Electronic Journal of Differential Equations, Vol. **2002**(2002), No. 88, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

Existence of global solutions to reaction-diffusion systems with nonhomogeneous boundary conditions via a Lyapunov functional *

Said Kouachi

Abstract

Most publications on reaction-diffusion systems of m components ($m \geq 2$) impose m inequalities to the reaction terms, to prove existence of global solutions (see Martin and Pierre [10] and Hollis [4]). The purpose of this paper is to prove existence of a global solution using only one inequality in the case of 3 component systems. Our technique is based on the construction of polynomial functionals (according to solutions of the reaction-diffusion equations) which give, using the well known regularizing effect, the global existence. This result generalizes those obtained recently by Kouachi [6] and independently by Malham and Xin [9].

1 Introduction

We consider the reaction-diffusion system

$$\partial_t u - a\Delta u = f(u,v,w) \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{1.1}$$
$$\partial_t v - b\Delta v = g(u,v,w) \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{1.2}$$
$$\partial_t w - c\Delta w = h(u,v,w) \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{1.3}$$

with the boundary conditions

$$\begin{aligned} &\lambda_1 u + (1-\lambda_1)\partial_\eta u = \beta_1, \\ &\lambda_2 v + (1-\lambda_2)\partial_\eta v = \beta_2, \quad \text{on } \mathbb{R}^+ \times \partial\Omega \\ &\lambda_3 w + (1-\lambda_3)\partial_\eta w = \beta_3, \end{aligned} \tag{1.4}$$

and the initial data

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad w(0,x) = w_0(x) \quad \text{in } \Omega. \tag{1.5}$$

The boundary conditions are specified as follows:

Mathematics Subject Classifications: 35K45, 35K57.
Key words: Reaction diffusion systems, Lyapunov functionals, global existence.

Submitted December 13, 2001. Published October 16, 2002.

(i) For nonhomogeneous Robin boundary conditions, we use $0 < \lambda_1$, λ_2, $\lambda_3 < 1$, $\beta_1 \geq 0$, β_2 and $\beta_3 \geq 0$.

(ii) For homogeneous Neumann boundary conditions, we use $\lambda_i = \beta_i = 0$, $i = 1, 2, 3$.

(iii) For homogeneous Dirichlet boundary conditions, we use $1 - \lambda_i = \beta_i = 0$, $i = 1, 2, 3$.

(iv) For a mixture of homogeneous Dirichlet with nonhomogeneous Robin boundary conditions, we use $1 - \lambda_i = \beta_i = 0$, $i = 1$, or 2 or 3 and $0 < \lambda_j < 1$, $\beta_j \geq 0$, $j = 1, 2, 3$, with $i \neq j$.

Here Ω is an open bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, ∂_η denotes the outward normal derivative on $\partial\Omega$, a, b and c are positive constants, $0 \leq \lambda_1, \lambda_2, \lambda_3 \leq 1$ and β_1, β_2 and $\beta_3 \geq 0$ are in $C^1(\partial\Omega; \mathbb{R}$.

The initial data are assumed to be nonnegative. The functions f, g and h are continuously differentiable on $\mathbb{R}^3_+$ satisfying $f(0, v, w) \geq 0$, $g(u, 0, w) \geq 0$ and $h(u, v, 0) \geq 0$ for all $u, v, w \geq 0$ which imply, via the maximum principle (see Smoller [14]), the positivity of the solution on its interval of existence. We suppose that the functions f, g and h are of polynomial growth and satisfy

$$Df(u, v, w) + Eg(u, v, w) + h(u, v, w) \leq C_1(u + v + w + 1), \tag{1.6}$$

for all $u, v, w \geq 0$ and all constants $D \geq \overline{D}$ and $E \geq \overline{E}$, where $\overline{D}$ and $\overline{E}$ are positive constants.

Morgan [11] generalized the results of Hollis, Martin and Pierre [4] to establish global existence for solutions of m-components systems ($m \geq 2$) with the boundary conditions (1.4), where

$$0 < \lambda_1, \lambda_2, \lambda_3 < 1 \quad \text{or} \quad \lambda_1 = \lambda_2 = \lambda_3 = 1, \quad \beta_1, \beta_2, \beta_3 \geq 0, \tag{1.7}$$

or

$$\lambda_1 = \lambda_2 = \lambda_3 = \beta_1 = \beta_2 = \beta_3 = 0, \tag{1.8}$$

and where again the reaction terms are polynomially bounded and satisfy, in the case of our system, the conditions

$$\begin{aligned} a_{11} f(u, v, w) &\leq c_{11} u + c_{12} v + c_{13} w + d_1, \\ (a_{21} f + a_{22} g)(u, v, w) &\leq c_{21} u + c_{22} v + c_{23} w + d_2, \\ (a_{31} f + a_{32} g + a_{33} h)(u, v, w) &\leq c_{31} u + c_{32} v + c_{33} w + d_3, \end{aligned} \tag{1.9}$$

for all $u, v, w \geq 0$ where a_{ij}, c_{ij} and d_i, $1 \leq i, j \leq 3$ are positive reals. Martin and Pierre [4] and Hollis [3] extended the results, under the same conditions, to the boundary conditions (1.4) where in (1.7), they took

$$0 \leq \lambda_1, \lambda_2, \lambda_3 \leq 1, \quad \beta_1, \beta_2, \beta_3 \geq 0, \tag{1.10}$$

but they imposed conditions of the form (1.9), at the same time, to the reaction terms whose corresponding components of the solution satisfy Neumann boundary conditions and to the others which satisfy Dirichlet boundary conditions. In other terms they imposed to the reaction terms to satisfy m inequalities. For example if $1-\lambda_3=\beta_3=0$, they took in (1.9) $a_{31}=a_{32}=0$. In this paper we show the global existence of a unique solution to problem (1.1)-(1.5) without using the first and the second conditions in (1.9), but only under the last one which is (1.6) and without distinguishing between the components of the solution which satisfy the one or the other type of boundary conditions; for this purpose, we use Lyapunov technique (see Kirane and Kouachi[5], Kouachi[6] and Kouachi and Youkana[7]).

2 Preliminary observations

The usual norms in spaces $L^p(\Omega)$,$L^\infty(\Omega)$ and $C(\overline{\Omega})$ are denoted respectively by

$$\|u\|_p^p=\frac{1}{|\Omega|}\int_\Omega |u(x)|^p dx, \tag{2.1}$$

$$\|u\|_\infty=\max_{x\in\Omega}|u(x)|. \tag{2.2}$$

It is well known that to prove global existence of solutions to (1.1)-(1.5) (see Henry [2], pp. 35-62), it suffices to derive a uniform estimate of $\|f(u,v,w)\|_p$, $\|g(u,v,w)\|_p$and $\|h(u,v,w)\|_p$on $[0,T_{\max}[$ in the space $L^p(\Omega)$ for some $p>N/2$. Our aim is to construct polynomial Lyapunov functionals allowing us to obtain L^p-bounds on u,v and w that lead to global existence.

Since the functions f,g and h are continuously differentiable on $\mathbb{R}_+^3$, then for any initial data in $C(\overline{\Omega})$, it is easy to check directly their Lipschitz continuity on bounded subsets of the domain of a fractional power of the operator

$$\begin{pmatrix} -a\Delta & 0 & 0\\ 0 & -b\Delta & 0\\ 0 & 0 & -c\Delta \end{pmatrix} \tag{2.3}$$

Under these assumptions, the following local existence result is well known (see Friedman [1] and Pazy [12]).

Proposition 2.1 *The system (1.1)-(1.5) admits a unique, classical solution (u,v,w) on $(0,T_{\max}[\times\Omega$. If $T_{\max}<\infty$ then*

$$\lim_{t\nearrow T_{\max}}\{\|u(t,.)\|_\infty+\|v(t,.)\|_\infty+\|w(t,.)\|_\infty\}=\infty, \tag{2.4}$$

where $T_{\max}(\|u_0\|_\infty,\|v_0\|_\infty,\|w_0\|_\infty)$ denotes the eventual blow-up time.

3 Results

Put $A = \sqrt{\frac{(\frac{b+c}{2})^2}{bc}}$, $B = \sqrt{\frac{(\frac{a+c}{2})^2}{ac}}$ and $C = \sqrt{\frac{(\frac{a+b}{2})^2}{ab}}$. Let θ and σ be two positive constants such that

$$(\sigma^2 - A^2)(\theta^2 - B^2) - (C - AB)^2 > 0 \quad \text{and} \quad \theta > C, \tag{3.1}$$

and let

$$\theta_q = \theta^{q^2} \quad \text{and} \quad \sigma_p = \sigma^{p^2}, \quad \text{for } q = 1, \dots, p \text{ and } p = 1, \dots, n, \tag{3.2}$$

where n is a positive integer. The main result of the paper reads as follows.

Theorem 3.1 *Suppose that the functions f, g and h are of polynomial growth and satisfy condition (1.6) for some positive constants D and E sufficiently large. Let $(u(t,.), v(t,.), w(t,.))$ be a solution of (1.1)-(1.5) and let*

$$L(t) = \int_\Omega H_n(u(t,x), v(t,x), w(t,x))dx, \tag{3.3}$$

where

$$H_n(u,v,w) = \sum_{p=0}^{n} \sum_{q=0}^{p} C_n^p C_p^q \theta_q \sigma_p u^q v^{p-q} w^{n-p}. \tag{3.4}$$

Then the functional L is uniformly bounded on the interval $[0, T_{\max}]$.

Corollary 3.2 *Under the hypotheses of theorem 3.1 all solutions of (1.1)-(1.5) with positive bounded initial data are global.*

Proposition 3.3 *If $\beta_1 = \beta_2 = \beta_3 = C_1 = 0$, then under the hypotheses of theorem 3.1 all solutions of (1.1)-(1.5) with positive bounded initial data are global and uniformly bounded on Ω.*

4 Proofs

For the proof of theorem 3.1, we need some preparatory Lemmata.

Lemma 4.1 *Let H_n be the homogeneous polynomial defined by (3.4). Then*

$$\partial_u H_n = n \sum_{p=0}^{n-1} \sum_{q=0}^{p} C_{n-1}^p C_p^q \theta_{q+1} \sigma_{p+1} u^q v^{p-q} w^{(n-1)-p}, \tag{4.1}$$

$$\partial_v H_n = n \sum_{p=0}^{n-1} \sum_{q=0}^{p} C_{n-1}^p C_p^q \theta_q \sigma_{p+1} u^q v^{p-q} w^{(n-1)-p}, \tag{4.2}$$

$$\partial_w H_n = n \sum_{p=0}^{n-1} \sum_{q=0}^{p} C_{n-1}^p C_p^q \theta_q \sigma_p u^q v^{p-q} w^{(n-1)-p}. \tag{4.3}$$

Proof. Differentiating H_n with respect to u yields

$$\partial_u H_n = \sum p = 0^n \sum_{q=0}^{p} q C_n^p C_p^q \theta_q \sigma_p u^{q-1} v^{p-q} w^{n-p}.$$

Using the fact that

$$qC_p^q = pC_{p-1}^{q-1} \quad \text{and} \quad pC_n^p = nC_{n-1}^{p-1}, \tag{4.4}$$

for $q = 1, \dots, p$ and $p = 1, \dots, n$, we get

$$\partial_u H_n = n \sum_{p=1}^{n} \sum_{q=1}^{p} C_{n-1}^{p-1} C_{p-1}^{q-1} \theta_q \sigma_p u^{q-1} v^{p-q} w^{n-p},$$

while changing in the sums the indexes $q-1$ by q and $p-1$ by p, we deduce (4.1). For the formula (4.2), differentiating H_n with respect to v gives

$$\partial_v H_n = \sum_{p=1}^{n} \sum_{q=0}^{p-1} (p-q) C_n^p C_p^q \theta_q \sigma_p u^q v^{p-q-1} w^{n-p}.$$

Taking account of

$$C_p^q = C_p^{p-q}, q = 1, \dots, p \quad \text{and} \quad p = 1, \dots, n, \tag{4.5}$$

using (4.4) and changing the index $p-1$ by p, we get (4.2).

Finally, we have

$$\partial_w H_n = \sum_{p=0}^{n} \sum_{q=0}^{p} (n-p) C_n^p C_p^q \theta_q \sigma_p u^q v^{p-q} w^{n-p-1}.$$

Since $(n-p)C_n^p = (n-p)C_n^{n-p} = nC_{n-1}^{n-p-1} = nC_{n-1}^p$, then we get (4.3).

Lemma 4.2 *The second partial derivatives of H_n are given by*

$$\partial_{u^2} H_n = n(n-1) \sum_{p=0}^{n-2} \sum_{q=0}^{p} C_{n-2}^p C_p^q \theta_{q+2} \sigma_{p+2} u^q v^{p-q} w^{(n-2)-p}, \tag{4.6}$$

$$\partial_{uv} H_n = n(n-1) \sum_{p=0}^{n-2} \sum_{q=0}^{p} C_{n-2}^p C_p^q \theta_{q+1} \sigma_{p+2} u^q v^{p-q} w^{(n-2)-p}, \tag{4.7}$$

$$\partial_{uw} H_n = n(n-1) \sum_{p=0}^{n-2} \sum_{q=0}^{p} C_{n-2}^p C_p^q \theta_{q+1} \sigma_{p+1} u^q v^{p-q} w^{(n-2)-p}, \tag{4.8}$$

$$\partial_{v^2} H_n = n(n-1) \sum_{p=0}^{n-2} \sum_{q=0}^{p} C_{n-2}^p C_p^q \theta_q \sigma_{p+2} u^q v^{p-q} w^{(n-2)-p}, \tag{4.9}$$

$$\partial_{vw} H_n = n(n-1) \sum_{p=0}^{n-2} \sum_{q=0}^{p} C_{n-2}^p C_p^q \theta_q \sigma_{p+1} u^q v^{p-q} w^{(n-2)-p}, \tag{4.10}$$

$$\partial_{w^2} H_n = n(n-1) \sum_{p=0}^{n-2} \sum_{q=0}^{p} C_{n-2}^p C_p^q \theta_q \sigma_p u^q v^{p-q} w^{(n-2)-p}. \tag{4.11}$$

Proof. Differentiating $\partial_u H_n$, given by the formula (4.1), with respect to u yields

$$\partial_{u^2} H_n = n \sum_{p=0}^{n-1} \sum_{q=0}^{p} q C_{n-1}^p C_p^q \theta_{q+1} \sigma_{p+1} u^{q-1} v^{p-q} w^{(n-1)-p}.$$

Using (4.4) we get (4.6)

$$\partial_{uv} H_n = \partial_v(\partial_u H_n) = n \sum_{p=0}^{n-1} \sum_{q=o}^{p} \sum (p-q) C_{n-1}^p C_p^q \theta_{q+1} \sigma_{p+1} u^q v^{p-q-1} w^{(n-1)-p}.$$

Applying (4.5) and then (4.4) we get (4.7).

$$\partial_{uw} H_n = \partial_w(\partial_u H_n) = n \sum_{p=0}^{n-1} \sum_{q=0}^{p} ((n-1)-p) C_{n-1}^p C_p^q \theta_{q+1} \sigma_{p+1} u^q v^{p-q-1} w^{n-2-p}.$$

Applying successively (4.5), (4.4) and (4.5) a second time we deduce (4.8).

$$\partial_{v^2} H_n = n \sum_{p=0}^{n-1} \sum_{q=0}^{p} (p-q) C_{n-1}^p C_p^q \theta_q \sigma_{(p+1)} u^q v^{p-q-1} w^{(n-1)-p}.$$

An application of (4.5) and then (4.4) yields (4.9).

$$\partial_{vw} H_n = \partial_v(\partial_w H_n) = n \sum_{p=0}^{n-1} \sum_{q=0}^{p} (p-q) C_{n-1}^p C_p^q \theta_q \sigma_p u^q v^{p-q-1} w^{(n-1)-p}.$$

One applies (4.5) and then (4.4), (4.10) yields. Finally we get (4.11), by differentiating $\partial_w H_n$ with respect to w and applying successively (4.5), (4.4) and (4.5) a second time. □

Proof of Theorem 3.1. Differentiating L with respect to t yields

$$\begin{aligned} L'(t) &= \int_\Omega \big[\partial_u H_n \frac{\partial u}{\partial t} + \partial_v H_n \frac{\partial v}{\partial t} + \partial_w H_n \frac{\partial w}{\partial t}\big] dx \\ &= \int_\Omega (a \partial_u H_n \Delta u + b \partial_v H_n \Delta v + c \partial_w H_n \Delta w) dx \\ &\quad + \int_\Omega (f \partial_u H_n + g \partial_v H_n + h \partial_w H_n) dx \\ &= I + J. \end{aligned}$$

Using Green's formula and applying Lemma 4.1 we get $I = I_1 + I_2$, where

$$I_1 = \int_{\partial\Omega} (a\partial_u H_n \partial_\eta u + b\partial_v H_n \partial_\eta v + c\partial_w H_n \partial_\eta w)\, dx,$$

$$I_2 = -n(n-1) \int_\Omega \sum_{p=0}^{n-2} \sum_{q=0}^{p} C_{n-2}^p C_p^q [(A_{pq} z).z]\, dx, \tag{4.12}$$

where

$$A_{pq} = \begin{pmatrix} a\theta_{q+2}\sigma_{p+2} & (\frac{a+b}{2})\theta_{q+1}\sigma_{p+2} & (\frac{a+c}{2})\theta_{q+1}\sigma_{p+1} \\ (\frac{a+b}{2})\theta_{q+1}\sigma_{p+2} & b\theta_q\sigma_{p+2} & (\frac{b+c}{2})\theta_q\sigma_{p+1} \\ (\frac{a+c}{2})\theta_{q+1}\sigma_{p+1} & (\frac{b+c}{2})\theta_q\sigma_{p+1} & c\theta_q\sigma_p \end{pmatrix} \tag{4.13}$$

for $q = 1, \dots, p$, $p = 1, \dots, n-2$, and $z = (\nabla u, \nabla v, \nabla w)^t$.

We prove that there exists a positive constant C_2 independent of $t \in [0, T_{\max}[$ such that

$$I_1 \le C_2 \text{ for all } t \in [0, T_{\max}[\tag{4.14}$$

and that

$$I_2 \le 0 \tag{4.15}$$

for several boundary conditions.

(i) If $0 < \lambda_1, \lambda_2, \lambda_3 < 1$, using the boundary conditions (1.4) we get

$$I_1 = \int_{\partial\Omega} (a\partial_u H_n(\gamma_1 - \alpha_1 u) + b\partial_v H_n(\gamma_2 - \alpha_2 v) + c\partial_w H_n(\gamma_3 - \alpha_3 w))dx,$$

where $\alpha_i = \lambda_i/(1-\lambda_i)$ and $\gamma_i = \beta_i/(1-\lambda_i)$, $i = 1, 2, 3$. Since $H(u, v, w) = a\partial_u H_n(\gamma_1 - \alpha_1 u) + b\partial_v H_n(\gamma_2 - \alpha_2 v) + c\partial_w H_n(\gamma_3 - \alpha_3 w) = P_{n-1}(u, v, w) - Q_n(u, v, w)$, where P_{n-1} and Q_n are polynomials with positive coefficients and respective degrees n and $n-1$ and since the solution is positive, then

$$\limsup_{(|u|+|v|+|w|)\to+\infty} H(u, v, w) = -\infty, \tag{4.16}$$

which prove that H is uniformly bounded on $\mathbb{R}_+^3$ and consequently (4.12).

(ii) If $\lambda_1 = \lambda_2 = \lambda_3 = 0$, then $I_1 = 0$ on $[0, T_{\max}[$.

(iii) The case of homogeneous Dirichlet conditions is trivial, since in this case the positivity of the solution on $[0, T_{\max}[\times\Omega$ implies $\partial_\eta u \le 0, \partial_\eta v \le 0$ and $\partial_\eta w \le 0$ on $[0, T_{\max}[\times\partial\Omega$. Consequently one gets again (4.12) with $C_2 = 0$.

(iv) If one or two of the components of the solution satisfy homogeneous Dirichlet boundary conditions and the other (others) satisfies the nonhomogeneous Robin conditions; for example $u = 0$, $\lambda_2 v + (1-\lambda_2)\partial_\eta v = \beta_2$ and $\lambda_3 w + (1-\lambda_3)\partial_\eta w = \beta_3$ on $[0, T_{\max}[\times\partial\Omega$ with $0 < \lambda_2, \lambda_3 < 1$ and $\beta_2, \beta_3 \ge 0$. Then, following the same reasoning as above we get

$$\limsup_{(|v|+|w|)\to+\infty} H(0, v, w) = -\infty, \tag{4.17}$$

and then (4.12).

Now we prove (4.15). The quadratic forms (with respect to ∇u, ∇v and ∇w) associated with the matrices A_{pq}, $q = 1, \dots, p$ and $p = 1, \dots, n-2$ are positive since their main determinants Δ_1, Δ_2 and Δ_3 are too according to Sylvister criterium. To see this, we have

1. $\Delta_1 = a\theta_{q+2}\sigma_{p+2} > 0$, for $q = 1, \dots, p$ and $p = 1, \dots, n-2$,

2.

$$\begin{aligned}\Delta_2 =&\begin{vmatrix} a\theta_{q+2}\sigma_{p+2} & (\frac{a+b}{2})\theta_{q+1}\sigma_{p+2} \\ (\frac{a+b}{2})\theta_{q+1}\sigma_{p+2} & b\theta_q\sigma_{p+2}\end{vmatrix}\\ =&ab((\theta_{q+2}\sigma_{p+2})(\ \theta_q\sigma_{(p+2)}) - (C\theta_{q+1}\sigma_{p+2})^2)\\ =&ab\sigma_{p+2}^2(\frac{\theta_q\theta_{q+2}}{\theta_{q+1}^2} - C^2)\theta_{q+1}^2 = ab\sigma_{p+2}^2(\theta^2 - C^2)\theta_{q+1}^2,\end{aligned}$$

for $q = 1, \dots, p$ and $p = 1, \dots, n-2$. Using (3.1), we get $\Delta_2 > 0$.

3.

$$\begin{aligned}c\theta_q\sigma_p\Delta_3 =&c\theta_q\sigma_p\begin{vmatrix} a\theta_{q+2}\sigma_{p+2} & (\frac{a+b}{2})\theta_{q+1}\sigma_{p+2} & (\frac{a+c}{2})\theta_{q+1}\sigma_{p+1} \\ (\frac{a+b}{2})\theta_{q+1}\sigma_{p+2} & b\theta_q\sigma_{p+2} & (\frac{b+c}{2})\theta_q\sigma_{p+1} \\ (\frac{a+c}{2})\theta_{q+1}\sigma_{p+1} & (\frac{b+c}{2})\theta_q\sigma_{p+1} & c\theta_q\sigma_p\end{vmatrix}\\ =&-(c(\frac{a+b}{2}) - (\frac{a+c}{2})(\frac{b+c}{2}))^2(\theta_q\sigma_{(p+2)}\theta_{q+1}\sigma_p)^2\\ &+ abc^2(\sigma_p\sigma_{p+2} - A^2\sigma_{p+1}^2)(\theta_q\theta_{q+2} - B^2\theta_{q+1}^2)\theta_q^2\sigma_p^2\\ =&\big[(\frac{\sigma_p\sigma_{p+2}}{\sigma_{p+1}^2} - A^2)(\frac{\theta_q\theta_{q+2}}{\theta_{q+1}^2} - B^2) - (C - AB)^2\big]\sigma_{(p+1)}^2\theta_{q+1}^2\\ =&\big[(\sigma^2 - A^2)(\theta^2 - B^2) - (C - AB)^2\big]\sigma_{(p+1)}^2\theta_{q+1}^2 > 0\,,\end{aligned}$$

for $q = 1, \dots, p$ and $p = 1, \dots, n-2$. Using again (3.1), we get $\Delta_3 > 0$. Consequently we have (4.15). Substituting the expressions of the partial derivatives given by lemma 4.1 in the second integral, yields

$$\begin{aligned}J =& \int_\Omega \Big[n\sum_{p=0}^{n-1}\sum_{q=0}^{p} C_{n-1}^p C_p^q u^q v^{p-q} w^{(n-1)-p}\Big]\big(\theta_{q+1}\sigma_{p+1}f + \theta_q\sigma_{p+1}g + \theta_q\sigma_p h\big)dx\\ =& \int_\Omega \Big[n\sum_{p=0}^{n-1}\sum_{q=0}^{p} C_{n-1}^p C_p^q u^q v^{p-q} w^{(n-1)-p}\Big]\big(\frac{\theta_{q+1}}{\theta_q}\frac{\sigma_{p+1}}{\sigma_p}f + \frac{\sigma_{p+1}}{\sigma_p}g + h\big)\theta_q\sigma_p dx.\end{aligned}$$

Using condition (1.6), we deduce

$$J \le C_3\int_\Omega \sum_{p=0}^{n-1}\sum_{q=0}^{p} C_{n-1}^p C_p^q u^q v^{p-q} w^{(n-1)-p}\,[(u+v+1)]\,dx.$$

Applying Holder's inequality to the integrals

$$\int_\Omega u^q v^{p-q} w^{(n-1)-p}\,[(u+v+1)]\,dx, \quad q = 1, \dots, p \text{ and } p = 1, \dots, n-1,$$

and following the same reasoning as in [7], one gets that there exist positive constants C_4 and C_5 such that the functional L satisfies the differential inequality

$$L'(t) \leq C_4 L(t) + C_5 L^{(n-1)/n}(t),$$

which can be written

$$nZ' \leq C_4 Z + C_5,$$

if $Z = L^{1/n}$. A simple integration of the later inequality gives the uniform bound of the functional L on the interval $[0, T^*]$; this ends the proof of the theorem.

Proof of corollary 3.2. The proof is an immediate consequence of theorem 3.1, the preliminary observations and the inequality

$$\int_\Omega (u(t,x) + v(t,x) + w(t,x))^n dx \leq C_6 L(t) \quad \text{on } [0, T^*[, \tag{4.18}$$

for some $n > N/2$.

Proof of proposition 3.3. In this case the functional L is of Lyapunov and then gives

$$L(t) \ \leq L(0) \quad \text{on } [0, T^*[.$$

Using (4.18), we get the uniform boundedness of the solution on $[0, T^*[\times\Omega$.

5 Applications

In this section we apply corollary 3.2 and proposition 3.3 to some particular biochemical and chemical reaction models. Throughout this section we assume that all reactions take place in a bounded domain Ω with smooth boundary $\partial\Omega$.

Let us begin with the general three-component reaction

$$lU + qV \underset{k}{\overset{h}{\leftrightarrows}} rW, \tag{5.1}$$

which leads to the reaction diffusion system

$$\frac{\partial u}{\partial t} - a\Delta u = -hu^l v^q + kw^r \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{5.2}$$

$$\frac{\partial v}{\partial t} - b\Delta v = -hu^l v^q + kw^r \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{5.3}$$

$$\frac{\partial w}{\partial t}\text{-}c\Delta w = hu^l v^q - kw^r \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{5.4}$$

with boundary conditions (1.4) and positive initial data in $\mathbb{L}^\infty(\Omega)$, where h, k, l, q, and r are positive constants such that $r \leq 1$ or $l + q \leq 1$. The special case $l = q = r = 1$ has been studied by Rothe [13] under homogeneous Neumann boundary conditions where he showed that $T_{\max} = \infty$ if $N \leq 5$. Morgan [11] generalized the results of Rothe for every integer $N \geq 1$ and when all the components satisfy the same boundary conditions (Neumann or Dirichlet). Hollis [3] completed the work of Morgan and established global existence if w satisfies the same type of boundary conditions as either u or v. But if boundary conditions of different types are imposed on u and v, global existence follows regardless of the type of boundary condition that is imposed on w. Recently we have proved, in [6], global existence of solutions to system (5.2)-(5.4), under homogeneous Neumann boundary conditions, by studying the two coupled systems (5.2)-(5.4) and (5.3)-(5.4) when $r \leq 1$ or $l + q \leq 1$. However we have

Proposition 5.1 *Solutions of (5.2)-(5.4) with nonnegative uniformly bounded initial data and nonhomogeneous boundary conditions (1.4) are positive and exist globally for every positive constants l, q and r such that $r \leq 1$ or $l + q \leq 1$.*

Proof. Conditions (1.6) is trivial when $r \leq 1$ by choosing $D + E \gg 1$. In the case $l + q \leq 1$, it is also satisfied by studying the system in the order (5.4)-(5.3)-(5.2), choosing $E + 1 \gg D$ and by applying the Young inequality to the term $u^l v^q$ (see [7] for more of details). Then corollary 3.2 implies that all components of the solution are in $\mathbb{L}^\infty(0, T^*; \mathbb{L}^n(\Omega))$ for all $n \geq 1$. Since the reaction terms are of polynomial growth, then $T_{\max} = +\infty$. Another example, is as follows:

$$\frac{\partial u_1}{\partial t} - d_1 \Delta u_1 = -k_1 u_1 u_2 + k_2 u_3 - k_3 u_1 u_4 + k_4 u_5, \tag{5.5}$$

$$\frac{\partial u_2}{\partial t} - d_2 \Delta u_2 = -k_1 u_1 u_2 + k_2 u_3, \tag{5.6}$$

$$\frac{\partial u_3}{\partial t} - d_3 \Delta u_3 = k_1 A u_1 u_2 B - k_2 u_3, \tag{5.7}$$

$$\frac{\partial u_4}{\partial t} - d_4 \Delta u_4 = -k_3 u_1 u_4 + k_4 u_5, \tag{5.8}$$

$$\frac{\partial u_5}{\partial t} - d_5 \Delta u_5 = k_3 u_1 u_4 - k_4 u_5. \tag{5.9}$$

Hollis [3] established global existence provided that: (1) u_3 satisfies the same type of boundary conditions as either u_1 or u_2, and (2) u_5 satisfies the same type of boundary condition as either u_1 or u_4. Our generalization is summarized as

Proposition 5.2 *Solutions of (5.5)-(5.9) with nonnegative uniformly bounded initial data and boundary conditions (1.4) exist globally.*

Proof. Condition (1.6) is satisfied for the three component system (5.5)-(5.7)-(5.9) while choosing $u_3 \gg u_4 \gg 1$. Then corollary 3.2 implies that u_1, u_3 and u_5 are in $\mathbb{L}^\infty(0, T^*; \mathbb{L}^n(\Omega))$ for all $n \geq 1$. So, global existence of u_3 and u_4 is a trivial consequence of the maximum principle (see J. Smoller [14]) applied successively to equations (5.6) and (5.8). $\square$

Finally we illustrate our results with the system

$$\frac{\partial u}{\partial t} - a\Delta u = -u^{\alpha}v^{\beta} - u^{\gamma}w^{\rho} + \mu_1 v + \mu_2 w, \tag{5.10}$$

$$\frac{\partial v}{\partial t} - b\Delta v = -u^{\alpha}v^{\beta} + u^{\gamma}w^{\rho}, \tag{5.11}$$

$$\frac{\partial w}{\partial t} - c\Delta w = u^{\alpha}v^{\beta} + u^{\gamma}w^{\rho}\,, \tag{5.12}$$

where $\alpha, \beta, \gamma, \rho \geq 1$ and $\mu_1, \mu_2 \geq 0$. S. L. Hollis [3] established global existence provided that either: (1) $\mu_1 = 0$ and u satisfies the same type of boundary conditions as w, or else (2) $\mu_2 = 0$, $\rho = 1$, and u satisfies the same type of boundary conditions as v.

It is trivial to verify that condition (1.6) is satisfied for system (5.10)-(5.12) by choosing $D + 1 \gg E \gg 1$. the result of corollary 3.2 applied to this system is summarized in the following proposition

Proposition 5.3 *Solutions of (5.5)-(5.9) with nonnegative uniformly bounded initial data and boundary conditions (1.4) exist for all $t > 0$ and all constants $\alpha, \beta, \gamma, \rho \geq 1$, $\mu_1, \mu_2 \geq 0$.*

If $\beta_1 = \beta_2 = \beta_3 = \mu_1 = \mu_2 = 0$, then proposition 3.3 applied to (5.5)-(5.9) permits us to give the following result.

Corollary 5.4 *All solutions of (5.5)-(5.9) with positive initial data in $\mathbb{L}^{\infty}(\Omega)$ are global and uniformly bounded on $[0, +\infty[\times\Omega$ provided that $\beta_1 = \beta_2 = \beta_3 = \mu_1 = \mu_2 = 0$.*

Another example, is

$$\frac{\partial u}{\partial t} - a\Delta u = -a_{11}u^{r_1} + a_{12}v^{r_2} + a_{13}w^{r_3} - c_{11}u + c_{12}v + c_{13}w + d_1 \text{ in } \mathbb{R}^+ \times \Omega, \tag{5.13}$$

$$\frac{\partial v}{\partial t} - b\Delta v = a_{21}u^{r_1} - a_{22}v^{r_2} + a_{23}w^{r_3} + c_{21}u - c_{22}v + c_{23}w + d_2 \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{5.14}$$

$$\frac{\partial v}{\partial t} - c\Delta w = a_{31}u^{r_1} + a_{32}v^{r_2} - a_{33}w^{r_3} + c_{31}u + c_{32}v - c_{33}w + d_3 \quad \text{in } \mathbb{R}^+ \times \Omega, \tag{5.15}$$

where $r_i > 1$, a_{ij}, c_{ij}, d_i are positive for $1 \leq i \leq j \leq 3$. It is clear that conditions of the form (1.9) are not satisfied and then techniques used by Hollis [3] are not applicable here, nevertheless the method of invariant regions (see Smoller [14]) can give the global existence of positive solutions under some complicated conditions on the constants a_{ij}, c_{ij} and d_i, $1 \leq i \leq j \leq 3$. However our technique is applicable and if

$$A = \begin{pmatrix} -a_{11} & a_{12} & a_{13} \\ a_{21} & -a_{22} & a_{23} \\ a_{31} & a_{32} & -a_{33} \end{pmatrix}, \tag{5.16}$$

then we have the following statement.

Proposition 5.5 *Suppose that for some $k = \{1,2,3\}$, the cofactor of $-a_{kk}$ is strictly positive, then solutions of (5.13)-(5.15) with nonnegative uniformly bounded initial data and boundary conditions (1.4) are positive and exist for all $t > 0$ for positive reals a_{ij}, c_{ij} and $d_i, 1 \leq i \leq j \leq 3$, provided that (1) $r_k \leq 1$ or (2) $r_i > 1$, $1 \leq i \leq 3$ and $\det A < 0$ or (3) $r_i > 1$, $1 \leq i \leq 3$ and a_{kk} is sufficiently large.*

Proof. Take for example $k = 3$
(1) Suppose that $a_{11}a_{22} - a_{12}a_{21} > 0$ and $r_3 \leq 1$, then (1.6) is satisfied if there exists D and E sufficiently large such that

$$-a_{11}D + a_{21}E + a_{31} \leq 0 \text{ and } a_{12}D - a_{22}E + a_{32} \leq 0. \tag{5.17}$$

The study of these two inequalities implies (1.6) if we choose D and E satisfying

$$D \geq \frac{a_{21}a_{32} + a_{31}a_{22}}{a_{11}a_{22} - a_{12}a_{21}} \quad \text{and} \quad E \geq \frac{a_{11}a_{32} + a_{12}a_{31}}{a_{11}a_{22} - a_{12}a_{21}}. \tag{5.18}$$

(2) Now, suppose that $r_i > 1$, $1 \leq i \leq 3$ and a_{33} is sufficiently large, then (1.6) is satisfied if there exists D and E sufficiently large such that (5.17) is too and

$$a_{13}D + a_{23}E - a_{33} \leq 0. \tag{5.19}$$

But if we develop the determinant of A with regard to the third column, then $\det A < 0$ is equivalent to

$$-a_{33}(a_{11}a_{22}-a_{12}a_{21})+a_{13}(a_{21}a_{32}+a_{31}a_{22})-a_{23}(-a_{11}a_{32}-a_{12}a_{31}) < 0, \tag{5.20}$$

and this inequality is equivalent to

$$a_{13}\Big(\frac{a_{21}a_{32} + a_{31}a_{22}}{a_{11}a_{22} - a_{12}a_{21}}\Big) + a_{23}\Big(\frac{a_{11}a_{32} + a_{12}a_{31}}{a_{11}a_{22} - a_{12}a_{21}}\Big) < a_{33}. \tag{5.21}$$

Then we can choose D and E such that (5.18) and (5.19) are satisfied. Consequently we get (1.6).
(3) This case is trivial by using (5.18) and (5.21).

References

[1] A. Friedman, Partial Differential Equations of Parabolic Type. Prentice Hall Englewood Chiffs. N. J. 1964.

[2] D. Henry, Geometric Theory of Semi-linear Parabolic Equations. Lecture Notes in Mathematics 840, Springer-Verlag, New-York, 1984.

[3] S. L. Hollis, On the Question of Global Existence for Reaction-Diffusion Systems with Mixed Boundary Conditions. Quarterly of Applied Mathematics LI, number 2, June 1993, 241-250.

[4] S. L. Hollis, R. H. Martin and M. Pierre, Global Existence and Boundedness in Reaction Diffusion Systems. SIAM. J. Math. Anal, Vol. 18, number 3, May 1987.

[5] M. Kirane and S. Kouachi, Global solutions to a system of strongly coupled reaction-diffusion equations. Nonlinear Analysis Theory, Methods and Applications. Volume 26, number 8 (1996).

[6] S. Kouachi, Global existence of solutions to reaction-diffusion systems via a Lyapunov functional. Electron. J. Diff. Eqns Vol. 2001(2001), No. 68, pp. 1-10.

[7] S. Kouachi and A. Youkana, Global existence for a class of reaction-diffusion systems. Bulletin of the Polish Academy of Sciences, Vol. 49, Number 3, (2001).

[8] S. Malham, Private communication.

[9] S. Malham and J. Xin, Global solutions to a reactive Boussinesq system with front data on an infinite domain. Comm. Math. Phys. 193 (1998), no. 2, 287–316.

[10] R. H. Martin and M. Pierre, Nonlinear reaction-diffusion systems. Nonlinear equations in the applied sciences, 363–398, Math. Sci. Engrg., 185, Academic Press, Boston, MA, 1992.

[11] J. Morgan, Global Existence for Semilinear Parabolic Systems, SIAM J. Math. Anal. 20, 1128-1144 (1989).

[12] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations. Applied Math. Sciences 44, Springer-Verlag, New York (1983).

[13] F. Rothe, Global Solutions of Reaction-Diffusion Systems, Lecture Notes in Math. 1072, Springer-Verlag, Berlin (1984).

[14] J. Smoller, Shock Waves and Reaction-Diffusion Equations, Springer-Verlag, New York (1983).

Said Kouachi
Centre Univ. Tébessa, Dépt. Mathématiques, 12002,
Tébessa, Algérie
and
Lab. Mathémathiques, Université d'Annaba,
B. P. 12, Annaba, 23200, Algérie
e-mail: kouachi.said@caramail.com