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Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping *

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Abstract

In this paper we obtain an exponential rate of decay for the solution of the viscoelastic nonlinear wave equation

$$u_{tt} - \Delta u + f(x,t,u) + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + a(x)u_t = 0 \quad \text{in } \Omega \times (0,\infty).$$

Here the damping term $a(x)u_t$ may be null for some part of the domain Ω. By assuming that the kernel g in the memory term decays exponentially, the damping effect allows us to avoid compactness arguments and and to reduce number of the energy estimates considered in the prior literature. We construct a suitable Liapunov functional and make use of the perturbed energy method.

1 Introduction

This manuscript is devoted to the study of the exponential decay of the solutions of the viscoelastic nonlinear wave equation

$$\begin{gathered} u_{tt} - \Delta u + f(x,t,u) + \int_0^t g(t-\tau)\Delta u(\tau)\,d\tau + a(x)u_t = 0 \quad \text{in } \Omega \times (0,\infty) \\ u = 0 \quad \text{on } \Gamma \times (0,\infty) \\ u(x,0) = u^0(x); \quad u_t(x,0) = u^1(x), \quad x \in \Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ whose boundary Γ is assumed regular. Let $x^0 \in \mathbb{R}^n$ be an arbitrary point of $\mathbb{R}^n$ and we set

$$\Gamma(x^0) = \{x \in \Gamma;\ m(x)\cdot\nu(x) > 0\} \tag{1.2}$$

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where ν represents the unit normal vector pointing towards the exterior of Ω and

$$m(x) = x - x^0. \tag{1.3}$$

Let ω be a neighborhood of $\Gamma(x^0)$ in Ω and consider $\delta > 0$ sufficiently small such that

$$Q_0 = \{x \in \Omega;\ d(x, \Gamma(x^0)) < \delta\} \subset \omega, \tag{1.4}$$

$$Q_1 = \{x \in \Omega;\ d(x, \Gamma(x^0)) < 2\delta\} \subset \omega. \tag{1.5}$$

Here, if $A \subset \mathbb{R}^n$, and $x \in \mathbb{R}^n$, $d(x, A) = \inf_{y \in A} |x - y|$. Then, $Q_0 \subset Q_1 \subset \omega$.

Next, we make some remarks about early works in connection with problem (1.1). When $g = 0$ and $f \neq 0$ and the feedback term depends on the velocity in a linear way, as in the present paper, Zuazua [16] proved that the energy related to problem (1.1) decays exponentially if the damping region contains a neighbourhood of the boundary Γ or, at least, contains a neibourhood of the particular part given by (1.1). In the same direction and considering $g = f = 0$, it is important to mention the result of Bardos , Lebeau and Rauch [2], based on microlocal analysis, that ensures a necessary and sufficient condition to obtain exponential decay, namely, the damping region satisfies the well known *geometric control condition.* The classical example of an open subset ω verifying this condition is when ω is a neighbourhood of the boundary. Later, still considering $g = 0$ and $f = 0$, Nakao [13, 14] extended the results of Zuazua [16] treating first the case of a linear degenerate equation, and then the case of a nonlinear dissipation $\rho(x, u_t)$ assuming, as usually, that the function ρ has a polynomial growth near the origin. More recently, Martinez [11] improved the previous results mentioned above in what concerns the linear wave equation subject to a nonlinear dissipation $\rho(x, u_t)$, avoiding the polynomial growth of the function $\rho(x, s)$ in zero. His proof is based on the piecewise multiplier technique developed by Liu [10] combined with nonlinear integral inequalities to show that the energy of the system decays to zero with a precise decay rate estimate if the damping region satisfies some geometrical conditions. It is important to mention that Lasiecka and Tataru [9] studied the nonlinear wave equation subject to a nonlinear feedback acting on a part of the boundary of the system and they were the first to prove that the energy decays to zero as fast as the solution of some associated differential equation and without assuming that the feedback has a polynomial growth in zero, although no decay rate has been showed.

On the other hand, when $a = 0$, that is, when the unique damping mechanism is given by the memory term, then, following ideas introduced by Munoz Rivera [12], it is possible to prove that the exponential decay holds for small initial data. Indeed, we presume that the result above mentioned is a direct consequence of his proof (see [12] for details). In this context we can cite the works of Dafermos [4], Dafermos and Nohel [4, 5], Jiang and Munoz Rivera [7], among others.

To end, we would like to mention the works from the authors Cavalcanti, Domingos Cavalcanti, Prates Filho and Soriano [3] in connection with the viscoelastic linear wave equation and nonlinear boundary dissipation and Aassila, Cavalcanti and Soriano [1] concerned with the wave equation subject to viscoelastic effects on the boundary and nonlinear boundary feedback.

The purpose of this paper is to obtain an exponential decay rate to the solutions of a viscoelastic nonlinear wave equation subject to a locally distributed dissipation as in Zuazua [16]. We would like to emphasize that the additional damping effect given by the kernel of the memory of the material allows us to avoid the compactness arguments and the amount of estimates and fields considered in the prior literature by constructing a suitable Liapunov functional and making use of the perturbed energy method. For simplicity, we will consider the linear localized effect, although we could consider the nonlinear one $\rho(x, u_t)$. In this direction, our work complements the previous ones for the viscoelastic wave equation.

Our paper is organized as follows. In section 2 we state the notation, assumptions and the main result and in section 3 we give the proof of the uniform decay.

2 Notation and Statment of Results

We begin by introducing some notation that will be used throughout this work. For the standard $L^p(\Omega)$ space we write

$$(u, v) = \int_\Omega u(x)v(x)\,dx, \quad \|u\|_p^p = \int_\Omega |u(x)|^p dx.$$

Next, we give the precise assumptions on the functions $a(x)$, $f(x, t, u)$ and on the memory term $g(t)$.

(A.1) Assume that $a : \Omega \to \mathbb{R}$ is a nonnegative and bounded function such that

$$a(x) \geq a_0 > 0 \quad \text{a.e.} \quad \text{in} \quad \omega. \tag{2.1}$$

(A.2) Assume that $f : \overline{\Omega} \times [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is an element of the space $C^1(\overline{\Omega} \times [0, \infty) \times \mathbb{R})$ and satisfies

$$|f(x, t, \xi)| \leq C_0(1 + |\xi|^{\gamma+1}) \tag{2.2}$$

where C_0 is a positive constant.

Let γ be a constant such that $\gamma > 0$ for $n = 1, 2$ and $0 < \gamma \leq 2/(n - 2)$ for $n \geq 3$ and let C_0' be a positive constant such that

$$f(x, t, \xi)\eta \geq |\xi|^\gamma \xi\eta - C_0'|\xi||\eta|; \quad \forall \eta \in \mathbb{R}. \tag{2.3}$$

Assume that there exist positive constants C_1 and C_2 such that

$$|f_t(x, t, \xi)| \leq C_1(1 + |\xi|^{\gamma+1}), \tag{2.4}$$

$$|f_\xi(x, t, \xi)| \leq C_2(1 + |\xi|^\gamma). \tag{2.5}$$

We also assume that there is a positive constant D_1 such that for all η , $\hat{\eta}$ in $\mathbb{R}^n$ and for all ξ, $\hat{\xi}$ in $\mathbb{R}$,

$$(f(x,t,\xi) - f(x,t,\hat{\xi}))(\eta - \hat{\eta}) \geq -D_1(|\xi|^\gamma + |\hat{\xi}|^\gamma)|\xi - \hat{\xi}||\eta - \hat{\eta}|. \tag{2.6}$$

A simple example of a function f that satisfies the above conditions is given by $f(x,t,\xi) = |\xi|^\gamma \xi + \varphi(x,t) sin(\xi)$, where $\varphi \in W^{1,+\infty}(\Omega \times \infty)$.

Remark: The variational formulation associated with problem (1.1) leads us to the identity

$$\begin{aligned}&(u''(t), v) + (\nabla u(t), \nabla v) + (f(x,t,u(t)), v)\\&- \int_0^t g(t-\tau)(\nabla u(\tau), \nabla v) d\tau + (au'(t), v) = 0 \quad \text{for all } v \in H_0^1(\Omega).\end{aligned}$$

Assumption (2.2) is required to prove that the nonlinear term in the above identity is well defined. The hypothesis (2.3) is used in the first a priori estimate and also and in the asymptotic stability. Now, hypotheses (2.4)-(2.5) are needed for the second a priori estimate while assumption (2.6) is necessary for proving the uniqueness of solutions.

For simplicity, we will consider the asymptotic stability for the simple case when $f(x,t,\xi) = |\xi|^\gamma \xi$, although the general case follows exactly making use of the same procedure.

To obtain the existence of regular and weak solutions we assume that kernel g satisfies the following.

(A.3) The function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is in $W^{2,1}(0,\infty) \cap W^{1,\infty}(0,\infty)$. Now, to obtain the uniform decay, we suppose that

$$1 - \int_0^\infty g(s)\, ds = l > 0, \tag{2.7}$$

and there exist $\xi_1, \xi_2 > 0$ such that

$$g(0) > 0 \quad \text{and} \quad -\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t); \quad \forall t \geq 0. \tag{2.8}$$

Note that (2.8) implies

$$g(0)e^{-\xi_1 t} \leq g(t) \leq g(0)e^{-\xi_2 t}; \quad \forall t \geq 0. \tag{2.9}$$

Consequently from (2.9) we have that the kernel $g(t)$ is between two exponential functions.

We observe that given $\{u^0, u^1\} \in H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega)$, problem (1.1) possesses a unique solution in the class

$$u \in L^\infty_{\text{loc}}(0,\infty; H_0^1(\Omega) \cap H^2(\Omega)), u' \in L^\infty_{\text{loc}}(0,\infty; H_0^1(\Omega)),\ u'' \in L^\infty_{\text{loc}}(0,\infty; L^2(\Omega))$$

which can easily obtained making use, for instance, of the Faedo-Galerkin method. Now, if $\{u^0, u^1\} \in H_0^1(\Omega) \times L^2(\Omega)$ and considering standard arguments of density, we can prove that problem (1.1) has a unique solution

$$u \in C^0([0,\infty); H_0^1(\Omega)) \cap C^1([0,\infty); L^2(\Omega)). \tag{2.10}$$

To see the proof of pexitence and uniqueness of a similar problem to the viscoelastic wave equation subject to nonlinear boundary feedback, we refer the reader to [3].

The energy related to problem (1.1) is

$$E(t) = \frac{1}{2}\|u'(t)\|_2^2 + \frac{1}{2}\|\nabla u(t)\|_2^2 + \frac{1}{\gamma+2}\|u(t)\|_{\gamma+2}^{\gamma+2}. \tag{2.11}$$

Now, we are in a position to state our main result.

Theorem 2.1 *Given $\{u^0, u^1\} \in H_0^1(\Omega) \times L^2(\Omega)$ and assuming that Hypotheses (A.1)-(A.3) hold, then, the unique solution of problem (1.1) in the class given in (2.10) decays exponentially, that is, there exist C, β positive constants such that*

$$E(t) \le Ce^{-\beta t} \tag{2.12}$$

provided that $\int_0^\infty g(s)ds$ is sufficiently small.

3 Uniform decay

In this section we prove the exponential decay for regular solutions of problem (1.1), and by using standard arguments of density we also can extend the same result to weak solutions. From (1.1) we deduce that

$$E'(t) \le -\int_\Omega a(x)|u'(x,t)|^2 dx + \int_0^t g(t-\tau)(\nabla u(\tau), \nabla u'(t)) d\tau. \tag{3.1}$$

A direct computation shows that

$$\begin{aligned} &\int_0^t g(t-\tau)(\nabla u(\tau), \nabla u'(t)) d\tau \\ &= \frac{1}{2}(g' \diamond \nabla u)(t) - \frac{1}{2}(g \diamond \nabla u)'(t) + \frac{d}{dt}\{\frac{1}{2}\int_0^t g(s)\,ds \|\nabla u(t)\|_2^2\} - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2, \end{aligned} \tag{3.2}$$

where $(g \diamond v)(t) = \int_0^t g(t-\tau)\|v(t) - v(\tau)\|_2^2 d\tau$. Define the modified energy as

$$e(t) = \frac{1}{2}\|u'(t)\|_2^2 + \frac{1}{2}(1 - \int_0^t g(s)\,ds)\|\nabla u(t)\|_2^2 + \frac{1}{\gamma+2}\|u(t)\|_{\gamma+2}^{\gamma+2} + \frac{1}{2}(g \diamond \nabla u)(t). \tag{3.3}$$

From (2.7) we deduce that $e(t) \ge 0$. Moreover,

$$E(t) \le l^{-1} e(t); \quad \forall t \ge 0. \tag{3.4}$$

Consequently, the uniform decay of $E(t)$ is a consequence of the decay of $e(t)$. On the other hand, from (2.11), (3.1), (3.2) and (3.3) we deduce

$$e'(t) \le -\int_\Omega a(x)|u'(x,t)|^2 dx + \frac{1}{2}(g' \diamond \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2. \tag{3.5}$$

Considering the assumptions (2.1) and (2.8), from (3.5) we obtain

$$e'(t) \le -a_0\|u'(t)\|_{L^2(\omega)}^2 - \frac{\xi_2}{2}(g \diamond \nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2. \tag{3.6}$$

Then $e'(t) \le 0$ and consequently $e(t)$ are non-increasing functions. Having in mind (1.4)-(1.5), we consider $\psi \in C_0^\infty(\mathbb{R}^n)$ such that

$$\begin{gathered} 0 \le \psi \le 1 \\ \psi = 1 \quad \text{in } \overline{\Omega}\backslash Q_1 \\ \psi = 0 \quad \text{in } Q_0. \end{gathered} \tag{3.7}$$

For an arbitrary $\varepsilon > 0$, define the peturbed energy

$$e_\varepsilon(t) = e(t) + \varepsilon\rho(t) \tag{3.8}$$

where

$$\rho(t) = 2\int_\Omega u'(h\cdot\nabla z)dx + \theta\int_\Omega u'z dx, \tag{3.9}$$

$$h = m\psi, \tag{3.10}$$

$$z(t) = u(t) - \int_0^t g(t-\tau)u(\tau)\,d\tau, \tag{3.11}$$

and $\theta \in]n-2, n[$, $\theta > \frac{2n}{\gamma+2}$. For short notation, put

$$k_1 = \min\big\{2(\theta - n + 2), 2(n-\theta), (\gamma+2)(\theta - \frac{2n}{\gamma+2})\big\} > 0. \tag{3.12}$$

Proposition 3.1 *There exists $\delta_0 > 0$ such that*

$$|e_\varepsilon(t) - e(t)| \le \varepsilon\delta_0 e(t) \quad \forall t \ge 0 \quad \textit{and} \quad \forall \varepsilon > 0.$$

Proof. From the Cauchy-Schwarz inequality

$$|\rho(t)| \le 2R(x^0)\|u'(t)\|_2\|\nabla z(t)\|_2 + \theta\lambda\|u'(t)\|_2\|\nabla z(t)\|_2, \tag{3.13}$$

where $\lambda > 0$ and satisfies $\|v|\|_2 \le \lambda\|\nabla v\|_2$ for all $v \in H_0^1(\Omega)$, and

$$R(x^0) = \max_{x\in\overline{\Omega}} |x - x^0|. \tag{3.14}$$

From (3.13) we obtain

$$|\rho(t)| \le (2R(x^0) + \theta\lambda)\big\{\frac{1}{2}\|u'(t)\|_2^2 + \frac{1}{2}\|\nabla z(t)\|_2^2\big\}. \tag{3.15}$$

We note that

$$\begin{aligned}\|\nabla z(t)\|_2^2 =&\|\nabla u(t)\|_2^2 - 2\int_0^t g(t-\tau)(\nabla u(\tau),\nabla u(t))d\tau\\&+\int_0^t g(t-\tau)\Big(\int_0^t g(t-s)(\nabla u(\tau),\nabla u(s))ds\Big)d\tau\end{aligned}$$

Now we estimate each term on the right-hand side of this inequality.
Estimate for $J_1 := -2\int_0^t g(t-\tau)(\nabla u(\tau),\nabla u(t))d\tau$. We have,

$$\begin{aligned}|J_1| \leq&\|\nabla u(t)\|_2^2 + (\int_0^t g(t-\tau)\|\nabla u(\tau)\|_2)^2\\ \leq&\|\nabla u(t)\|_2^2 + 2\|g\|_{L^1(0,\infty)}\int_0^t g(t-\tau)\big[\|\nabla u(\tau)-\nabla u(t)\|_2^2+\|\nabla u(t)\|_2^2\big]d\tau.\end{aligned}$$

Having in mind that $\|g\|_{L^1(0,\infty)} < 1$, we obtain

$$|J_1| \leq 3\|\nabla u(t)\|_2^2 + 2(g\diamond\nabla u)(t). \tag{3.17}$$

Estimate for $J_2 := \int_0^t g(t-\tau)(\int_0^t g(t-s)(\nabla u(\tau),\nabla u(s))ds)d\tau$. We infer

$$\begin{aligned}|J_2| \leq&(\int_0^t g(t-\tau)\|\nabla u(\tau)\|_2^2)^2\\ \leq&2\|g\|_{L^1(0,\infty)}\int_0^t g(t-\tau)\big[\|\nabla u(\tau)-\nabla u(t)\|_2^2+\|\nabla u(t)\|_2^2\big]d\tau.\end{aligned}$$

Consequently

$$|J_2| \leq 2(g\diamond\nabla u)(t) + 2\|\nabla u(t)\|_2^2. \tag{3.18}$$

Combining (3.15)-(3.18) taking (3.4) into account, it follows that

$$|\rho(t)| \leq 20(l^{-1}+1)(2R(x^0)+\theta\lambda)e(t)$$

which, in view of (3.8) allow us to conclude the desired result. This completes the proof.

Proposition 3.2 *There exists $\delta_1 = \delta_1(\varepsilon)$ such that*

$$e'_\varepsilon(t) \leq -\delta_1 e(t) \quad \forall t\geq 0$$

provided that $\|g\|_{L^1(0,\infty)}$ is small enough.

Proof. Getting the derivative of $\rho(t)$ given in (3.9) with respect to t and substituting $u'' = \Delta z - f(x,t,u) - a(x)u'$ in the obtained expression,

$$\begin{aligned}\rho'(t) =&2(\Delta z(t), h\cdot\nabla z(t)) - 2(f(x,t,u(t)), h\cdot\nabla z(t))\\&-2(au'(t), h\cdot\nabla z(t)) + 2(u'(t), h\cdot\nabla z'(t)) + \theta(\Delta z(t), z(t))\\&-\theta(f(x,t,u(t)),z(t)) - \theta(au'(t),z(t)) + \theta\|u'(t)\|_2^2\\&-\theta g(0)(u'(t),u(t)) - \theta\int_0^t g'(t-\tau)(u'(t),u(\tau))d\tau.\end{aligned}\tag{3.19}$$

Next, we will estimate some terms on the right-hand side of identity (3.19).
Estimate for $I_1 := 2(\Delta z(t), h\cdot\nabla z(t))$. Employing Gauss and Green formulas, we deduce

$$\begin{aligned} I_1 =& -2\sum_{i,k=1}^n \int_\Omega \frac{\partial z}{\partial x_k}\frac{\partial h_i}{\partial x_k}\frac{\partial z}{\partial x_i}dx + \int_\Omega (\operatorname{div} h)|\nabla z|^2 dx \\ & - \int_\Gamma (h\cdot\nu)|\nabla z|^2 d\Gamma + \int_\Gamma \frac{\partial z}{\partial \nu}(2h\cdot\nabla z)d\Gamma. \end{aligned} \tag{3.20}$$

Estimate for $I_2 := -2(u'(t), h\cdot\nabla z'(t))$. Making use of Gauss formula and noting that $u' = 0$ on Γ,

$$I_2 = -\int_\Omega (\operatorname{div} h)|u'|^2 dx - 2g(0)(u'(t), h\cdot\nabla u(t)) - 2\int_0^t g'(t-\tau)(u'(t), h\cdot\nabla u(\tau))d\tau. \tag{3.21}$$

Estimate for $I_3 := \theta(\Delta z(t), z(t))$. Considering Green formula and observing that $z = 0$ on Γ,

$$I_3 = -\theta\|\nabla z(t)\|_2^2. \tag{3.22}$$

Estimate for $I_4 := -2(f(x,t,u(t)), h\cdot\nabla z(t))$. Considering assumption (2.3), taking (3.11) and Gauss formula into account and noting that $u = 0$ on Γ, it follows that

$$\begin{aligned} I_4 \le& \frac{2n}{\gamma+2}\|u(t)\|_{\gamma+2}^{\gamma+2} + \frac{2}{\gamma+2}\int_\Omega (\nabla\psi\cdot m)|u|^{\gamma+2}dx \\ & + 2\int_0^t g(t-\tau)\int_\Omega |u(x,t)|^{\gamma+1}\psi\sum_{k=1}^n |m_k||\frac{\partial u}{\partial x_k}(x,\tau)|dx\,d\tau. \end{aligned} \tag{3.23}$$

Estimate for $I_5 := -\theta(f(x,t,u(t)), z(t))$. In view of assumption (2.3),

$$I_5 \le -\theta\|u(t)\|_{\gamma+2}^{\gamma+2} + \theta\int_0^t g(t-\tau)(|u(t)|^\gamma u(t), u(\tau))d\tau. \tag{3.24}$$

Then, combining (3.19)-(3.24) we arrive at

$$\begin{aligned} \rho'(t) \le& -2\sum_{i,k=1}^n \int_\Omega \frac{\partial z}{\partial x_k}\frac{\partial h_i}{\partial x_k}\frac{\partial z}{\partial x_i}dx + \int_\Omega (\operatorname{div} h)|\nabla z|^2 dx \\ & - \int_\Omega (\operatorname{div} h)|u'|^2 dx - 2g(0)(u'(t), h\cdot\nabla u(t)) \\ & - 2\int_0^t g'(t-\tau)(u'(t), h\cdot\nabla u(\tau))d\tau - \theta\|\nabla z(t)\|_2^2 \\ & + (\frac{2n}{\gamma+2}-\theta)\|u(t)\|_{\gamma+2}^{\gamma+2} + \frac{2}{\gamma+2}\int_\Omega (\nabla\psi\cdot m)|u|^{\gamma+2}dx \\ & + 2\int_0^t g(t-\tau)\int_\Omega |u(x,t)|^{\gamma+1}\psi\sum_{k=1}^n |m_k||\frac{\partial u}{\partial x_k}(x,\tau)\|dx\,d\tau \end{aligned} \tag{3.25}$$

$$
\begin{aligned}
&- 2(au'(t), h\cdot\nabla z(t)) + \theta\int_0^t g(t-\tau)(|u(t)|^\gamma u(t), u(\tau))d\tau\\
&- \theta(au'(t), z(t)) + \theta\|u'(t)\|_2^2\\
&- \theta g(0)(u'(t), u(t)) - \theta\int_0^t g'(t-\tau)(u'(t), u(\tau))d\tau\\
&- \int_\Gamma (h\cdot\nu)|\nabla z|^2 d\Gamma + \int_\Gamma \frac{\partial z}{\partial\nu}(2h\cdot\nabla z)d\Gamma.
\end{aligned}
$$

Next we handle the boundary terms. Observe that since $z = 0$ on Γ we have $\frac{\partial z}{\partial x_k} = \frac{\partial z}{\partial \nu}\nu_k$ which implies

$$
h\cdot\nabla z = (h\cdot\nu)\frac{\partial z}{\partial\nu} \quad\text{and}\quad |\nabla z|^2 = (\frac{\partial z}{\partial\nu})^2 \quad \text{on } \Gamma.
$$

From the above expressions and taking (1.2), (1.4) and (3.7) into account,

$$
\begin{aligned}
&- \int_\Gamma (h\cdot\nu)|\nabla z|^2 d\Gamma + \int_\Gamma \frac{\partial z}{\partial\nu}(2h\cdot\nabla z)d\Gamma\\
&= \int_\Gamma (h\cdot\nu)(\frac{\partial z}{\partial\nu})^2 d\Gamma\\
&= \int_{\Gamma(x^0)} (m\cdot\nu)\psi(\frac{\partial z}{\partial\nu})^2 d\Gamma + \int_{\Gamma\backslash\Gamma(x^0)} (m\cdot\nu)\psi(\frac{\partial z}{\partial\nu})^2 d\Gamma \le 0.
\end{aligned} \tag{3.26}
$$

Then, from (3.25)-(3.26) and after some computations, we conclude that

$$
\begin{aligned}
\rho'(t) \le&(n-2-\theta)\|\nabla z(t)\|_2^2 + (\theta - n)\|u'(t)\|_2^2 + (\frac{2n}{\gamma+2} - \theta)\|u(t)\|_{\gamma+2}^{\gamma+2}\\
&+ n\int_{Q_1}[1-\psi]|u'|^2dx + (n-2)\int_{Q_1}[\psi-1]|\nabla z|^2 dx\\
&- 2\sum_{i,k=1}^n \int_\Omega \frac{\partial z}{\partial x_k} m_i \frac{\partial\psi_i}{\partial x_k}\frac{\partial z}{\partial x_i}dx + \int_{Q_1\backslash Q_0}(\nabla\psi\cdot m)|\nabla z|^2 dx\\
&- \int_{Q_1\backslash Q_0}(\nabla\psi\cdot m)|u'|^2dx + \frac{2}{\gamma+2}\int_{Q_1\backslash Q_0}(\nabla\psi\cdot m)|u|^{\gamma+2}dx\\
&+ 2\int_0^t g(t-\tau)\int_\Omega |u(x,t)|^{\gamma+1}\psi\sum_{k=1}^n |m_k||\frac{\partial u}{\partial x_k}(x,\tau)|dx\,d\tau\\
&- 2g(0)(u'(t), h\cdot\nabla u(t)) - 2\int_0^t g'(t-\tau)(u'(t), h\cdot\nabla u(\tau))d\tau\\
&- 2(au'(t), h\cdot\nabla z(t)) + \theta\int_0^t g(t-\tau)(|u(t)|^\gamma u(t), u(\tau))d\tau\\
&- \theta(au'(t), z(t)) - \theta g(0)(u'(t), u(t)) - \theta\int_0^t g'(t-\tau)(u'(t), u(\tau))d\tau.
\end{aligned} \tag{3.27}
$$

Having in mind (3.12), (3.16) and adding and subtracting suitable terms in order to obtain $-k_1 e(t)$ from (3.27), and supposing without loss of generality that $g(0) \leq 1$, we infer

$$
\begin{aligned}
\rho'(t) \leq & - k_1 e(t) - \frac{k_1}{2}\Big(\int_0^t g(s)ds\Big)\|\nabla u(t)\|_2^2 + \frac{k_1}{2}(g \diamond \nabla u)(t) \\
& + 2\int_0^t g(t-\tau)\|\nabla u(\tau)\|_2 \|\nabla u(t)\|_2 d\tau + 2\Big(\int_0^t g(t-\tau)\|\nabla u(\tau)\|_2 d\tau\Big)^2 \\
& + 2n\int_{Q_1} |u'|^2 dx + 3R(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)| \int_{Q_1\backslash Q_0} |\nabla z|^2 dx \\
& - \theta g(0)(u'(t), u(t)) - \theta\int_0^t g'(t-\tau)(u'(t), u(\tau))d\tau. \\
& + R(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)| \int_{Q_1\backslash Q_0} |u'|^2 dx \\
& + \frac{2}{\gamma+2}R(x^0)\max_{x\in\overline{\Omega}}|\nabla\psi(x)| \int_{Q_1\backslash Q_0} |u|^{\gamma+2} dx \\
& + 2R(x^0)\int_0^t g(t-\tau)\|u(t)\|_{2(\gamma+1)}^{\gamma+1}\|\nabla u(\tau)\|_2 d\tau + 2|(u'(t), h\cdot\nabla u(t))| \\
& + 2\int_0^t |g'(t-\tau)|\|u'(t)\|_2\|h\cdot\nabla u(\tau)\|_2 d\tau + 2|(au'(t), h\cdot\nabla z(t))| \\
& + \theta\int_0^t g(t-\tau)\|u(t)\|_{2(\gamma+1)}^{\gamma+1}\|u(\tau)\|_2 d\tau + \theta|(au'(t), z(t))| \\
& + \theta|(u'(t), u(t))| + \theta\int_0^t |g'(t-\tau)|\|u'(t)\|_2\|u(\tau)\|_2 d\tau.
\end{aligned}
\tag{3.28}
$$

Next, we analyze some terms on the right hand side in the above the inequality. Estimate for $I_5 := 2\int_0^t g(t-\tau)\|\nabla u(\tau)\|_2\|\nabla u(t)\|_2\, d\tau$. Considering Cauchy-Schwarz inequality and also employing the inequality $ab \leq \frac{1}{4\eta}a^2 + \eta b^2$, for an arbitrary $\eta > 0$, we obtain

$$
|I_5| \leq \frac{1}{2\eta}\|\nabla u(t)\|_2^2 + \eta\|g\|_{L^1(0,\infty)}\int_0^t g(t-\tau)\|\nabla u(\tau)\|_2^2 d\tau.
$$

Having in mind that $\|g\|_{L^1(0,\infty)} < 1$, the last inequality yields

$$
\begin{aligned}
|I_5| \leq & (2\eta)^{-1}\|\nabla u(t)\|_2^2 + 2\eta(g\diamond\nabla u)(t) + 2\eta\Big(\int_0^t g(s)ds\Big)\|\nabla u(t)\|_2^2 \\
\leq & (2\eta)^{-1}\|\nabla u(t)\|_2^2 + 4\eta e(t) + 2\eta\Big(\int_0^t g(s)ds\Big)\|\nabla u(t)\|_2^2.
\end{aligned}
\tag{3.29}
$$

Estimate for $I_6 := 2(\int_0^t g(t-\tau)\|\nabla u(\tau)\|_2 d\tau)^2$. Considering the Cauchy-Schwarz inequality it holds that

$$
|I_6| \leq \|g\|_{L^1(0,\infty)}\int_0^t g(t-\tau)\|\nabla u(\tau)\|_2^2 d\tau \leq 4(g\diamond\nabla u)(t) + 4\|\nabla u(t)\|_2^2 \tag{3.30}
$$

where in the last inequality we used $\|g\|_{L^1(0,\infty)} < 1$.
Estimate for $I_7 := 3R(x^0)\max_{x\in\overline{\Omega}} \|\nabla\psi(x)| \int_{Q_1\setminus Q_0} |\nabla z|^2 dx$.

Setting

$$A = 3R(x^0)\max_{x\in\overline{\Omega}} |\nabla\psi(x)| \tag{3.31}$$

and taking (3.16) into account, we obtain, as in (3.29) and (3.30), the estimate

$$\begin{aligned}|I_7| \leq &(3A + A(4\eta)^{-1})\|\nabla u(t)\|_2^2 + 4\eta A e(t) + 2A(g\diamond\nabla u)(t)\\ &+ 2A\eta(\int_o^t g(s)ds)\|\nabla u(t)\|_2^2,\end{aligned} \tag{3.32}$$

where $\eta > 0$ is an arbitrary number.
Estimate for $I_8 := 2A(\gamma+2)^{-1}\int_{Q_1\setminus Q_0} |u|^{\gamma+2} dx$. Considering $k_0 > 0$ such that $\|v\|_{\gamma+2} \leq k_0\|\nabla v\|_2$ for all $v \in H_0^1(\Omega)$, and taking (3.4) into account, we deduce

$$|I_8| \leq 2^{(\gamma+2)/2} A k_0^{\gamma+2}(\gamma+2)^{-1} l^{-\gamma/2}[E(0)]^{\gamma/2}\|\nabla u(t)\|_2^2. \tag{3.33}$$

Estimate for $I_9 := 2R(x^0)\int_0^t g(t-\tau)\|u(t)\|_{2(\gamma+1)}^{\gamma+1}\|\nabla u(\tau)\|_2 d\tau$. Considering $k_1 > 0$ such that $\|v\|_{2(\gamma+1)} \leq k_1\|\nabla v\|_2$ for all $v \in H_0^1(\Omega)$, making use of the inequality $ab \leq \eta a^2 + \frac{1}{4\eta}b^2$ for an arbitrary $\eta > 0$ and also the Cauchy-Schwarz one, we obtain

$$\begin{aligned}|I_9| \leq &k_1^{2(\gamma+1)} R^2(x^0) 2^\gamma \eta^{-1} l^{-\gamma}[E(0)]^\gamma\|\nabla u(t)\|_2^2 + 2\eta l^{-1} e(t)\\ &+ 2\eta(\int_o^t g(s)ds)\|\nabla u(t)\|_2^2.\end{aligned} \tag{3.34}$$

Estimate for $I_{10} := 2|(u'(t), h\cdot\nabla u(t))|$. We have

$$|I_{10}| \leq 4\eta e(t) + R^2(x^0)\eta^{-1}\|\nabla u(t)\|_2^2. \tag{3.35}$$

Estimate for $I_{11} := 2\int_0^t |g'(t-\tau)|\|u'(t)\|_2\|h\cdot\nabla u(\tau)\|_2 d\tau$. Analogously we have done before and taking the assumption (2.8) into account, we arrive at

$$|I_{11}| \leq 2\eta e(t) + 2\xi_1 R^2(x^0)\eta^{-1}(g\diamond\nabla u)(t) + 2\xi_1 R^2(x^0)\eta^{-1}\|\nabla u(t)\|_2^2. \tag{3.36}$$

Estimate for $I_{12} := 2|(au'(t), h\cdot\nabla z(t))|$. From Cauchy-Schwarz inequality, making use of the inequality $ab \leq \eta a^2 + \frac{1}{4\eta}b^2$ ($\eta > 0$ arbitrary) and considering (3.16), (3.17) and (3.18), we deduce

$$|I_{12}| \leq (2\eta)^{-1}\|a\|_{L^\infty(\Omega)}\|u'(t)\|_{L^2(Q_1\setminus Q_0)}^2 + 4\eta R(x^0)(6l^{-1}+3)e(t). \tag{3.37}$$

Estimate for $I_{13} := \theta\int_0^t g(t-\tau)\|u(t)\|_{2(\gamma+1)}^{\gamma+1}\|u(\tau)\|_2 d\tau$. Analogously, we have

$$I_{13} \leq \theta^2 k_1^{2(\gamma+1)} 2^\gamma l^{-\gamma}(4\eta)^{-1}\|\nabla u(t)\|_2^2 + 4\eta\lambda^2 e(t) + 2\eta\lambda^2\int_o^t g(s)ds\|\nabla u(t)\|_2^2, \tag{3.38}$$

where $\lambda > 0$ comes from the Poincaré inequality $\|v\|_2 \leq \lambda\|\nabla v\|_2$ for all $v \in H_0^1(\Omega)$.
Estimate for $I_{14} := \theta|(au'(t), z(t))|$. Similarly, we deduce

$$I_{14} \leq 4\eta\theta^2\lambda^2\|a\|_{L^\infty(\Omega)}e(t) + 3(4\eta)^{-1}\|\nabla u(t)\|_2^2. \tag{3.39}$$

Estimate for $I_{15} := \theta|(u'(t), u(t))|$. We have

$$|I_{15}| \leq 2\eta e(t) + \theta^2\lambda^2(4\eta)^{-1}\|\nabla u(t)\|_2^2. \tag{3.40}$$

Estimate for $I_{16} := \theta\int_0^t |g'(t-\tau)|\|u'(t)\|_2\|u(\tau)\|_2 d\tau$. It holds that

$$|I_{16}| \leq 2\eta e(t) + \theta^2\lambda^2\xi_1(2\eta)^{-1}(g \diamond \nabla u)(t) + \theta^2\lambda^2\xi^2(2\eta)^{-1}\|\nabla u(t)\|_2^2. \tag{3.41}$$

Combining (3.28)-(3.41),

$$\begin{aligned}\rho'(t) \leq &- [k_1 - \eta L]e(t)\\ &- [2^{-1}k_1 - 4\eta(1+A)](\int_o^t g(s)ds)\|\nabla u(t)\|_2^2 + M(\eta)(g \diamond \nabla u)(t)\\ &+ N(\eta)\|\nabla u(t)\|_2^2 + K(\eta)\|u'(t)\|_{L^2(\omega)}^2,\end{aligned} \tag{3.42}$$

where

$$\begin{aligned}L &= 10 + 2l^{-1} + 4R(x^0)(6l^{-1} + 3) + 4\lambda^2 + 4\theta\lambda^2\|a\|_{L^\infty(\Omega)},\\ M(\eta) &= 2^{-1}k_1 + 4 + 2A + 2\eta^{-1}\xi_1^2R^2(x^0) + \theta^2\lambda^2\xi_1^2(2\eta)^{-1},\\ N(\eta) =&3A + 4 + (\gamma+2)^{-1}2^{(\gamma+2)/2}k_0^{\gamma+2}l^{-\gamma/2}[E(0)]^{l/2} + \theta^2\lambda^2(4\eta)^{-1}\\ &+ \theta^2\lambda^2\xi_1^2(2\eta)^{-1} + (4\eta)^{-1}\Big(4(1+A)\\ &+ R^2(x^0)\big(4k_1^{2(\gamma+1)}2^\gamma l^{-\gamma}[E(0)]^\gamma + 2\xi_1^2 + 1\big) + \theta^2k_1^{2(\gamma+1)}2^\gamma l^{-\gamma}\Big),\\ K(\eta) &= (2\eta)^{-1} + 2\eta + 3^{-1}A.\end{aligned}$$

Choosing $\eta > 0$ sufficiently small such that

$$k_2 = k_1 - \eta L > 0 \quad \text{and} \quad 2^{-1}k_1 - 4\eta(1+A) \geq 0,$$

from (3.6), (3.8), (3.9), (3.42) and considering the assumption (2.8), we obtain

$$\begin{aligned}e_\varepsilon'(t) =&e'(t) + \varepsilon\rho'(t)\\ \leq &- \varepsilon k_2e(t) + \xi_1 2^{-1}\int_0^t g(s)ds\|\nabla u(t)\|_2^2 - (a_0 - \varepsilon K)\|u'(t)\|_{L^2(\omega)}^2 - (2^{-1}\xi_2\\ &- \varepsilon M)(g \diamond \nabla u)(t) - (2^{-1}g(0) - \varepsilon N)\|\nabla u(t)\|_2^2.\end{aligned} \tag{3.43}$$

Choosing $\varepsilon > 0$ small enough such that

$$a_0 - \varepsilon K \geq 0, \quad 2^{-1}g(0) - \varepsilon N \geq 0 \quad \text{and} \quad 2^{-1}\xi_2 - \varepsilon M \geq 0,$$

from (3.43) we deduce

$$e'_\varepsilon(t) \leq -\big(\varepsilon k_2 - \xi_1 \|g\|_{L^1(0,\infty)} l^{-1}\big) e(t). \tag{3.44}$$

For a fixed $\varepsilon > 0$ sufficiently small such that the Propositions 3.1 and 3.2 hold and considering $\|g\|_{L^1(0,\infty)}$ small enough, from (3.44) we conclude that $e'_\varepsilon(t) \leq -\delta_1 e(t)$ which completes the proof of Proposition 3.2.

Combining the Propositions 3.1 and 3.2, we deduce the exponential rate of decay.

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