Uniqueness for radial Ginzburg-Landau type minimizers

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Abstract

We prove the uniqueness of radial minimizers of a Ginzburg-Landau type functional. We present also an analysis of the location of the zeros of the radial minimizer.

1 Introduction

For \( n \geq 2 \), let \( B = \{ x \in \mathbb{R}^n; |x| < 1 \} \) and \( \partial B \) its boundary. On this domain, we find minimizers to the Ginzburg-Landau-type functional

\[
E_\varepsilon(u, B) = \frac{1}{p} \int_B |\nabla u|^p + \frac{1}{4\varepsilon^p} \int_B (1 - |u|^2)^2, \quad (p \geq n)
\]

on the class of functions

\[
W = \{ u(x) = f(r) \frac{x}{|x|} \in W^{1,p}(B, \mathbb{R}^n); f(1) = 1, r = |x| \}.
\]

Such minimizer is denoted by \( u_\varepsilon \) and is called a radial minimizer.

Many authors have studied the existence, uniqueness and asymptotic behaviour of \( u_\varepsilon \) as \( \varepsilon \to 0 \). For \( p = n = 2 \), studies of asymptotic behaviour can be found in \([1, 11]\), studies of uniqueness in \([7]\), and other related topics in \([2, 3, 5, 10]\). For \( p = n > 2 \) and \( p > n = 2 \), the asymptotic behaviour was studied in \([6]\) and \([9]\), respectively. However, uniqueness was not mentioned there. In this paper, we prove the following results for \( p \geq n \).

**Theorem 1.1** Assume \( u_\varepsilon \) is a radial minimizer of \( E_\varepsilon(u, B) \). Then for any given \( \eta \in (0, 1/2) \) there exists a positive constant \( h = h(\eta) \) such that

\[
Z_\varepsilon = \{ x \in B; |u_\varepsilon(x)| < 1 - \eta \} \subset B(0, h\varepsilon) = \{ x \in \mathbb{R}^n; |x| < h\varepsilon \}.
\]

**Theorem 1.2** The radial minimizer of \( E_\varepsilon(u, B) \) on \( W \) is unique.

This article is organized as follows: In §2, we present some basic properties of minimizers. In §3, we prove Theorem 1.1 which implies, in particular, that the zeros of \( u_\varepsilon \) are contained in \( B(0, h\varepsilon) \). We conclude with the proof of Theorem 1.2 in §4.

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2 Preliminaries

For a function \( u(x) = f(r) \frac{x}{|x|} \), written in polar coordinates, we have

\[
|\nabla u| = (f'_r + (n-1)r^{-2}f^2)^{1/2}, \quad \int_B |u|^p = |S^{n-1}| \int_0^1 r^{n-1} |f|^p \, dr,
\]

\[
\int_B |\nabla u|^p = |S^{n-1}| \int_0^1 r^{n-1} (f'_r + (n-1)r^{-2}f^2)^{p/2} \, dr.
\]

It is easily seen that \( f(r) \frac{x}{|x|} \in W^{1,p}(B, \mathbb{R}^n) \) implies \( f(r) r^{\frac{n-1}{p}} \) and \( f(r) r^{\frac{n-1}{p}} \) are in \( L^p(0,1) \). Conversely, if \( f(r) \) is in \( W^{1,p}_{\text{loc}}(0,1) \), and \( f(r) r^{\frac{n-1}{p}} \), \( f(r) r^{\frac{n-1}{p}} \) are in \( L^p(0,1) \), then \( f(r) \frac{x}{|x|} \in W^{1,p}(B, \mathbb{R}^n) \). Thus if we denote

\[
V = \{ f \in W^{1,p}_{\text{loc}}(0,1); r^{\frac{n-1}{p}} f_r \in L^p(0,1), r^{(n-1-p)/p} f \in L^p(0,1), f(1) = 1 \},
\]

then \( V = \{ f(r); u(x) = f(r) \frac{x}{|x|} \in W \} \).

**Proposition 2.1** The set \( V \) defined above is a subset of \( \{ f \in C[0,1]; f(0) = 0 \} \).

**Proof.** Let \( f \in V \). If \( p > n \), let \( h(r) = f(r) r^{\frac{n-1}{p}} \). Then

\[
\int_0^1 |h'(r)|^p \, dr = \left( \frac{p-1}{p-n} \right)^p \int_0^1 |f'(r) r^{\frac{n-1}{p}}| r^{\frac{n-1}{p}} \, dr
\]

\[
= \left( \frac{p-1}{p-n} \right)^p \int_0^1 s^{n-1} |f'(s)|^p \, ds < \infty
\]

by noting \( f_s(s)s^{(n-1)/p} \in L^p(0,1) \).

If \( p = n \), let \( h(r) = f(r^2) \) with \( x > 1 \) to be determined later. Then for any \( y \in (1,p) \),

\[
\int_0^1 |h'(r)|^y \, dr = x^y \int_0^1 |f'(r^2)| r^{(y-1)/2} \, dr
\]

\[
= x^{y-1} \int_0^1 |f'(s)| s^{(y-1)/2} \, ds,
\]

where \( s = r^2 \). Choose \( x, y \) such that \( (1 - \frac{1}{2})(1 - \frac{1}{y}) = \frac{n-1}{n} \). Hence

\[
\int_0^1 |h'(r)|^y \, dr = x^{y-1} \int_0^1 |f'(s)| s^{(n-1)/n} \, ds
\]

\[
\leq x^{y-1} (\int_0^1 |f'(s)|^n s^{n-1} \, ds)^{y/n} < \infty.
\]

Using an interpolation inequality and Young inequality, \( \| h \|_{W^{1,p}((0,1), R)} < \infty \) which implies that \( h(r) \in C[0,1] \) and hence \( f(r) \in C[0,1] \).
Suppose \( f(0) > 0 \), then \( f(r) \geq s > 0 \) for \( r \in [0,t) \) with \( t > 0 \) small enough since \( f \in C[0,1] \). We have
\[
\int_0^1 r^{n-1-p} f^p \, dr \geq s^p \int_0^t r^{n-1-p} \, dr = \infty,
\]
which contradicts \( r^{(n-1)/p-1} f \in L^p(0,1) \). Therefore \( f(0) = 0 \) and the proof is complete.

**Proposition 2.2** The functional \( E_\varepsilon(u,B) \) achieves its minimum on \( W \) by a function \( u_\varepsilon(x) = f_\varepsilon(r) \frac{x}{|x|} \).

**Proof.** Note that \( W^{1,p}(B,\mathbb{R}^n) \) is a reflexive Banach space and \( E_\varepsilon(u,B) \) is weakly lower-semicontinuous. To prove the existence of minimizers of \( E_\varepsilon(u,B) \) in \( W \), it suffices to verify that \( W \) is a weakly closed subset of \( W^{1,p}(B,\mathbb{R}^n) \). Clearly \( W \) is a convex subset of \( W^{1,p}(B,\mathbb{R}^n) \). Now we prove that \( W \) is a closed subset of \( W^{1,p}(B,\mathbb{R}^n) \). Let \( u_k = f_k(r) \frac{x}{|x|} \in W \) such that \( \lim_{k \to \infty} u_k = u \), in \( W^{1,p}(B,\mathbb{R}^n) \). By the embedding theorem there exists a subsequence \( u_k = f_k(r) \frac{x}{|x|} \) such that \( \lim_{k \to \infty} f_k = f \), in \( C(0,1] \) and \( u = f(r) \frac{x}{|x|} \). Combining this with \( f_k(1) = 1 \), we see that \( f(1) = 1 \). Thus \( u \in W \).

**Proposition 2.3** The minimizer \( u_\varepsilon \) is a weak radial solution of
\[
- \text{div}(|\nabla u|^{p-2} \nabla u) = \frac{1}{\varepsilon^p} u(1-|u|^2), \quad \text{on } B, \tag{2.1}
\]
\[
u|_{\partial B} = x. \tag{2.2}
\]

**Proof.** Denote \( u_\varepsilon \) by \( u \). For any \( t \in [0,1] \) and \( \phi = f(r) \frac{x}{|x|} \in C^{\infty}_0(B,\mathbb{R}^n) \), we have \( u + t\phi \in W \) as long as \( t \) is small sufficiently. Since \( u \) is a minimizer we obtain
\[
\frac{dE_\varepsilon(u + t\phi,B)}{dt}|_{t=0} = 0,
\]
namely,
\[
0 = \frac{d}{dt}|_{t=0} \int_B \left( \frac{1}{p} |\nabla (u + t\phi)|^p + \frac{1}{4\varepsilon^p} (1-|u|^2)^2 \right) dx
\]
\[
= \int_B |\nabla u|^{p-2} \nabla u \nabla \phi dx - \frac{1}{\varepsilon^p} \int_B u \phi (1-|u|^2) dx.
\]
By a limit process we see that the test function \( \phi \) can be any member of \( \{ \phi = f(r) \frac{x}{|x|} \in W^{1,p}(B,\mathbb{R}^n) ; \phi|_{\partial B} = 0 \} \).

As in [6, Lemma 2.2], we also have the following statement.

**Proposition 2.4** Let \( u_\varepsilon \) be a weak radial solution of (2.1)-(2.2). Then \( |u_\varepsilon| \leq 1 \) on \( \overline{B} \).
Proof. Taking $\phi = u - \frac{u}{|u|} \min(1, |u|)$. Let $B_+ = \{ x \in B : |u| > 1, a.e. \text{ on } B \}$.

Noting
\[ \nabla \phi = 0, a.e. \text{ on } B \setminus B_+; \quad \nabla \phi = \nabla u \left(1 - \frac{1}{|u|}\right) + \frac{u \nabla u}{|u|^3}, a.e. \text{ on } B_+ , \]
we have
\[ \int_{B_+} |\nabla|^p \left(1 - \frac{1}{|u|}\right) + \int_{B_+} |\nabla|^p \left(\frac{u \nabla u}{|u|^3}\right)^2 + \frac{1}{\varepsilon^p} \int_{B_+} |u||u|^2 - 1(|u| - 1) = 0. \]
This implies that $|B_+| = 0$. Thus $|u_\varepsilon| \leq 1$.

Proposition 2.5 Assume $u_\varepsilon$ is a weak radial solution of (2.1)-(2.2). Then there exist positive constants $C_1, \rho$ which are independent of $\varepsilon$, such that
\[ \|\nabla u_\varepsilon(x)\|_{L(B(x, \rho \varepsilon/y))} \leq C_1 \varepsilon^{-1}, \quad \text{if } x \in B(0, 1 - \rho \varepsilon), \quad (2.3) \]
\[ |u_\varepsilon(x)| \geq \frac{10}{11}, \quad \text{if } x \in \overline{B} \setminus B(0, 1 - 2 \rho \varepsilon). \quad (2.4) \]

Proof. Let $y = x \varepsilon^{-1}$ in (2.1) and denote $v(y) = u(x), B_\varepsilon = B(0, \varepsilon^{-1})$. Then
\[ \int_{B_\varepsilon} |\nabla v|^{p-2} \nabla v \nabla \phi = \int_{B_\varepsilon} v(1 - |v|^2) \phi, \quad \phi \in W^{1,p}_0(B_\varepsilon, \mathbb{R}). \quad (2.5) \]
This implies that $v(y)$ is a weak solution of (2.5). By using the standard discuss of the H"{o}lder continuity of weak solution of (2.5) on the boundary (for example see Theorem 1.1 and Line 19-21 of Page 104 in [4]) we can see that for any $y_0 \in \partial B_\varepsilon$ and $y \in B(y_0, \rho_0)$ (where $\rho_0 > 0$ is a constant independent of $\varepsilon$), there exist positive constants $C = C(\rho_0)$ and $\alpha \in (0, 1)$, both independent of $\varepsilon$, such that
\[ |v(y) - v(y_0)| \leq C(\rho_0)|y - y_0|^\alpha. \]
Choose $\rho > 0$ sufficiently small such that
\[ y \in B(y_0, 2 \rho) \subset B(y_0, \rho_0), \quad \text{and } C(\rho_0)|y - y_0|^\alpha \leq \frac{1}{11}, \quad (2.6) \]
then
\[ |v(y)| \geq |v(y_0)| - C(\rho_0)|y - y_0|^\alpha = 1 - C(\rho_0)|y - y_0|^\alpha \geq \frac{10}{11}. \]
Let $x = y \varepsilon$. Thus $|u_\varepsilon(x)| \geq 10/11$, if $x \in B(x_0, 2 \rho \varepsilon)$, where $x_0 \in \partial B$. This implies (2.4). Taking $\phi = v \zeta^p, \zeta \in C_0^\infty(B_\varepsilon, R)$ in (2.5), we obtain
\[ \int_{B_\varepsilon} |\nabla v|^p \zeta^p \leq p \int_{B_\varepsilon} |\nabla v|^{p-1} \zeta^p |\nabla \zeta||v| + \int_{B_\varepsilon} |v|^2 (1 - |v|^2) \zeta^p. \]
For $\rho$ as in (2.6), setting $y \in B(0, \varepsilon^{-1} - \rho), B(y, \rho/2) \subset B_\varepsilon$,
\[ \zeta = \begin{cases} 1 & \text{in } B(y, \rho/4), \\ 0 & \text{in } B_\varepsilon \setminus B(y, \rho/2) \end{cases} \]
and $|\nabla \zeta| \leq C(\rho)$, we have

$$\int_{B(y,\rho/2)} |\nabla v|^p \zeta^p \leq C(\rho) \int_{B(y,\rho/2)} |\nabla v|^{p-1} \zeta^{p-1} + C(\rho).$$

Using Holder’s inequality, we can derive $\int_{B(y,\rho/4)} |\nabla v|^p \leq C(\rho)$. Combining this with the Tolksdroff theorem in [12] yields

$$\|\nabla v\|_{L^\infty(B(y,\rho/8))} \leq C(\rho) \int_{B(y,\rho/4)} (1 + |\nabla v|)^p \leq C(\rho).$$

which implies

$$\|\nabla u\|_{L^\infty(B(x,\varepsilon\rho/8))} \leq C(\rho) \varepsilon^{-1}.$$

Proposition 2.6 Let $u_\varepsilon$ be a radial minimizer of $E_\varepsilon(u, B)$. Then

$$E_\varepsilon(u_\varepsilon, B) \leq C \varepsilon^{n-p} + C, \quad \text{when } p > n,$$

$$E_\varepsilon(u_\varepsilon, B) \leq \frac{1}{n} (n-1)^{n/2} |S^{n-1}| \ln \varepsilon + C \quad \text{when } p = n,$$

with a constant $C$ independent of $\varepsilon \in (0, 1)$.

Proof. Denote

$$I(\varepsilon, R) = \min \left\{ \int_{B(0,R)} \frac{1}{p} |\nabla u|^p + \frac{1}{\varepsilon^p} (1 - |u|^2)^2; u \in W_R \right\},$$

where

$$W_R = \{ u(x) = f(r) \frac{x}{|x|} \in W^{1,p}(B(0,R), \mathbb{R}^n); r = |x|, f(R) = 1 \}.$$

Then

$$I(\varepsilon, 1) = E_\varepsilon(u_\varepsilon, B)$$

$$= \frac{1}{p} \int_B |\nabla u_\varepsilon|^p dx + \frac{1}{4\varepsilon^p} \int_B (1 - |u_\varepsilon|^2)^2 dx$$

$$= \varepsilon^{n-p} \left[ \frac{1}{p} \int_{B(0,\varepsilon^{-1})} |\nabla u_\varepsilon|^p dy + \frac{1}{4} \int_{B(0,\varepsilon^{-1})} (1 - |u_\varepsilon|^2)^2 dy \right]$$

$$= \varepsilon^{n-p} I(1, \varepsilon^{-1}).$$

Let $u_1$ be a solution of $I(1, 1)$ and define

$$u_2 = \begin{cases} 
    u_1, & \text{if } 0 < |x| < 1; \\
    \frac{x}{|x|}, & \text{if } 1 \leq |x| \leq \varepsilon^{-1}.
\end{cases}$$
Thus $u_2 \in W_{\varepsilon - 1}$ and when $p > n$,

$$I(1, \varepsilon^{-1}) \leq \frac{1}{p} \int_{B(0, \varepsilon^{-1})} |\nabla u_2|^p + \frac{1}{4} \int_{B(0, \varepsilon^{-1})} (1 - |u_2|^2)^2$$

$$= \frac{1}{p} \int_B |\nabla u_1|^p + \frac{1}{4} \int_B (1 - |u_1|^2)^2 + \frac{1}{p} \int_{B(0, \varepsilon^{-1}) \setminus B} |\nabla \frac{x}{|x|}|^p$$

$$= I(1, 1) + \frac{(n - 1)p/2|S^{n-1}|}{p} \int_1^{\varepsilon^{-1}} r^{n-p-1} dr$$

$$= I(1, 1) + \frac{(n - 1)p/2|S^{n-1}|}{p(p - n)} (1 - \varepsilon^{p-n}) \leq C.$$  

Similarly, when $p = n$,

$$I(1, \varepsilon^{-1}) \leq I(1, 1) + \frac{1}{n} (n - 1)^{n/2}|S^{n-1}| \ln \varepsilon + C.$$  

Substituting these into (2.9) yields (2.7) and (2.8).

### 3 Location of zeros of minimizers

**Proposition 3.1** Let $u_\varepsilon$ be a radial minimizer of $E_\varepsilon(u, B)$. Then there exists a constant $C$ independent of $\varepsilon \in (0, 1]$ such that

$$\frac{1}{\varepsilon^n} \int_B (1 - |u_\varepsilon|^2)^2 \leq C. \quad (3.1)$$

**Proof.** When $p > n$, (3.1) can be derived by multiplying (2.7) by $\varepsilon^{p-n}$. When $p = n$, as in [6, eqn.(3.6)], we derive that

$$\int_B |\nabla u_\varepsilon|^n dx \geq (n - 1)^{n/2}|S^{n-1}| \ln \varepsilon - C,$$

where $C$ is independent of $\varepsilon$. Combining this with (2.8) we obtain (3.1).

**Proposition 3.2** Let $u_\varepsilon$ be a radial minimizer of $E_\varepsilon(u, B)$. Then for any $\eta \in (0, 1/2)$, there exist positive constants $\lambda, \mu$ independent of $\varepsilon \in (0, 1)$ such that if

$$\frac{1}{\varepsilon^n} \int_{B(0, 1 - \rho \varepsilon) \cap B^{2\varepsilon}} (1 - |u_\varepsilon|^2)^2 \leq \mu, \quad (3.2)$$

where $B^{2\varepsilon}$ is the ball of radius $2\varepsilon$ with $l \geq \lambda$, then

$$|u_\varepsilon(x)| \geq 1 - \eta, \quad \forall x \in B(0, 1 - \rho \varepsilon) \cap B^{l\varepsilon}. \quad (3.3)$$
Proof. First we observe that there exists a constant $C_2 > 0$ which is independent of $\varepsilon$ such that for any $x \in B$ and $0 < \rho \leq 1$,
\[
|B(0, 1 - \rho \varepsilon) \cap B(x, r)| \geq C_2 r^n.
\]
To prove this proposition, we choose
\[
\lambda = \frac{\eta}{2C_1}, \quad \mu = \frac{C_2}{C_1} \left(\frac{\eta}{2}\right)^{n+2},
\]
where $C_1$ is the constant in (2.3). Suppose that there is a point $x_0 \in B(0, 1 - \rho \varepsilon) \cap B(0, \lambda \varepsilon)$ such that
\[
|u_\varepsilon(x_0)| < 1 - \eta.
\]
Then applying (2.3) we have
\[
|u_\varepsilon(x) - u_\varepsilon(x_0)| \leq C_1 \varepsilon^{-1} |x - x_0| \leq C_1 \varepsilon^{-1} (\lambda \varepsilon)
\]
\[
= C_1 \lambda = \frac{\eta}{2}, \quad \forall x \in B(x_0, \lambda \varepsilon),
\]
hence $(1 - |u_\varepsilon(x)|^2)^2 > \frac{\eta^2}{4}, \quad \forall x \in B(x_0, \lambda \varepsilon)$. Thus
\[
\int_{B(x_0, \lambda \varepsilon) \cap B(0, 1 - \rho \varepsilon)} (1 - |u_\varepsilon|^2)^2 > \frac{\eta^2}{4} |B(0, 1 - \rho \varepsilon) \cap B(x_0, \lambda \varepsilon)|
\]
\[
\geq C_2 \frac{\eta^2}{4} (\lambda \varepsilon)^n = C_2 \frac{\eta^2}{4} \left(\frac{\eta}{2C_1}\right)^n \varepsilon^n = \mu \varepsilon^n.
\]
Since $x_0 \in B(0, \lambda \varepsilon)$, and $(B(x_0, \lambda \varepsilon) \cap B(0, 1 - \rho \varepsilon)) \subset (B(2 \lambda \varepsilon) \cap B(0, 1 - \rho \varepsilon))$, (3.5) implies
\[
\int_{B(2 \lambda \varepsilon) \cap B(0, 1 - \rho \varepsilon)} (1 - |u_\varepsilon|^2)^2 > \mu \varepsilon^n,
\]
which contradicts (3.2) and thus (3.3) is proved.

Let $u_\varepsilon$ be a radial minimizer of $E_\varepsilon(u, B)$. Given $\eta \in (0, 1/2)$. Let $\lambda, \mu$ be constants in Proposition 3.2 corresponding to $\eta$. If
\[
\frac{1}{\varepsilon^n} \int_{B(x, 2 \lambda \varepsilon) \cap B(0, 1 - \rho \varepsilon)} (1 - |u_\varepsilon|^2)^2 \leq \mu,
\]
then $B(x, \lambda \varepsilon)$ is called the ball of type I. Otherwise it is called the ball of type II.

Now suppose that $\{B(x_i^+, \lambda \varepsilon), i \in I\}$ is a family of balls satisfying
\[
x_i^+ \in B(0, 1 - \rho \varepsilon), i \in I;
B(0, 1 - \rho \varepsilon) \subset \bigcup_{i \in I} B(x_i^+, \lambda \varepsilon);
B(x_i^+, \lambda \varepsilon/4) \cap B(x_j^+, \lambda \varepsilon/4) = \emptyset, i \neq j.
\]
Denote $J_\varepsilon = \{i \in I; B(x_i^+, \lambda \varepsilon) \text{ is a ball of type II}\}$.

Proposition 3.3 There exists an upper bound for the number of balls of type II, i.e., there exists a positive integer $N$ such that $\text{Card} J_\varepsilon \leq N$. 

Proof. Since (3.7) implies that every point in $B$ can be covered by finite, say $m$ (independent of $\varepsilon$) balls, from (3.6) and the definition of balls of type II, we have

\[
\mu \varepsilon^n \text{Card} J_\varepsilon \leq \sum_{i \in J_\varepsilon} \int_{B(x^\varepsilon_i, 2\lambda \varepsilon) \cap B(0, 1 - \rho \varepsilon)} (1 - |u_\varepsilon|^2)^2 \\
\leq m \int_{\cup_{i \in J_\varepsilon} B(x^\varepsilon_i, 2\lambda \varepsilon) \cap B(0, 1 - \rho \varepsilon)} (1 - |u_\varepsilon|^2)^2 \\
\leq m \int_B (1 - |u_\varepsilon|^2)^2 \leq m C \varepsilon^n
\]

and hence for some $n$, Card $J_\varepsilon \leq \frac{mC}{n} \leq N$.

Proposition 3.3 is an important result since the number of balls of type II is always finite as $\varepsilon$ becomes sufficiently small.

Similar to the argument of [1, Theorem IV.1], we have

**Proposition 3.4** There exist a subset $J \subset J_\varepsilon$ and a constant $h \geq \lambda$ such that

\[
\cup_{i \in J_\varepsilon} B(x^\varepsilon_i, \lambda \varepsilon) \subset \cup_{i \in J} B(x^\varepsilon_i, h \varepsilon), \\
|x^\varepsilon_i - x^\varepsilon_j| > 8h \varepsilon, \quad i, j \in J, \quad i \neq j. \\
\tag{3.8}
\]

Proof. If there are two points $x_1, x_2$ such that (3.8) is not true with $h = \lambda$, we take $h_1 = 9\lambda$ and $J_1 = J_\varepsilon \setminus \{1\}$. In this case, if (3.8) holds we are done. Otherwise we continue to choose a pair points $x_3, x_4$ which does not satisfy (3.8) and take $h_2 = 9h_1$ and $J_2 = J_\varepsilon \setminus \{1, 3\}$. After at most $N$ steps we may choose $\lambda \leq h \leq \lambda 9^N$ and conclude this proposition.

Applying Proposition 3.4, we may modify the family of balls of type II such that the new one, denoted by $\{B(x^\varepsilon_i, h \varepsilon); i \in J\}$, satisfies

\[
\cup_{i \in J_\varepsilon} B(x^\varepsilon_i, \lambda \varepsilon) \subset \cup_{i \in J} B(x^\varepsilon_i, h \varepsilon), \\
\text{Card } J \leq \text{Card } J_\varepsilon, \\
|x^\varepsilon_i - x^\varepsilon_j| > 8h \varepsilon, \quad i, j \in J, \quad i \neq j.
\]

The last condition implies that every two balls in the new family are disjoint.

Now we prove the main result of this section.

**Theorem 3.5** Let $u_\varepsilon$ be a radial minimizer of $E_\varepsilon(u, B)$. Then for any $\eta \in (0, 1/2)$, there exists a constant $h = h(\eta)$ independent of $\varepsilon \in (0, 1)$ such that $Z_\varepsilon = \{x \in B; |u_\varepsilon(x)| < 1 - \eta\} \subset B(0, h \varepsilon)$. In particular the zeros of $u_\varepsilon$ are contained in $B(0, h \varepsilon)$.

Proof. Suppose there exists a point $x_0 \in Z_\varepsilon$ such that $x_0 \in B(0, h \varepsilon)$. Then all points on the circle $S_\varepsilon = \{x \in B; |x| = |x_0|\}$ satisfy $|u_\varepsilon(x)| < 1 - \eta$ and hence by virtue of Proposition 3.2 and (2.4), all points on $S_\varepsilon$ are contained in balls of type II. However, since $|x_0| \geq h \varepsilon, S_\varepsilon$ can not be covered by a single ball of type
II. $S_0$ can be covered by at least two balls of type II. However this is impossible.

This theorem plays the key role in proving the uniqueness of radial minimizers. Furthermore, it implies that all the zeros of the radial minimizer locate near the singularity $0$ of $\frac{x}{|x|}$, which is not mentioned in [6] when $p = n > 2$.

Using Theorem 3.5 and (2.11), we can see that there exists a positive constant $\gamma$ given $\varepsilon$

Fix $\varepsilon \in (0,1)$. Suppose $u_1(x) = f_1(r) \frac{x}{|x|}$ and $u_2(x) = f_2(r) \frac{x}{|x|}$ are both radial minimizers of $E_\varepsilon(u, B)$ on $W$, then they are weak radial solutions of (2.1) (2.2).

Thus

$$\int_B (|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2) \nabla \phi dx = \frac{1}{\varepsilon^p} \int_B [(u_1 - u_2) - (u_1|u_1|^2 - u_2|u_2|^2)] \phi dx.$$

Taking $\phi = u_1 - u_2 = (f_1 - f_2) \frac{x}{|x|}$, we have

$$\int_B (|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2) \nabla (u_1 - u_2) dx$$

$$= \frac{1}{\varepsilon^p} \int_B (f_1 - f_2)^2 dx - \frac{1}{\varepsilon^p} \int_B (f_1 - f_2)^2 (f_1^2 + f_2^2 + f_1 f_2) dx$$

$$= \frac{1}{\varepsilon^p} \int_{B\setminus B(0,h\varepsilon)} (f_1 - f_2)^2 [1 - (f_1^2 + f_2^2 + f_1 f_2)] dx$$

$$+ \frac{1}{\varepsilon^p} \int_{B(0,h\varepsilon)} (f_1 - f_2)^2 dx - \frac{1}{\varepsilon^p} \int_{B(0,h\varepsilon)} (f_1 - f_2)^2 (f_1^2 + f_2^2 + f_1 f_2) dx.$$

Letting $\eta < 1 - \frac{1}{\sqrt{2}}$ in (3.9), we have $f_1, f_2 \geq 1/\sqrt{2}$, on $B \setminus B(0,h\varepsilon)$ for any given $\varepsilon \in (0,1)$. Hence

$$\int_B (|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2) \nabla (u_1 - u_2) dx \leq \frac{1}{\varepsilon^p} \int_{B(0,h\varepsilon)} (f_1 - f_2)^2 dx.$$

Applying [12, eqn.(2.11)], we can see that there exists a positive constant $\gamma$ independent of $\varepsilon$ and $h$ such that

$$\gamma \int_B |\nabla (u_1 - u_2)|^2 dx \leq \frac{1}{\varepsilon^p} \int_{B(0,h\varepsilon)} (f_1 - f_2)^2 dx,$$

which implies

$$\int_B |\nabla (f_1 - f_2)|^2 dx \leq \frac{1}{\gamma \varepsilon^p} \int_{B(0,h\varepsilon)} (f_1 - f_2)^2 dx.$$
Denote $G = B(0, h\varepsilon)$. Applying [8, Theorem 2.1], we have $\|f\|_{2n}^{2n} \leq \beta \|\nabla f\|_2$, where $\beta = 2(n - 1)/(n - 2)$. Taking $f = f_1 - f_2$ and applying (4.2), we obtain $f(|x|) = 0$ as $x \in \partial B$ and

$$\int_B |f|^\frac{2n}{n-2} dx \leq \beta^2 \int_B |\nabla f|^2 dx \leq \beta^2 \gamma^{-1} \int_G |f|^2 dx \varepsilon^{-p}.$$

Using Holder’s inequality, we derive

$$\int_G |f|^2 dx \leq |G|^{1-\frac{n-2}{n}} \left[ \int_G |f|^{\frac{2n}{n-2}} dx \right]^{\frac{n-2}{n}} \leq |B|^{1-\frac{n-2}{n}} h^2 \varepsilon^{2 - p} \frac{\beta^2}{\gamma} \int_G |f|^2 dx.$$

Hence for any given $\varepsilon \in (0, 1)$,

$$\int_G |f|^2 dx \leq C(\beta, |B|, \gamma, \varepsilon) h^2 \int_G |f|^2 dx. \quad (4.3)$$

Denote $F(\eta) = \int_{B(0,h(\eta)\varepsilon)} |f|^2 dx$, then $F(\eta) \geq 0$ and (4.3) implies that

$$F(\eta)(1 - C(\beta, |B|, \gamma, \varepsilon) h^2) \leq 0. \quad (4.4)$$

On the other hand, since $C(\beta, |B|, \gamma, \varepsilon)$ is independent of $\eta$, we may take $0 < \eta < 1 - \frac{1}{\sqrt{2}}$ so small that $h = h(\eta) \leq c9^N = 9^N \frac{9}{\sqrt{2}}$ (which is implied by (3.4)) satisfies

$$0 < 1 - C(\beta, |B|, \gamma, \varepsilon) h^2$$

for the fixed $\varepsilon \in (0, 1)$, which and (4.4) imply that $F(\eta) = 0$. Namely $f = 0$ a.e. on $G$, or $f_1 = f_2$, a.e. on $B(0, h\varepsilon)$.

Substituting this into (4.1), we know that $u_1 - u_2 = C$ a.e. on $B$. Noticing the continuity of $u_1, u_2$ which is implied by Proposition 2.1, and $u_1 = u_2 = x$ on $\partial B$, we can see at last that

$$u_1 = u_2, \quad \text{on } \overline{B}.$$

References


functional in higher dimensions associated with n-harmonic maps, Adv. in 

tion differentielle liee a l'équation de Ginzburg-Landau, Ann. IHP. Analyse 

[8] O. Ladyzhenskaya, N. Uraltseva: Linear and quasilinear elliptic equations, 


[12] P. Tolksdorff: Everywhere regularity for some quasilinear systems with a lack 

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