P_1) \ge& \frac{\sigma}{\xi_{m-2}}\frac{\eta}{2} + \sigma\frac{\alpha}{\beta}\frac{\eta}{2} - \frac{\sigma\xi_{m-2}}{2}K^*|y'(\bar t)| - \eta \\ \ge& \frac{\sigma}{\xi_{m-2}}\frac{\eta}{2} + \sigma\frac{\alpha}{\beta}\frac{\eta}{2} - \frac{\sigma\xi_{m-2}}{2}K^*\hat\eta - \eta, \end{aligned}$$

where $\hat\eta := \max\{\eta, \frac{\alpha\eta}{2\beta}\}$. In this way, by the assumption (2.3) and the choice of K^* at (2.24), we get

$$W(P_1) \ge 0.$$

Thus as at the superlinear case, we obtain a point $P \in [P_0, P_1]$ such that $W(P) = 0$ and this clearly completes the proof.

Remark 2.2 By the choice of $m^* \in \big(\sum_{i=1}^{m-2}\alpha_i\xi_i + \frac{\beta}{\alpha}\big\{\sum_{i=1}^{m-2}\alpha_i\frac{\xi_i}{\xi_1} - 1\big\},\ 1\big)$ and following Ma [21], we may easily show that for

$$\sum_{i=1}^{m-2}\alpha_i\xi_i \geq 1,$$

there is not (positive) solution $y \in \mathcal{X}(P)$ of the BVP (1.5)-(1.6). Indeed, if there is one, then

$$y(1) = \sum_{i=1}^{m-2}\alpha_i y(\xi_i) = \sum_{i=1}^{m-2}\alpha_i\xi_i\frac{y(\xi_i)}{\xi_i} \geq \sum_{i=1}^{m-2}\alpha_i\xi_i\frac{y(\xi^*)}{\xi^*} > \frac{y(\xi^*)}{\xi^*},$$

where clearly $\xi^* = \xi_{m-2}$ and this contradicts the concavity of the solution $y = y(t)$. Furthermore we must seek the monotone (obviously increasing) solutions of (1.5)-(1.6), only for the case $\sum_{i=1}^{m-2}\alpha_i \geq 1$, since otherwise we get

$$0 = W(P) = \sum_{i=1}^{m-2}\alpha_i y(\xi_i) - y(1) < \Big[\sum_{i=1}^{m-2}\alpha_i - 1\Big]y(1) < 0.$$

The question of existence of such a monotone solution remains open. However we can obtain a strictly decreasing solution for the boundary-value problem

$$\begin{gathered} y''(t) = -f(t, y(t), y'(t)), \quad 0 \leq t \leq 1, \\ \alpha y(0) + \beta y'(0) = 0, \quad y(1) = \sum_{i=1}^{m-2}\alpha_i y(\xi_i). \end{gathered} \tag{2.27}$$

where $\alpha \geq 0$ and $\beta > 0$.

Remark 2.3 Suppose that the concept of jointly sublinearity is modified to

$$\begin{gathered} f_0 := \lim_{y\to 0+}\min_{0\leq t\leq 1}\frac{f(t,y,y')}{y} = +\infty, \quad \text{for } |y'| \leq M. \\ f_{\infty,-\infty} := \lim_{(y,y')\to(+\infty,-\infty)}\max_{0\leq t\leq 1}\frac{f(t,y,y')}{y} = 0. \end{gathered} \tag{2.28}$$

Then, following almost the same line as above (under the obvious modifications) we may prove the next theorem.

Theorem 2.4 *Assume that (1.7) holds and further*

$$\sigma^* = \sum_{i=1}^{m-2}\alpha_i\xi_i + \frac{\beta}{\alpha}\Big\{\sum_{i=1}^{m-2}\alpha_i\frac{\xi_i}{\xi_1} - 1\Big\} < 1.$$

Then the boundary-value problem (2.27) has a positive strictly decreasing solution provided that:

- *The function f is superlinear (see (1.8)) along with (1.10), or*
- *The function f is sublinear (see (2.28)), (1.11) is true and in addition,*

$$\sum_{i=1}^{m-2} \alpha_i \xi_i \Big[\frac{1}{\xi_{m-2}} - \frac{\alpha}{\beta}\Big] > 1.$$

Furthermore there exists a positive number H such that

$$0 < y(t) \leq H \quad \text{and} \quad -2H \leq y'(t) \leq -\frac{\alpha}{\beta} H, \quad 0 \leq t \leq 1,$$

for any such solution.

Remark 2.5 Again, as in Remark 2.2, we may show that for

$$\sum_{i=1}^{m-2} \alpha_i \xi_i \geq 1,$$

there is no (positive) solution $y \in \mathcal{X}(P)$ of the BVP (2.27). Furthermore we must seek the possible solutions of (2.27) only for the case

$$\sum_{i=1}^{m-2} \alpha_i \leq 1,$$

since otherwise, by the monotonicity of $y(t)$, we get the contradiction

$$0 = W(P) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i) \geq \Big[\sum_{i=1}^{m-2} \alpha_i - 1\Big] y(1) > 0.$$

Finally consider the boundary-value problem

$$y'' + f(t, y, y') = 0, y'(0) = 0, \quad y(1) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i). \tag{2.29}$$

Then following almost the same lines as above, we may prove the next theorem.

Theorem 2.6 *Assume that (1.7) holds and*

$$\sum_{i=1}^{m-2} \alpha_i \frac{\xi_i}{\xi_1} < 1.$$

Then the boundary-value problem (2.29) has a positive strictly decreasing solution provided that

- *The function f is superlinear (see (1.8)) along with (1.10), or*

- *The function f is sublinear (see (2.28)), (1.11) holds and in addition*

$$\sum_{i=1}^{m-2} \alpha_i \frac{\xi_i}{\xi_{m-2}} > 1.$$

Furthermore there exists a positive number H such that

$$0 < y(t) \leq H \quad \textit{and} \quad -2H \leq y'(t) \leq 0, \quad 0 \leq t \leq 1,$$

for any such solution.

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