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# Resolvent kernel for the Kohn Laplacian on Heisenberg groups *

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**Abstract**

We present a formula that relates the Kohn Laplacian on Heisenberg groups and the magnetic Laplacian. Then we obtain the resolvent kernel for the Kohn Laplacian and find its spectral density. We conclude by obtaining the Green kernel for fractional powers of the Kohn Laplacian.

## 1 Introduction

The Heisenberg group can be described as the set $H_n = \mathbb{R}^{2n} \times \mathbb{R}$ equipped with the group law

$$(x, y, t)(x', y', t') = (x + x', y + y', t + t' + 2(x'y - y'x))\,.$$

Its infinitesimal generators are the left invariant vector fields

$$X_k = \frac{\partial}{\partial x_k} + 2y_k \frac{\partial}{\partial t}, \quad Y_k = \frac{\partial}{\partial y_k} - 2x_k \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}$$

which satisfy the canonical relations $[Y_j, X_k] = 4\delta_{jk}T$. The self-adjoint operators $\frac{1}{2i}X_k$ and $\frac{1}{2i}Y_k$ correspond to the position and momentum operators in quantum mechanics. One also considers the following combinations of $X_k$ and $Y_k$:

$$Z_k = \frac{1}{2}(X_k - iY_k) = \frac{\partial}{\partial z_k} + i\overline{z}_k \frac{\partial}{\partial t}, \quad \overline{Z}_k = \frac{1}{2}(X_k + iY_k) = \frac{\partial}{\partial \overline{z}_k} - iz_k \frac{\partial}{\partial t}$$

where $z_k = x_k + iy_k$. The operators $Z_k, \overline{Z}_k$ are related to the creation or annihilation operators. The model for the Kohn Laplacian on the Heisenberg group is

$$\Box_b = -\sum_{k=1}^{n} (Z_k \overline{Z}_k + \overline{Z}_k Z_k)$$

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which can be written as

$$\Box_b = -2\sum_{k=1}^{n}\frac{\partial^2}{\partial z_k\partial\overline{z}_k} - 2i\sum_{k=1}^{n}(\overline{z}_k\frac{\partial}{\partial\overline{z}_k} - z_k\frac{\partial}{\partial z_k})\frac{\partial}{\partial t} - 2\sum_{k=1}^{n}|z_k|^2\frac{\partial^2}{\partial t^2}. \tag{1.1}$$

This operator with $D(\Box_b) = C_0^\infty(\mathbb{H}_n, \mathbb{C})$, the space of complex-valued $C^\infty$-functions with a compact support in $\mathbb{H}_n$, as its natural regular domain in the Hilbert space $X = L^2(\mathbb{H}_n, d\mu \otimes dt)$, is essentially self-adjoint. Here $d\mu$ and $dt$ denote respectively the Lebesgue measure on $\mathbb{C}^n$ and $\mathbb{R}$. It should be noted that the operator $\Box_b$ is subelliptic but not elliptic [6, p.374]. Its spectrum in $X$ is the set $[0, +\infty[$. As it will be shown in lemma 3.1 below, the operator $\Box_b$ is connected to the magnetic Laplacian

$$\widetilde{\Delta} = -\sum_{j=1}^{n}\frac{\partial^2}{\partial z_j\partial\overline{z}_j} + \sum_{j=1}^{n}\overline{z}_j\frac{\partial}{\partial\overline{z}_j}, \tag{1.2}$$

acting on the Hilbert space $L^2(\mathbb{C}^n, e^{-|z|^2}d\mu)$. Here $|z|^2 = \langle z, z\rangle$ denotes the Euclidean norm square and $\langle z, w\rangle = \sum_{j=1}^{n} z_j\overline{w}_j$ the Hermitian product of $\mathbb{C}^n$.

This paper is organized as follows. In section 2 we set some notation for special functions and state some formulas. In section 3, we give the resolvent of the Kohn Laplacian $\Box_b$. In section 4, we obtain the Green kernel for fractional powers of $\Box_b^\alpha$. In section 5, we give the spectral density of $\Box_b$. Section 6 will be the Appendix.

## 2 Notation and formulas

Here we list some special functions and formulas to be used later. The reader can proceed to section 3 and refer back to this section as necessary.

The confluent hypergeometric function [10, p.204] is denoted by

$${}_1F_1(a, c, x) = \frac{\Gamma(c)}{\Gamma(a)}\sum_{j=1}^{\infty}\frac{\Gamma(a+j)}{\Gamma(c+j)}\frac{x^j}{j!}.$$

As in [10, p.264], we define the function

$$G(a, c, x) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\ {}_1F_1(a, c; x) + \frac{\Gamma(c-1)}{\Gamma(a)}x^{1-c}\ {}_1F_1(a-c+1, 2-c, x) \tag{2.1}$$

Note note that at $c = m \in \mathbb{Z}_+$, the second term of (2.1) has a removable singularity. More precisely,

$$\lim_{c\to m} G(a, c, x) = \frac{(-1)^m}{(m-1)!\Gamma(c-m+1)}\Big\{\sum_{k=1}^{m-1}\frac{(-1)^{k-1}(k-1)!}{(a-k)_k(m-k)_k}x^{-k} + \sum_{k=0}^{\infty}\frac{(a)_k x^k}{k!(m)_k}[\ln x + \Psi(a+k) - \Psi(m+k) - \Psi(k+1)]\Big\}, \tag{2.2}$$

where $\Psi(x)$ is the logarithmic derivative of $\Gamma(x)$ and $(x)_j =: x(x+1)\cdots(x+j-1)$. For $m=1$, one makes the convention that the first series in the second term of (2.2) vanishes [10, p.213].

The Whittaker function [10, p.225] is denoted by

$$W_{\beta,\alpha}(x) = x^{\alpha+\frac{1}{2}}e^{-\frac{x}{2}}G(\alpha-\beta+\frac{1}{2}, 2\alpha+1, x) \tag{2.3}$$

The Macdonald functions [10, p.159] are defined as

$$K_\nu(x) = \frac{1}{2}\frac{\pi}{\sin\pi\nu}(I_{-\nu}(x) - I_\nu(x)), \tag{2.4}$$

where $I_\nu(x)$ is the modified Bessel function of indice $\nu$ given by the series

$$I_\nu(x) = \sum_{m=0}^{\infty}\frac{(\frac{1}{2}x)^{\nu+2m}}{m!\Gamma(\nu+m+1)}. \tag{2.5}$$

We have also the following formulas for $\nu \notin \mathbb{Z}_+$:

$$K_\nu(x) = \frac{\pi}{2\sin\nu\pi}\Big\{\sum_{m=0}^{\infty}\frac{(\frac{1}{2}x)^{-\nu+2m}}{m!\Gamma(-\nu+m+1)} - \sum_{m=0}^{\infty}\frac{(\frac{1}{2}x)^{\nu+2m}}{m!\Gamma(\nu+m+1)}\Big\}, \tag{2.6}$$

$$\begin{aligned} K_\nu(x) = (-1)^{\nu+1}I_\nu(x)\ln(\frac{x}{2}) + \frac{1}{2}\sum_{k=0}^{\nu-1}\frac{(-1)^k(\nu-k-1)!}{k!}(\frac{x}{2})^{2k-\nu} \\ + (-1)^\nu\sum_{k=0}^{\infty}\frac{(\frac{x}{2})^{2k+\nu}}{k!(k+\nu)!}[\Psi(\nu+k+1) - \Psi(k+1)]. \end{aligned} \tag{2.7}$$

For $\nu = 0$, we assume that the first series in (2.7) vanishes, see [10, p.159]. The Laguerre polynomials [7, p.1037] are defined as

$$L_k^m(x) = \sum_{j=0}^{k}(-1)^k\binom{k+m}{k-j}\frac{x^j}{j!}. \tag{2.8}$$

# 3 The resolvent of the operator $\Box_b$

First, we establish the connection between the Kohn Laplacian $\Box_b$ and the operator $\widetilde{\Delta}$ given in (1.1). For this, we denote by $\mathcal{F}$ the map from $L^2(\mathbb{H}_n, d\mu\otimes dt)$ to $L^2(\mathbb{C}^n, d\mu)$ given by the partial Fourier transform with respect to the variable $t$:

$$\mathcal{F}\varphi(z,\lambda) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\varphi(z,t)e^{-i\lambda t}dt = \lim_{\rho\to+\infty}\int_{|t|\le\rho}\varphi(z,t)e^{-i\lambda t}dt.$$

considered as the limit in the $L^2(\mathbb{R})$-norm sense.

**Lemma 3.1** *For $\lambda > 0$, the operators $\Box_b$ and $\widetilde{\Delta}$ are related by*

$$T_\lambda \circ (\frac{1}{2}\mathcal{F} \circ \Box_b - n\lambda) \circ T_\lambda^{-1} = 2\lambda\widetilde{\Delta}, \tag{3.1}$$

*where $T_\lambda : L^2(\mathbb{C}^n, d\mu) \to\ L^2(\mathbb{C}^n, e^{-|z|^2}d\mu)$ is the map*

$$T_\lambda f(z) = 2^{-n}\lambda^{-n}e^{\frac{1}{2}|z|^2}f(\frac{z}{\sqrt{2\lambda}}).$$

**Proof** Let $f(z,t) \in D(\Box_b) = C_0^\infty(\mathbb{H}_n)$ and let $g(z,t) = \Box_b f(z,t)$. i.e.,

$$g(z,t) = [-2\sum_{k=1}^n \frac{\partial^2}{\partial z_k \partial \overline{z}_k} - 2i\sum_{k=1}^n (\overline{z}_k \frac{\partial}{\partial \overline{z}_k} - z_k \frac{\partial}{\partial z_k})\frac{\partial}{\partial t} - 2\sum_{k=1}^n |z_k|^2 \frac{\partial^2}{\partial t^2}]f(z,t). \tag{3.2}$$

Then, applying $\mathcal{F}$ to both sides of this equation and using the property

$$\mathcal{F}(\frac{\partial^\alpha f}{\partial t^\alpha})(z,\lambda) = (i\lambda)^\alpha \mathcal{F}(f)(z,\lambda),$$

equation (3.2) becomes

$$\mathcal{F}g(z,\lambda) = 2[-\sum_{k=1}^n \frac{\partial^2}{\partial z_k \partial \overline{z}_k} + \lambda\sum_{k=1}^n (\overline{z}_k \frac{\partial}{\partial \overline{z}_k} - z_k \frac{\partial}{\partial z_k}) + \lambda^2 \sum_{k=1}^n |z_k|^2]\mathcal{F}f(z,\lambda).$$

Denoting by $\mathcal{F} \circ \Box_b$ the operator

$$2\Big[-\sum_{k=1}^n \frac{\partial^2}{\partial z_k \partial \overline{z}_k} + \lambda\sum_{k=1}^n (\overline{z}_k \frac{\partial}{\partial \overline{z}_k} - z_k \frac{\partial}{\partial z_k}) + \lambda^2 \sum_{k=1}^n |z_k|^2\Big],$$

formula (3.1) follows by a direct computation. $\diamondsuit$

**Theorem 3.2** *Let $\xi \in \mathbb{C}$ such that $\mathrm{Re}(\xi) < 0$. Then, the resolvent operator $(\xi - \Box_b)^{-1}$ is given by*

$$(\xi - \Box_b)^{-1} f(z,t) = \int_{\mathbb{H}_n} R(\xi; (z,t), (w,s)) f(w,s) d\mu(w) \otimes ds$$

*where*

$$R(\xi, (z,t), (w,s)) = \frac{-2^{n-1}}{\pi^{n+\frac{1}{2}}} \int_0^\infty x^{n-1}\Gamma(\frac{-\xi}{4x} + \frac{n}{2}) G(\frac{-\xi}{4x} + \frac{n}{2}, n, 2x|z-w|^2) \times e^{-x|z-w|^2} \cos\big(x(t-s) + 2x\,\mathrm{Im}\langle z,w\rangle\big) dx \tag{3.3}$$

**Proof** Let $\xi \in \mathbb{C}$ be such that $\mathrm{Re}(\xi) < 0$ and $f \in C_0^\infty(\mathbb{H}_n, \mathbb{C})$. Then we need to solve the equation

$$(\xi - \Box_b)g(z,t) = f(z,t). \tag{3.4}$$

Applying the map $\mathcal{F}$ to both sides of this equation, we obtain

$$(\xi - \mathcal{F} \circ \Box_b)\mathcal{F}g(z,\lambda) = \mathcal{F}f(z,\lambda) \tag{3.5}$$

where $\lambda$ is the dual variable of $t$. Because of lemma 3.1, for $\lambda > 0$, equation (3.4) becomes

$$T_\lambda^{-1} \circ 4\lambda(\frac{\xi}{4\lambda} - \frac{n}{2} - \widetilde{\Delta}) \circ T_\lambda \mathcal{F}g(z,\lambda) = \mathcal{F}f(z,\lambda). \tag{3.6}$$

Now, for $\mathrm{Re}(\xi) < 0$, the operator $\frac{\xi}{4\lambda} - \frac{n}{2} - \widetilde{\Delta}$ is invertible because $\frac{\xi}{4\lambda} - \frac{n}{2}$ does not belong to the set $\mathbb{Z}_+$ which is the spectrum of $\widetilde{\Delta}$. Thus, we can write

$$\mathcal{F}g(z,\lambda) = T_\lambda^{-1} \circ (4\lambda)^{-1}(\frac{\xi}{4\lambda} - \frac{n}{2} - \widetilde{\Delta})^{-1} \circ T_\lambda \mathcal{F}f(z,\lambda). \tag{3.7}$$

On the other hand, for $\zeta \in \mathbb{C}\backslash\mathbb{Z}_+$, we have the following formula, [1, p.6938],

$$(\zeta - \widetilde{\Delta})^{-1}f(z) = -\pi^{-n}\Gamma(-\zeta)\int_{\mathbb{C}^n} e^{\langle z,w\rangle} G(-\zeta, n, |z-w|^2) f(w) e^{-|w|^2} d\mu(w). \tag{3.8}$$

Next, after a computation using (3.8), equation (3.7) becomes

$$\begin{aligned}\mathcal{F}g(z,\lambda) = -\pi^{-n}2^{n-2}\lambda^{n-1}\Gamma(\frac{-\xi}{4\lambda} + \frac{n}{2})\int_{\mathbb{C}^n} G(\frac{-\xi}{4\lambda} + \frac{n}{2}, n, 2\lambda|z-w|^2) \\ \times e^{2\lambda\langle z,w\rangle - \lambda|z|^2 - \lambda|w|^2}\mathcal{F}f(w,\lambda)d\mu(w).\end{aligned} \tag{3.9}$$

For the case $\lambda < 0$, it suffices to change the variable $z$ to $i\overline{z}$ in (3.5). This yields (3.5) again with $-\lambda$ instead of $\lambda$. Hence, in view of (3.9), we get

$$\begin{aligned}\mathcal{F}g(z,\lambda) = -\pi^{-n}2^{n-2}(-\lambda)^{n-1}\Gamma(\frac{\xi}{4\lambda} + \frac{n}{2})\int_{\mathbb{C}^n} G(\frac{\xi}{4\lambda} + \frac{n}{2}, n, -2\lambda|z-w|^2) \\ \times e^{-2\lambda\langle w,z\rangle + \lambda|z|^2 + \lambda|w|^2}\mathcal{F}f(w,\lambda)d\mu(w).\end{aligned} \tag{3.10}$$

In summary, for $\lambda \neq 0$, we obtain

$$\mathcal{F}g(z,\lambda) = \int_{\mathbb{C}^n} Q_\lambda(\xi; z, w)\mathcal{F}f(w,\lambda)d\mu(w), \tag{3.11}$$

where

$$\begin{aligned}Q_\lambda(\xi; z, w) = -\pi^{-n}2^{n-2}|\lambda|^{n-1}\Gamma(\frac{-\xi}{4|\lambda|} + \frac{n}{2})G(\frac{-\xi}{4|\lambda|} + \frac{n}{2}, n, 2|\lambda||z-w|^2) \\ \times e^{2|\lambda|\langle z,w\rangle - |\lambda||w|^2 - |\lambda||z|^2}.\end{aligned}$$

Now, by the Parseval-Plancherel theorem [11, p.39], from (3.11) we get the integral representation

$$\begin{aligned} g(z,t) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{i\lambda t} \Big( \int_{\mathbb{C}^n} Q_\lambda(\xi; z, w) \mathcal{F} f(w,\lambda) d\mu(w) \Big) d\lambda \\ &= \int_{\mathbb{C}^n} \Big( \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{i\lambda t} Q_\lambda(\xi, z, w) \mathcal{F} f(w,\lambda) d\lambda \Big) d\mu(w). \end{aligned} \tag{3.12}$$

On the other hand, square integrability of the function $\lambda \to e^{i\lambda t} Q_\lambda(\xi; z, w)$ (see section 6 below) enables us to write

$$\begin{aligned} &\int_{-\infty}^{\infty} e^{i\lambda t} Q_\lambda(\xi, z, w) \mathcal{F} f(w,\lambda) d\lambda \\ &= \int_{-\infty}^{\infty} \Big( \int_{-\infty}^{\infty} e^{is(t-\lambda)} Q_s(\xi, z, w) ds) f(w,\lambda) \Big) d\lambda \end{aligned} \tag{3.13}$$

(see [11, p.49] for the general theory). Hence, by (3.12) and (3.13) one can write

$$g(z,t) = \int_{\mathbb{C}^n} R(\xi; (z,t), (w,\lambda)) f(w,\lambda) d\mu(w) \otimes d\lambda$$

where

$$R(\xi; (z,t), (w,\lambda)) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{i(t-\lambda)} Q_s(\xi; z, w) ds.$$

Finally, a direct computation yields

$$\begin{aligned} R(\xi; (z,t), (w,\lambda)) = \frac{-2^{n-1}}{\pi^{n+\frac{1}{2}}} \int_0^\infty & s^{n-1} \Gamma(\frac{-\xi}{4s} + \frac{n}{2}) G(\frac{-\xi}{4s} + \frac{n}{2}, n, 2s|z-w|^2) \\ & \times e^{-s|z-w|^2} \cos(s(t-\lambda) + 2s \operatorname{Im}\langle z, w\rangle) ds. \end{aligned}$$

This completes the proof of theorem 3.2. ♢

**Remark 3.3** Considering the limit value $\xi = 0$ in (3.3), we obtain the kernel function

$$\begin{aligned} R_0((z,t),(w,\lambda)) = \frac{-2^{n-1}\Gamma(\frac{n}{2})}{\pi^{n+\frac{1}{2}}} \int_0^\infty & s^{n-1} \Gamma(\frac{n}{2}) G(\frac{n}{2}, n, 2s|z-w|^2) \\ & \times e^{-s|z-w|^2} \cos(s(t-\lambda) + 2s \operatorname{Im}\langle z, w\rangle) ds. \end{aligned} \tag{3.14}$$

which corresponds to a right inverse of $\Box_b$. That is,

$$\Box_b^{-1} f(z,t) = \int_{\mathbb{H}_n} -R_0((z,t),(w,s)) f(w,\lambda) d\mu(w) \otimes d\lambda\,.$$

In other words, $-R_0((z,t),(w,\lambda))$ is a Green kernel of $\Box_b$. Another Green kernel for $\Box_b$ is given by

$$F = \frac{1}{2} c_n \rho^{-2n}$$

where $\rho = (t^2 + |z|^4)^{\frac{-1}{4}}$ and $c_n = n(n+2)\int_{\mathbb{H}_n} |z|^2(1+\rho(z,t)^4)^{\frac{-(n+4)}{2}} d\mu(z)dt$, which was obtained by Folland using an analogous fact to $\|x\|^{2-n}$ being (a constant multiple of) the fundamental solution of the Laplacian on $R^n$ with source at 0, see [6, theorem 2, p.375].

# 4 Green kernel for fractional powers of $\Box_b$

As application of the formula obtained for the resolvent kernel of $\Box_b$, we give the Green kernel of the fractional power operator $\Box_b^\alpha$ for $\alpha \in ]0,1[$.

**Proposition 4.1** *Let $\alpha \in ]0,1[$. Then the Green kernel of the fractional power operator $\Box_b^\alpha$ is*

$$\begin{aligned} H_\alpha((z,t),(w,s)) = &-\frac{4^{1-\alpha}\Gamma(1-\alpha+n)}{2\pi^{n+\frac{1}{2}}\Gamma(\alpha)}\int_0^\infty (\sinh\frac{\lambda}{2})^{-n}\lambda^{\alpha-1} \\ &\times ((|z-w|\coth\frac{\lambda}{2})^2 + t - s + 2\operatorname{Im}\langle z,w\rangle)^{-\frac{1-\alpha+n}{2}} \\ &\times \cos((n+1-\alpha)\arctan(\frac{t-s+2\operatorname{Im}\langle z,w\rangle}{|z-w|}\tanh\frac{\lambda}{2}))d\lambda. \end{aligned}$$

**Proof** Since $\Box_b$ is a positive self-adjoint operator, its resolvent [8, p.21] satisfies

$$\|R(-s)\| \le \frac{1}{s}. \tag{4.1}$$

This estimate enables us to define the fractional powers $\Box_b^\alpha, \alpha \in ]0,1[$ according to the formula, [8, p.127],

$$\Box_b^\alpha g = \frac{\sin\pi\alpha}{\pi}\int_0^{+\infty} s^{\alpha-1}R(-s)\Box_b g ds,\ g \in D(\Box_b).$$

Thanks to Kato's formula [8, pp.123-125], the resolvent operator $R_\alpha(\gamma) = (\gamma - \Box_b^\alpha)^{-1}$, $\alpha\pi < |\arg\gamma| \le \pi$ , is given by

$$R_\alpha(\gamma) = \frac{\sin\pi\alpha}{\pi}\int_0^\infty \frac{\xi^\alpha R(-\xi)}{\xi^{2\alpha} - 2\xi^\alpha\gamma\cos\pi\alpha + \gamma^2}d\xi. \tag{4.2}$$

The action of $R_\alpha(\gamma)$ on a function $f \in L^2(\mathbb{H}_n)$ is

$$R_\alpha(\gamma)f(z,t) = \frac{\sin\pi\alpha}{\pi}\int_0^\infty \frac{\xi^\alpha R(-\xi)f(z,t)}{\xi^{2\alpha} - 2\xi^\alpha\gamma\cos\pi\alpha + \gamma^2}d\xi,\ \text{almost every where.} \tag{4.3}$$

Then the resolvent kernel of $\Box_b^\alpha$ is

$$I_\alpha(\gamma;(z,t),(w,s)) = \frac{\sin\pi\alpha}{\pi}\int_0^\infty \frac{\xi^\alpha R(-\xi;(z,t),(w,s))}{\xi^{2\alpha} - 2\xi^\alpha\gamma\cos\pi\alpha + \gamma^2}d\xi. \tag{4.4}$$

The limit value $\gamma = 0$ in (4.4) gives a Green kernel of $\Box_b^\alpha$:

$$I_\alpha := I_\alpha(0;(z,t),(w,s)) = \frac{\sin\pi\alpha}{\pi}\int_0^\infty \xi^{-\alpha}R(-\xi,(z,t),(w,s))d\xi, \tag{4.5}$$

Using theorem 3.2, $I_\alpha$ in (4.5) becomes

$$\begin{aligned} I_\alpha = \frac{-2^{n-1}\sin\pi\alpha}{\pi^{n+\frac{3}{2}}}\int_0^\infty & x^{n-1}\cos(\tau x)e^{-x|z-w|^2} \\ & \times\Big(\int_0^\infty \xi^{-\alpha}\Gamma(\frac{\xi}{4x}+\frac{n}{2})G(\frac{\xi}{4x}+\frac{n}{2},n;2x\rho)d\xi\Big)dx, \end{aligned} \tag{4.6}$$

where

$$\tau = t - s + 2\operatorname{Im}\langle z,w\rangle, \quad \rho = |z-w|^2. \tag{4.7}$$

Making the change of variable $u = \frac{1}{4x}\xi$, equation (4.6) can be rewritten as

$$I_\alpha = \frac{-2^{n-1}4^{-\alpha+1}\sin\pi\alpha}{\pi^{n+\frac{3}{2}}}\int_0^\infty x^{n-\alpha}\cos(\tau x)e^{-x\rho}N_{\alpha,n,\rho}(x)dx, \tag{4.8}$$

where

$$N_{\alpha,n,\rho}(x) = \int_0^\infty u^{-\alpha}\Gamma(u+\frac{n}{2})G(u+\frac{n}{2},n;2x\rho)du. \tag{4.9}$$

The formula

$$\Gamma(a)G(a,c,x) = \int_0^\infty (1-e^{-\lambda})^{-c}\exp(\frac{-x}{e^\lambda - 1})e^{-\lambda a}d\lambda$$

is obtained by combining (2.3) with the integral representation

$$\Gamma(s-\nu+1)e^{\frac{1}{2}\alpha}\alpha^{\frac{1}{2}\nu-\frac{1}{2}}W_{\frac{1}{2}\nu-\frac{1}{2}-s,\frac{1}{2}\nu} = \int_0^\infty e^{-st}(e^t-1)^{\nu-1}\exp(\frac{-\alpha}{e^t-1})dt$$

(see [4, p.146]. Then, the integral (4.9) becomes

$$\begin{aligned} N_{\alpha,n,\rho}(x) &= \int_0^\infty u^{-\alpha}\Big(\int_0^\infty \frac{1}{(1-e^{-\lambda})^n}\exp(\frac{2\rho x}{1-e^\lambda})e^{-\lambda u}e^{-\frac{n}{2}\lambda}d\lambda\Big)du \\ &= \int_0^\infty \frac{e^{-\frac{n}{2}\lambda}}{(1-e^{-\lambda})^n}\exp(\frac{2\rho}{1-e^\lambda}x)\Big(\int_0^\infty u^{-\alpha}e^{-\lambda u}du\Big)d\lambda \\ &= \int_0^\infty \frac{2^{-n}\Gamma(1-\alpha)}{(\sinh\frac{\lambda}{2})^n}\exp(\frac{2\rho}{1-e^\lambda}x)\lambda^{\alpha-1}d\lambda. \end{aligned}$$

So, (4.8) becomes

$$\begin{aligned} I_\alpha =& \frac{-2^{1-2\alpha}}{\pi^{n+\frac{1}{2}}\Gamma(\alpha)}\int_0^\infty x^{n-\alpha}e^{-x\rho}\cos\tau x \\ & \times\Big(\int_0^\infty \frac{1}{(\sinh\frac{\lambda}{2})^n}\exp(\frac{2\rho}{1-e^\lambda}x)\lambda^{-1+\alpha}d\lambda\Big)dx \\ =& \frac{-2^{1-2\alpha}}{\pi^{n+\frac{1}{2}}\Gamma(\alpha)}\int_0^\infty \frac{1}{(\sinh\frac{\lambda}{2})^n}\lambda^{\alpha-1}\Big(\int_0^\infty x^{n-\alpha}\exp(-x\rho\coth\frac{\lambda}{2})\cos\tau x dx\Big)d\lambda. \end{aligned} \tag{4.10}$$

By using the formula

$$\int_0^\infty x^{\nu-1}e^{-ax}\cos(xy)dx = \Gamma(\nu)(a^2+y^2)^{-\frac{\nu}{2}}\cos(\nu\arctan(\frac{y}{a}))$$

(see [7], p. 490) for $\nu = 1-\alpha+n$, $a = \rho\coth\frac{t}{2}$ and $y = \tau$, the fundamental solution $I_\alpha$ in (4.10) becomes

$$I_\alpha = -\frac{4^{1-\alpha}\Gamma(1-\alpha+n)}{2\pi^{n+\frac{1}{2}}\Gamma(\alpha)}\int_0^\infty (\sinh\frac{\lambda}{2})^{-n}\lambda^{\alpha-1}((\rho\coth\frac{\lambda}{2})^2+\tau^2)^{-\frac{1-\alpha+n}{2}} \times\cos\big((n+1-\alpha)\arctan(\frac{\tau}{\rho}\tanh\frac{\lambda}{2})\big)d\lambda.$$

So, the proof of proposition 4.1 is complete. $\diamondsuit$

**Remark 4.2** Proposition 4.1 extends the result obtained by Benson et al in [3, p.457] by providing the Green kernel of powers $\Box_b^p$ with $1 \le p \le n, p \in \mathbb{Z}_+$.

# 5 Spectral density of the operator $\Box_b$

In this section, we give the spectral density of the Kohn Laplacian $\Box_b$. The extension of $\Box_b$ given by its adjoint will be also denoted by $\Box_b$. The domain of the extension $\Box_b$ will be denoted by $\chi$. This extension $\Box_b$ admits a spectral decomposition $\{E_\lambda\}_\lambda$. The $\{E_\lambda\}_\lambda$ is an increasing family of projectors that satisfy

$$I = \int_{-\infty}^\infty dE_\lambda$$

where $I$ is the identity operator, and $\Box_b = \int_{-\infty}^\infty \lambda dE_\lambda$ in the weak sense; that is,

$$(\Box_b f, g) = \int_{-\infty}^\infty \lambda d(E_\lambda f, g)$$

for $f \in \chi$ and $g \in L^2(\mathbb{H}_n)$, where $(f,g)$ is the inner product of $L^2(\mathbb{H}_n)$. The spectral density

$$e_\lambda = \frac{dE_\lambda}{d\lambda}$$

is understood as an operator-valued distribution; i.e., an element of the space $\mathcal{D}'(\mathbf{R}, L(\chi, L^2(\mathbb{H}_n)))$. Here $L(\chi, L^2(\mathbb{H}_n))$ is the space of bounded operators from $\chi$ to $L^2(\mathbb{H}_n)$.

**Proposition 5.1** *The spectral density $e_\lambda = \frac{dE_\lambda}{d\lambda}$ of $\Box_b$ is the operator valued distribution $\varphi \to \langle e_\lambda, \varphi\rangle$ from $\mathcal{D}(\mathbb{R})$ to $L(\chi, L^2(\mathbb{H}_n))$ given by*

$$\langle e_\lambda, \varphi\rangle f(z,t) = \int_{H_n}\Big[\int_0^\infty e(\lambda,(z,t),(w,s))\varphi(\lambda)d\lambda\Big]f(w,s)d\mu(w)\otimes ds$$

*where*

$$e(\lambda,(z,t),(w,s)) = -\frac{1}{\pi^{n+\frac12}}\sum_{j=0}^{\infty} L_j^{n-1}\big(\frac{\lambda|z-w|^2}{(2j+n)}\big)\cos\big(\frac{\lambda(t-s)+2\operatorname{Im}\langle z,w\rangle}{2(2j+n)}\big) \times \frac{\lambda^n}{(2j+n)^{n+1}}e^{-\frac{\lambda|z-w|^2}{2(2j+n)}} \quad (5.1)$$

**Proof** We can write the resolvent kernel of $\Box_b$, given in (3.3), as

$$\begin{aligned} &R(\xi;(z,t),(w,s)) \\ &= \frac{-2^{n-1}}{\pi^{n+\frac12}}\int_0^\infty x^{n-1}\Gamma(\eta(x))G(\eta(x),n,2x|z-w|^2)e^{-x\rho}\cos(x\tau)dx \end{aligned} \quad (5.2)$$

where $\rho$ , $\tau$ are defined in (4.7) and $\eta(x) = \frac{-\xi}{4x} + \frac{n}{2}$. Next, using the summation formula

$$\Gamma(a)G(a,c,y) = \sum_{j=0}^{\infty}\frac{1}{j+a}L_j^{c-1}(y)$$

(understood in the distributional sense; [1, theroem 3.1]) for $a = \eta(x)$, $c = n$ and $y = 2x\rho$, we get successively

$$\begin{aligned} &R(\xi;(z,t),(w,s)) \\ &= \frac{-2^{n-1}}{\pi^{n+\frac12}}\sum_{j=0}^{\infty}\int_0^\infty x^{n-1}\frac{1}{j+\eta(x)}L_j^{n-1}(2x\rho)e^{-x\rho}\cos(x\tau)dx \\ &= \frac{-2^{n-1}}{\pi^{n+\frac12}}\sum_{j=0}^{\infty}\int_0^\infty \frac{2^{-n+1}\lambda^n}{(2j+n)^{n+1}}L_j^{n-1}\big(\frac{\lambda}{2j+n}\rho\big)e^{\frac{-\rho\lambda}{2(2j+n)}}\cos\big(\frac{\tau\lambda}{2(2j+n)}\big)\frac{d\lambda}{\lambda-\xi} \\ &= \frac{-1}{\pi^{n+\frac12}}\int_0^\infty \Big[\sum_{j=0}^{\infty}\frac{\lambda^n}{(2j+n)^{n+1}}L_j^{n-1}\big(\frac{\lambda}{2j+n}\rho\big)e^{\frac{-\rho\lambda}{2(2j+n)}}\cos\big(\frac{\tau\lambda}{2(2j+n)}\big)\Big]\frac{d\lambda}{\lambda-\xi}. \end{aligned} \quad (5.3)$$

Setting

$$\begin{aligned} &e((\lambda,(z,t),(w,s))) \\ &= \frac{-1}{\pi^{n+\frac12}}\sum_{j=0}^{\infty}\frac{\lambda^n}{(2j+n)^{n+1}}L_j^{n-1}\big(\frac{\lambda}{2j+n}\rho\big)e^{\frac{-\rho\lambda}{2(2j+n)}}\cos\big(\frac{\tau\lambda}{2(2j+n)}\big) \end{aligned} \quad (5.4)$$

Then, we can write

$$R(\xi;(z,t),(w,s)) = \int_0^\infty \frac{e(\lambda,(z,t),(w,s))}{\xi-\lambda}d\lambda \quad (5.5)$$

The convergence of the series given in (5.4) can be seen as follows. For $j$ sufficiently large, we have

$$\begin{aligned}&|\frac{\lambda^n}{(2j+n)^{n+1}}L_j^{n-1}(\frac{\lambda}{2j+n}\rho)e^{\frac{-\rho\lambda}{2(2j+n)}}\cos(\frac{\tau\lambda}{2(2j+n)})|\\&\qquad\qquad\le c(n;\lambda,\tau,\rho)j^{-n-1}L_j^{n-1}(\frac{\lambda}{2j+n}\rho),\end{aligned}$$

where $c(n;\lambda,\tau,\rho)$ is a positive constant. And by using the asymptotic formula

$$L_j^{n-1}(\frac{\rho\lambda}{2j+1})=O(j^{n-1})$$

(see, [9], p.248) the inequality (5.7) becomes

$$|\frac{\lambda^n}{(2j+n)^{n+1}}L_j^{n-1}(\frac{\lambda}{2j+n}\rho)e^{\frac{-\rho\lambda}{2(2j+n)}}\cos(\frac{\tau\lambda}{2(2j+n)})|\le c(n;\lambda,\tau,\rho)\frac{1}{j^2}.$$

Now, in view of (5.6) for $\xi\in\mathbb{C}\backslash\mathbb{R}$ and $f,g\in L^2(\mathbb{H}_n)$, we have

$$\begin{aligned}&(R(\xi)f,g)\\&=\int_0^\infty\frac{d\lambda}{\xi-\lambda}\int_{\mathbb{H}_n}[\int_{\mathbb{H}_n}e(\lambda,(z,t),(w,s))f(w,s)d\mu(w)\otimes ds]\overline{g(z,t)}d\mu(z)\otimes dt\,.\end{aligned}$$

On the other hand, recalling the formula, [2, p. 134],

$$(R(\xi)f,g)=\int_0^\infty\frac{1}{\xi-\lambda}d(E_\lambda f,g).$$

By uniqueness of the spectral measure, we get

$$\frac{d(E_\lambda f,g)}{d\lambda}=(K_\lambda f,g),$$

where $K_\lambda$ the operator

$$K_\lambda f(z,t)=\int_{\mathbb{H}_n}e(\lambda,(z,t),(w,s))f(w,s)d\mu(w)ds,\ \ f\in L^2(\mathbb{H}_n)\,.$$

Finally, an interpretation argument of the operator $K_\lambda$ in terms of the operators valued distribution see [5, p.9] completes the proof. $\diamondsuit$

# 6 Appendix

**Proposition 6.1** *The function $\lambda\to e^{i\lambda t}Q_\lambda(\xi;z,w)$ belongs to $L^2(\mathbb{R},d\lambda)$, where*

$$\begin{aligned}Q_\lambda(\xi;z,w)=-\pi^{-n}2^{n-2}|\lambda|^{n-1}\Gamma(\frac{-\xi}{4|\lambda|}+\frac{n}{2})G(\frac{-\xi}{4|\lambda|}+\frac{n}{2},n,2|\lambda||z-w|^2)\\\times e^{2|\lambda|\langle z,w\rangle-|\lambda||w|^2-|\lambda||z|^2}.\end{aligned}$$

To prove this proposition, we need the following lemma.

**Lemma 6.2** *For $\lambda \in \mathbb{R}$, $\mathrm{Re}(\xi) < 0$, and $w \neq z$, we have the following estimates:*
*(i) For $|\lambda| > 0$,*

$$|Q_\lambda(\xi; z, w)| \leq \pi^{-n-\frac{1}{2}} 2^{\frac{n}{2}-\frac{3}{2}} |z-w|^{1-n} |\lambda|^{\frac{n-1}{2}} K_{\frac{n-1}{2}}(|\lambda||z-w|^2).$$

*(ii) For $|\lambda| > \frac{-\mathrm{Re}(\xi)}{4}$,*

$$|Q_\lambda(\xi; z, w)| \leq \pi^{-n} 2^{n-2} |\lambda|^{n-1} C_n(\xi, |z-w|^2) e^{-t|\lambda||z-w|^2}.$$

**Proof** Let $\mathrm{Re}(\xi) < 0$ and

$$S(\lambda) = \Gamma(\frac{-\xi}{4|\lambda|} + \frac{n}{2}) G(\frac{-\xi}{4|\lambda|} + \frac{n}{2}, n, 2|\lambda||z-w|^2).$$

This function can be expressed using the kernel $Q_\lambda(\xi; z, w)$ as

$$S(\lambda) = -\pi^n 2^{2-n} |\lambda|^{1-n} e^{-2|\lambda|\langle z,w\rangle + |\lambda||w|^2 + |\lambda||z|^2} Q_\lambda(\xi; z, w). \tag{6.1}$$

Using the integral representation

$$G(a, c, x) = \frac{1}{\Gamma(a)} \int_0^{+\infty} t^{a-1}(1+t)^{c-a-1} e^{-tx} dt, \mathrm{Re}(a) > 0, \ \ \mathrm{Re}(x) > 0$$

[9, p.277], $S(\lambda)$ can be written as

$$S(\lambda) = \int_0^\infty t^{\frac{-\xi}{4|\lambda|} + \frac{n}{2} - 1} (1+t)^{\frac{\xi}{4|\lambda|} + \frac{n}{2} - 1} e^{-2|\lambda|t|z-w|^2} dt, \quad w \neq z\,.$$

Then we have the estimate

$$|S(\lambda)| \leq \int_0^\infty \phi_{\xi,\lambda,n}(t) dt,$$

where

$$\phi_{\xi,\lambda,n}(t) = \exp((\frac{-\mathrm{Re}(\xi)}{4|\lambda|} + \frac{n}{2} - 1)\ln(t) + (\frac{\mathrm{Re}(\xi)}{4|\lambda|} + \frac{n}{2} - 1)\ln(1+t) - 2|\lambda|t|z-w|^2).$$

On the other hand, for $t > 0$ it follows that

$$(\ln(1+t) - \ln(t)) \frac{\mathrm{Re}(\xi)}{4|\lambda|} + (\frac{n}{2} - 1)\ln(t) + (\frac{n}{2} - 1)\ln(1+t) \leq (\frac{n}{2} - 1)\ln(t^2 + t).$$

Therefore,

$$\int_0^\infty \phi_{\xi,\lambda,n}(t) dt \leq \int_0^\infty (t(1+t))^{\frac{n}{2}-1} e^{-2|\lambda|t|z-w|^2} dt. \tag{6.2}$$

To compute the integral in (6.2), we make use of the formula

$$\int_0^\infty (2\beta x + x^2)^{\nu-1} e^{-px} dx = \frac{\Gamma(\nu)}{\sqrt{\pi}} (\frac{2\beta}{p})^{\nu-\frac{1}{2}} e^{\beta p} K_{\nu-\frac{1}{2}}(\beta p),$$

$|\arg(\beta)| < \pi$ and $\mathrm{Re}(\nu) > 0$ (see [7, p.322]). With $\nu = n/2$, $\beta = 1/2$ and $p = 2|\lambda|\,|z-w|^2$, we obtain the estimate (i).

For (ii), we proceed as follows. For $\lambda \neq 0$ and $w \neq z$, we have that

$$|S(\lambda)| \leq \int_0^1 \phi_{\xi,\lambda,n}(t)dt + \int_1^\infty \phi_{\xi,\lambda,n}(t)dt.$$

Then it is easy to see that

$$\int_0^1 \phi_{\xi,\lambda,n}(t)dt \leq \int_0^1 (t(1+t))^{\frac{n}{2}-1} dt.$$

On other hand for $|\lambda| \geq \frac{-\mathrm{Re}(\xi)}{4}$, we have

$$\int_1^\infty \phi_{\xi,\lambda,n}(t)dt \leq \int_1^\infty (t(1+t))^{\frac{n}{2}-1} e^{\frac{\mathrm{Re}(\xi)}{2} t|z-w|^2} dt.$$

Consequently for $|\lambda| \geq \frac{-\mathrm{Re}(\xi)}{4}$, one gets the estimate

$$\begin{aligned} |S(\lambda)| \leq & \int_0^1 (t(1+t))^{\frac{n}{2}-1} dt + \int_1^\infty (t(1+t))^{\frac{n}{2}-1} e^{t\frac{\mathrm{Re}(\xi)}{2}|z-w|^2} dt \\ = & C_{n,\xi,|z-w|^2} < +\infty\,. \end{aligned} \tag{6.3}$$

Then, in view of (6.1) and (6.3) we get estimate (ii).

**Proof of proposition 6.1** First, we shall prove that the function $\lambda \to \varphi(\lambda) = |\lambda|^{\frac{n-1}{2}} K_{\frac{n-1}{2}}(|\lambda||z-w|^2)$ is $L^2$-integrable for $|\lambda| \leq \frac{-\mathrm{Re}(\xi)}{4}$. We will discuss two cases.
Case $\frac{n-1}{2} \notin \mathbb{Z}_+$: in view of (2.6) we have

$$\varphi(\lambda) = \frac{\pi}{2\sin\frac{(n-1)\pi}{2}} (\sum_{k=0}^\infty \frac{(\frac{\rho}{2})^{\frac{1-n}{2}+2k}(|\lambda|)^{2k}}{k!\Gamma(k-\frac{n-1}{2}+1)} - \sum_{k=0}^\infty \frac{(\frac{\rho}{2})^{\frac{n-1}{2}+2k}(|\lambda|)^{n-1+2k}}{k!\Gamma(k+\frac{n-1}{2}+1)})$$

where $\rho = |z-w|^2$. It is not difficult to show uniform convergence of the involved series for $|\lambda| \leq \frac{-\mathrm{Re}(\xi)}{4}$. Thus, the function $\varphi(\lambda)$ is continuous on the compact set $B_\xi = \{\lambda, |\lambda| \leq \frac{-\mathrm{Re}(\xi)}{4}\}$ , consequently, $\varphi(\lambda)$ is $L^2$-Integrable on $B_\xi$.

Case $\frac{n-1}{2} \in \mathbb{Z}_+$: by (2.7) we can write

$$\begin{aligned}\varphi(\lambda) =& (-1)^{\frac{n-1}{2}} I_{\frac{n-1}{2}}(\rho|\lambda|)|\lambda|^{\frac{n-1}{2}} \ln(\frac{\rho|\lambda|}{2}) \\ &+ \frac{1}{2}\sum_{k=0}^{\frac{n-1}{2}-1} \frac{(-1)^k(\frac{n-1}{2}-k-1)!}{k!}(\frac{\rho}{2})^{2k-\frac{n-1}{2}}|\lambda|^{2k} \\ &+ \frac{1}{2}(-1)^{\frac{n-1}{2}}\sum_{k=0}^{\infty} \frac{(\frac{\rho}{2})^{2k+n-1}}{k!(k+\frac{n-1}{2})!}[\Psi(\frac{n-1}{2}+k+1)+\Psi(k+1)]|\lambda|^{2k+n-1}.\end{aligned} \tag{6.4}$$

Note that for $n \neq 1$ the last series converges uniformly in the compact set $B_\xi$ where we have use the asymptotic behavior of the function $\Psi(z)$ as $|z| \to +\infty$, [9, p.18],

$$\Psi(z) = \ln(z) - \frac{1}{2z} - \frac{1}{12z^2} + \frac{1}{120z^4} - \frac{1}{252z^6} + O(\frac{1}{z^8}).$$

Then, using the same argument as in the first case, we deduce that $\varphi(\lambda)$ is $L^2$-integrable on the set $B_\xi$. For $n = 1$, (6.4) can be rewritten as

$$\varphi(\lambda) = I_0(\rho|\lambda|)\ln(\frac{\rho|\lambda|}{2}) + \sum_{k=0}^{\infty} \frac{(\frac{\rho}{2})^{2k}}{(k!)^2} 2\Psi(k+1)|\lambda|^{2k}$$

which is also $L^2$-integrable on $B_\xi$. Thus, we have

$$\int_{|\lambda| \leq \frac{-\operatorname{Re}(\xi)}{4}} |Q_\lambda(\xi; z, w)|^2 d\lambda < +\infty.$$

By (ii) of lemma (6.1) we get that

$$\int_{|\lambda| > \frac{-\operatorname{Re}(\xi)}{4}} |Q_\lambda(\xi; z, w)|^2 d\lambda < +\infty.$$

Therefore,

$$\int_{-\infty}^{\infty} |Q_\lambda(\xi; z, w)|^2 d\lambda < +\infty.$$

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