On the eigenvalue problem for the Hardy-Sobolev operator with indefinite weights *

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Abstract

In this paper we study the eigenvalue problem

$$\Delta_p u - a(x)|u|^{p-2} u = \lambda|u|^{p-2} u, \quad u \in W^{1,p}_0(\Omega),$$

where $1 < p \leq N$, $\Omega$ is a bounded domain containing 0 in $\mathbb{R}^N$, $\Delta_p$ is the $p$-Laplacean, and $a(x)$ is a function related to Hardy-Sobolev inequality. The weight function $V(x) \in L^s(\Omega)$ may change sign and has nontrivial positive part. We study the simplicity, isolatedness of the first eigenvalue, nodal domain properties. Furthermore we show the existence of a nontrivial curve in the Fučik spectrum.

1 Introduction

Let $\Omega$ be a bounded domain containing 0 in $\mathbb{R}^N$. Then the Hardy-Sobolev inequality for $1 < p < N$ states that

$$\int_{\Omega} |\nabla u|^p dx \geq \left( \frac{N-p}{p} \right)^p \int_{\Omega} |u|^p |x|^p dx$$  \quad (1.1)

for all $u \in W^{1,p}_0(\Omega)$. It is known that $(N-p)^p p$ is the best constant in (1.1). In a recent work Adimurthi, Choudhuri and Ramaswamy [2] improved the above inequality. In particular, when $p = N$ their inequality reads

$$\int_{\Omega} |\nabla u|^N dx \geq \left( \frac{N-1}{N} \right)^N \int_{\Omega} \frac{|u|^N}{|x|^N (\log \frac{R}{|x|})^N} dx, \quad \forall u \in W^{1,N}_0(\Omega),$$  \quad (1.2)

where $R > e^{2/N} \sup_{\Omega} |x|$. Subsequently it was shown in [4] that $(\frac{N-1}{N})^N$ is the best constant in (1.2). In view of the above two inequalities we define the Hardy-Sobolev Operator $L_\mu$ on $W^{1,p}_0(\Omega)$ as

$$L_\mu u := -\Delta_p u - \mu a(x)|u|^{p-2} u$$

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where

\[
a(x) = \begin{cases} 
  \frac{1}{|x|^p} & 1 < p < N \\
  \frac{1}{(|x| \log R/|x|)^N} & p = N 
\end{cases}
\]

and \(0 \leq \mu < \frac{(N-p)p}{p} \) or \(\frac{N-1}{N} \) depends on the value of \(p\). Here \(\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)\) denotes the \(p\)-Laplacian. In the present work we consider the following eigenvalue problem:

\[
L_{\mu} u = \lambda V(x)|u|^{p-2} u \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial \Omega.
\]

(1.3)

We assume that \(V \in L_{\text{loc}}^1(\Omega)\), \(V^+ = V_1 + V_2 \not\equiv 0\) with \(V_1 \in L_{\text{loc}}^p(\Omega)\) and \(V_2\) is such that

\[
\lim_{x \to y, x \in \Omega} |x-y|^p V_2(x) = 0 \quad \forall y \in \overline{\Omega} \quad \text{for } p < N
\]

\[
\lim_{x \to y, x \in \Omega} |x-y|^p (\log \frac{R}{|x-y|})^p V_2(x) = 0 \quad \forall y \in \overline{\Omega} \quad \text{for } p = N.
\]

(1.4)

where \(V^+(x) = \max\{V(x), 0\}\). We also assume

(H) There exists \(r > \frac{N}{p}\) and a closed subset \(S\) of measure zero in \(\mathbb{R}^N\) such that \(\Omega \backslash S\) is connected and \(V \in L_{\text{loc}}^r(\Omega \backslash S)\).

We define the functional \(J_{\mu}\) on \(W_0^{1,p}(\Omega)\) as

\[
J_{\mu}(u) := \int_{\Omega} |\nabla u|^p - \mu \int_{\Omega} a(x)|u|^{p-2} u.
\]

Then \(J_{\mu}\) is \(C^1\) on \(W_0^{1,p}(\Omega)\). Our goal here is to study the eigenvalue problem and some main properties (simplicity, isolatedness) of

\[
\lambda_1 := \inf \left\{ J_{\mu}(u); u \in W_0^{1,p}(\Omega) \quad \text{and} \quad \int_{\Omega} V|u|^p dx = 1 \right\}
\]

We use the following results in Section 2.

Proposition 1.1 ([5]) Let \(\Omega \subset \mathbb{R}^n\) is bounded domain and suppose \((u_n) \in W^{1,p}(\Omega)\) such that \(u_n \rightharpoonup u\) weakly in \(W_0^{1,p}(\Omega)\) satisfies

\[-\Delta_p u_n = f_n + g_n \text{ in } \mathcal{D}'(\Omega)\]

where \(f_n \rightharpoonup f\) in \(W^{-1,p'}\) and \(g_n\) is a bounded sequence of Radon measures, i.e.,

\[
\langle g_n, \phi \rangle \leq C_K \|
\phi\|_{\infty}
\]

for all \(\phi \in C_c^\infty(\Omega)\) with support in \(K\). Then there exists a subsequence \((u_{n_k})\) of \((u_n)\) such that \(\nabla u_{n_k}(x) \rightharpoonup \nabla u(x)\) a.e. in \(\Omega\).
Proposition 1.2 ((Brezis-Lieb[6])) Suppose $f_n \to f$ a.e. and $\|f_n\|_p \leq C < \infty$ for all $n$ and for some $0 < p < \infty$. Then

$$\lim_{n \to \infty} \{\|f_n\|_p^p - \|f_n - f\|_p^p\} = \|f\|_p^p.$$ 

In section 2 we study the eigenvalue problem for $L_\mu$ and show that the first eigenvalue is simple and the eigenfunctions corresponding to other eigenvalues changes sign. In section 3 we study the existence of nontrivial curve in the Fučík spectrum of $L_\mu$. Finally in the last section we study some nodal domain properties of $L_\mu$ with a stronger assumption on $V$ that $V \in L^r(\Omega)$ for some $r > \frac{N}{p}$.

We now provide a brief account of what is known about the problems of type (1.3). In case of $\mu = 0$, the above properties are well known when $V$ is bounded[see[1]]. For indefinite weights with different integrability conditions see[3] and [14]. In [14] the problem of simplicity and sign changing nature of other eigen functions are left open. In Theorem 2.1 below we prove the above properties. In a recent work Cuesta [7] proved above properties with stronger assumption that $V \in L^s(\Omega)$ for some $s > \frac{N}{p}$. When $\mu \neq 0$ and $V = 1$ the above properties are studied in [11],[12].

2 Eigenvalue Problem

In this section we show that the first eigenvalue is simple and the eigenfunctions corresponding to other eigenvalues changes sign. We prove the following theorem.

Theorem 2.1 The first eigenvalue, $\lambda_1$, is simple and the eigenfunctions corresponding to the other eigenvalues changes sign.

The next theorem is proven with the help of a deformation lemma for $C^1$ manifolds.

Theorem 2.2 There exists a sequence $\{\lambda_n\}$ of eigenvalues of $L_\mu$ such that $\lambda_n \to \infty$.

Let us define the operators

$$L(u, v) := |\nabla u|^p - (p - 1) \frac{u^p}{v^{p-1}} |\nabla v|^p - p \frac{u^{p-1}}{v^{p-1}} \nabla u |\nabla v|^{p-2} \nabla v$$

$$R(u, v) := |\nabla u|^p - |\nabla v|^{p-2} \nabla v \cdot \nabla \left( \frac{u^p}{v^{p-1}} \right)$$

Then $R(u, v) = L(u, v) \geq 0$ for all $u, v \in C^1(\Omega \setminus \{0\}) \cap W^{1,p}(\Omega)$ with $u \geq 0, v > 0$ and equal to 0 if and only if $u = kv$ for some constant $k$ [3, Theorem 1.1]. We need following lemmas to prove our results.

Lemma 2.3 The mapping $u \mapsto \int_\Omega V^+ |u|^p dx$ is weakly continuous.
Proof: In case the $1 < p < N$, the proof follows as in [14]. Here we give the proof when $p = N$. Clearly $u \rightarrow \int_\Omega V_1 |u|^p$ is weakly continuous. Since $\Omega$ is compact, there is a finite covering of $\Omega$ by closed balls $B(x_i, r_i)$ such that, for $1 \leq i \leq k$,

$$|x - x_i| \leq r_i \implies |x - x_i|^N (\log \frac{R}{|x - x_i|})^N V_2(x) \leq \epsilon.$$  \hspace{1cm} (2.1)

There exists $r > 0$ such that, for $1 \leq i \leq k$,

$$|x - x_i| \leq r \implies |x - x_j|^N (\log \frac{R}{|x - x_i|})^N V_2(x) \leq \epsilon/k.$$  \hspace{1cm} (2.2)

Define $A := \cup_{j=1}^k B(x_j, r)$. Then by inequality (1.2)

$$\int_A V_2 |u|^N dx \leq c N, \quad \int_A V_2 |u|^N dx \leq c^N$$

where $c = \frac{N}{N-1} \sup_n \|u_n\|$. It follows from (2.1) that $V_2 \in L^1(\Omega \setminus A)$ so that

$$\int_{\Omega \setminus A} V_2 |u_n|^N dx \longrightarrow \int_{\Omega \setminus A} V_2 |u|^N dx$$

Now the conclusion follows from (2.2) and (2.3). \hspace{1cm} \Box

Define $M := \left\{ u \in W_0^{1,p}(\Omega); \int_\Omega V|u|^p = 1 \right\}$

**Lemma 2.4** The eigenvalue $\lambda_1$ is attained.

Proof: Let $u_n$ be a sequence in $M$ such that $J_\mu(u_n) \rightarrow \lambda_1$. Since $W_0^{1,p}(\Omega)$ is reflexive, there exists a subsequence $\{u_n\}$ of $\{u_n\}$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}$ and a.e. in $\Omega$. Now for $n \in \mathbb{N}$ choose $u_n$ such that $J_\mu(u_n) \leq \inf_M J_\mu + \frac{1}{n}$. Now by the Ekeland Variational Principle, there exists a sequence $\{v_n\}$ such that

$$J_\mu(v_n) \leq J_\mu(u_n),$$

$$\|u_n - v_n\| \leq \frac{1}{n},$$

$$J_\mu(v_n) \leq J_\mu(u) + \frac{1}{n} \|v_n - u\| \quad \forall u \in M$$

Now standard calculations from above three equations, as in [10], gives

$$|J_\mu'(v_n)w - J_\mu'(v_n)| \int_\Omega V|v_n|^{p-2}v_nw| \leq C \frac{1}{n} \|w\|.$$  \hspace{1cm} (2.4)

By Proposition 1.1, there exists a subsequence of $\{v_n\}$, which we still denote by $\{v_n\}$ such that $v_n \rightarrow v$ weakly in $W_0^{1,p}(\Omega)$ and $\nabla v_n \rightarrow \nabla v$ a.e. in $\Omega$. Since
$|\nabla v_n|^p - \nabla v_n$ is bounded in $(L^p'(\Omega))^N$, $1/p + 1/p' = 1$, and $\nabla v_n \to \nabla v$ a.e. in $\Omega$, we have

$|\nabla v_n|^p - \nabla v_n \to |\nabla v|^p - \nabla v$ a.e. in $\Omega$

$|\nabla v_n|^p - \nabla v_n \to |\nabla v|^p - \nabla v$ weakly in $(L^p'(\Omega))^N$

which allows us to pass the limit as $n \to \infty$ in (2.4), obtaining

$-\Delta_p v - a(x)|v|^{p-2}v - \lambda_1|v|^{p-2}v = 0$ in $\mathcal{D}'(\Omega)$.

Observe that

$$\int_\Omega V^- |v_n|^p dx = \int_\Omega V^+ |v_n|^p dx - 1 \to \int_\Omega V^+ |v|^p dx - 1$$

as $n \to \infty$. Now using Fatou’s lemma we can conclude that $v \not\equiv 0$. □

**Lemma 2.5** The eigenvalue $\lambda_1$ is simple.

**Proof:** This is an adaptation from a proof in [3]. Let $\{\psi_n\}$ be a sequence of functions such that $\psi_n \in C^\infty_c(\Omega), \psi_n \geq 0, \psi_n \to \phi_1$ in $W^{1,p}$, a.e. in $\Omega$ and $\nabla \psi_n \to \nabla \phi_1$ a.e. in $\Omega$. Then we have

$$0 = \int_\Omega (|\nabla \phi_1|^p - (\mu a(x) + \lambda_1 V)\phi_1^p) \, dx$$

$$= \lim_{n \to \infty} \int_\Omega (|\nabla \psi_n|^p - (\mu a(x) + V \lambda_1)\psi_n^p) \, dx.$$  (2.5)

Consider the function $w_1 := \psi_n^p/(u_2 + \frac{1}{n})^{p-1}$. Then $w_1 \in W^{1,p}_0(\Omega)$. So testing the equation satisfied by $u_2$ with $w_1$ we get,

$$\int_\Omega (\lambda_1 V + \mu a(x))\psi_n^p \left(\frac{u_2}{u_2 + \frac{1}{n}}\right)^{p-1} = \int_\Omega |\nabla u_2|^{p-2} \nabla u_2 \cdot \nabla \left(\frac{\psi_n^p}{(u_2 + \frac{1}{n})^{p-1}}\right).$$  (2.6)

Now from (2.5) and (2.6) we obtain

$$0 = \lim_{n \to \infty} \int_\Omega (|\nabla \psi_n|^p - |\nabla u_2|^{p-2} \nabla u_2 \cdot \nabla \left(\frac{\psi_n^p}{(u_2 + \frac{1}{n})^{p-1}}\right))$$

$$= \lim_{n \to \infty} \int_\Omega L(\psi_n, u_2) \geq \int_\Omega L(\phi_1, u_2) \geq 0$$

by Fatou’s lemma. Now by assumption (H), $\phi_1, u_2$ are in $C^1(\Omega \setminus S \cup \{0\})$ [9, 15]. Therefore $\phi_1 = ku_2$ for some constant $k$. □

**Proof of Theorem 2.1, completed:** Let $\phi_1, u$ be the eigenfunctions corresponding to $\lambda_1$ and $\lambda$ respectively. Then $\phi_1, u$ satisfies

$$-\Delta_p \phi_1 - \mu a(x)\phi_1^{p-1} = \lambda_1 V(x)\phi_1^{p-1} \text{ in } \mathcal{D}'(\Omega),$$  (2.7)

$$-\Delta_p u - \mu a(x)|u|^{p-2}u = \lambda V(x)|u|^{p-2}u \text{ in } \mathcal{D}'(\Omega)$$  (2.8)
respectively. Suppose $u$ does not change sign. We may assume $u \geq 0$ in $\Omega$. Let 
\{$\psi_n$\} be a sequence in $C_0^\infty$ such that $\psi_n \to \phi_1$ as $n \to \infty$. Now consider the
test functions $w_1 = \phi_1$, $w_2 = \frac{\psi_n}{(u + \frac{k}{n})^{p-1}}$. Then $w_1, w_2 \in W_0^{1,p}(\Omega)$. Testing (2.7)
with $w_1$ and (2.8) with $w_2$ we get
\[
\int_\Omega |\nabla \phi_1|^p dx - \int_\Omega (\lambda_1 V(x) + \mu a(x)) \phi_1^p dx = 0 \tag{2.9}
\]
\[
\int_\Omega |\nabla u|^p - 2|u| \nabla u \nabla \left( \frac{\psi_n^p}{(u + \frac{k}{n})^{p-1}} \right) dx - \int_\Omega (\lambda V(x) + \mu a(x)) \psi_n^p \left( \frac{u}{u + \frac{k}{n}} \right)^{p-1} dx = 0
\]
Since $R(u,v) \geq 0$, we get
\[
\int_\Omega |\nabla \psi_n|^p dx - \int_\Omega (\lambda V(x) + \mu a(x)) \psi_n^p \left( \frac{u}{u + \frac{k}{n}} \right)^{p-1} dx \geq 0. \tag{2.10}
\]
Subtracting (2.9) from (2.10) and taking the limit as $n \to \infty$ we get,
\[
(\lambda - \lambda_1) \int_\Omega V(x) \phi_1^p \leq 0
\]
This is a contradiction to the fact that $\lambda > \lambda_1$. \hfill \Box

**Proof of Theorem 2.2:** Let $\tilde{J}_\mu$ be the restriction of $J_\mu$ to the set $M$. Define
\[
\lambda_\mu = \inf_{\gamma(A) \geq n} \sup_{u \in A} J_\mu(u)
\]
where $A$ is a closed subset of $M$ such that $A = -A$, and $\gamma(A)$ is the Krasnosel’skiĭ genus of $A$. Now we show that $\tilde{J}_\mu$ satisfies (P.S.) condition at level $\lambda_\mu$. Let \{$u_n$\} be a sequence in $M$ such that $J_\mu(u_n) \to \lambda_\mu$ and
\[
\langle J_\mu(u_n), \phi \rangle - J_\mu(u_n) \int_\Omega |u_n|^{p-2} u_n \phi V dx = o(1). \tag{2.11}
\]
Since $u_n$ is bounded, there exists a subsequence \{$u_n$\}, $u$ such that $u_n \to u$ weakly in $W_0^{1,p}(\Omega)$. Since $\lambda_\mu > 0$ we may assume that $J_\mu(u_n) \geq 0$. Using Lemma 2.3 and (2.11), we get
\[
\langle J_\mu(u_n) - J_\mu(u), u_n - u \rangle + J_\mu(u_n) \int_\Omega |u_n|^{p-2} u_n - |u|^{p-2} u \ (u_n - u) V^- dx = o(1).
\]
But
\[
\int_\Omega |u_n|^{p-2} u_n - |u|^{p-2} u \ |u_n - u| V^- \geq 0.
\]
By Propositions 1.1 and 1.2, we have
\[
\|u_n - u\|_{1,p} = \|u_n\|_{1,p} - \|u\|_{1,p} + o(1)
\]
\[
\|\frac{u_n - u}{|x|}\|_{0,p} = \|\frac{u_n}{|x|}\|_{0,p} - \|\frac{u}{|x|}\|_{0,p} + o(1)
\]
Therefore
\[ o(1) = (J_\mu(u_n) - J_\mu(u), (u_n - u)) + J_\mu(u_n) \int \Omega \left| \nabla u_n \right|^p - \left| \nabla u \right|^p dx \]
\[ \geq \int \Omega |\nabla u_n - \nabla u|^p - \int \Omega \mu a(x) |u_n - u|^p + o(1) \]
\[ \geq C \|u_n - u\|_{1,p} + o(1). \]

Now by the classical critical point theory for \( C^1 \) manifolds [13], it follows that \( \lambda_k \)'s are critical points of \( J_\mu \) on \( M \). Since \( \lambda_k \geq c \lambda_0^k \), where \( \lambda_0^k \) are eigenvalues of \( L_0 \), we have \( \lambda_k \to \infty \). \( \square \)

### 3 Fučík Spectrum

In this section we study the existence of a non-trivial curve in the Fučík spectrum \( \sum_{p,\mu}^\ast \) of \( L_\mu \). The Fučík spectrum of \( L_\mu \) is defined as the set of \((\alpha, \beta) \in \mathbb{R}^2 \) such that
\[ L_\mu u = \alpha V(u^+)^{p-1} + \beta V(u^-)^{p-1} \text{ in } \Omega, \]
\[ u = 0 \text{ on } \partial \Omega, \]
has a nontrivial solution \( u \in W_0^{1,p}(\Omega) \). The variational approach that we follow here is same as that of [8, 12]. We prove the following statement.

**Theorem 3.1** There exists a nontrivial curve \( C \) in \( \sum_{p,\mu}^\ast \).

Let us consider the functional
\[ J_s(u) = \int \Omega |\nabla u|^p - \int \Omega \mu a(x) |u|^p - s \int \Omega V u^{p+} \]
\( J_s \) is a \( C^1 \) functional on \( W_0^{1,p}(\Omega) \). We are interested in the critical points of the restriction \( J_s \) of \( J_s \) to \( M \). By Lagrange multiplier rule, \( u \in M \) is a critical point of \( J_s \) if and only if there exist \( t \in \mathbb{R} \) such that \( J'_s(u) = t J'(u) \), i.e., for all \( v \in W_0^{1,p} \) we have
\[ \int \Omega |\nabla u|^{p-2} \nabla u \nabla v - \int \Omega \mu a(x) |u|^{p-2} u v - s \int \Omega V u^{p+} v = t \int \Omega V |u|^{p-2} u v, (\Omega) \]
This implies that
\[ -\Delta_p u - \mu a(x) |u|^{p-2} u = (s + t) V(x) (u^+)^{p-1} - t V(x) (u^-)^{p-1} \text{ in } \Omega \]
\[ u = 0 \text{ on } \partial \Omega \]
holds in the weak sense. i.e., \( (s + t, t) \in \sum_{p,\mu} \), taking \( v = u \) in (3.1), we get \( t \) as a critical value of \( J_s \). Thus the points in \( \sum_{p,\mu} \) on the parallel to the diagonal
passing through $(s,0)$ are exactly of the form $(s + \tilde{J}_s(u), \tilde{J}_s(u))$ with $u$ a critical point of $\tilde{J}_s$.

A first critical point of $\tilde{J}_s$ comes from global minimization. Indeed

$$\tilde{J}_s(u) \geq \lambda_1 \int_\Omega |u|^p - s \int_\Omega u^{p^*} \geq \lambda_1 - s$$

for all $u \in M$, and $\tilde{J}_s(u) = \lambda_1 - s$ for $u = \phi_1$.

**Proposition 3.2** The function $\phi_1$ is a global minimum of $\tilde{J}_s$ with $\tilde{J}_s(\phi_1) = \lambda_1 - s$, the corresponding point in $\sum_{p,\mu}$ is $(\lambda_1, \lambda_1 - s)$ which lies on the vertical line through $(\lambda_1, \lambda_1)$.

**Lemma 3.3** Let $0 \neq v_n \in W_0^{1,p}(\Omega)$ satisfy $v_n \geq 0$ a.e. and $|v_n| > 0$ to 0, then $\int_\Omega |\nabla v_n|^p - \mu a(x)|v_n|^p dx / \int_\Omega V|v_n|^p \to +\infty$.

**Proof:** Let $w_n = v_n / |v_n|^p_{\Omega}$ and assume by contradiction that $\int_\Omega |\nabla w_n|^p - \int_\Omega \mu a(x)|w_n|^p$ has a bounded subsequence. By (1.1) or (1.2), we get $w_n$ bounded in $W_0^{1,p}(\Omega)$. Then for a further subsequence, $w_n \to w$ in $L^p(\Omega, V^+)$.

Now observe that

$$\int_\Omega V^-(x)|w|^p \leq \lim_{n \to \infty} \int_\Omega V^-(x)|w_n|^p = \lim_{n \to \infty} \int_\Omega V^+|w_n|^p - 1 = \int_\Omega V^+|w|^p - 1.$$

Then $w \geq 0$ and $\int_\Omega V^+(x)|w|^p \geq 1$. So for some $\epsilon > 0$, $\delta = |w| > \epsilon > 0$, we deduce that $|w_n > \epsilon/2| > \delta$ for $n$ sufficiently large, which contradicts the assumption $|v_n| > 0$ to 0. \qed

A second critical point of $\tilde{J}_s$ comes next.

**Proposition 3.4** $-\phi_1$ is a strict local minimum of $\tilde{J}_s$, and $\tilde{J}_s(-\phi_1) = \lambda_1$, the corresponding point in $\sum_{p,\mu}$ is $(\lambda_1 + s, \lambda_1)$.

**Proof:** We follow the ideas in [8, Prop. 2.3]. Assume by contradiction that there exist a sequence $u_n \in M$ with $u_n \neq -\phi_1$, $u_n \to -\phi_1$ in $W_0^{1,p}(\Omega)$ and $\tilde{J}_s(u_n) \leq \lambda_1$.

Claim: $u_n$ changes sign for $n$ sufficiently large. Since $u_n \to -\phi_1$, $u_n$, it must follow that $u_n \leq 0$ some where. If $u_n \leq 0$ a.e., in $\Omega$, then

$$\tilde{J}_s(u_n) = \int_\Omega |\nabla u_n|^p - \int_\Omega \mu a(x)|u_n|^p > \lambda_1$$

since $u_n \neq \pm \phi_1$, and this contradicts $\tilde{J}_s(u_n) \leq \lambda_1$. This completes the proof of claim. Let $r_n = [\int_\Omega |\nabla u_n^+|^p - \int_\Omega \mu a(x)|u_n^+|^p] / \int_\Omega V u_n^+ |v|^p$, we have

$$\tilde{J}_s(u_n) = \int_\Omega |\nabla u_n^+|^p + \int_\Omega |\nabla u_n^-|^p - \int_\Omega \mu a(x)|u_n^+|^p$$

$$\geq (r_n - s) \int_\Omega V u_n^+ + \lambda_1 \int_\Omega V u_n^-$$
on the other hand
\[ \tilde{J}_s(u_n) \leq \lambda_1 = \lambda_1 \int_\Omega V u_n + \int_\Omega V u_n^p \]
combining the two inequalities, we get \( r_n \leq \lambda_1 + s \). Now since, \( u_n \to -\phi_1 \)
in \( L^p(\Omega) \), \( |u_n| > 0 \to 0 \). The Lemma 3.3 then implies \( r_n \to +\infty \), which
contradicts \( r_n \leq \lambda_1 + s \). \( \square \)

Now as in the proof of Theorem 2.2, one can show that \( \tilde{J}_s \) satisfies the P.S.
condition at any positive level.

**Lemma 3.5** Let \( \epsilon_0 > 0 \) be such that \( \tilde{J}_s(u) > \tilde{J}_s(-\phi_1) \) \( \forall u \in B(-\phi_1,\epsilon_0) \cap M \) (3.2)
with \( u \neq -\phi_1, B \subset W^{1,p}_0 \). Then for any \( 0 < \epsilon < \epsilon_0 \)
\[ \inf \{ \tilde{J}_s(u); u \in M \text{ and } \|u - (-\phi_1)\|_{1,p} = \epsilon \} > \tilde{J}_s(-\phi_1). \] (3.3)

The proof of this lemma follows from the Ekeland variational principle. Therefore, we omit it. For details we refer the reader to [8]. Let
\[ \Gamma = \{ \gamma \in C([-1,1]; M) : \gamma(-1) = -\phi_1, \gamma(1) = \phi_1 \} \neq \emptyset \]
and the geometric assumptions of Mountain-pass Lemma are satisfied by previous Lemma. Therefore, there exists \( u \in W^{1,p}_0 \) such that \( \tilde{J}_s'(u) = 0 \) and \( J_s(u) = c \),
where \( c \) is given by
\[ c(s) = \inf \sup_\Gamma J_s(u). \] (3.4)

Proceeding in this manner for each \( s \geq 0 \) we get a non-trivial curve \( C: s \in \mathbb{R}^+ \to (s + c(s), c(s)) \in \mathbb{R}^2 \) in \( \Sigma_{p,\mu} \), which completes the proof of Theorem 3.1.

## 4 Nodal Domain Properties

In this section we show that \( \lambda_1 \) is isolated in the spectrum under the assumption on \( V \) that \( V \in L^s(\Omega) \) for some \( s > \frac{N}{p} \). By the regularity results in [15, 9] the
solutions of (1.3) are \( C^1(\Omega \setminus \{0\}) \). In [11] it is shown that the positive solutions
of (1.3) when \( V = 1 \) tends to \( +\infty \) as \( |x| \to 0 \). We prove the following theorem.

**Theorem 4.1** The eigenvalue \( \lambda_1 \) is isolated in the spectrum provided that \( V \in L^s(\Omega) \) for some \( s > \frac{N}{p} \). Moreover, for \( v \) an eigenfunction corresponding to an
eigenvalue \( \lambda \neq \lambda_1 \) and \( O \) be a nodal domain of \( v \), then
\[ |O| \geq (C\lambda\|V\|_s)^{-\gamma} \] (4.1)
where \( \gamma = \frac{N}{sp-N} \) and \( C \) is a constant depending only on \( N \) and \( p \).

**Lemma 4.2** Let \( u \in C(\Omega \setminus \{0\}) \cap W^{1,p}_0(\Omega) \) and let \( O \) be a component of \( \{ x \in \Omega; u(x) > 0 \} \). Then \( u|_O \in W^{1,p}_0(O) \)
Proof: case (i): $1 < p < N$.
Let $u_n \in C_c(\Omega) \cap W^{1,p}_0(\Omega)$ such that $u_n \to u$ in $W^{1,p}_0(\Omega)$. Then $u_n^+ \to u^+$ in $W^{1,p}_0(\Omega)$. Let $v_n = \min(u_n, u)$ and let $\psi_r \in C(\Omega)$ be a cutoff function such that

$$
\psi_r(x) = \begin{cases} 
0 & \text{if } |x| \leq r/2 \\
1 & \text{if } |x| \geq r
\end{cases}
$$

and $|\nabla \psi_r(x)| \leq C/r$ for some constant $C$. Now consider the sequence $w_{n,r}(x) = \psi_r v_n(x)$. Since $\psi_r v_n \in C(\Omega)$, we have $w_{n,r} \in C(\Omega)$ and vanishes on the boundary $\partial \Omega$. Indeed for $x \in \partial \Omega$ and $x = 0$ then $\psi_r = 0$ and so $w_{n,r} = 0$. If $x \in \partial \Omega \cap \Omega$ and $x \not= 0$ then $u(x) = 0$(since $u$ is continuous except at 0) and so $w_n(x) = 0$. If $x \in \partial \Omega$ then $u_n(x) = 0$ and hence $v_n(x) = 0$. So in all the cases $w_{n,r}(x) = 0$ for $x \in \partial \Omega$. Therefore $w_{n,r} \in W^{1,p}_0(\Omega)$ and

$$
\int_{\Omega} |\nabla (w_{n,r}) - \nabla (\psi_r u)|^p = \int_{\Omega} |(\nabla \psi_r)v_n + \psi_r \nabla v_n - (\nabla \psi_r)u - \psi_r \nabla u|^p dx
\leq \|\nabla \psi_r v_n - \nabla \psi_r u\|_{L^p(\Omega)}^p + \|\psi_r \nabla v_n - \psi_r \nabla u\|_{L^p(\Omega)}^p
$$

which approaches 0 as $n \to \infty$, i.e., $w_{n,r} \to \psi_r u|_{\Omega}$ in $W^{1,p}_0(\Omega)$. Now

$$
\int_{\Omega} |\nabla \psi_r u + \psi_r \nabla u - u|^p \leq \int_{\Omega} |\psi_r \nabla u - \nabla u|^p + \int_{\Omega \cap \{r/2 < |x| < r\}} |\nabla \psi_r|^p u
$$

which approaches 0 as $r \to 0$ by (1.1). Therefore, $u|_{\Omega} \in W^{1,p}_0(\Omega)$.

Case(ii): $p = N$. In this case we use the following cut-off function which are introduced in [11]

$$
\psi_r(x) = \begin{cases} 
0 & \text{if } |x| \leq r \\
2 \log \left( \frac{r}{|x|} \right) / \log(r) & \text{if } r \leq |x| \leq r^{1/2} \\
1 & \text{if } |x| \geq r^{1/2}.
\end{cases}
$$

and we can proceed as in the previous case.

\[ \square \]

Proof of Theorem 4.1: The proof follows as in [1, 7]. Let $\mu_n$ be a sequence of eigenvalues such that $\mu_n > \lambda_1$ and $\mu_n \to \lambda_1$. Let the corresponding eigenfunctions $u_n$ converge to $\phi_1$. such that $\|u_n\|_{L^p(V)} = 1$. i.e., $u_n$ satisfies

$$
-\Delta_p u_n - \mu_n |x||u_n|^{p-2} u_n = \lambda_n V(x)|u_n|^{p-2} u_n. \tag{4.2}
$$

Testing (4.2) with $u_n$ and applying weighted Hardy-Sobolev inequality we get $u_n$ to be bounded. Therefore by Proposition 1.1, there exists a subsequence $(u_{n_k})$ of $(u_n)$ such that $u_{n_k} \to u$ weakly in $W^{1,p}_0(\Omega)$, strongly in $L^p(\Omega)$ and $\nabla u_{n_k} \to \nabla u$ a.e. in $\Omega$. Taking limit $n \to \infty$ in (4.2) we get

$$
-\Delta_p u - \mu |x||u|^{p-2} u = \lambda_1 V(x)|u|^{p-2} u \quad \text{in} \quad \mathcal{D}'(\Omega).
$$
Therefore $u = \pm \phi_1$. By Theorem 2.1, $u_n$ changes sign. Without loss of generality, we can assume that $u = +\phi_1$, then
\[
|\{x; u_n < 0\}| \to 0. \tag{4.3}
\]

Testing (4.2) with $u_n^-$, we get
\[
\int_\Omega |\nabla u_n^-|^p - \int_\Omega \mu a(x)u_n^- = \int_\Omega \lambda_n V(x)u_n^-.
\]
By Hardy-Sobolev and Sobolev inequalities, we get
\[
C_1 \|u_n\|^p_{1,p} \leq \int_\Omega V(x)|u_n|^p \leq C \|V\|_s \|u_n\|^p_{1,p} |\Omega_n^-|^\gamma \leq C_3 \|u_n\|^p_{1,p} |\Omega_n^-|^\gamma \|V\|_s, 
\]
for some positive $\gamma > 0$. This implies that
\[
|\Omega_n^-| \geq C_4^{1/\gamma}, \quad \Omega_n^- = \{x \in \Omega; u_n < 0\}. 
\]
This contradicts (4.3).

Next we prove the estimate (4.1). Assume that $v > 0$ in $O$, the case $v < 0$ being treated similarly. We observe by Lemma 4.2, that $v \in W^{1,p}_0(O)$. Hence the function defined as $w(x) = v(x)$ if $x \in O$ and $w(x) = 0$ if $x \in \Omega \setminus O$ belongs to $W^{1,p}_0(\Omega)$. Using $w$ as test function in the equation satisfied by $v$, we find
\[
\int_O |\nabla v|^p dx - \int_\Omega \mu a(x)|v|^p dx = \lambda \int_\Omega V|x|^p dx \leq \lambda \|V\|_s \|v\|^p_{p,\Omega} \frac{\gamma - p'}{p-1},
\]
by Holder inequality. On the other hand by Sobolev and Hardy-Sobolev inequalities we have that $\int_O |\nabla v|^p dx \geq C \|v\|^p_{p,\Omega}$ for some constant $C = C(N,p)$. Hence
\[
C \leq \lambda \|V\|_s |\Omega| \frac{\gamma - p'}{p-1}.
\]

\textbf{Corollary 4.3} Each eigenfunction has a finite number of nodal domains.

\textbf{Proof:} Let $O_j$ be a nodal domain of an eigenfunction associated to some positive eigenvalue $\lambda$. It follows from (4.1) that
\[
|\Omega| \geq \sum_j |O_j| \geq (C \lambda \|V\|_s)^{-\gamma} \sum_j 1
\]
and the proof follows.

\textbf{References}


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