Electronic Journal of Differential Equations, Vol. **2002**(2002), No. 20, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

$\mathcal{L}^{2,\Phi}$ regularity for nonlinear elliptic systems of second order *

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Abstract

This paper is concerned with the regularity of the gradient of the weak solutions to nonlinear elliptic systems with linear main parts. It demonstrates the connection between the regularity of the (generally discontinuous) coefficients of the linear parts of systems and the regularity of the gradient of the weak solutions of systems. More precisely: If abovementioned coefficients belong to the class $L^\infty(\Omega)\cap\mathcal{L}^{2,\Psi}(\Omega)$ (generalized Campanato spaces), then the gradient of the weak solutions belong to $\mathcal{L}^{2,\Phi}_{loc}(\Omega,\mathbb{R}^{nN})$, where the relation between the functions Ψ and Φ is formulated in Theorems 3.1 and 3.2 below.

1 Introduction

In this paper, we consider the problem of the regularity of the first derivatives of weak solutions to the nonlinear elliptic system

$$-D_\alpha a_i^\alpha(x,u,Du)=a_i(x,u,Du),\quad i=1,\dots,N, \tag{1.1}$$

where a_i^α, a_i are Caratheodorian mappings from $(x,u,z)\in\Omega\times\mathbb{R}^N\times\mathbb{R}^{nN}$ into $\mathbb{R}$, $N>1$, $\Omega\subset\mathbb{R}^n$, $n\ge 3$ is a bounded open set. A function $u\in W^{1,2}_{\rm loc}(\Omega,\mathbb{R}^N)$ is called a weak solution of (1.1) in Ω if

$$\int_\Omega a_i^\alpha(x,u,Du)D_\alpha\varphi^i(x)\,dx=\int_\Omega a_i(x,u,Du)\varphi^i(x)\,dx,\quad \forall\,\varphi\in C_0^\infty(\Omega,\mathbb{R}^N).$$

We use the summation convention over repeated indices.

As it is known, in case of a general system (1.1), only partial regularity can be expected for $n>2$ (see e.g. [2, 6, 9]). Under the assumptions below we will prove $\mathcal{L}^{2,\Phi}$-regularity of gradient of weak solutions for the system (1.1) whose coefficients a_i^α have the form

$$a_i^\alpha(x,u,Du)=A_{ij}^{\alpha\beta}(x)D_\beta u^j+g_i^\alpha(x,u,Du), \tag{1.2}$$

* *Mathematics Subject Classifications:* 49N60, 35J60.
Key words: Nonlinear equations, regularity, Morrey-Campanato spaces.

Submitted May 31, 2001. Published February 19, 2002.
J. Daněček was partially supported by the research project MSM no. 261100006

where $i,\ j = 1,\dots,N,\ \alpha,\ \beta = 1,\dots,n,\ A_{ij}^{\alpha\beta}$ is a matrix of functions, and the following condition of strong ellipticity

$$A_{ij}^{\alpha\beta}(x)\xi_\alpha^i\xi_\beta^j \geq \nu|\xi|^2, \quad \text{a.e. } x\in\Omega,\ \forall\,\xi\in\mathbb{R}^{nN};\nu>0 \tag{1.3}$$

holds, and g_i^α are functions with sublinear growth in z. In what follows, we formulate the conditions on the smoothness and the growth of the functions $A_{ij}^{\alpha\beta}$, g_i^α and a_i precisely.

It is well known (see [2]) that, in the case of linear elliptic systems with continuous coefficients $A_{ij}^{\alpha\beta}$, the gradient of weak solutions has the $L^{2,\lambda}$-regularity and, if the coefficients $A_{ij}^{\alpha\beta}$ belong to some Hölder class, then the gradient of weak solutions belongs to the BMO-class (functions with bounded mean oscillations, see Definition 2.1). These results were generalized in [3] where the first author has proved the $L^{2,\lambda}$-regularity of the gradient of weak solutions to (1.1)-(1.3) in the situation where the coefficients $A_{ij}^{\alpha\beta}$ are continuous and the BMO-regularity of gradient in the case where coefficients $A_{ij}^{\alpha\beta}$ are Hölder continuous.

In the case of linear elliptic systems when the coefficients $A_{ij}^{\alpha\beta}$ are "small multipliers of $BMO(\Omega)$", a class neither containing nor contained in $C(\overline{\Omega})$, Acquistapace in [1] proved global (under Dirichlet boundary condition) and local BMO-regularity for the gradient of solutions. In [1] the local BMO-regularity does not follow in a standard way from the global one, because there are no regularity results in the Morrey spaces $L^{2,\lambda}$, $0<\lambda<n$. The last mentioned fact was a motive for [4] and [5]. In [4, 5] the Morrey regularity for the gradient of weak solutions for nonlinear elliptic systems of type (1.1) is proved when the coefficients $A_{ij}^{\alpha\beta}$ are generally discontinuous (not necessarily "small multipliers of $BMO(\Omega)$").

The purpose of this paper is a generalization of the results from [4] and [5]. Result of this paper may open a way to proving the BMO-regularity for the gradient of solutions of (1.1).

If we want to sketch our method of proof, we must say that its crucial point is the assumption on $A_{ij}^{\alpha\beta}$: $A_{ij}^{\alpha\beta}\in L^\infty(\Omega)\cap\mathcal{L}^{2,\Psi}(\Omega)$ (for the definition see below). Taking into account higher integrability of gradient Du (for some $r>2$), we obtain $\mathcal{L}^{2,\Phi}$-regularity of the gradient.

2 Notation and definitions

We consider the bounded open set $\Omega\subset\mathbb{R}^n$ with points $x=(x_1,\dots x_n)$, $n\geq 3$, $u\,\Omega\to\mathbb{R}^N$, $N\geq 1$, $u(x)=(u^1(x),\dots,u^N(x))$ is a vector-valued function, $Du=(D_1u,\dots,D_nu)$, $D_\alpha=\partial/\partial x_\alpha$. The meaning of $\Omega_0\subset\subset\Omega$ is that the closure of Ω_0 is contained in Ω, i.e. $\overline{\Omega}_0\subset\Omega$. For the sake of simplicity we denote by $|\cdot|$ the norm in $\mathbb{R}^n$ as well as in $\mathbb{R}^N$ and $\mathbb{R}^{nN}$. If $x\in\mathbb{R}^n$ and r is a positive real number, we write $B_r(x)=\{y\in\mathbb{R}^n : |y-x|<r\}$, i.e., the open ball in $\mathbb{R}^n$, $\Omega(x,r)=\Omega\cap B_r(x)$. Denote by $u_{x,r}=|\Omega(x,r)|_n^{-1}\int_{\Omega(x,r)}u(y)\,dy = ⨍_{\Omega(x,r)}u(y)\,dy$ the mean value of the function $u\in L^1(\Omega,\mathbb{R}^N)$ over the set $\Omega(x,r)$,

where $|\Omega(x,r)|_n$ is the n-dimensional Lebesgue measure of $\Omega(x,r)$. The bounded domain $\Omega \subset \mathbb{R}^n$ is said to be of type $\mathcal{A}$ if there exists a constant $\mathcal{A} > 0$ such that, for every $x \in \overline{\Omega}$ and all $0 < r < \operatorname{diam}\Omega$, it holds $|\Omega(x,r)|_n \geq \mathcal{A}r^n$. Beside the usually used space $C_0^\infty(\Omega,\mathbb{R}^N)$, the Hölder spaces $C^{0,\alpha}(\Omega,\mathbb{R}^N)$, $C^{0,\alpha}(\overline{\Omega},\mathbb{R}^N)$ and the Sobolev spaces $W^{k,p}(\Omega,\mathbb{R}^N)$, $W^{k,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$, $W^{k,p}_0(\Omega,\mathbb{R}^N)$ (see, e.g.[8]), we use the following Morrey and Campanato spaces.

Definition 2.1 Let $\lambda \in [0,n]$, $q \in [1,\infty)$. A function $u \in L^q(\Omega,\mathbb{R}^N)$ is said to belong to Morrey space $L^{q,\lambda}(\Omega,\mathbb{R}^N)$ if

$$||u||^q_{L^{q,\lambda}(\Omega,\mathbb{R}^N)} = \sup_{x\in\Omega, r>0} \frac{1}{r^\lambda}\int_{\Omega(x,r)} |u(y)|^q\,dy < \infty.$$

Let $\lambda \in [0,n+q]$, $q \in [1,\infty)$. The Campanato space $\mathcal{L}^{q,\lambda}(\Omega,\mathbb{R}^N)$ is the subspace of such functions $u \in L^q(\Omega,\mathbb{R}^N)$ for which

$$[u]^q_{\mathcal{L}^{q,\lambda}(\Omega,\mathbb{R}^N)} = \sup_{r>0, x\in\Omega} \frac{1}{r^\lambda}\int_{\Omega(x,r)} |u(y) - u_{x,r}|^q\,dy < \infty.$$

Let $Q_0 \subset \mathbb{R}^n$ is a cube whose edges are parallel with the coordinate axes. The $BMO(Q_0,\mathbb{R}^N)$ space (bounded mean oscillation space) is the subspace of such functions $u \in L^1(Q_0,\mathbb{R}^N)$ for which

$$\langle u\rangle_{Q_0} = \sup_{Q\subset Q_0} \frac{1}{|Q|}\int_Q |u(y) - u_Q|\,dy < \infty,$$

where $u_Q = ⨍_Q u(y)\,dy$ and $Q \subset Q_0$ is the cube homotetic with Q_0.

Remark $u \in L^{q,\lambda}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ if and only if $u \in L^{q,\lambda}(\Omega_0,\mathbb{R}^N)$ for each $\Omega_0 \subset\subset \Omega$.

Proposition 2.1 *For a domain $\Omega \subset \mathbb{R}^n$ of the class $\mathcal{C}^{0,1}$ we have the following*

(a) *With the norms $\|u\|_{L^{q,\lambda}}$ and $\|u\|_{\mathcal{L}^{q,\lambda}} = \|u\|_{L^q} + [u]_{L^{q,\lambda}}$, $\|u\|_{BMO} = \|u\|_{L^1} + \langle u\rangle_Q$, $L^{q,\lambda}(\Omega,R^N)$, $\mathcal{L}^{q,\lambda}(\Omega,R^N)$ and $BMO(Q_0,\mathbb{R}^N)$ are Banach spaces.*

(b) *$L^{q,\lambda}(\Omega,\mathbb{R}^N)$ is isomorphic to the $\mathcal{L}^{q,\lambda}(\Omega,\mathbb{R}^N)$, $1 \leq q < \infty$, $0 \leq \lambda < n$.*

(c) *$L^{q,n}(\Omega,\mathbb{R}^N)$ is isomorphic to the $L^\infty(\Omega,\mathbb{R}^N) \subsetneq L^{q,n}(\Omega,\mathbb{R}^N)$, $1 \leq q < \infty$.*

(d) *$\mathcal{L}^{2,n}(\Omega,\mathbb{R}^N)$ is isomorphic to the $\mathcal{L}^{q,n}(\Omega,\mathbb{R}^N)$ and $\mathcal{L}^{q,n}(Q,\mathbb{R}^N) = BMO(Q,\mathbb{R}^N)$, Q being a cube, $1 \leq q < \infty$.*

(e) *if $u \in W^{1,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ and $Du \in L^{2,\lambda}_{\mathrm{loc}}(\Omega,\mathbb{R}^{nN})$, $n-2 < \lambda < n$, then $u \in C^{0,(\lambda+2-n)/2}(\Omega,\mathbb{R}^N)$.*

(f) *$\mathcal{L}^{q,\lambda}(\Omega,R^N)$ is isomorphic to the $C^{0,(\lambda-n)/q}(\overline{\Omega},R^N)$ for $n < \lambda \leq n+q$.*

For more details see [2, 6, 8, 9].

The generalization of Campanato spaces $\mathcal{L}^{q,\lambda}$ (see [2]) are the classes $\mathcal{L}^{2,\Psi}$ introduced by Spanne [10] and [11].

Definition 2.2 Let Ψ be a positive function on $(0, \operatorname{diam}\Omega]$. A function $u \in L^2(\Omega, \mathbb{R}^N)$ is said to belong to $\mathcal{L}^{2,\Psi}(\Omega, \mathbb{R}^N)$ if

$$[u]_{2,\Psi,\Omega} = \sup_{x\in\Omega, r\in(0,\operatorname{diam}\Omega]} \frac{1}{\Psi(r)} \Big(\int_{\Omega(x,r)} |u(y) - u_{x,r}|^2 \, dy \Big)^{1/2} < \infty$$

and by $l^{2,\Psi}(\Omega, \mathbb{R}^N)$ we denote the subspace of all $u \in \mathcal{L}^{2,\Psi}(\Omega, \mathbb{R}^N)$ such that

$$[u]_{2,\Psi,\Omega,r_0} = \sup_{x\in\Omega, r\in(0,r_0]} \frac{1}{\Psi(r)} \Big(\int_{\Omega(x,r)} |u(y) - u_{x,r}|^2 \, dy \Big)^{1/2} = o(1) \text{ as } r_0 \searrow 0.$$

Some basic properties of the above-mentioned spaces are formulated in the following proposition (for the proofs see [1, 10, 11]).

Proposition 2.2 *For a domain $\Omega \subset \mathbb{R}^n$ of the class $\mathcal{C}^{0,1}$ we have the following*

(a) $\mathcal{L}^{2,\Psi}(\Omega, \mathbb{R}^N)$ *is a Banach space with norm* $\|u\|_{\mathcal{L}^{2,\Psi}(\Omega,\mathbb{R}^N)} = \|u\|_{L^2(\Omega,\mathbb{R}^N)} + [u]_{\mathcal{L}^{2,\Psi}(\Omega,\mathbb{R}^N)}$.

(b) Let $\Psi(r) = r^{n/2}/(1 + |\ln r|)$. *Then* $C^0(\overline{\Omega}, \mathbb{R}^N) \setminus \mathcal{L}^{2,\Psi}(\Omega, \mathbb{R}^N)$ *and* $(L^\infty(\Omega, \mathbb{R}^N) \cap l^{2,\Psi}(\Omega, \mathbb{R}^N)) \setminus C^0(\overline{\Omega}, \mathbb{R}^N)$ *are not empty.*

In the sequel we assume that $\Psi : (0, d] \to (0, \infty)$ has the form

$$\Psi(r) = r^{\zeta/2}\psi(r), \quad 0 \le \zeta \le n+2, \tag{2.1}$$

where ψ is a continuous, non-decreasing function such that $\lim_{r\to 0+} \psi(r) = 0$ and $r \to \psi(r)/r^\xi$ for some $\xi > 0$ is almost decreasing, i.e. there exists $k_\psi \ge 1$ and $d_0 \le d$ such that

$$k_\psi \frac{\psi(r)}{r^\xi} \ge \frac{\psi(R)}{R^\xi}, \quad \forall\, 0 < r < R \le d_0. \tag{2.2}$$

Remark The function $\psi(r) = 1/(1 + |\ln r|)$ satisfies (2.2) with an arbitrary $\xi > 0$.

3 Main results

Suppose that for all $(x, u, z) \in \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN}$ the following conditions hold:

$$|a_i(x, u, z)| \le f_i(x) + L|z|^{\gamma_0}, \tag{3.1}$$

$$|g_i^\alpha(x, u, z)| \le f_i^\alpha(x) + L|z|^\gamma, \tag{3.2}$$

$$g_i^\alpha(x, u, z) z_\alpha^i \ge \nu_1 |z|^{1+\gamma} - f^2(x) \tag{3.3}$$

for almost all $x \in \Omega$ and all $u \in \mathbb{R}^N$, $z \in \mathbb{R}^{nN}$. Here L, ν_1 are positive constants, $1 \le \gamma_0 < (n+2)/n$, $0 \le \gamma < 1$, f, $f_i^\alpha \in L^{\sigma,\lambda}(\Omega)$, $\sigma > 2$, $0 < \lambda \le n$, $f_i \in L^{\sigma q_0, \lambda q_0}(\Omega)$, $q_0 = n/(n+2)$. We set $A = (A_{ij}^{\alpha\beta})$, $g = (g_i^\alpha)$, $a = (a_i)$, $\widetilde{f} = (f_i)$, $\widetilde{\widetilde{f}} = (f_i^\alpha)$.

The next theorem is slightly generalizing the main result from [4].

Theorem 3.1 *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a weak solution to the system (1.1) and the conditions (1.2), (1.3), (3.1), (3.2) and (3.3) be satisfied. Suppose further that $A^{\alpha\beta}_{ij} \in L^\infty(\Omega) \cap \mathcal{L}^{2,\Psi}(\Omega)$, $i,\ j = 1, \dots, N$, $\alpha,\ \beta = 1, \dots, n$ and Ψ is a function satisfying the condition (2.1) with $\zeta = n$. Then*

$$Du \in \begin{cases} L^{2,\lambda}_{\mathrm{loc}}(\Omega, R^{nN}) & \text{if } \lambda < n \\ L^{2,\lambda'}_{\mathrm{loc}}(\Omega, R^{nN}) \text{ with arbitrary } \lambda' < n & \text{if } \lambda = n. \end{cases}$$

Therefore,

$$u \in \begin{cases} C^{0,(\lambda-n+2)/2}(\Omega, \mathbb{R}^N) & \text{if } n-2 < \lambda < n \\ C^{0,\vartheta}(\Omega, R^N) \text{ with arbitrary } \vartheta < 1 & \text{if } \lambda = n. \end{cases}$$

To obtain $\mathcal{L}^{2,\Phi}$-regularity for the first derivatives of the weak solution we strengthen the conditions on the coefficients g^α_i and a_i. Namely suppose that

$$|a_i(x, u, z)| \le f_i(x) + L|z|^{\gamma_0} \tag{3.4}$$

$$|g^\alpha_i(x, u, z_1) - g^\alpha_i(y, v, z_2)| \le L(|f^\alpha_i(x) - f^\alpha_i(y)| + |z_1 - z_2|^\gamma) \tag{3.5}$$

$$g^\alpha_i(x, u, z) z^i_\alpha \ge \nu_1 |z|^{1+\gamma} - f^2(x). \tag{3.6}$$

for a.e. $x \in \Omega$ and all $u,\ v \in \mathbb{R}^N$, $z_1,\ z_2 \in \mathbb{R}^{nN}$. Here L, ν_1 are positive constants, $1 \le \gamma_0 < (n+2)/n$, $0 \le \gamma < 1$, $f,\ f^\alpha_i \in \mathcal{L}^{2,n}(\Omega)$, $f_i \in L^{\sigma q_0, n q_0}(\Omega)$, $\sigma > 2$, $q_0 = n/(n+2)$. It is not difficult to see that from assumptions (3.4)–(3.6) follow (3.1)–(3.3) with $\lambda = n$.

We can now formulate the main result of this paper.

Theorem 3.2 *Let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a weak solution to the system (1.1) and suppose that the conditions (1.2), (1.3), (3.4), (3.5) and (3.6) hold. Let further $A^{\alpha\beta}_{ij} \in L^\infty(\Omega) \cap \mathcal{L}^{2,\Psi}(\Omega)$, for each $i,\ j = 1, \dots, N$, $\alpha,\ \beta = 1, \dots, n$ and Ψ be a function satisfying the conditions (2.1) and (2.2) with $\zeta = n$ and $0 < \xi \le 2$. Then $Du \in \mathcal{L}^{2,\Phi}_{\mathrm{loc}}(\Omega, \mathbb{R}^{nN})$ with $\Phi(R) = R^{n/2}$ in the case when the function ψ has a form of some power function and $\Phi(R) = R^{\lambda/2} \psi^{(r-2)/2r}(R)$ for some $r > 2$ and arbitrary $\lambda < n$ in another cases.*

Remark The conditions (2.1) and (2.2) in Theorem 3.2 are for example satisfied with the function $\psi(r) = 1/(1 + |\ln r|)$ (see also Proposition 2.2(b)).

4 Some lemmas

In this section we present the results needed for the proof of the main theorem. In $B_R(x) \subset \mathbb{R}^n$ we consider a linear elliptic system

$$-D_\alpha(A^{\alpha\beta}_{ij} D_\beta u^j) = 0 \tag{4.1}$$

with constant coefficients for which (1.3) holds.

Lemma 4.1 ([2, pp. 54-55]) *Let $u \in W^{1,2}(B_R(x), \mathbb{R}^N)$ be a weak solution to the system (4.1). Then, for each $0 < \sigma \leq R$,*

$$\int_{B_\sigma} |Du(y)|^2\, dy \leq c\Big(\frac{\sigma}{R}\Big)^n \int_{B_R} |Du(y)|^2\, dy,$$

$$\int_{B_\sigma} |Du(y) - (Du)_\sigma|^2\, dy \leq c\Big(\frac{\sigma}{R}\Big)^{n+2} \int_{B_R} |Du(y) - (Du)_R|^2\, dy$$

hold with a constant c independent of the homotethie.

The following lemma generalizes [7, Lemma 3.1] and is fundamental for proving Theorem 3.2.

Lemma 4.2 *Let ψ be a function from the condition (2.1). Further let ϕ be a nonnegative function on $(0, d]$ and A, B, C, α, β be nonnegative constants. Suppose that for all $0 < \sigma < R \leq d$, we have:*

$$\phi(\sigma) \leq \Big[A\Big(\frac{\sigma}{R}\Big)^\alpha + B\Big]\, \phi(R) + C\, R^\beta \psi(R), \tag{4.2}$$

$$\phi(d) < \infty. \tag{4.3}$$

Further let the constant $0 < K < 1$ exist such that $\varepsilon = k_\psi(AK^{\alpha-\beta-\xi} + BK^{-\beta-\xi}) < 1$. Then $\phi(\sigma) \leq c\,\sigma^\beta \psi(\sigma)$, for $0 < \sigma \leq d$, where

$$c = \max\Big\{\frac{Ck_\psi}{(1-\varepsilon)K^{\beta+\xi}}, \sup_{r\in[Kd,d]} \frac{\phi(r)}{r^\beta \psi(r)}\Big\}.$$

Proof From (4.2) and (4.3), it follows that $\sup_{r\in[\sigma,d]} \phi(r) < \infty$. We set

$$c_n = \sup_{r\in[1/n,d]} \frac{\phi(r)}{r^\beta \psi(r)}.$$

It is obvious that $c_n \leq c_0 = \sup_{r\in(0,d]} \phi(r)/r^\beta\psi(r)$.
When $c_0 = \sup_{r\in[Kd,d]} \phi(r)/r^\beta\psi(r)$, we have the result. Also

$$c_0 > \sup_{r\in[Kd,d]} \frac{\phi(r)}{r^\beta \psi(r)}$$

and there exists a sequence $\{r_n\}_{n=n_0}^\infty$ such that $1/n < r_n < Kd$ and

$$\Big|\frac{\phi(r_n)}{r_n^\beta \psi(r_n)} - c_n\Big| < \frac{c_n}{n}.$$

Put $\sigma = r_n$, $R = r_n/K$ in (4.2) and using (2.2) we get

$$\frac{K^\xi}{k_\psi}\frac{\phi(r_n)}{r_n^\beta \psi(r_n)} \leq \frac{\phi(r_n)}{r_n^\beta \psi(\frac{r_n}{K})} \leq (AK^{\alpha-\beta} + BK^{-\beta})\frac{\phi(\frac{r_n}{K})}{(\frac{r_n}{K})^\beta \psi(\frac{r_n}{K})} + CK^{-\beta}$$

and thus

$$\frac{\phi(r_n)}{r_n^\beta\psi(r_n)} \le k_\psi(AK^{\alpha-\beta-\xi}+BK^{-\beta-\xi})\frac{\phi(\frac{r_n}{K})}{(\frac{r_n}{K})^\beta\psi(\frac{r_n}{K})} + Ck_\psi K^{-\beta-\xi}.$$

As $r_n/K \in [1/n, d]$, we have

$$\frac{\phi(\frac{r_n}{K})}{(\frac{r_n}{K})^\beta\psi(\frac{r_n}{K})} \le c_n$$

and also

$$c_n - \frac{c_n}{n} \le \frac{\phi(r_n)}{(r_n)^\beta\psi(r_n)} \le \varepsilon c_n + Ck_\psi K^{-\beta-\xi}.$$

Then

$$\Big(1-\varepsilon-\frac{1}{n}\Big)c_n \le Ck_\psi K^{-\beta-\xi}.$$

For $n \to \infty$, we get the statement of this lemma. □

The following lemma is a special case of [3, Lemma 3.4].

Lemma 4.3 ([3, pp. 757-758]) *(i) Let $u \in W^{1,2}_{\rm loc}(\Omega,\mathbb{R}^N)$, $Du \in L^{2,\tau}(\Omega,\mathbb{R}^{nN})$, $0 \le \tau < n$ and (3.1) be satisfied with $f_i \in L^{2q_0,\mu_0 q_0}(\Omega)$, $0 < \mu_0 \le n$. Then $a_i \in L^{2q_0,\lambda_0}(\Omega)$ and for each ball $B_R(x) \subset \Omega$ we have*

$$\int_{B_R(x)} |a_i(x,u,Du)|^{2q_0}\,dy \le c\,R^{\lambda_0}, \tag{4.4}$$

where $c = c(n, L, \gamma_0, \operatorname{diam}\Omega, \|\widetilde{f}\|_{L^{2q_0,\mu_0q_0}(\Omega,\mathbb{R}^N)}, \|Du\|_{L^2(\Omega,\mathbb{R}^{nN})})$ and $\lambda_0 = \min\{\mu_0 q_0, n-(n-\tau)q_0\gamma_0\}$.

(ii) Let $u \in W^{1,2}_{\rm loc}(\Omega,\mathbb{R}^N)$ and (3.2) be satisfied with $f_i^\alpha \in L^{2,\lambda}(\Omega)$, $0 < \lambda \le n$. Then, for each $\varepsilon \in (0,1)$ and all $B_R(x) \subset \Omega$,

$$\int_{B_R(x)} |g_i^\alpha(x,u,Du)|^2\,dy \le c(L)\,\varepsilon\int_{B_R(x)}|Du|^2\,dy + c\,R^\lambda. \tag{4.5}$$

where $c = c(n, L, \varepsilon, \gamma, \operatorname{diam}\Omega, \|\widetilde{\widetilde{f}}\|_{L^{2,\lambda}(\Omega,\mathbb{R}^{nN})}, \|Du\|_{L^2(\Omega,\mathbb{R}^{nN})})$.

For the proof of (i) can be found in [2, pp. 106-107] and the proof (ii) in [5].

In the following considerations we will use a result about higher integrability of the gradient of a weak solution of the system (1.1).

Proposition 4.4 ([6, p. 138]) *Suppose that (1.2), (1.3), (3.1)–(3.3) or (3.4)–(3.6) are fulfilled and let $u \in W^{1,2}_{\rm loc}(\Omega,\mathbb{R}^N)$ be a weak solutions of (1.1). Then there exists an exponent $r > 2$ such that $u \in W^{1,r}_{\rm loc}(\Omega,\mathbb{R}^N)$. Moreover there exists a constant $c = c(\nu,\nu_1,L,\|A\|_{L^\infty})$ and $\widetilde{R} > 0$ such that, for all balls $B_R(x) \subset \Omega$, $R < \widetilde{R}$, the following inequality is satisfied*

$$\begin{aligned}\Big(\fint_{B_{R/2}(x)}|Du|^r\,dy\Big)^{1/r} \le c\Big\{&\Big(\fint_{B_R(x)}|Du|^2\,dy\Big)^{1/2}\\ &+\Big(\fint_{B_R(x)}(|f|^r+|\widetilde{\widetilde{f}}|^r)\,dy\Big)^{1/r} + R\Big(\fint_{B_R(x)}|\widetilde{f}|^{rq_0}dy\Big)^{1/rq_0}\Big\}.\end{aligned}$$

5 Proof of Theorems

Proof of Theorem 3.1. Let $B_{R/2}(x_0) \subset B_R(x_0) \subset \Omega$ be an arbitrary ball and let $w \in W_0^{1,2}(B_{R/2}(x_0), \mathbb{R}^N)$ be a solution of the following system

$$\begin{aligned}
&\int_{B_{R/2}(x_0)} (A_{ij}^{\alpha\beta})_{x_0,R/2} D_\beta w^j D_\alpha \varphi^i \, dx \\
&\quad = \int_{B_{R/2}(x_0)} \big((A_{ij}^{\alpha\beta})_{x_0,R/2} - A_{ij}^{\alpha\beta}(x)\big) D_\beta u^j D_\alpha \varphi^i \, dx \\
&\qquad - \int_{B_{R/2}(x_0)} g_i^\alpha(x,u,Du) D_\alpha \varphi^i \, dx + \int_{B_{R/2}(x_0)} a_i(x,u,Du)\varphi^i \, dx
\end{aligned} \tag{5.1}$$

for all $\varphi \in W_0^{1,2}(B_{R/2}(x_0), \mathbb{R}^N)$. It is known that, under the assumption of this theorem, such solution exists and it is unique for all $R < R'$ (R' is sufficiently small). We can put $\varphi = w$ in (5.1) and, using ellipticity, Hölder and Sobolev inequalities, we obtain

$$\begin{aligned}
\nu \int_{B_{R/2}(x_0)} |Dw|^2 \, dx &\le c\Big(\int_{B_{R/2}(x_0)} |A_{x_0,R/2} - A(x)|^2 |Du|^2 \, dx \\
&\quad + \int_{B_{R/2}(x_0)} |g(x,u,Du)|^2 \, dx + \big(\int_{B_{R/2}(x_0)} |a(x,u,Du)|^{2q_0} \, dx \big)^{1/q_0} \Big) \\
&= c\,(I + II + III).
\end{aligned}$$

From Proposition 4.4 with $r > 2$, Hölder inequality ($r' = r/(r-2)$) and from the fact that, for a BMO-function, all L^r norms, $1 \le r < \infty$ are equivalent (see Proposition 2.1(d)) we obtain

$$I \le c\Big(\int_{B_{R/2}(x_0)} |A(x) - A_{x_0,R/2}|^{2r'} \, dx \Big)^{1/r'} \Big(\int_{B_{R/2}(x_0)} |Du|^r \, dx \Big)^{2/r} \tag{5.2}$$

From the assumptions of this theorem and taking into account the properties of matrix $A = (A_{ij}^{\alpha\beta})$ we can estimate the first term on the right hand side of (5.2)

$$\begin{aligned}
&\int_{B_{R/2}(x_0)} |A(x) - A_{x_0,R/2}|^{2r'} \, dx \le c\Big(\int_{B_{R/2}(x_0)} |A(x) - A_{x_0,R/2}|^2 \, dx \Big)^{1/2} \times \\
&\times \Big(\int_{B_{R/2}(x_0)} |A(x) - A_{x_0,R/2}|^{2(2r'-1)} \, dx \Big)^{1/2} \\
&\le \; c(n, [A]_{2,\Psi,\Omega}) \|A\|^{2r'-1}_{L^\infty(\Omega,\mathbb{R}^{n^2N^2})} \, R^n \, \psi(R).
\end{aligned} \tag{5.3}$$

To estimate the last integral in (5.2) we use Proposition 4.4 obtaining

$$\begin{aligned}\Big(\int_{B_{R/2}(x_0)}|Du|^r\,dx\Big)^{2/r} &\le c\Big\{\frac{1}{R^{n(1-2/r)}}\int_{B_R(x)}|Du|^2\,dy\\ &\quad+\Big(\int_{B_R(x)}(|f|^r+|\widetilde{\widetilde{f}}|^r)\,dy\Big)^{2/r}+R^{2(1-2/r)}\Big(\int_{B_R(x)}|\widetilde{f}|^{rq_0}\,dy\Big)^{2/rq_0}\Big\}\\ &\le c\Big(\frac{1}{R^{n(1-2/r)}}\int_{B_R(x)}|Du|^2\,dy+R^{2\lambda/r}+R^{2(r-2+\lambda)/r)}\Big),\end{aligned} \tag{5.4}$$

where $c=c(r,\|f\|_{L^{r,\lambda}(\Omega)},\|\widetilde{\widetilde{f}}\|_{L^{r,\lambda}(\Omega)},\|\widetilde{f}\|_{L^{rq_0,\lambda q_0}(\Omega)})$. From (5.2), (5.3) and (5.4) we obtain

$$\begin{aligned}I &\le c\Big(\psi^{1/r'}(R)\int_{B_R(x_0)}|Du|^2dx+(R^{2\lambda/r}+R^{2(r-2+\lambda)/r)})R^{n/r'}\psi^{1/r'}(R)\Big)\\ &\le c\Big(\psi^{1/r'}(R)\int_{B_R(x_0)}|Du|^2dx+\ R^{n-2(n-\lambda)/r}\psi^{1/r'}(R)\Big),\end{aligned}$$

where $c=c(n,r,[A]_{2,\Psi,\Omega},\|A\|_{L^\infty(\Omega,\mathbb{R}^{n^2N^2})},\|f\|_{L^{r,\lambda}(\Omega)},\|\widetilde{\widetilde{f}}\|_{L^{r,\lambda}(\Omega)},\|\widetilde{f}\|_{L^{rq_0,\lambda q_0}(\Omega)})$. We can estimate II and III by means of Lemma 4.3 (with $\tau=0$) and we have

$$\nu^2\int_{B_{R/2}(x_0)}|Dw|^2\,dx\le c\Big\{(\varepsilon+\psi^{1/r'}(R))\int_{B_R(x_0)}|Du|^2\,dx+R^\mu\Big\}, \tag{5.5}$$

where $\mu=\min\{n,n-2(n-\lambda)/r,n+2-n\gamma_0\}$.

The function $v=u-w\in W^{1,2}(B_{R/2}(x_0),\mathbb{R}^N)$ is the solution of the system

$$\int_{B_{R/2}(x_0)}(A_{ij}^{\alpha\beta})_{x_0,R/2}D_\beta v^jD_\alpha\varphi^i\,dx=0,\qquad\forall\varphi\in W_0^{1,2}(B_{R/2}(x_0),\mathbb{R}^N). \tag{5.6}$$

From Lemma 4.1 we have, for $0<\sigma\le R/2$,

$$\int_{B_\sigma(x_0)}|Dv|^2\,dx\le c\Big(\frac{\sigma}{R}\Big)^n\int_{B_{R/2}(x_0)}|Dv|^2\,dx.$$

By means of (5.5) and the last estimate we obtain, for all $0<\sigma\le R$ and $\varepsilon\in(0,1)$, the following estimate

$$\int_{B_\sigma(x_0)}|Du|^2\,dx\le c_1\Big[\Big(\frac{\sigma}{R}\Big)^n+\varepsilon+\psi^{1/r'}(R)\Big]\int_{B_R(x_0)}|Du|^2\,dx+c_2\,R^\mu,$$

where the constants c_1 and c_2 only depend on the above-mentioned parameters. Now, in a way analogous to that from [5], we obtain the result.

Proof of Theorem 3.2. By Theorem 3.1, $Du\in L^{2,\lambda}_{\mathrm{loc}}(\Omega,\mathbb{R}^{nN})$ for arbitrary $\lambda<n$. Let $B_{R/2}(x_0)\subset B_R(x_0)\subset\Omega$ be an arbitrary ball and let

$w \in W_0^{1,2}(B_{R/2}(x_0), \mathbb{R}^N)$ be a solution of the following system

$$\begin{aligned}
&\int_{B_{R/2}(x_0)} (A_{ij}^{\alpha\beta})_{x_0,R/2} D_\beta w^j D_\alpha \varphi^i \, dx \\
&\quad = \int_{B_{R/2}(x_0)} \big((A_{ij}^{\alpha\beta})_{x_0,R/2} - A_{ij}^{\alpha\beta}(x)\big) D_\beta u^j D_\alpha \varphi^i \, dx \\
&\quad - \int_{B_{R/2}(x_0)} \big[g_i^\alpha(x,u,Du) - (g_i^\alpha(x,u,Du))_{x_0,R/2}\big] D_\alpha \varphi^i \, dx \\
&\quad + \int_{B_{R/2}(x_0)} a_i(x,u,Du)\varphi^i \, dx
\end{aligned} \tag{5.7}$$

for all $\varphi \in W_0^{1,2}(B_{R/2}(x_0), \mathbb{R}^N)$. It is known that, under the assumption of Theorem 3.2, such solution exists and, it is unique for all $R < R'$ (R' is sufficiently small, $R' \le 1$). We can put $\varphi = w$ in (5.7) and using the ellipticity, the Hölder and the Sobolev inequalities, we obtain

$$\begin{aligned}
\nu^2 \int_{B_{R/2}(x_0)} |Dw|^2 \, dx \le c \Big(&\int_{B_{R/2}(x_0)} |A_{x_0,R/2} - A(x)|^2 |Du|^2 \, dx \\
&+ \int_{B_{R/2}(x_0)} |g_i^\alpha(x,u,Du) - (g_i^\alpha(x,u,Du))_{x_0,R/2}|^2 \, dx \\
&+ \Big(\int_{B_{R/2}(x_0)} |a(x,u,Du)|^{2q_0} \, dx \Big)^{1/q_0} \Big) = c(I + II + III).
\end{aligned} \tag{5.8}$$

The estimate of I is analogous to that in Theorem 3.1 and we have

$$\begin{aligned}
I &\le c\,\psi^{1/r'}(R) \int_{B_R(x_0)} |Du|^2 dx + c\,\big(R^{2\lambda/r} + R^{2(r-2+\lambda)/r)}\big) R^{n/r'} \psi^{1/r'}(R) \\
&\le c\,\Big(\int_{B_R(x_0)} |Du|^2 dx + c\,R^{n-2(n-\lambda)/r} \Big) \psi^{1/r'}(R) \\
&\le c\,\big(R^\lambda + R^{n-2(n-\lambda)/r}\big) \psi^{1/r'}(R) \le c\,R^\lambda \psi^{1/r'}(R),
\end{aligned}$$

where $c = c(n, r, \|A\|_{L^\infty(\Omega, \mathbb{R}^{n^2N^2})}, \|f\|_{L^{r,\lambda}(\Omega)}, \|\widetilde{\widetilde{f}}\|_{L^{r,\lambda}(\Omega)}, \|\widetilde{f}\|_{L^{rq_0,\lambda q_0}(\Omega)})$.

Further, we estimate the second integral on the right hand side of (5.8). From the assumption (3.5) and by means of Young inequality, we obtain

$$\begin{aligned}
II &\le \fint_{B_{R/2}(x_0)} \Big(\int_{B_{R/2}(x_0)} |g_i^\alpha(x,u(x),Du(x)) - g_i^\alpha(y,u(y),Du(y))|^2 \, dy \Big) \, dx \\
&\le c\Big(\int_{B_{R/2}(x_0)} |\widetilde{\widetilde{f}} - (\widetilde{\widetilde{f}})_{x_0,R/2}|^2 \, dx + \int_{B_{R/2}(x_0)} |Du - (Du)_{x_0,R/2}|^{2\gamma} \, dx \Big) \\
&\le c\Big(\varepsilon \int_{B_R(x_0)} |Du - (Du)_{x_0,R}|^2 \, dx + c(\varepsilon, \gamma, \|\widetilde{\widetilde{f}}\|^2_{L^{2,n}(\Omega, \mathbb{R}^{nN})}) R^n \Big),
\end{aligned}$$

where $\varepsilon \in (0,1)$ is arbitrary.

We can estimate III by means of Lemma 4.3 (with $\tau = \lambda$, $\mu_0 = n$) and, using the estimate I, II, we have

$$\begin{aligned}\nu^2 \int_{B_{R/2}(x_0)} |Dw|^2\,dx \le & c\,\varepsilon \int_{B_R(x_0)} |Du(y) - (Du)_{x_0,R}|^2\,dy \\ & + c\left(R^n + R^\lambda \psi^{1/r'}(R) + R^{n+2-(n-\lambda)\gamma_0}\right) \\ \le & c\,\varepsilon \int_{B_R(x_0)} |Du(y) - (Du)_{x_0,R}|^2\,dy + c\,\Phi^2(R),\end{aligned} \tag{5.9}$$

where Φ is defined in the formulation of Theorem 3.2.

The function $v = u - w \in W^{1,2}(B_{R/2}(x_0), \mathbb{R}^N)$ is the solution of the system

$$\int_{B_{R/2}(x_0)} (A_{ij}^{\alpha\beta})_{x_0,R/2} D_\beta v^j D_\alpha \varphi^i\,dx = 0, \quad \forall \varphi \in W_0^{1,2}(B_{R/2}(x_0), \mathbb{R}^N).$$

From Lemma 4.1, we have, for $0 < \sigma \le R/2$

$$\int_{B_\sigma(x_0)} |Dv - (Dv)_{x_0,\sigma}|^2\,dx \le c\left(\frac{\sigma}{R}\right)^{n+2} \int_{B_{R/2}(x_0)} |Dv - (Dv)_{x_0,R/2}|^2\,dx. \tag{5.10}$$

By means of (5.9) and (5.10) we obtain for all $0 < \sigma \le R$ and $\varepsilon \in (0,1)$, the following estimate

$$\begin{aligned}&\int_{B_\sigma(x_0)} |Du(x) - (Du)_{x_0,\sigma}|^2\,dx \\ &\quad \le c_1\left[\left(\frac{\sigma}{R}\right)^{n+2} + \varepsilon\right] \int_{B_R(x_0)} |Du(x) - (Du)_{x_0,R}|^2\,dx + c_2\,\Phi^2(R),\end{aligned}$$

where the constants c_1 and c_2 only depend on the above-mentioned parameters.

Now from Lemma 4.2 we get the result in the following manner. In the case $\Phi(R) = R^{n/2}$, the result is obvious. In other cases if we put $\phi(R) = \int_{B_R(x_0)} |Du(x) - (Du)_{x_0,R}|^2\,dx$, $\alpha = n+2$, $\beta = \lambda$, $A = c_1$, $B = c_1\varepsilon$ and $C = c_2$, we can choose $0 < K < 1$ such that $Ak_\psi K^{n+2-\lambda-\xi} < 1/2$. It is obvious that a constant $\varepsilon > 0$ exists such that $Bk_\psi K^{-\lambda-\xi} = c_1\varepsilon k_\psi K^{-\lambda-\xi} < 1/2$ and then, for all $0 < \sigma \le R < R_0$, $R < R_0$, the assumptions of Lemma 4.2 are satisfied and therefore

$$\int_{B_R(x_0)} |Du(x) - (Du)_{x_0,R}|^2\,dx \le c\,\Phi^2(R).$$

From this follows that $Du \in \mathcal{L}^{2,\Phi}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$.

Remark In [2] for a linear system and in [3] for a nonlinear system (1.1), (1.2), it is proved that the gradient of solution $Du \in BMO(\Omega_0, \mathbb{R}^{nN})$, $\Omega_0 \subset\subset \Omega$ in a situation where the coefficients $A_{ij}^{\alpha\beta} \in C^{0,\gamma}(\overline{\Omega})$, $\gamma \in (0,1)$. Taking into account that for $\Psi(R) = R^{\gamma+n/2}$ we have $\mathcal{L}^{2,\Psi} = C^{0,\gamma}$, one may prove by the method used in the proof of Theorem 3.2 (which is different from the methods in [2] and [3]) the above results too.

Remark In [1] the local BMO-regularity for the gradient of weak solutions of linear elliptic systems is proved. This result was obtained using the global BMO-regularity result and the L^p-regularity result of gradient for all $1 < p < \infty$. Using the global BMO-regularity result from [1] and Theorem 3.2 one may obtain the local BMO-regularity of the gradient too.

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