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A NONLOCAL PROBLEM FOR FOURTH ORDER HYPERBOLIC EQUATIONS WITH MULTIPLE CHARACTERISTICS

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Abstract. In this paper, we study fourth order differential equations with multiple characteristics and dominated low terms. We prove the existence and uniqueness of a Riemann function for this equation, and then provide an integral representation of the general solution of the Goursat problem. We also provide sufficient conditions for the solvability of a nonlocal problem.

1. Introduction

Partial differential equations of higher order with dominated low terms are encountered when studying mathematical models for certain natural and physical processes. As an example of such type of equations, is the equation of moisture transfer [2]

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}(D\frac{\partial w}{\partial x} + A\frac{\partial^2 w}{\partial x \partial t}),$$

where w is the concentration of moisture per unit, D is the coefficient of diffusivity, and $A > 0$ is the varying coefficient of Hallaire. Under the proper schematization of the process of absorbing the soil moisture by the roots of plants, the pressure $u(x,t)$ in the area of root absorption satisfies the equation of form [4]

$$(\frac{\partial}{\partial x} + \frac{1}{x})(u_{xt} + \lambda u_x) = \mu u_t.$$

Obviously, the equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^4 u}{\partial x^2 \partial t^2} = 0,$$

which describes the longitudinal waves in a thin elastic stem taking into account the effects of transversal inertia, is of the same type [5].

In the present work, a class equations with fourth order partial derivatives and dominated lower order terms is considered.

In the space $\mathbb{R}^3$ of the independent variables x_1, x_2 and x_3 let

$$\Pi := \{(x_1, x_2, x_3) \in R^3 : a_i < x_i < b_i\}; \quad \Pi_i :=]a_i; b_i[; \quad \Pi_{ij} := \Pi_i \times \Pi_j$$

for $i, j = 1, 2, 3$. For the class of functions φ, continuous in $\bar\Pi$ with partial derivatives $D^i_{x_1}\varphi$, $D^j_{x_2}\varphi$, $D^k_{x_3}\varphi$, $0 \le i \le m$, $0 \le j \le n$, $0 \le k \le l$, we use the symbol $C^{i,j,k}(\bar\Pi)$.

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Consider the Goursat problem

$$\frac{\partial^4}{\partial x_1^2\partial x_2\partial x_3}u(x)+\sum_{i,j,k}a^{i,j,k}(x)\frac{\partial^{i+j+k}}{\partial x_1^i\partial x_2^j\partial x_3^k}u(x)=f(x), \tag{1.1}$$

$$\begin{gathered} u(x_1,x_2,x_3^0)=\varphi_{12}(x_1,x_2),\quad u(x_1,x_2^0,x_3)=\varphi_{13}(x_1,x_3),\\ u(x_1^0,x_2,x_3)=\varphi_{23}(x_2,x_3),\quad u_{x_1}(x_1^0,x_2,x_3)=\widetilde{\varphi}_{23}(x_2,x_3), \end{gathered} \tag{1.2}$$

where $i=0,1,2$; $j,k=0,1$; $i+j+k\neq 4$, $x,x^0\in\bar\Pi$ and the functions φ_{ij} satisfy the following compatibility conditions

$$\begin{gathered} \varphi_1(x_1):=\varphi_{12}(x_1,x_2^0)=\varphi_{13}(x_1,x_3^0),\quad \varphi_2(x_2):=\varphi_{12}(x_1^0,x_2)=\varphi_{23}(x_2,x_3^0),\\ \varphi_3(x_3):=\varphi_{13}(x_1^0,x_3)=\varphi_{23}(x_2^0,x_3),\quad \varphi_0:=\varphi_1^0=\varphi_2(x_2^0)=\varphi_3(x_3^0). \end{gathered} \tag{1.3}$$

Theorem 1.1. *For any $f\in C(\bar\Pi)$, $a^{i,j,k}\in C^{i,j,k}(\bar\Pi)$ and $\varphi_{12}\in C^{2,1}(\bar\Pi_{12})$, $\varphi_{13}\in C^{2,1}(\bar\Pi_{13})$, $\varphi_{23}\in C^{1,1}(\bar\Pi_{23})$, $\widetilde\varphi_{23}\in C^{1,1}(\bar\Pi_{23})$ satisfying the compatibility conditions (1.3) the Goursat problem (1.1), (1.2) has one and only one solution $u\in C^{2,1,1}(\bar\Pi)$.*

Lemma 1.2. *Let $a(x)$ and $b(x)$ be continuous functions. An arbitrary solution of equation*

$$y''+a(x)y'+b(x)y=0,\quad x\in[\alpha,\beta] \tag{1.4}$$

is monotonous if and only if $b(x)=0$ everywhere in $[\alpha,\beta]$.

Let

$$\begin{gathered} D:=\{x=(x_1,x_2,x_3)\in R^3: 0<x_i<x_i^0\},\\ D_i:=]0;x_i^0[;\ D_{ij}:=D_i\times D_j;\quad i,j=1,2,3. \end{gathered}$$

For equation (1.1) consider the boundary conditions

$$\begin{gathered} u(x_1,x_2,0)=\varphi_{12}(x_1,x_2),\quad u(x_1,0,x_3)=\varphi_{13}(x_1,x_3),\\ u(0,x_2,x_3)=\varphi_{23}(x_2,x_3),\quad u(x_1^0,x_2,x_3)=\psi(x_2,x_3), \end{gathered} \tag{1.5}$$

where the functions φ_{ij}, ψ satisfy the compatibility conditions

$$\begin{gathered} \varphi_{12}(x_1,0)=\varphi_{13}(x_1,0),\quad \varphi_{12}(0,x_2)=\varphi_{23}(x_2,0),\\ \varphi_{13}(0,x_3)=\varphi_{23}(0,x_3),\quad \varphi_{12}(0,0)=\varphi_{13}(0,0)=\varphi_{23}(0,0),\\ \varphi_{12}(0,x_2)=\psi(x_2,0),\quad \varphi_{13}(0,x_3)=\psi(0,x_3). \end{gathered} \tag{1.6}$$

Theorem 1.3. *Assume that $f\in C(\bar D)$, $a^{i,j,k}\in C^{i,j,k}(\bar D)$, $\varphi_{12}\in C^{2,1}(\bar D_{12})$, $\varphi_{13}\in C^{2,1}(\bar D_{13})$, $\varphi_{23},\psi\in C^{1,1}(\bar D_{23})$. If there holds the condition*

$$(a^{0,1,1}-a^{1,1,1}_{x_1})(x)=0,\ x\in D \tag{1.7}$$

then problem (1.1), (1.5), (1.6) is uniquely solvable in the class $C^{2,1,1}(\bar D)$.

2. The Riemann function and the solution of (1.1)

Following the scheme in [1, 3], we define the Riemann function $v(x;\xi)$, $(x;\xi)\in\Pi\times\Pi$ as a solution of the Goursat problem

$$\frac{\partial^4}{\partial x_1^2\partial x_2\partial x_3}v(x)+\sum_{i,j,k}(-1)^{i+j+k}\frac{\partial^{i+j+k}}{\partial x_1^i\partial x_2^j\partial x_3^k}(a^{i,j,k}(x)v(x))=0, \tag{2.1}$$

$$\begin{aligned} &\big[v_{x_1x_1x_2}-(a^{2,0,1}v)_{x_1x_1}-(a^{1,1,1}v)_{x_1x_2}+(a^{1,0,1}v)_{x_1}\\ &\quad+(a^{0,1,1}v)_{x_2}-a^{0,0,1}v\big](x_1,x_2,\xi_3)=0,\quad (x_1,x_2)\in\bar\Pi_{12}; \end{aligned}$$

$$
\begin{gathered}
\big[v_{x_1x_1x_3}-(a^{2,1,0}v)_{x_1x_1}-(a^{1,1,1}v)_{x_1x_3}+(a^{1,1,0}v)_{x_1} \\
+(a^{0,1,1}v)_{x_3}-a^{0,1,0}v\big](x_1,\xi_2,x_3)=0,\quad (x_1,x_3)\in\bar\Pi_{13}; \\
\big[v_{x_1x_2x_3}-(a^{2,1,0}v)_{x_1x_2}-(a^{2,0,1}v)_{x_1x_3}+(a^{2,0,0}v)_{x_1}\big](\xi_1,x_2,x_3)=0,\ (x_2,x_3)\in\bar\Pi_{23}; \\
\big[v_{x_1x_1}-(a^{1,1,1}v)_{x_1}+a^{0,1,1}v\big](x_1,\xi_2,\xi_3)=0,\quad x_1\in\bar\Pi_1; \\
\big[v_{x_1x_2}-(a^{2,0,1}v)_{x_1}\big](\xi_1,x_2,\xi_3)=0,\quad x_2\in\bar\Pi_2; \\
\big[v_{x_1x_3}-(a^{2,1,0}v)_{x_1}\big](\xi_1,\xi_2,x_3)=0,\quad x_3\in\bar\Pi_3; \\
v_{x_1}(\xi)=1;\quad v(\xi_1,x_2,x_3)=0,\quad (x_2,x_3)\in\bar\Pi_{23},
\end{gathered}
\tag{2.2}
$$

where $i=0,1,2;\ j,k=0,1;\ i+j+k\neq 4$. For simplicity, we have omitted the second triplet of arguments of the Riemann function.

Remark 2.1. The boundary conditions (2.2) for the Riemann function can be received from the certain consideration of the integral

$$
\int_{x^0}^{x}(vLu-uL^*v)(y)dy \tag{2.3}
$$

Further, by integration of equation (2.1) twice on the variable y_1 and once on the variables y_2 and y_3 in corresponding segments of integration ($y_i\in[\xi_i;x_i], i=1,2,3$), and taking into account the differential relations (2.2), we have the following Volterra integral equation of the second kind, with respect to the first triplet of arguments of the Riemann function $v(x;\xi)$

$$
\begin{aligned}
&v(x)-\int_{\xi_1}^{x_1}[(a^{1,1,1}-(x_1-y_1)a^{0,1,1})v](y_1,x_2,x_3)dy_1 \\
&-\int_{\xi_2}^{x_2}(a^{2,0,1}v)(x_1,y_2,x_3)dy_2-\int_{\xi_3}^{x_3}(a^{2,1,0}v)(x_1,x_2,y_3)dy_3 \\
&+\int_{\xi_1}^{x_1}\int_{\xi_2}^{x_2}[(a^{1,0,1}-(x_1-y_1)a^{0,0,1})v](y_1,y_2,x_3)dy_1dy_2 \\
&+\int_{\xi_1}^{x_1}\int_{\xi_3}^{x_3}[(a^{1,1,0}-(x_1-y_1)a^{0,1,0})v](y_1,x_2,y_3)dy_1dy_3 \\
&+\int_{\xi_2}^{x_2}\int_{\xi_3}^{x_3}(a^{2,0,0}v)(x_1,y_2,y_3)dy_2dy_3 \\
&-\int_{\xi_1}^{x_1}\int_{\xi_2}^{x_2}\int_{\xi_3}^{x_3}[(a^{1,0,0}-(x_1-y_1)a^{0,0,0})v](y_1,y_2,y_3)dy_1dy_2dy_3=x_1-\xi_1.
\end{aligned}
$$

The last equation unconditionally has an unique solution and therefore the existence and uniqueness of the solution of the problem (2.1), (2.2) is proved.

Now, integration (2.3) and taking into account the differential relations (2.2), for the regular solution of problem (1.1), (1.2), (1.3), we have

$$
\begin{aligned}
u(x_1,x_2,x_3)&=[v_{x_1}-a^{1,1,1}v](x_1^0,x_2^0,x_3^0;x)\varphi_0 \\
&+\int_{x_1^0}^{x_1}([v_{x_1}-a^{1,1,1}v]\varphi_1'-a^{0,1,1}v\varphi_1)(y_1,x_2^0,x_3^0;x)dy_1 \\
&+\int_{x_2^0}^{x_2}([v_{x_1}-a^{1,1,1}v]\varphi_2'+[(a^{2,0,1}v)_{x_1}-a^{1,0,1}v]\varphi_2)(x_1^0,y_2,x_3^0;x)dy_2
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{x_3^0}^{x_3}([v_{x_1}-a^{1,1,1}v]\varphi_3'+[(a^{2,1,0}v)_{x_1}-a^{1,1,0}v]\varphi_3)(x_1^0,x_2^0,y_3;x)dy_3\\
&+\int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2}\big(v_{x_1}\big[\frac{\partial^2\varphi_{12}}{\partial y_1\partial y_2}+a^{2,0,1}\frac{\partial\varphi_{12}}{\partial y_1}\big]\\
&-v\big[a^{1,1,1}\frac{\partial^2\varphi_{12}}{\partial y_1\partial y_2}-(a^{2,0,1}_{x_1}-a^{1,0,1})\frac{\partial\varphi_{12}}{\partial y_1}+a^{0,1,1}\frac{\partial\varphi_{12}}{\partial y_2}+a^{0,0,1}\varphi_{12}\big]\big)(y_1,y_2,x_3^0;x)dy_1dy_2\\
&+\int_{x_1^0}^{x_1}\int_{x_3^0}^{x_3}\big(v_{x_1}\big[\frac{\partial^2\varphi_{13}}{\partial y_1\partial y_3}+a^{2,1,0}\frac{\partial\varphi_{13}}{\partial y_1}\big]\\
&-v\big[a^{1,1,1}\frac{\partial^2\varphi_{13}}{\partial y_1\partial y_3}-(a^{2,1,0}_{x_1}-a^{1,1,0})\frac{\partial\varphi_{13}}{\partial y_1}+a^{0,1,1}\frac{\partial\varphi_{13}}{\partial y_3}+a^{0,1,0}\varphi_{13}\big]\big)(y_1,x_2^0,y_3;x)dy_1dy_3\\
&+\int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3}\big(v_{x_1}\big[\frac{\partial^2\varphi_{23}}{\partial y_2\partial y_3}+a^{2,1,0}\frac{\partial\varphi_{23}}{\partial y_2}+a^{2,0,1}\frac{\partial\varphi_{23}}{\partial y_3}+a^{2,0,0}\varphi_{23}\big]\\
&-v\big[a^{1,1,1}\frac{\partial^2\varphi_{23}}{\partial y_2\partial y_3}-(a^{2,1,0}_{x_1}-a^{1,1,0})\frac{\partial\varphi_{23}}{\partial y_2}-(a^{2,0,1}_{x_1}-a^{1,0,1})\frac{\partial\varphi_{23}}{\partial y_3}-(a^{2,0,0}_{x_1}-a^{1,0,0})\varphi_{23}\\
&+\frac{\partial^2\widetilde{\varphi}_{23}}{\partial y_2\partial y_3}+a^{2,1,0}\frac{\partial\widetilde{\varphi}_{23}}{\partial y_2}+a^{2,0,1}\frac{\partial\widetilde{\varphi}_{23}}{\partial y_3}+a^{2,0,0}\widetilde{\varphi}_{23}\big]\big)(x_1^0,y_2,y_3;x)dy_2dy_3\\
&-\int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3}v(y;x)f(y)dy_1dy_2dy_3. \qquad (2.4)
\end{aligned}
$$

This proves the Theorem (1.1). □

Let $v(x;\xi)$, $(x;\xi)\in\overline{\Pi}\times\overline{\Pi}$ be the Riemann function for equation (1.1), and let $x^0\in\overline{\Pi}$ be an arbitrary point. Assuming that u is the regular solution of equation (1.1) in $\overline{\Pi}$ which satisfies homogenous boundary conditions $u(x_1^0,x_2,x_3)=u(x_1,x_2^0,x_3)=u(x_1,x_2,x_3^0)=u_{x_1}(x_1^0,x_2,x_3)=0$, then, as it is easy to see, from formula (2.4) it follows next representation

$$
u(x_1,x_2,x_3)=-\int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3}v(y_1,y_2,y_3;x)f(y_1,y_2,y_3)dy_1dy_2dy_3,\ x\in\overline{\Pi},
$$

for an arbitrary continuous function f.

Using the last representation and arbitrariness of the choices of point x^0 and function f, from equation (1.1) one can get following relations:

$$
\begin{gathered}
[v_{\xi_1\xi_1\xi_2}+a^{2,0,1}v_{\xi_1\xi_1}+a^{1,1,1}v_{\xi_1\xi_2}+a^{1,0,1}v_{\xi_1}+a^{0,1,1}v_{\xi_2}+a^{0,0,1}v](x;\xi_1,\xi_2,x_3)=0,\\
[v_{\xi_1\xi_1\xi_3}+a^{2,1,0}v_{\xi_1\xi_1}+a^{1,1,1}v_{\xi_1\xi_3}+a^{1,1,0}v_{\xi_1}+a^{0,1,1}v_{\xi_3}+a^{0,1,0}v](x;\xi_1,x_2,\xi_3)=0,\\
[v_{\xi_1\xi_2\xi_3}+a^{2,1,0}v_{\xi_1\xi_2}+a^{2,0,1}v_{\xi_1\xi_3}+a^{2,0,0}v_{\xi_1}](x;x_1,\xi_2,\xi_3)=0,\\
[v_{\xi_1\xi_1}+a^{1,1,1}v_{\xi_1}+a^{0,1,1}v](x;\xi_1,x_2,x_3)=0,\\
[v_{\xi_1\xi_2}+a^{2,0,1}v_{\xi_1}](x;x_1,\xi_2,x_3)=0,[v_{\xi_1\xi_3}+a^{2,1,0}v_{\xi_1}](x;x_1,x_2,\xi_3)=0,\\
v_{\xi_1}(x;x)=1,\ v(x;x_1,\xi_2,\xi_3)=0.
\end{gathered}
$$

These relations are dual to relations (1.2) in the certain sense (the left sides of (1.1) and (2.1), considered as differential operators, are conjugated) , so, the definition of the Riemann function as the solution of the Goursat problem (2.1),(2.2) is logically correct.

3. Proof of the Lemma 1.2 and the Theorem 1.3

The if - part is obvious, therefore only the only if - part has to be proved. Let us assume the contrary: there exists $x_0 \in [\alpha, \beta]$ satisfying $b(x_0) \neq 0$ whereas an arbitrary solution of

$$y'' + a(x)y' + b(x)y = 0, \quad x \in [\alpha, \beta] \tag{3.1}$$

is monotonous. Certainly, because of continuity of $b(x)$ there exists the segment $[\alpha_1, \beta_1]$ such that it contains the point x_0 and $b(x) \neq 0,\ \ x \in [\alpha_1, \beta_1]$. Proceeding from the well-known fact that any solution of class $C^2[\alpha_1, \beta_1]$ can be uniquely prolonged till the solution of (3.1) of class $C^2[\alpha, \beta]$ on whole $[\alpha, \beta]$ we shall not restrict the generality of reasoning if assume that $b(x) \neq 0, \ \ x \in [\alpha, \beta]$.

Let $y(x) = c_1 y_1(x) + c_2 y_2(x)$ be an arbitrary solution of equation (3.1) and $y'(x) = c_1 y_1'(x) + c_2 y_2'(x)$ be a constant-signed function where $y_1(x)$ and $y_2(x)$ form a fundamental system of solutions of (3.1).

Consider the sets $K_i := \{x \in [\alpha, \beta] : y_i' = 0\}, \ \ i = 1, 2$. Obviously, the sets K_1 and K_2 are closed. Let us see that there hold the following properties

$$A. \ \ K_1 \cap K_2 = \emptyset, \quad B. \ \ K_1 \cup K_2 = [\alpha, \beta].$$

The property A is obvious since assuming the opposite implies the existence of a point $x_0 \in [\alpha, \beta]$ such that $y_1'(x_0) = y_2'(x_0) = 0$ and therefore for Wronsky's determinant we have $(W[y_1, y_2])(x_0) = 0$ which contradicts to the fundamentality of system $y_1(x), \ \ y_2(x)$.

Now suppose that the property B is not true. This implies the existence of a point $x_0 \in [\alpha, \beta]$ such that $y_1'(x_0) \neq 0$ and $y_2'(x_0) \neq 0$. Without restriction of a reasoning generality we assume that $y_1'(x_0) = y_2'(x_0)$ since in opposite case instead the pair $y_1(x), \ \ y_2(x)$ one may consider the pair $\frac{y_2'(x_0)}{y_1'(x_0)} y_1(x), \ \ y_2(x)$. It is easy to note that $y_1''(x_0) \neq y_2''(x_0)$ because in other case from (3.1) we would have $y_1(x_0) = y_2(x_0)$, and according to $y_1'(x_0) = y_2'(x_0)$ and uniqueness of Cauchy's problem solution we would get $y_1(x) = y_2(x), \ \ x \in [\alpha, \beta]$ contradicting to the condition of linear independence of functions $y_1(x), \ \ y_2(x)$. Therefore $y_1''(x_0) \neq y_2''(x_0)$ and as it is easy to verify for $c_1 = 1$ and $c_2 = -1$ the condition of sign-constancy of the function $y'(x) = c_1 y_1'(x) + c_2 y_2'(x)$ is violated in a neighborhood of the point x_0. This proves the property B.

Now, considering the segment $[\alpha, \beta]$ as a topological space with the relative topology induced from R, which is obviously connected, we have from the properties A and B that one of the sets $K_1, \ \ K_2$ is empty, whereas another coincides with $[\alpha, \beta]$, say $K_1 = [\alpha, \beta]$. This means that $y_1'(x) = 0, \ \ x \in [\alpha, \beta]$, whence from (3.1) $b(x) y_1(x) = 0, \ \ x \in [\alpha, \beta]$. According to our assumption $b(x) \neq 0$ and therefore $y_1(x) = 0, \ \ x \in [\alpha, \beta]$. The last contradicts to the linear independence of the functions $y_1(x), \ \ y_2(x)$ and so the lemma is proven.

Now, let us prove Theorem (1.3). Consider unknown function $\tau(x_2, x_3)$ assuming that $\tau(x_2, x_3) = u_{x_1}(0, x_2, x_3)$. Then, according to (2.4) the regular solution of equation (1.1) with boundary conditions

$$\begin{gathered} u(x_1, x_2, 0) = \varphi_{12}(x_1, x_2), \quad u(x_1, 0, x_3) = \varphi_{13}(x_1, x_3), \\ u(0, x_2, x_3) = \varphi_{23}(x_2, x_3), \quad u_{x_1}(0, x_2, x_3) = \tau(x_2, x_3), \end{gathered}$$

and the compatibility conditions

$$\varphi_{12}(x_1, 0) = \varphi_{13}(x_1, 0) = \varphi_1(x_1), \quad \varphi_{12}(0, x_2) = \varphi_{23}(x_2, 0) = \varphi_2(x_2),$$

$$\varphi_{13}(0,x_3)=\varphi_{23}(0,x_3)=\varphi_3(x_3),\quad \varphi_1(0)=\varphi_2(0)=\varphi_3(0)=\varphi_0,$$

are given by formula

$$\begin{aligned}
u(x_1,x_2,x_3)&=[v_{x_1}-a^{1,1,1}v](0,0,0;x)\varphi_0\\
&+\int_0^{x_1}([v_{x_1}-a^{1,1,1}v]\varphi_1'-a^{0,1,1}v\varphi_1)(y_1,0,0;x)dy_1\\
&+\int_0^{x_2}([v_{x_1}-a^{1,1,1}v]\varphi_2'+[(a^{2,0,1}v)_{x_1}-a^{1,0,1}v]\varphi_2)(0,y_2,0;x)dy_2\\
&+\int_0^{x_3}([v_{x_1}-a^{1,1,1}v]\varphi_3'+[(a^{2,1,0}v)_{x_1}-a^{1,1,0}v]\varphi_3)(0,0,y_3;x)dy_3\\
&+\int_0^{x_1}\int_0^{x_2}\big(v_{x_1}\big[\frac{\partial^2\varphi_{12}}{\partial y_1\partial y_2}+a^{2,0,1}\frac{\partial\varphi_{12}}{\partial y_1}\big]\\
&-v\big[a^{1,1,1}\frac{\partial^2\varphi_{12}}{\partial y_1\partial y_2}-(a^{2,0,1}_{x_1}-a^{1,0,1})\frac{\partial\varphi_{12}}{\partial y_1}+a^{0,1,1}\frac{\partial\varphi_{12}}{\partial y_2}+a^{0,0,1}\varphi_{12}\big]\big)(y_1,y_2,0;x)dy_1dy_2\\
&+\int_0^{x_1}\int_0^{x_3}\big(v_{x_1}\big[\frac{\partial^2\varphi_{13}}{\partial y_1\partial y_3}+a^{2,1,0}\frac{\partial\varphi_{13}}{\partial y_1}\big]\\
&-v\big[a^{1,1,1}\frac{\partial^2\varphi_{13}}{\partial y_1\partial y_3}-(a^{2,1,0}_{x_1}-a^{1,1,0})\frac{\partial\varphi_{13}}{\partial y_1}+a^{0,1,1}\frac{\partial\varphi_{13}}{\partial y_3}+a^{0,1,0}\varphi_{13}\big]\big)(y_1,0,y_3;x)dy_1dy_3\\
&+\int_0^{x_2}\int_0^{x_3}\big(v_{x_1}\big[\frac{\partial^2\varphi_{23}}{\partial y_2\partial y_3}+a^{2,1,0}\frac{\partial\varphi_{23}}{\partial y_2}+a^{2,0,1}\frac{\partial\varphi_{23}}{\partial y_3}+a^{2,0,0}\varphi_{23}\big]\\
&-v\big[a^{1,1,1}\frac{\partial^2\varphi_{23}}{\partial y_2\partial y_3}-(a^{2,1,0}_{x_1}-a^{1,1,0})\frac{\partial\varphi_{23}}{\partial y_2}-(a^{2,0,1}_{x_1}-a^{1,0,1})\frac{\partial\varphi_{23}}{\partial y_3}-(a^{2,0,0}_{x_1}-a^{1,0,0})\varphi_{23}\\
&+\frac{\partial^2\tau}{\partial y_2\partial y_3}+a^{2,1,0}\frac{\partial\tau}{\partial y_2}+a^{2,0,1}\frac{\partial\tau}{\partial y_3}+a^{2,0,0}\tau\big]\big)(0,y_2,y_3;x)dy_2dy_3\\
&-\int_{x_1^0}^{x_1}\int_{x_2^0}^{x_2}\int_{x_3^0}^{x_3}v(y;x)f(y)dy_1dy_2dy_3.
\end{aligned}$$

Now, putting $x_1=x_1^0$ in the last expression and taking into account that $u(x_1^0,x_2,x_3)=\psi(x_2,x_3)$ we come after some transformations to the Volterra integral equation with regard to the function $\tau(x_2,x_3)$:

$$\begin{aligned}
&v(0,x_2,x_3;x_1^0,x_2,x_3)\tau(x_2,x_3)+\int_0^{x_3}\theta_1(0,x_2,y_3;x_1^0,x_2,x_3)\tau(x_2,y_3)dy_3\\
&+\int_0^{x_2}\int_0^{x_3}\theta_2(0,y_2,y_3;x_1^0,x_2,x_3)\tau(y_2,y_3)dy_2dy_3=\chi(x_2,x_3),
\end{aligned}\tag{3.2}$$

where θ_1, θ_2 and χ are known functions. As it is well-known the last equation is solvable if

$$v(0,x_2,x_3;x_1^0,x_2,x_3)\neq 0,\quad 0\le x_2\le x_2^0,\ 0\le x_3\le x_3^0.$$

Further, according to the fourth condition of (2.2) for the Riemann function we have

$$\begin{gathered}
[v_{x_1x_1}-(a^{1,1,1}v)_{x_1}+a^{0,1,1}v](x_1,x_2,x_3;x_1^0,x_2,x_3)=0,\\
0\le x_1\le x_1^0,\ 0\le x_2\le x_2^0,\ 0\le x_3\le x_3^0.
\end{gathered}$$

Consider the last expression as an ordinary differential equation with respect to x_1, for fixed x_2 and x_3, and rewrite it as

$$\begin{aligned} &v_{x_1x_1}(x_1,x_2,x_3;x_1^0,x_2,x_3) - a^{1,1,1}(x_1,x_2,x_3)v_{x_1}(x_1,x_2,x_3;x_1^0,x_2,x_3) \\ &\quad + [a^{0,1,1}(x_1,x_2,x_3) - a^{1,1,1}_{x_1}(x_1,x_2,x_3)]v(x_1,x_2,x_3;x_1^0,x_2,x_3) = 0. \end{aligned} \tag{3.3}$$

Now, if we assume (1.7) holds, then the solution of (3.3) is monotonous. Taking into account that due to the last differential relations of (2.2)

$$v(x_1^0,x_2,x_3;x_1^0,x_2,x_3) = 0, \quad v_{x_1}(x_1^0,x_2,x_3;x_1^0,x_2,x_3) = 1$$

we have

$$v(0,x_2,x_3;x_1^0,x_2,x_3) \neq 0, \quad 0 \le x_2 \le x_2^0,\ 0 \le x_3 \le x_3^0.$$

Further, assuming (1.7) holds, (3.2) is uniquely solvable with regard to the function $\tau(x_2,x_3)$. Replacing the last condition of (1.5) by $u_{x_1}(0,x_2,x_3) = \tau(x_2,x_3)$ we come to the problem (1.1), (1.2), (1.3) which solution will satisfy conditions (1.5). This proves the Theorem (1.3).

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