A spectral mapping theorem for evolution semigroups on asymptotically almost periodic functions defined on the half line *

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Abstract

We prove that the evolution semigroup on \( \text{AAP}_0(\mathbb{R}_+, X) \) is strongly continuous. Then we prove some properties of the generator of this evolution semigroup and show some applications in the theory of inequalities.

1 Introduction

Let \( X \) be a complex Banach space and \( \mathcal{L}(X) \) the Banach algebra of all linear and bounded operators acting on \( X \). The norms in \( X \) and in \( \mathcal{L}(X) \) will be denoted by \( \| \cdot \| \). Let \( A \) be a linear and bounded operator acting on \( X \). We consider the system

\[
\dot{u}(t) = Au(t) \quad t \geq 0
\]

and the Cauchy problem

\[
\begin{align*}
\dot{u}(t) &= Au(t) + e^{i\mu t}x \quad t \geq 0 \\
u(0) &= 0
\end{align*}
\]

where \( \mu \in \mathbb{R} \) and \( x \in X \). It is well-known [12, 2] that the system (1.1) is exponentially stable; that is, there exist the constants \( N > 0 \) and \( \nu > 0 \) such that

\[
\| e^{tA} \| \leq Ne^{-\nu t} \quad \text{for all } t \geq 0,
\]

if and only if the solution of the Cauchy problem (1.2) is bounded for every \( \mu \in \mathbb{R} \) and any \( x \in X \), i.e., if and only if

\[
\sup_{t > 0} \left\| \int_0^t e^{-i\mu \xi} e^{\xi A} x d\xi \right\| < \infty, \quad \forall \mu \in \mathbb{R} \text{ and } \forall x \in X.
\]

For unbounded linear operators, the above result is false, see e.g. [28, Example 3.1]. However some weaker results, described as follows, hold.

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Let $\mathbf{T} = \{ T(t) : t \geq 0 \} \subset \mathcal{L}(X)$ be a strongly continuous semigroup on $X$ and $A : D(A) \subset X \to X$ its infinitesimal generator. It is well-known that the Cauchy problem

$$\dot{u}(t) = Au(t)$$
$$u(0) = x \in X$$

is well-posed and the mild solution of (1.3) is defined by

$$u(t) = T(t)x, \quad t \geq 0.$$  (1.4)

For well-posedness of equations we refer the reader to [29, 30] and the references therein. The mild solution of the non-homogeneous Cauchy problem

$$\dot{u}(t) = Au(t) + f(t) \quad t \geq 0$$
$$u(0) = x$$

is

$$u_f(t) = T(t)x + \int_0^t T(t - \xi)f(\xi)d\xi, \quad t \geq 0.$$  (1.6)

Particularly for $x = 0$, $y \in X$, and $f(t) := e^{i\mu t}y$, the solution $u_f(\cdot)$ can be written as

$$u_{\mu y}(t) = \int_0^t T(t - \xi)e^{i\mu \xi}yd\xi = e^{i\mu t} \int_0^t e^{-i\mu \xi}T(\xi)yd\xi.$$  (1.7)

In [28], it is shown that if $u_{\mu y}(\cdot)$ is bounded on $\mathbb{R}_+$ for every $\mu \in \mathbb{R}$ and all $y \in X$ then

$$\sigma(A) \subset \{ \lambda \in \mathbb{C} : \text{Re}(\lambda) < 0 \}.$$  (1.8)

Conversely if (1.7) holds and $\mathbf{T}$ is uniformly bounded (i.e. sup$_{t \geq 0} \| T(t) \| < \infty$) then $u_{\mu y}(\cdot)$ is bounded on $\mathbb{R}_+$ for every $\mu \in \mathbb{R}$ and all $y \in X$. This last result is proven in [4, Proposition 2]. Another result of this type is due to Arendt and Batty in [1].

For $x \in X$, let $\omega(x)$ the infimum of all $\omega \in \mathbb{R}$ for which there exists $M_\omega > 0$ such that $\| T(t)x \| \leq M_\omega e^{\omega t}$ for all $t \geq 0$. Let $\omega_1(\mathbf{T})$ the supremum of all $\omega(x)$ with $x \in D(A)$. Frank Neubrander [25] proved that $\omega_1(\mathbf{T})$ is the infimum of all $\omega \in \mathbb{R}$ with the property that

$$\{ \text{Re}(\lambda) > \omega \} \subset \rho(A) \quad \text{and there is} \quad R(\lambda, A)x = \lim_{t \to \infty} \int_0^t e^{-\lambda s}T(s)xds$$

for every $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) > \omega$ and any $x \in X$. Neerven [23, 24] has shown that if

$$\sup_{\mu \in \mathbb{R}} \sup_{t > 0} \int_0^t e^{-i\mu \xi}T(\xi)y\,d\xi = M(x) < \infty, \quad \forall \mu \in \mathbb{R} \text{ and } \forall x \in X$$

then $\omega_1(\mathbf{T}) < 0$; that is, if (1.8) holds then every solution of the system (1.1), starting in $D(A)$, is exponentially stable.
However, there can be solutions of the system (1.1) starting in $X \setminus D(A)$ which are not exponentially stable, even if (1.8) holds, see e.g. [6, Example 2]. Moreover, in [23, Corollary 5 and the proof of Theorem 4] it is shown that if (1.8) holds then the operator resolvent $R(\lambda, A)$ exists and the function $\lambda \mapsto R(\lambda, A)$ is uniformly bounded on $\{Re(\lambda) > 0\}$. Combining this fact with the Gearhart’s famous stability theorem [14] (see also Herbst [15], Howland [16], Huang [17], Prüss [27] Weiss [32]) follows that if $X$ is a complex Hilbert space and (1.8) holds then

$$\omega_0(T) := \lim_{t \to \infty} \frac{\ln \|T(t)\|}{t}$$

is negative, i.e. in these conditions every solution of the system (1.1) is exponentially stable. This and related results are explicitly presented in a very recent paper of Phong [31]. It seems that the last stability result, having (1.8) as hypothesis, cannot be extended for periodic evolution families, but a weaker result holds, see Theorem 2.4 below.

For a well-posed, non-autonomous Cauchy problem

$$\dot{u}(t) = A(t)u(t), \quad t \geq 0$$

$$u(0) = x \in X$$

(1.9)

with (possibly unbounded) linear operators $A(t)$, the mild solutions lead to an evolution family on $\mathbb{R}_+$, $\mathcal{U} = \{U(t, s) : t \geq s \geq 0\} \subset \mathcal{L}(X)$; that is:

(e1) $U(t, r) = U(t, s)U(s, r)$ for all $t \geq s \geq r \geq 0$ and $U(t, t) = I$ for any $t \geq 0$, ($I$ is the identity operator in $\mathcal{L}(X)$)

(e2) The maps $(t, s) \mapsto U(t, s)x : \{(t, s) : t \geq s \geq 0\} \to X$ are continuous for each $x \in X$.

An evolution family is **exponentially bounded** if there exist $\omega \in \mathbb{R}$ and $M_\omega > 0$ such that

$$\|U(t, s)\| \leq M_\omega e^{\omega(t-s)}, \quad \forall t \geq s \geq 0.$$ 

(1.10)

An evolution family is **exponentially stable** if (1.10) holds with some negative $\omega$.

If the evolution family $\mathcal{U}$ verifies the condition

(e3) $U(t, s) = U(t-s, 0)$ for all $t \geq s \geq 0$,

then the family $\mathbf{T} = \{U(t, 0) : t \geq 0\} \subset \mathcal{L}(X)$ is a strongly continuous semigroup on $X$. In this case the estimate (1.10) holds automatically.

If the Cauchy problem (1.9) is $q$-periodic, i.e. $A(t+q) = A(t)$ for $t \geq 0$, then the corresponding evolution family $\mathcal{U}$ is $q$-periodic, that is,

(e4) $U(t+q, s+q) = U(t, s)$ for all $t \geq s \geq 0$.

Every $q$-periodic evolution family is exponentially bounded [9, Lemma 4.1]. For a locally Bochner integrable function $f : \mathbb{R}_+ \to X$, the mild solution of the well-posed, inhomogeneous Cauchy problem

$$\dot{u}(t) = A(t)u(t) + f(t), \quad t \geq 0$$

$$u(0) = x$$

(1.11)
is
\[ u_f(t, x) := U(t, 0)x + \int_0^t U(t, \tau)f(\tau)d\tau, \quad (t \geq 0). \quad (1.12) \]

We also consider evolution families on the line. We shall use the same notation as in the case of evolution families on \( \mathbb{R}_+ \) with the mention that variables \( s \) and \( t \) can take any value in \( \mathbb{R} \). For more details about the strongly continuous semigroups and evolution families we refer to [13].

We recall the notion of evolution semigroup. For more details we refer the reader to [10, 11] and references therein. Let us consider the following spaces:

- \( BUC(\mathbb{R}, X) \) is the space of all \( X \)-valued, bounded and uniformly continuous functions on the real line endowed with the sup-norm.
- \( C_0(\mathbb{R}, X) \) is the subspace of \( BUC(\mathbb{R}, X) \) consisting of all functions \( f \) such that \( \lim_{|t| \to \infty} f(t) = 0 \).
- \( AP(\mathbb{R}, X) \) is the space of all almost periodic functions, that is, the smallest closed subspace of \( BUC(\mathbb{R}, X) \) containing the functions of the form, [20],
  \[ t \mapsto e^{i\mu t}x, \quad \mu \in \mathbb{R} \text{ and } x \in X. \]

Let \( \mathcal{U} = \{ U(t, s) : t \geq s \in \mathbb{R} \} \) be a strongly continuous and exponentially bounded evolution family of bounded linear operators on \( X \). For every \( t \geq 0 \) and each \( F \in C_0(\mathbb{R}, X) \) the function
\[
s \mapsto (T(t)F)(s) := U(s, s-t)F(s-t) : \mathbb{R} \to X \tag{1.13}
\]

belongs to \( C_0(\mathbb{R}, X) \) and the family \( \mathcal{T} = \{ T(t) : t \geq 0 \} \) is a strongly continuous semigroup on \( C_0(\mathbb{R}, X) \), [19]. If \( \mathcal{U} = \{ U(t, s) : t \geq s \in \mathbb{R} \} \) is a \( q \)-periodic evolution family, \( t \geq 0 \), and \( G \in AP(\mathbb{R}, X) \) then the function given by
\[
s \mapsto (S(t)G)(s) := U(s, s-t)G(s-t) : \mathbb{R} \to X, \tag{1.14}
\]

belongs to \( AP(\mathbb{R}, X) \) and the one-parameter family \( \mathcal{S} = \{ S(t) : t \geq 0 \} \) is a strongly continuous semigroup on \( AP(\mathbb{R}, X) \), [21]. \( \mathcal{T} \) and \( \mathcal{S} \) are called evolution semigroups on \( C_0(\mathbb{R}, X) \) and \( AP(\mathbb{R}, X) \), respectively. In the following we will consider spaces consisting of functions defined on \( \mathbb{R}_+ \). \( AP(\mathbb{R}_+, X) \) and \( C_0(\mathbb{R}_+, X) \) are the spaces consisting of all functions \( g : \mathbb{R}_+ \to X \) for which there exists \( G \in AP(\mathbb{R}, X) \), respectively \( G \in C_0(\mathbb{R}, X) \), such that \( G(s) = g(s) \) for all \( s \geq 0 \). \( C_{00}(\mathbb{R}_+, X) \) is the subspace of \( C_0(\mathbb{R}_+, X) \) consisting of all functions \( f \) for which \( f(0) = 0 \), and \( AAP_0(\mathbb{R}_+, X) \) is the space of all \( X \)-valued functions \( h \) such that \( h(0) = 0 \) and there exist \( f \in C_0(\mathbb{R}_+, X) \) and \( g \in AP(\mathbb{R}_+, X) \) such that \( h = f + g \). For each \( h \in AAP_0(\mathbb{R}_+, X) \) and every \( t \geq 0 \) consider the function \( T(t)h \) given by
\[
[T(t)h](s) = \begin{cases} U(s, s-t)h(s-t), & s \geq t \\ 0, & 0 \leq s < t. \end{cases} \tag{1.15}
\]
2 Results

Lemma 2.1 Let $T = \{T(t) : t \geq 0\}$ be a strongly continuous semigroup and $A : D(A) \subset X \rightarrow X$ its infinitesimal generator. If $T$ is uniformly stable, i.e. there exists a positive constant $M$ such that $\sup_{t \geq 0} \|T(t)\| = M < \infty$, then

$$\|Ax\|^2 \leq 4M^2\|A^2x\|\|x\|, \quad \text{for all } x \in D(A^2).$$

The proof of this lemma can be found in [18].

Lemma 2.2 The semigroup $T = \{T(t) : t \geq 0\}$ described in (1.15) is defined on $AAP_0(\mathbb{R}_+, X)$ and is strongly continuous. This semigroup is called evolution semigroup associated to $U$ on the space $AAP_0(\mathbb{R}_+, X)$.

Proof. Let $h = f + g$ with $f \in C_0(\mathbb{R}_+, X)$ and $g \in AP(\mathbb{R}_+, X)$ such that $h(0) = 0$ and let $F \in C_0(\mathbb{R}, X)$ and $G \in AP(\mathbb{R}, X)$ such that $F(s) = f(s)$ and $G(s) = g(s)$ for all $s \geq 0$. It is easy to see that for each $t \geq 0$, we have

$$T(t)h = (1_{[0,\infty)}S(t)G) + (1_{[t,\infty)}T(t)f - 1_{[0, t)}S(t)G).$$

Here $\{S(t)\}_{t \geq 0}$ is the evolution semigroup on $AP(\mathbb{R}, X)$ given in (1.14) and $1_J$ is the characteristic function of the interval $J$. If we put $g_1 := 1_{[0,\infty)}S(t)G$ and $f_1 := 1_{[t,\infty)}T(t)f - 1_{[0, t)}S(t)G$ then $f_1 \in C_0(\mathbb{R}_+, X)$, $g_1 \in AP(\mathbb{R}_+, X)$ and $(f_1 + g_1)(0) = 0$, so $T(t)$ is defined on $AAP_0(\mathbb{R}_+, X)$ for every $t \geq 0$. Moreover, for all $h \in AAP_0(\mathbb{R}_+, X)$, we have:

$$\sup_{s \geq 0} \|T(t)h - h\| \leq \sup_{s \geq t} \|(T(t)h - h)(s)\| + \sup_{s \in [0, t]} \|(T(t)h - h)(s)\|$$

$$\leq \sup_{s \geq t} \|(S(t)G - G)(s)\| + \sup_{s \geq t} \|(T(t)F - F)(s)\| + \sup_{s \in [0, t]} \|h(s)\|$$

$$\leq \|S(t)G - G\|_{AP(\mathbb{R}, X)} + \|T(t)F - F\|_{C_0(\mathbb{R}, X)} + \sup_{s \in [0, t]} \|h(s)\|.$$
Theorem 2.4  Let $\mathcal{U}, \mathcal{T}$ and $(A,D(A))$ as in Lemma 2.3. The following five statements are equivalent.

(i) $\mathcal{U}$ is uniformly exponentially stable

(ii) $A$ is an invertible operator

(iii) For every $f \in \text{AAP}_0(\mathbb{R}_+,X)$ the function $t \mapsto u_f(t,0) = \int_0^t U(t,s)f(s)ds$ belongs to $\text{AAP}_0(\mathbb{R}_+,X)$

(iv) For every $f \in \text{AAP}_0(\mathbb{R}_+,X)$ the function $u_f(\cdot,0)$ is bounded on $\mathbb{R}_+$

(v) For every $f \in P^0_q(\mathbb{R}_+,X)$ and $\mu \in \mathbb{R}$ the function $t \mapsto \int_0^t U(t,s)e^{-i\mu s}f(s)ds$ is bounded on $\mathbb{R}_+$.

Proof. (i) $\Rightarrow$ (ii) Let $X := \text{AAP}_0(\mathbb{R}_+,X)$. Then

$$\|\mathcal{T}(t)\|_{\mathcal{L}(X)} = \sup\{\sup_{s \geq t} \|U(s,s-t)h(s-t)\| : \|h\|_X = 1\} \leq \sup\{M_\omega e^{\omega t} \sup_{s \geq t} \|h(s-t)\| : \|h\|_X = 1\} \leq M_\omega e^{\omega t},$$

for all $t \geq 0$. Thus $\omega_0(\mathcal{T}) := \lim_{t \to \infty} \frac{\ln\|\mathcal{T}(t)\|}{t} \leq \omega < 0$ and by the general theory of linear semigroups [26, p. 4-5] it follows that $0 \in \rho(A)$; that is, $A$ is an invertible operator.

(ii) $\Rightarrow$ (iii) follows from Lemma 2.3.

(iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (v) are obvious.

(v) $\Rightarrow$ (i) follows as in [5, Theorem 4]; see also [8, Theorem 2.1], or [21].

Remarks: 1. Let $A(\cdot)$ be a $q$-periodic operator-valued function on $\mathbb{R}_+$ and $\{U(t,s) : t \geq s \geq 0\}$ the $q$-periodic evolution family associated to it. Since the function $t \mapsto \int_0^t U(t,s)f(s)ds$ is the mild solution of the abstract Cauchy problem

$$\dot{u}(t) = A(t)u(t) + f(t) \quad (t \geq 0)$$

$$u(0) = 0,$$

the equivalence between (i) and (iii) from Theorem 2.4 can be interpreted in the following way:

The system $\dot{u}(t) = A(t)u(t)$ is exponentially stable if and only if for every input $f \in \text{AAP}_0(\mathbb{R}_+,X)$ the mild solution of the Cauchy problem (2.1), belongs to $\text{AAP}_0(\mathbb{R}_+,X)$.

2. A related result on the individual stability, where the space $\text{AAP}(\mathbb{R}_+,X)$ is also involved, can be found in [3, Proposition 2.9].

3. If every solution of the $q$-periodic homogeneous system $\dot{v}(t) = A(t)v(t)$, $t \in \mathbb{R}_+$ is exponentially stable then for each $f \in P^0_q(\mathbb{R}_+,X)$ and every $\mu \in \mathbb{R}$
there is a mild solution $u(\cdot)$ of the Cauchy problem
\[
\dot{v}(t) = A(t)v(t) + e^{i\mu t}f(t), \quad (t \geq 0)
v(0) = x
\]
such that the function $t \mapsto e^{-i\mu t}u(t)$ is $q$-periodic on $\mathbb{R}_+$. Indeed the family $U = \{U(t,s) : t \geq s \geq 0\}$ is exponentially stable, so the resolvent set $\rho(V)$ contains all complex number $z$ with $|z| \geq 1$. Here $V := U(q,0)$ denotes the monodromy operator associated to $U$. Then for each $\mu \in \mathbb{R}$ we have that $e^{i\mu q} \in \rho(V)$. Let $y := \int_0^t U(q,\tau)e^{i\mu \tau}f(\tau)d\tau$ and $x = (e^{i\mu q} - V)^{-1}(y)$. Using (1.12) follows that
\[
\begin{align*}
u(t) &= U(t,0)x + \int_0^t U(t,\tau)e^{i\mu \tau}f(\tau)d\tau.
\end{align*}
\]
In the end we obtain that $e^{-i\mu q}u(t + q) = u(t)$ for every $t \geq 0$, and now it is easy to see that the function $e^{-i\mu}u(\cdot)$ is $q$-periodic.

4. The proof of Theorem 2.4 depends on Lemma 2.3 which also depends on the strongly continuity of the evolution semigroup $T$, defined in (1.15). Thus the condition, $h(0) = 0$, which appears in the definition of the space $AAP_0(\mathbb{R}_+,X)$, is essentially in the proof of Theorem 2.4, because it is involved in the proof of Lemma 2.2.

An immediate consequence of Theorem 2.4 is the spectral mapping theorem for the evolution semigroup $T$ on $AAP_0(\mathbb{R}_+,X)$. Similar results, but for the evolution semigroup on $C_{00}(\mathbb{R}_+,X)$, can be found in [22, Theorem 2.2, Corollary 2.4]. Recall that $\sigma(L)$ denotes the spectrum of the linear operator $L$ acting on $X$, and $\rho(L) := C \setminus \sigma(L)$ is the resolvent set of $L$. The spectral radius of $L$ is $r(L) := \sup\{ |\lambda| : \lambda \in \sigma(L) \}$ and the spectral bound is $s(L) := \sup\{ \Re(\lambda) : \lambda \in \sigma(L) \}$.

**Theorem 2.5** Let $U$ be a $q$-periodic evolution family of bounded linear operators on the Banach space $X$. Then the evolution semigroup $T$ on $AAP_0(\mathbb{R}_+,X)$ satisfies the spectral mapping theorem, as follows:
\[
e^{t\sigma(A)} = \sigma(T(t)) \setminus \{0\}, \quad t \geq 0.
\]
Moreover, $\sigma(A) = \{ \lambda \in C : \Re(\lambda) \leq s(A) \}$ and
\[
\sigma(T(t)) = \{ \lambda \in C : |\lambda| \leq r(T(t)) \}, \quad \text{for all } t > 0.
\]

The proof of this theorem follows from Theorem 2.4 using an argument given in [22, Corollary 2.4].

Another application of Theorem 2.4 is the following inequality of Landau’s type. For more details about theorems of this form, see [7].
Theorem 2.6 Let $\mathcal{U} = \{U(t,s) : t \geq s \geq 0\}$ be a $q$-periodic evolution family of bounded linear operators acting on $X$ and let $f \in X := \mathcal{AAP}_0(\mathbb{R}_+, X)$. Suppose that the following two conditions are satisfied:

(i) $u_f(\cdot, 0) = \int_0^\cdot U(\cdot, s)f(s)ds$ belongs to $X$

(ii) $v_f(\cdot) := \int_0^\cdot (\cdot - s)U(\cdot, s)f(s)ds$ belongs to $X$.

If $\sup\{\|U(t,s)\| : t \geq s \geq 0\} = M < \infty$ then

$$\|u_f(\cdot, 0)\|_X^2 \leq 4M^2\|f\|_X \cdot \|v_f(\cdot)\|_X.$$  \hspace{1cm} (2.2)

Proof. Let $T$ the evolution semigroup associated to $\mathcal{U}$ on the space $X$ and $(A, D(A))$ its infinitesimal generator. From Lemma 2.3 results that $u_f(\cdot, 0)$ belongs to $D(A)$ and $Au_f(\cdot, 0) = -f$. Using Fubini’s theorem it is easy to see that $v_f(\cdot) \in D(A^2)$ and $A^2v_f(\cdot) = f$. Now the inequality (2.2) can be easily obtained from Lemma 2.1. \hfill \Box

Proposition 2.7 Let $f$ be a $X$-valued, locally Bochner integrable function on $\mathbb{R}_+$ and $g, h$ the mappings on $\mathbb{R}_+$ given by

$$g(t) := \int_0^t f(s)ds \quad \text{and} \quad h(t) = \int_0^t (t - s)f(s)ds.$$

If $\sup\{|f(t)| : t \geq 0\} = M_1 < \infty$ and $\sup\{|h(t)| : t \geq 0\} = M_3 < \infty$ then

$$|g(r)|^2 \leq 4M_1M_3, \quad \forall r \geq 0.$$  \hspace{1cm} (2.3)

Proof. For every $t \geq 0$ and any $X$-valued function $F$ on $\mathbb{R}_+$ let us consider the function $F_t$ given by

$$F_t(s) = \begin{cases} F(s - t), & s \geq t \\ 0, & 0 \leq s < t. \end{cases}$$

With this notation, we have

$$h_t(r) - h(r) + tg(r) = \int_0^t (t - s)f_s(r)ds, \quad \forall t \geq 0, \quad \text{and} \quad \forall r \geq 0.$$  \hspace{1cm} (2.4)

Passing to the norm in this equation, we obtain

$$\|g(r)\| \leq \frac{2M_3}{t} + \frac{tM_1}{2}, \quad \forall t > 0.$$  \hspace{1cm} (2.5)

If $M_1 = 0$ or $M_3 = 0$ then $g = 0$ and (2.3) holds with equality. If $M_1 > 0$ and $M_3 > 0$ then (2.3) can be obtained from (2.5) with $t = \sqrt{4M_3/M_1}$. \hfill \Box
Remark. If \( f \) is a continuous function then Proposition 2.7 follows directly and easily by \([18]\), because \( g'(t) = f(t) \) and \( h'(t) = g(t) \) for all \( t \geq 0 \). The author thanks to the referee who brought to the author’s attention about this fact.

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