

Existence and multiplicity results for nonlinear elliptic problems in \mathbb{R}^N with an indefinite functional *

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Abstract

We prove the existence of a nontrivial solution for the nonlinear elliptic problem $-\Delta u = \lambda h(x)u + a(x)g(u)$ in \mathbb{R}^N , where g is superlinear near zero and near infinity, $a(x)$ changes sign, λ is positive, and $h(x) \geq 0$ is a weight function. For g odd, we prove the existence of an infinite number of solutions.

1 Introduction

We consider the elliptic problem

$$-\Delta u = \lambda h(x)u + a(x)g(u), \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \quad (1.1)$$

where $N \geq 3$, $\lambda > 0$ is a real parameter, $g \in C^1(\mathbb{R}; \mathbb{R})$, $a \in C^1(\mathbb{R}^N)$ changes sign, and

$$0 \leq h(x) \in L^{N/2} \cap L^\infty \cap C^1(\mathbb{R}^N), \quad h \not\equiv 0. \quad (1.2)$$

Here, we denote by $\mathcal{D}^{1,2}(\mathbb{R}^N)$ the closure of $\mathcal{D}(\mathbb{R}^N)$ with respect to the norm $(\int_{\mathbb{R}^N} |\nabla u|^2)^{1/2}$. The corresponding problem over a *bounded* domain Ω with Dirichlet boundary conditions on $\partial\Omega$ has been considered by several authors in recent years; we refer the reader to [1, 2, 3, 5, 7, 11] and references therein.

However, not so much seems to be known in such indefinite situation where the domain is unbounded. Here we consider the setting introduced in [6] and assume the following:

$$a(x) \in C^1(\mathbb{R}^N) \quad \text{with} \quad \limsup_{|x| \rightarrow \infty} a(x) < 0, \quad (1.3)$$

$$\nabla a(x) \neq 0, \quad \forall x : a(x) = 0. \quad (1.4)$$

We explicitly observe that $a(x)$ may be unbounded. Concerning the function g , we assume that g is *superlinear*, *subcritical* and a further *sign condition*. More

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precisely, we assume $g \in C^1(\mathbb{R}, \mathbb{R})$ and that for some positive constants ℓ and C , it holds

$$g(0) = g'(0) = 0, \quad (1.5)$$

$$\lim_{|s| \rightarrow \infty} \frac{g'(s)}{|s|^{p-2}} = \ell, \quad \text{with } 2 < p < 2^* = 2N/(N-2), \quad (1.6)$$

$$sg(s) \geq 0, \quad \forall s \in \mathbb{R}, \quad (1.7)$$

$$G(s) \leq Csg(s), \quad \forall s \in \mathbb{R}, \quad (1.8)$$

where $G(s) = \int_0^s g(\xi) d\xi$. We point out that in case (1.7) holds with a *strict* inequality (for $s \neq 0$) then (1.8) is a local condition, in the sense that it is implied by (1.6) together with $G(s) \leq Csg(s)$ for all $|s| \leq \delta$, for a small $\delta > 0$ (see also Remark 3.2 at the end of section 3).

To state Theorem 1.1, let us first recall the well-known fact that (1.2) implies the existence of a sequence $\mu_i(h)$ of eigenvalues of the linear problem

$$-\Delta u = \mu h(x)u, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$$

such that $0 < \mu_1(h) < \mu_2(h) \leq \mu_3(h) \leq \dots \rightarrow \infty$. This is due to the fact that the map $u \mapsto \int hu^2$ is compact in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

Theorem 1.1 *Consider problem (1.1), suppose that $a(x)$ changes sign and that (1.2)-(1.8) hold. If $\mu_k(h) < \lambda < \mu_{k+1}(h)$ for some $k \geq 1$ then (1.1) has a nonzero solution.*

We point out that the existence of *positive* solutions for (1.1) was proved in [6] for $\lambda < \Lambda$, with Λ lying in a right neighborhood of the *first* eigenvalue $\mu_1(h)$. Since *positive* solutions cannot exist in general if $\lambda > \Lambda$ (see [6]), it is natural to ask whether other (possibly sign-changing) solutions exist for any $\lambda > 0$. Theorem 1.1 answers this question in the affirmative whenever λ is not one of the eigenvalues $\mu_j(h)$. It should also be noted that, in contrast with the assumptions made in [6] (see also the references therein), we assume that the function $a(x)$ satisfies condition (1.4) instead of having a “thick” zero set (see Lemma 1.2 in [6]) and, for Theorem 1.1, we do not assume that $g(s)$ behaves like a superlinear power at zero. Condition (1.4) was introduced in [5] and later used in [7, 11].

Typically, Theorem 1.1 applies to nonlinearities such as, for example, $g(s) = |s|^{p-2}s + \theta(s)|s|^{q-2}s$ where $2 < q < p < 2^*$ and θ is C^1 , bounded, $\theta(s) \geq 0 \forall s$ and $s\theta'(s) \geq 0$ near the origin; on the other hand, if θ has compact support then we can take any $q \in (2, +\infty)$.

As a further result, we show that if g is an odd function then (1.1) has infinitely many (pairs of) solutions for any value of λ , provided that we strengthen conditions (1.2), (1.7) and (1.8). Namely, we assume that

$$0 < h(x) \in L^{N/2} \cap L^\infty \cap C^1(\mathbb{R}^N), \quad (1.9)$$

$$sg(s) > 0, \quad \forall s \in \mathbb{R}, s \neq 0, \quad (1.10)$$

$$\lim_{s \rightarrow 0} \frac{sg(s)}{|s|^q} = \ell_0 \in (0, \infty) \quad \text{for some } q > 0 \quad (1.11)$$

and prove the following.

Theorem 1.2 *Consider problem (1.1), suppose that $a(x)$ changes sign, g is odd and that (1.3)-(1.6) and (1.9)-(1.11) hold. Then, for any $\lambda \in \mathbb{R}$, problem (1.1) has infinitely many solutions.*

We observe that similar conclusions for *bounded* domains Ω were obtained in [2] for odd g and in [4, 14] for perturbations of an odd function g , under a different set of assumptions (again, the “thick zero set” assumption on $a(x)$ was used by those authors). Concerning (1.9), we mention that the assumption on the positivity can be replaced by the weaker assumption that $h > 0$ a.e. over the bounded open set $\{x : a(x) > 0\}$ (see Lemma 2.2 in section 2).

The proofs of Theorems 1.1 and 1.2 are given in sections 3 and 4, respectively. In order to prove our results, we have to face the lack of compactness due to the unboundedness of the domain and the fact that $a(x)$ changes sign. We overcome this by constructing an appropriate sequence of solutions of the equation in (1.1), lying in $H_0^1(\Omega_n)$, where $\Omega_n = B_{R_n}(0) \subset \mathbb{R}^N$ with $R_n \rightarrow \infty$; the estimates on the Morse indices of these solutions [7] insure the boundedness of the sequence and allows us to take its limit in the space $\mathcal{D}^{1,2}(\mathbb{R}^N)$ (weak limit, in case of Theorem 1.1; strong limit, in case of Theorem 2). Roughly speaking, our argument relies on the fact that the Morse index estimates provide Palais-Smale sequences for (1.1) with the additional property that the sequence is bounded in $L^\infty(\mathbb{R}^N)$; together with a version of Brezis-Lieb lemma proved in section 2, this yields compactness for the problem. Regarding the multiplicity result, it will follow from the observation that, under the assumptions of Theorem 2, these Palais-Smale sequences can be constructed at arbitrarily large levels of energy.

We mention that, although our results in section 2 suggest that perhaps one could work directly in a convenient Banach subspace of the Hilbert space $\mathcal{D}^{1,2}(\mathbb{R}^N)$, we prefer to use the approximated sequence of Hilbert spaces $H_0^1(\Omega_n)$. This is mainly because, in the latter case, Morse index estimates in Hilbert spaces and their connections with blow-up techniques (see [7]) can be directly applied to our problem without the need of additional theoretical developments.

2 Preliminary results

We recall that we denote by $\mathcal{D}^{1,2}(\mathbb{R}^N)$ the closure of $\mathcal{D}(\mathbb{R}^N)$ with respect to the norm $\|u\| = (\int_{\mathbb{R}^N} |\nabla u|^2)^{1/2}$. The notation $\|u\|_r$ ($1 \leq r \leq \infty$) stands for the norm in L^r spaces and the Sobolev continuous immersion of $\mathcal{D}^{1,2}(\mathbb{R}^N)$ into $L^{2^*}(\mathbb{R}^N)$ will be repeatedly used (see [16]).

As mentioned in the Introduction, we prove Theorems 1 and 2 through an approximation argument in bounded open balls of \mathbb{R}^N . Under assumption (1.2), we denote by $(\mu_i(h))_{i \in \mathbb{N}}$ and $(\mu_i^R(h))_{i \in \mathbb{N}}$ (for each $R > 0$) the sequence of eigenvalues of the problems

$$\begin{aligned} -\Delta u &= \mu h(x)u, & u &\in \mathcal{D}^{1,2}(\mathbb{R}^N) \quad \text{and} \\ -\Delta u &= \mu^R h(x)u, & u &\in H_0^1(B_R(0)). \end{aligned}$$

Lemma 2.1 *Given $k \in \mathbb{N}$ and $\varepsilon > 0$ there exists $R_0 > 0$ such that*

$$|\mu_k(h) - \mu_k^R(h)| < \varepsilon, \quad \forall R \geq R_0.$$

Proof. We first recall that the theory of compact symmetric operators on Hilbert spaces implies the following variational characterization of $\mu_k(h)$:

$$\mu_k(h) = \min_{\dim X=k} \max_{u \in X, u \neq 0} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\int_{\mathbb{R}^N} hu^2} = \max_{u \in X_k, u \neq 0} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\int_{\mathbb{R}^N} hu^2},$$

where we denote by X_k the eigenspace associated with the first k eigenvalues and X runs through the k -dimensional subspaces of $\mathcal{D}^{1,2}(\mathbb{R}^N)$. A similar formula holds for $\mu_k^R(h)$ where, now, X is a subspace of $H_0^1(B_R(0))$. Since $H_0^1(B_R(0)) \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$, this shows in particular that $\mu_k(h) \leq \mu_k^R(h)$.

On the other hand, let X_k be spanned by $\{\varphi_1, \dots, \varphi_k\}$ and denote by Ψ_R a smooth function $\Psi_R \in \mathcal{D}(\mathbb{R}^N)$ such that $0 \leq \Psi_R \leq 1$, $\Psi_R = 1$ in $B_R(0)$, $\Psi_R = 0$ in $\mathbb{R}^N \setminus B_{2R}(0)$ and $\|\nabla \Psi_R\|_\infty \leq CR^{-1}$ for every $R > 0$. By unique continuation, the space spanned by $\{\Psi_R \varphi_1, \dots, \Psi_R \varphi_k\}$ has dimension k . Therefore, in view of the above variational characterizations, the lemma will be proved if we show that, for every large R ,

$$\frac{\int |\nabla(u\Psi_R)|^2}{\int h(u\Psi_R)^2} \leq \varepsilon + \frac{\int |\nabla u|^2}{\int hu^2}, \quad \forall u \in X_k. \quad (2.1)$$

Except otherwise indicated, all integrals are taken over the whole space \mathbb{R}^N . On the other hand, (2.1) will follow once we show that

$$\frac{\int |\nabla(u\Psi_R)|^2}{\int h(u\Psi_R)^2} - \frac{\int |\nabla u|^2}{\int hu^2} \rightarrow 0 \quad \text{as } R \rightarrow \infty, \quad (2.2)$$

uniformly for $u \in X_k$, $\int |\nabla u|^2 = 1$. In order to prove (2.2), let R_n be any sequence such that $R_n \rightarrow \infty$, denote $\Psi_n = \Psi_{R_n}$ and let $u_n \in X_k$ be any sequence such that $\int |\nabla u_n|^2 = 1$. Since X_k is finite dimensional,

$$\varepsilon_n := \int_{\mathbb{R}^N \setminus B_{R_n}(0)} |u_n|^{2^*} \rightarrow 0.$$

Similarly,

$$\int |\nabla u_n|^2 \Psi_n^2 \rightarrow 1 \quad \text{and} \quad \liminf_{n \rightarrow \infty} \left(\int hu_n^2 - \int hu_n^2 \Psi_n^2 \right) > 0.$$

Recalling that $\int |\nabla u_n|^2 = 1$, it remains to prove that

$$\int |\nabla(u_n \Psi_n)|^2 - \int hu_n^2 - \int hu_n^2 \Psi_n^2 \rightarrow 0. \quad (2.3)$$

Observing that, by Hölder inequality,

$$\int hu_n^2 (1 - \Psi_n^2) \leq \int_{\mathbb{R}^N \setminus B_{R_n}(0)} hu_n^2 \leq \|h\|_{N/2} \varepsilon_n^{2/2^*} \rightarrow 0,$$

the expression in (2.3) can be written as

$$\left(\int |\nabla(u_n \Psi_n)|^2 - 1 \right) \int h u_n^2 + o(1),$$

where $o(1) \rightarrow 0$ as $n \rightarrow \infty$. Finally, we observe that

$$\begin{aligned} \left| \int |\nabla(u_n \Psi_n)|^2 - 1 \right| &\leq o(1) + \int u_n^2 |\nabla \Psi_n|^2 + 2 \int |u_n| |\Psi_n| |\nabla u_n| |\nabla \Psi_n| \\ &\leq o(1) + C R_n^{-2} \int_{\mathbb{R}^N \setminus B_{R_n}(0)} u_n^2 \\ &\quad + 2(R_n^{-2} \int_{\mathbb{R}^N \setminus B_{R_n}(0)} u_n^2)^{1/2} \left(\int |\nabla u_n|^2 \right)^{1/2} \\ &\leq o(1) + C'(\varepsilon_n^{2/2^*} + \varepsilon_n^{1/2^*}) \rightarrow 0 \end{aligned}$$

and the lemma follows. \square

For any $R > 0$ and $k \in \mathbb{N}$ we denote by $X_{k,R}$ the closure in $H_0^1(B_R(0))$ of the eigenspaces associated with the eigenvalues $\mu_i^R(h)$ for $i \geq k+1$.

Lemma 2.2 *Assume (1.9) and let $p \in [1, 2^*)$. Given $\varepsilon > 0$ and $k_1 \in \mathbb{N}$ there exist $R_0 > 0$ and $k \in \mathbb{N}$, $k \geq k_1$, such that*

$$\int_{B_1(0)} |u|^p \leq \varepsilon \left(\int_{B_R(0)} |\nabla u|^2 \right)^{p/2} \quad \forall R \geq R_0 \quad \forall u \in X_{k,R}.$$

Proof. Assuming the contrary, there exist sequences $k_n \rightarrow \infty$, $R_n \rightarrow \infty$, $u_n \in X_{k_n, R_n}$ such that $\int_{B_{R_n}(0)} |\nabla u_n|^2 = 1$ and $\int_{B_1(0)} |u_n|^p \geq \varepsilon$. Up to a subsequence, we may assume that (u_n) converges weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ to some function u and that $u_n \rightarrow u$ strongly in $L^p(B_1(0))$. In particular, $\int_{B_1(0)} |u|^p \geq \varepsilon$. Since $u_n \in X_{k_n, R_n}$, it follows from the definition of $\mu_{k_n}^{R_n}(h)$ that

$$1 = \int_{B_{R_n}(0)} |\nabla u_n|^2 \geq \mu_{k_n}^{R_n}(h) \int_{B_{R_n}(0)} h u_n^2. \quad (2.4)$$

On the other hand, as observed at the beginning of the proof of Lemma 2.1, for every n we have that

$$\mu_{k_n}^{R_n}(h) \geq \mu_{k_n}(h) \rightarrow \infty. \quad (2.5)$$

Combining this with (2.4) yields that

$$\int_{B_1(0)} h u^2 = \lim_{n \rightarrow \infty} \int_{B_1(0)} h u_n^2 \leq \lim_{n \rightarrow \infty} \int_{B_{R_n}(0)} h u_n^2 = 0.$$

Since $h > 0$ in $B_1(0)$, this implies $u = 0$ in $B_1(0)$, contradicting the fact that $\int_{B_1(0)} |u|^p \geq \varepsilon$. \square

We end this section with a version of the well-known Brezis-Lieb lemma which is suitable for our purposes. We first recall the following.

Lemma 2.3 (Brezis-Lieb lemma) *Let $H : \mathbb{R} \rightarrow \mathbb{R}$ be continuous, $H \geq 0$ and satisfy*

$$\forall \varepsilon \exists C_\varepsilon : |H(s+t) - H(s)| \leq \varepsilon |H(s)| + C_\varepsilon |H(t)|, \quad \forall s, t \in \mathbb{R}. \quad (2.6)$$

For a given sequence (w_n) of measurable functions in \mathbb{R}^N , suppose $w_n \rightarrow w$ a.e., $\sup_n \int H(w_n) < \infty$ and $\int H(w) < \infty$. Then $\sup_n \int H(w_n - w) < \infty$ and

$$\int |H(w_n) - H(w) - H(w_n - w)| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Proof. An inspection of the proof as given for example in Lemma 1.32 of [16] shows that condition (2.6) is sufficient to deduce the conclusion of the lemma. \square

The next lemma provides a sufficient condition for (2.6) to hold.

Proposition 2.4 *Let $H : \mathbb{R} \rightarrow \mathbb{R}$ be continuous, $H(s) > 0$ for all $s \neq 0$ and suppose that for some $0 < p, q < \infty$, it holds*

$$\lim_{s \rightarrow 0} \frac{H(s)}{|s|^q} = \ell_0 > 0 \quad \text{and} \quad \lim_{|s| \rightarrow \infty} \frac{H(s)}{|s|^p} = \ell_\infty > 0.$$

Then H satisfies condition (2.6).

Proof. Step 1. Given $\varepsilon > 0$, fix $0 < \varepsilon_0 < R_0$ and $C_2/C_1 < 1 + \varepsilon$, $C'_2/C'_1 < 1 + \varepsilon$, in such a way that

$$C_1 |s|^q \leq H(s) \leq C_2 |s|^q, \quad \forall |s| \leq 2\varepsilon_0, \quad C'_1 |s|^p \leq H(s) \leq C'_2 |s|^p, \quad \forall |s| \geq R_0.$$

Then condition (2.6) is trivially satisfied in case $|t| \leq \varepsilon_0$ and $|s| \leq \varepsilon_0$; and also in case $|t| \geq R_0$, $|s| \geq R_0$ and $|t+s| \geq R_0$.

Step 2. Take $\lambda > 0$ such that

$$H(t) \geq \lambda, \quad \forall t : \varepsilon_0 \leq |t| \leq R_0. \quad (2.7)$$

Next, choose $\delta \in]0, \varepsilon_0[$ in such a way that

$$|H(s+t) - H(s)| \leq \varepsilon \lambda, \quad \forall |s| \leq R_0, \quad \forall |t| \leq \delta, \quad (2.8)$$

$$|H(s+t) - H(s)| \leq \varepsilon H(s), \quad \forall |s| \geq R_0, \quad \forall |t| \leq \delta. \quad (2.9)$$

Step 3. We prove condition (2.6) in the case $|t| \leq \delta$. As mentioned in step 1 above, we may assume that $|s| \geq \varepsilon_0$. Thus, if $|s| \geq R_0$ the conclusion follows from (2.9) while if $|s| \leq R_0$ the conclusion follows from (2.7) and (2.8).

Step 4. It remains to check the case where $|t| \geq \delta$. First, we observe that there exists $C_\delta > 0$ and $C_3 > 0$ such that

$$C_\delta |t|^p \leq H(t), \quad \forall |t| \geq \delta \quad \text{and} \quad |H(t)| = H(t) \leq C_3(1 + |t|^p), \quad \forall t \in \mathbb{R}. \quad (2.10)$$

Suppose then that $|t| \geq \delta$. In case $|s| \leq R_0 + |t|$ we deduce from (2.10) that

$$|H(s+t) - H(s)| \leq H(s+t) + H(s) \leq C_4(1 + |t|^p) \leq C_\varepsilon H(t),$$

for some large constant $C_\varepsilon > 0$. Finally, in case $|s| \geq R_0 + |t|$ we have that $|s| \geq R_0$, $|s+t| \geq R_0$ and we can argue as in step 1, thanks to the first inequality in (2.10). \square

3 Proof of Theorem 1.1

Throughout this section we assume that the conditions in Theorem 1.1 are satisfied. As a consequence of Lemma 2.1, it follows from the assumptions of Theorem 1 that we can fix R so large that the constant λ appearing in (1.1) satisfies, for every large $R > 0$,

$$\mu_k^R(h) < \lambda < \mu_{k+1}^R(h). \quad (3.1)$$

Take any sequence $R_n \rightarrow \infty$ and let $\Omega_n = B_{R_n}(0)$. We denote by I the functional

$$I(u) = \frac{1}{2} \int (|\nabla u|^2 - \lambda h(x)u^2) - \int a(x)G(u). \quad (3.2)$$

Under assumptions (1.2), (1.3), (1.5), (1.6) and for each $n \in \mathbb{N}$, I is a C^2 functional over $H_0^1(\Omega_n)$ and its critical points in $H_0^1(\Omega_n)$ correspond to solutions of (1.1) lying in $H_0^1(\Omega_n)$. For any such critical point u , we denote by $m(u)$ the Morse index of u with respect to I , that is, the supremum of the dimensions of the linear subspaces of $H_0^1(\Omega_n)$ on which the quadratic form $D^2I(u)$ is negative definite.

Proposition 3.1 *Under the assumptions of Theorem 1.1, for every n the equation in (1.1) has a nonzero solution $u_n \in H_0^1(\Omega_n)$ and the following holds:*

- (i) (u_n) is bounded in $L^\infty(\Omega_n)$.
- (ii) Either $m(u_n) \leq k - 1$ for every n or else $\limsup_{n \rightarrow \infty} I(u_n) > 0$.

Proof. Step 1. Since the proof is based on [7, 11], we shall be sketchy. At first we observe that the existence of nonzero solutions (u_n) follows straightforwardly from the main theorem in [11], which was proved by means of a truncation argument and the use of a critical point theorem in [9, 10]. We point out that, since $a(x)$ changes sign, the truncation argument is needed in order to insure the Palais-Smale condition for the functional I over $H_0^1(\Omega_n)$ as well as to obtain the required geometric condition on I . At this point, assumptions (1.3), (1.7) and (1.8) are not used; regarding the unique continuation property mentioned in Lemma 2 of [11], we also observe that the equation $-\Delta u = \mu h(x)u$ can be written as $Ku = u/\mu$ where $Ku = (-\Delta)^{-1}(h(x)u)$ in $H_0^1(\Omega_n)$, so that the proof of the quoted lemma remains unchanged.

Step 2. By construction, the sequence of the Morse indices of these solutions is bounded ($m(u_n) \leq k$ for every n , see [7, 11]). Also, our regularity assumptions imply that $u_n \in C(\overline{\Omega_n}) \cap C^2(\Omega_n)$. Assume by contradiction that

$$M_n := \|u_n\|_\infty = \max_{\Omega_n} u_n = u_n(x_n) \rightarrow +\infty$$

for some $x_n \in \Omega_n$ (the case where $\|u_n\|_\infty = \max_{\Omega_n} (-u_n)$ is similar). Since $\Delta u_n(x_n) \leq 0$, the equation in (1.1) shows that

$$a^-(x_n)g(M_n) \leq CM_n + a^+(x_n)g(M_n),$$

where we denoted $a^+ := \max\{a, 0\}$ and $a^- := \max\{-a, 0\}$. From assumption (1.3) we see that (x_n) is bounded. Thus, up to a subsequence, we can assume that $x_n \rightarrow x_0 \in \mathbb{R}^N$ and $a(x_0) \geq 0$. At this point, the blow-up argument in section 3 of [11] can be applied, leading to a contradiction. Indeed, since $m(u_n) \leq k$ and $\|u_n\|_\infty \rightarrow \infty$, it is shown in [11] that the sequence $v_n(x) = u_n(\lambda_n x + x_n)/M_n$, with $\lambda_n = M_n^{(2-p)/2}$ or $\lambda_n = M_n^{(2-p)/3}$ depending on whether $a(x_0) > 0$ or $a(x_0) = 0$, respectively, converges uniformly in compact sets to 0. Since, by definition, $v_n(0) = 1$, this is impossible and therefore part (i) in Proposition 3.1 is proved.

Step 3. Finally, as explained in Proposition 2 of [7], each solution u_n can be chosen in such a way that either $m(u_n) \leq k - 1$ or else, for some small $r_n > 0$,

$$I(u_n) \geq \inf\{I(u) : u \in X_{k,n}, \|u\| = r_n\}, \quad (3.3)$$

where we denote by $X_{k,n}$ the closure of the eigenspaces associated with the eigenvalues $\mu_i^{R_n}(h)$ for $i \geq k + 1$. Actually, by the construction in [7], (3.3) holds for a modified functional

$$\tilde{I}(u) = \frac{1}{2} \int (|\nabla u|^2 - \lambda h(x)u^2) - \int a^+(x)G(u) + \int a^-(x)\tilde{G}(u),$$

where \tilde{G} is a truncation of the function G which still satisfies (1.7). Thus, (3.3) should be written as

$$I(u_n) \geq \inf\{\tilde{I}(u) : u \in X_{k,n}, \|u\| = r_n\}. \quad (3.4)$$

So, in order to prove (ii) in Proposition 3.1 it is enough to show that the right hand side of (3.4) can be bounded below by some positive constant which does not depend on n . Now, to show this, let u be any function in $X_{k,n}$. From (3.1) we see that, for some constant $\eta > 0$ independent of n ,

$$\begin{aligned} \tilde{I}(u) &\geq \eta\|u\|^2 - \int a^+G(u) + \int a^-(x)\tilde{G}(u) \\ &\geq \eta\|u\|^2 - \int a^+G(u) \end{aligned}$$

where we have used (1.7). To estimate the above integral term, we use the fact that (1.5) and (1.6) imply that $|G(s)| \leq \varepsilon s^2 + C_\varepsilon |s|^p$ for any $\varepsilon > 0$. As a consequence, and since a^+ has compact support (cf. (1.3)), we obtain

$$\int a^+u^2 \leq C \left(\int_{\{a>0\}} |u|^{2^*} \right)^{2/2^*} \leq C \left(\int_{\Omega_n} |u|^{2^*} \right)^{2/2^*} \leq C' \|u\|^2,$$

and a similar estimate for $\int a^+|u|^p$. Therefore,

$$\tilde{I}(u) \geq \|u\|^2 (\eta - \varepsilon C' - C'_\varepsilon \|u\|^{p-2}), \quad (3.5)$$

where C' and C'_ε are independent of n . By choosing ε small we see that we can select r_n independently of n . This proves the claim and completes the proof of the proposition. \square

Now we can complete the proof of Theorem 1.1.

Proof of Theorem 1.1 completed. Let (u_n) be given by Proposition 3.1. If we multiply the equation in (1.1) by u_n and integrate over Ω_n we obtain

$$\int |\nabla u_n|^2 + \int a^- g(u_n) u_n = \lambda \int h u_n^2 + \int a^+ g(u_n) u_n. \quad (3.6)$$

Except otherwise indicated, all integrals are taken over \mathbb{R}^N , by extending the solutions as zero outside Ω_n . Since $(\|u_n\|_\infty)$ is bounded, it follows from (3.6) and (1.3) that

$$\int |\nabla u_n|^2 + \int g(u_n) u_n \leq \lambda \int h u_n^2 + C. \quad (3.7)$$

We claim that $(\|u_n\|)$ is bounded. Indeed, suppose $t_n := \|u_n\| \rightarrow \infty$ and denote $v_n := u_n/t_n$. Up to a subsequence, $v_n \rightarrow v$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and a. e. Fix any function $\varphi \in \mathcal{D}(\mathbb{R}^N)$. If we multiply the equation in (1.1) by $\frac{a u_n \varphi}{t_n^2}$ and integrate we see that

$$\int a^2 \frac{g(u_n) u_n}{t_n^2} \varphi \leq C', \quad (3.8)$$

for some positive constant C' depending on φ . Since g is superlinear at infinity (cf. (1.6)), we deduce from (3.8) that $a^2 |v| \varphi = 0$. Since φ is arbitrary and since a vanishes on a set of measure zero, we obtain that $v = 0$. Going back to (3.7) and using the fact that the map $u \mapsto \int h u^2$ is compact in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, we conclude that $\|v_n\|^2 \rightarrow 0$ as $n \rightarrow \infty$, which is a contradiction. This proves that $(\|u_n\|)$ is bounded.

Up to a subsequence, let u be a weak limit of (u_n) in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $u_n \rightarrow u$ a. e. Using test functions in (1.1) we see that u is a solution of (1.1). It remains to show that $u \neq 0$.

Suppose by contradiction that (u_n) converges weakly to 0 in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Since a^+ has compact support, it follows from (3.6) and (1.5) – (1.6) that

$$\int |\nabla u_n|^2 + \int a^- g(u_n) u_n \rightarrow 0$$

and subsequently, using (1.3), that

$$\int |\nabla u_n|^2 \rightarrow 0 \quad \text{and} \quad \int |a| g(u_n) u_n \rightarrow 0. \quad (3.9)$$

Using (1.7) – (1.8), this implies that

$$I(u_n) = \frac{1}{2} \left(\int |\nabla u_n|^2 - \lambda \int h u_n^2 \right) - \int a G(u_n) \rightarrow 0 \quad (3.10)$$

as $n \rightarrow \infty$. So, Proposition 3.1 (ii) implies that $m(u_n) \leq k - 1$ for every n .

On the other hand, we have seen in the proof of Lemma 2.1 (see (2.1)) that, since $\lambda > \mu_k(h)$, there exist a k -dimensional space $X \subset H_0^1(B_R(0))$ (for some large $R > 0$) and a constant $\eta > 0$ such that

$$I''(0)(v, v) = \int |\nabla v|^2 - \lambda \int h v^2 \leq -2\eta \int |\nabla v|^2, \quad \forall v \in X. \quad (3.11)$$

Observe also that, by elliptic regularity, $X \subset L^\infty(B_R(0))$. Therefore, since $u_n \rightarrow 0$ a.e. and $(\|u_n\|_\infty)$ is bounded, Lebesgue's dominated convergence theorem implies that

$$\int g'(u_n)v^2 \rightarrow 0, \quad \text{uniformly in } v \in X : \int |\nabla v|^2 = 1. \quad (3.12)$$

Combining (3.11) and (3.12) we conclude that, for large n ,

$$I''(u_n)(v, v) = \int |\nabla v|^2 - \lambda \int hv^2 - \int ag'(u_n)v^2 \leq -\eta \int |\nabla v|^2, \quad \forall v \in X.$$

By definition, this says that $m(u_n) \geq k$ for large n , which contradicts the fact that $m(u_n) \leq k - 1$ and concludes the proof of Theorem 1.1. \square

Remark 3.2 Under the assumptions of Theorem 1.1, suppose $a(x)$ is bounded. In this case, it is sufficient to assume that the inequality in (1.8) holds for all $|s| \leq \delta$, for some $\delta > 0$. Indeed, if $a(x)$ is bounded the conclusion in (3.10) is a consequence of the fact that $\int |\nabla u_n|^2 \rightarrow 0$, $\int g(u_n)u_n \rightarrow 0$ (as follows from (3.9)) and

$$\begin{aligned} \int |G(u_n)| &= \int_{\{|u_n| \leq \delta\}} G(u_n) + \int_{\{|u_n| \geq \delta\}} G(u_n) \\ &\leq C \int g(u_n)u_n + C_\delta \int |u_n|^{2^*} \rightarrow 0. \end{aligned}$$

Remark 3.3 In the same manner, one can also treat the case where the equation in (1.1) contains an extra term of the form $-b(x)|u|^{r-2}u$ with $b \geq 0$, b bounded and $r > 2$ small with respect to p .

More generally, following [7], the conclusion of Theorem 1.1 holds for an equation of the form

$$-\Delta u = \lambda h(x)u + a(x)g(u) - b(x)f(u),$$

where λ , h , a , g are as in Theorem 1.1 and $b \geq 0$, $b \in C^1 \cap L^\infty(\mathbb{R}^N)$, f satisfies assumptions similar to (1.5), (1.6), (1.8) and, moreover,

$$|f'(s)| \leq C|s|^{r-2} \quad \text{and} \quad f(s)s - r \int_0^s f \leq Cs^2, \quad \forall |s| \geq 1,$$

where $C > 0$ and $2 < r < 2(p+1)/3$. This can be easily checked through an inspection of the proof of Theorem 1.1 (the condition on r is needed in the blow-up argument of [11]). We leave the details for the interested reader.

4 Proof of Theorem 1.2

Although the arguments here are similar to the ones in the preceding section, we need to go into some more details, as far as the truncation method in [7, 11] is concerned.

We introduce some notation. Following [7], fix any sequence of numbers $a_j \rightarrow +\infty$ and $p_j \in (2, p)$, $p_j \rightarrow p$. Define

$$g_j(s) = \begin{cases} A_j |s|^{p_j-2} s + B_j, & \text{for } s \geq a_j; \\ g(s), & \text{for } 0 \leq s \leq a_j; \\ -g_j(-s), & \text{for } s \leq 0. \end{cases}$$

The coefficients are chosen in such a way that g_j is C^1 . Observe that g_j is odd and $g_j = g$ in $[-a_j, a_j]$. We denote $G(s) := \int_0^s g(\xi) d\xi$, $G_j(s) := \int_0^s g_j(\xi) d\xi$. For any $j \in \mathbb{N}$ and $R > 0$ we consider the modified problem

$$-\Delta u = \lambda h(x)u + a^+(x)g(u) - a^-(x)g_j(u), \quad u \in H_0^1(B_R(0)), \quad (4.1)$$

where $a^\pm := \max\{\pm a, 0\}$. The corresponding energy functional is *even* and is given by

$$I_R^j(u) = \frac{1}{2} \int_{B_R(0)} (|\nabla u|^2 - \lambda h(x)u^2) - \int_{B_R(0)} a^+(x)G(u) + \int_{B_R(0)} a^-(x)G_j(u)$$

for $u \in H_0^1(B_R(0))$. For any $\ell \in \mathbb{N}$ and $R > 0$, we have the orthogonal sum

$$H_0^1(B_R(0)) = V_{\ell,R} \oplus X_{\ell,R},$$

where $V_{\ell,R}$ stands for the ℓ -dimensional eigenspace associated with the first ℓ eigenvalues $\mu_i^R(h)$, $i = 1, \dots, \ell$. Finally, for any critical point u of I_R^j we denote by $m_R^j(u)$ its Morse index.

Recall that I_R^j satisfies the Palais-Smale condition over $H_0^1(B_R(0))$ if every sequence $(u_n) \subset H_0^1(B_R(0))$ such that $(I_R^j(u_n))$ is bounded and $\|\nabla I_R^j(u_n)\| \rightarrow 0$ has a convergent subsequence.

The next lemma collects some facts that were proved in [11, Prop.3].

Lemma 4.1 *Assume (1.2), (1.4), and (1.6). Then*

- (a) *For any $j \in \mathbb{N}$ and $R > 0$, the functional I_R^j satisfies the Palais-Smale condition over $H_0^1(B_R(0))$.*
- (b) *For any $j, \ell \in \mathbb{N}$ and $R > 0$, $I_R^j(u) \rightarrow -\infty$ as $\|u\| \rightarrow \infty$, $u \in V_{\ell,R}$.*
- (c) *For any $\ell \in \mathbb{N}$ and $R > 0$ there exist $j_0 \in \mathbb{N}$ and $c > 0$ such that $\|u\|_\infty \leq c$ for every $j \geq j_0$ and every critical point u of I_R^j such that $m_R^j(u) \leq \ell$.*

Observe that (c) is an a-priori estimate in $L^\infty(B_R(0))$ for the solutions of $(P)_R^j$ having bounded Morse index; for each fixed R , the estimate depends on R but *not* on j .

Next, for every $j, \ell \in \mathbb{N}$ and $R > 0$ we let

$$b_{\ell,R}^j = \inf\{I_R^j(u) : u \in X_{\ell-1,R}, \|u\| = r_\ell\}, \quad (4.2)$$

where r_ℓ is a large constant to be chosen later.

Lemma 4.2 *Assume (1.9), (1.3), (1.5)–(1.7), and let $d \in \mathbb{R}$. Then there exist $\ell \in \mathbb{N}$ and $R_0 > 0$ such that*

$$b_{\ell,R}^j \geq d, \quad \forall j \in \mathbb{N} \quad \forall R \geq R_0.$$

Proof. The proof is similar to the one in (3.4) except that we now use Lemma 2.2. Without loss of generality (cf. (1.3)) we assume that $\{x : a(x) > 0\} \subset B_1(0)$. At first we recall from (2.5) that we can fix $\ell_1 \in \mathbb{N}$ such that $\mu_\ell^R(h) > \lambda$ for every $R > 0$ and every $\ell \geq \ell_1$, so that, for some $\eta > 0$,

$$\int_{B_R(0)} |\nabla u|^2 - \lambda \int_{B_R(0)} hu^2 \geq \eta \|u\|^2, \quad \forall \ell \geq \ell_1 \quad \forall R > 0 \quad \forall u \in X_{\ell-1,R}.$$

Since, by (1.6), $\int a^+ G(u) \leq c \left(\int_{B_1(0)} |u|^p + 1 \right)$, we deduce that, for some positive constants c_1, c_2 ,

$$I_R^j(u) \geq c_1(c_2 \|u\|^2 - \int_{B_1(0)} |u|^p - 1), \quad (4.3)$$

for all $j \in \mathbb{N}$, $R > 0$, $\ell \geq \ell_1$ and $u \in X_{\ell-1,R}$. Fix $\varepsilon > 0$ so small that

$$c_2 - \varepsilon r^{p-2} = \frac{c_2}{2} \quad (4.4)$$

where r is given by

$$c_1 c_2 \frac{r^2}{2} - c_1 \geq d. \quad (4.5)$$

For this value of ε , let $\ell \geq \ell_1$ and R_0 be given by Lemma 2.2. Thanks to that lemma, we can rewrite (4.3) as

$$I_R^j(u) \geq c_1 \|u\|^2 (c_2 - \varepsilon \|u\|^{p-2}) - c_1, \quad \forall j \in \mathbb{N} \quad \forall R \geq R_0 \quad \forall u \in X_{\ell-1,R}. \quad (4.6)$$

From (4.4)–(4.6) and the definition in (4.2) with $r_\ell = r$ we conclude that

$$b_{\ell,R}^j \geq c_1 r^2 (c_2 - \varepsilon r^{p-2}) - c_1 = c_1 c_2 \frac{r^2}{2} - c_1 \geq d$$

for every $j \in \mathbb{N}$ and $R \geq R_0$. □

In view of the above lemmas we can now prove the existence of a suitable sequence of approximating solutions to our original problem (1.1).

Proposition 4.3 *Assume (1.9), (1.3), (1.4), (1.5)–(1.7), and let $d \in \mathbb{R}$. Then there exist $R_0 > 0$ and $C > 0$ such that for every $R \geq R_0$ the problem*

$$-\Delta u = \lambda h(x)u + a(x)g(u), \quad u \in H_0^1(B_R(0)) \quad (4.7)$$

has a solution u_R satisfying $I(u_R) \geq d$ and $\|u_R\|_\infty \leq C$.

Proof. We recall from section 3 that I is the energy functional associated with (4.7) (cf. (3.2)). Let R_0 and ℓ be given by Lemma 4.2 and fix any $R \geq R_0$. For every $j \in \mathbb{N}$ we consider the modified problem $(P)_R^j$. In view of Lemma 4.1 (b), for any $j \in \mathbb{N}$ we can choose $\rho_\ell^j > r_\ell$ (r_ℓ as given in (4.2)) in such a way that

$$a_{\ell,R}^j := \sup\{I_R^j(u) : u \in V_{\ell,R} : \|u\| = \rho_\ell^j\} \leq d - 1.$$

Denote

$$\begin{aligned} D_{\ell,R} &= \{u \in V_{\ell,R} : \|u\| \leq \rho_\ell^j\}, \\ \Gamma_{\ell,R} &= \{\gamma \in C(D_{\ell,R}; H_0^1(B_R(0))) : \gamma \text{ is odd and } \gamma|_{\partial D_{\ell,R}} = \text{identity}\}, \\ c_{\ell,R}^j &= \inf_{\gamma \in \Gamma_{\ell,R}} \sup_{u \in D_{\ell,R}} I_R^j(\gamma(u)). \end{aligned}$$

Since $a_{\ell,R}^j < b_{\ell,R}^j$ and since I_R^j satisfies the Palais-Smale condition (see Lemma 4.1 (a)), it is known that $c_{\ell,R}^j$ is a critical value for I_R^j such that $c_{\ell,R}^j \geq b_{\ell,R}^j$ (see e.g. [13, Th. 5.2], [16, Th. 3.6]). In other words, there exists u_R^j such that

$$\nabla I_R^j(u_R^j) = 0 \quad \text{and} \quad I_R^j(u_R^j) = c_{\ell,R}^j \geq d.$$

On the other hand, since $V_{\ell,R}$ has dimension ℓ , we can choose u_R^j in such a way that its Morse index is not greater than ℓ ; this follows readily from the arguments in e.g. [8, 12]. From Lemma 4.1 (c) we conclude that $\|u_R^j\|_\infty$ is bounded independently of j . As a consequence, for j large we have that u_R^j is a solution of problem (4.7).

At this point, for every $R \geq R_0$ we have constructed a solution u_R of (4.7) such that $I(u_R) \geq d$. Moreover, the Morse indices $m(u_R)$ are bounded above by some fixed number ℓ . To finish the proof of Proposition 4.3 it remains to show that $\|u_R\|_\infty$ is bounded independently of R . Since a^+ has compact support, this follows as in step 2 of the proof of Proposition 3.1. \square

Proof of Theorem 2 completed. Step 1. Fix any $d \in \mathbb{R}$ and take any sequence $R_n \rightarrow \infty$. Let (u_n) be the corresponding solutions of $(P)_{\lambda,R_n}$ given by Proposition 4.3. As in the proof of Theorem 1.1, up to a subsequence, (u_n) has a weak limit u in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $u_n \rightarrow u$ a. e. and the sequence $(\int |a| g(u_n) u_n)$ is bounded. Clearly, u is a solution of (1.1) and $u \in L^\infty(\mathbb{R}^N)$. Step 2. By multiplying the equation in (1.1) by $u\Psi_R$ where Ψ_R is as in the proof of Lemma 2.1 and by letting $R \rightarrow \infty$ we readily see that

$$\int a^- g(u) u < \infty,$$

so that $\int |a| g(u) u$ is also finite. Since (1.6), (1.10), and (1.11) imply that $|G(s)| = G(s) \leq Cg(s)s$ for all $s \in \mathbb{R}$, we see that $\int |a| G(u)$ is finite and that $(\int |a| G(u_n))$ is bounded. In particular, $I(u)$ and $I'(u)u$ are finite numbers. Step 3. Denote $v_n = u_n - u$. In view of assumptions (1.6), (1.10), (1.11), and of Proposition 2.4 (with respect to the measures $a^\pm dx$), we see that

$$\int a^\pm (H(u_n) - H(u) - H(v_n)) \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (4.8)$$

where H is any of the functions $H(s) = G(s)$ or $H(s) = sg(s)$. As a consequence,

$$\begin{aligned} 0 &= I'(u_n)u_n = I'(u)u + I'(v_n)v_n + o(1) \\ &= I'(v_n)v_n + o(1) \\ &= \int |\nabla v_n|^2 - \lambda \int h(x)v_n^2 - \int a(x)g(v_n)v_n + o(1). \end{aligned}$$

Since $v_n \rightarrow 0$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and strongly in $L_{\text{loc}}^p(\mathbb{R}^N)$ and since a^+ has compact support, this implies that

$$\int |\nabla v_n|^2 \rightarrow 0, \quad \int |a|g(v_n)v_n \rightarrow 0 \quad \text{and} \quad \int |a|G(v_n) \rightarrow 0. \quad (4.9)$$

It follows from (4.9) and (4.8) with $H(s) = G(s)$ that $u_n \rightarrow u$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ and

$$I(u) = \lim_{n \rightarrow \infty} I(u_n) \geq d.$$

Since d is any real number, this clearly yields infinitely many solutions for problem (1.1).

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