Electronic Journal of Differential Equations, Vol. **2002**(2002), No. 89, pp. 1–21.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

Elliptic equations with one-sided critical growth *

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Abstract

We consider elliptic equations in bounded domains $\Omega \subset \mathbb{R}^N$ with nonlinearities which have critical growth at $+\infty$ and linear growth λ at $-\infty$, with $\lambda > \lambda_1$, the first eigenvalue of the Laplacian. We prove that such equations have at least two solutions for certain forcing terms provided $N \geq 6$. In dimensions $N = 3, 4, 5$ an additional lower order growth term has to be added to the nonlinearity, similarly as in the famous result of Brezis-Nirenberg for equations with critical growth.

1 Introduction

We consider the superlinear problem

$$\begin{gathered} -\Delta u = \lambda u + u_+^{2^\star-1} + g(x,u_+) + f(x) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ is a bounded domain with smooth boundary, $2^\star = 2N/(N-2)$ is the critical Sobolev exponent, $g(\cdot, s_+) \in C(\bar\Omega \times \mathbb{R}, \mathbb{R}^+)$ has a subcritical growth at infinity, and $s_+ = \max\{s, 0\}$. Furthermore, we assume that $\lambda > \lambda_1$, the first eigenvalue of the Laplacian. This means that the function

$$k(s) = \lambda s + s_+^{2^\star-1} + g(x, s_+)$$

"interferes" with all but a finite number of eigenvalues of the Laplacian, in the sense that

$$\lambda_1 < \lambda = \lim_{s\to-\infty} \frac{k(s)}{s} < \lim_{s\to+\infty} \frac{k(s)}{s} = +\infty$$

For subcritical nonlinearities, such problems have been treated by Ruf-Srikanth [11] and de Figueiredo [4], proving the existence of at least two solutions provided that the forcing term $f(x)$ has the form

$$f(x) = h(x) + te_1(x), \tag{1.2}$$

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* *Mathematics Subject Classifications:* 35J20.
Key words: Nonlinear elliptic equation, critical growth, linking structure.

Submitted March 01, 2002. Published October 18, 2002.

where $h \in L^r(\Omega)$, for some $r > N$, is given, e_1 is the (positive and normalized) first eigenfunction of the Laplacian, and $t > T$, for some sufficiently large number $T = T(h)$.

We remark that the search of solutions for forcing terms of the form (1.2) is natural if the nonlinearity crosses *all* eigenvalues, i.e. $\lambda < \lambda_1$ (then the problem is of so-called Ambrosetti-Prodi type); indeed, in this case there exists an obvious *necessary condition* of the form $\int_\Omega f e_1 dx < c$. In the present case there does not seem to exist a necessary condition for solvability, and therefore one might expect solutions for *any forcing term* in $L^2(\Omega)$. This is an *open problem*; the only positive result known is for the corresponding ODE with Neumann boundary conditions, cf. [7].

Equation (1.1) with nonlinearities with critical growth have recently been considered by de Figueiredo - Jianfu [6], proving a similar existence result as the one stated for the subcritical case, however with the restriction that the space dimension N satisfies $N \geq 7$. While it is known that in problems with critical growth the low dimensions may show different behavior (see Brezis - Nirenberg [3], where different behavior occurs in dimension $N = 3$), it is somewhat surprising to encounter difficulties already in the dimensions $4, 5$ and 6.

The present work is motivated by the mentioned work of de Figueiredo and Jianfu [6], who treated equation

$$\begin{gathered} -\Delta u = \lambda u + u_+^{2^\star - 1} + f(x) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.3}$$

They established the existence of at least two solutions under suitable conditions on $f = h + te_1$, more precisely they proved:

Theorem 1.1 (de Figueiredo - Jianfu [6]) *Suppose that*
i) $\lambda > \lambda_1$
ii) $h \in L^r(\Omega), r > N$, *is given, with* $h \in \ker(-\Delta - \lambda)^\perp$ *if* λ *is an eigenvalue.*
Then there exists $T_0 = T_0(h) > 0$ *such that if* $t > T_0$ *then problem (1.3) has a negative solution* $\phi_t \in W^{2,r} \cap W_0^{1,r} \subset C^{1,1-N/r}$.

Suppose in addition that
iii) λ *is* not *an eigenvalue of* $(-\Delta, H_0^1(\Omega))$
iv) $N \geq 7$
Then problem (1.3) has a second solution for $t > T_0$.

We remark that in [6] only $h \in L^2(\Omega)$ is assumed, however this seems not sufficient to get the stated results.

In this paper we improve and extend the result of de Figueiredo and Jianfu in the following ways:

Main Results:

1) $\lambda > \lambda_1$ and λ is not an eigenvalue:
$N \geq 6$: if $h \in L^r(\Omega)$ $(r > N)$, then equation (1.1) has a second solution for $f = h + te_1$, with $t > T_0(h)$.

$N = 3, 4, 5$: if $h \in L^r(\Omega)$ $(r > N)$ and $g(x, s_+) \geq cs^q$ for some $q = q(N) > 1$ and $c > 0$, then equation (1.1) has a second solution for $f = h + te_1$, with $t > T_0$.

2) λ is an eigenvalue, i.e. $\lambda = \lambda_k$, for some $k \geq 2$: If $h \in L^r(\Omega)$ $(r > N)$ satisfies $h \perp ker(-\Delta - \lambda_k, H^1_0(\Omega))$, then equation (1.1) has a second solution for $f = h + te_1$ with $t > T_0$ (in dimensions $N = 3, 4, 5$ the assumptions made in 1) have to be added).

3) λ in a left neighborhood of λ_k, $k \geq 2$: There exists a $\delta > 0$ such that if $\lambda \in (\lambda_k - \delta, \lambda_k)$ and $h \in L^r(\Omega)$ $(r > N)$ satisfies $h \perp ker(-\Delta, H^1_0(\Omega)$, then equation (1.1) has at least three solutions for $f = h+te_1$, with $t > T_0$ (in dimensions $N = 3, 4, 5$ the assumptions made in 1) have to be added).

For proving these statements, one proceeds as follows: the first (negative) solution is easily obtained (see [11], [4], [6]). To obtain a second solution, one uses the saddle point structure around the first solution and applies the generalized mountain pass theorem of Rabinowitz [10]. To prove the Palais-Smale condition, one proceeds as in [3], using a sequence of concentrating functions (obtained from the so-called Talenti function). However due to the presence of the first solution, lower order terms appear in the estimates. To handle these estimates, an "orthogonalization" procedure based on separating the supports of the Talenti sequence and the (approximate) first solution is used (this approach was introduced in [8]). With this method we obtain the results 1) and 2).

To prove 3), one shows that the "branch" of solutions with $\lambda \in (\lambda_k, \lambda_{k+1})$ can be extended to $\lambda = \lambda_k$ if $h \perp ker(-\Delta)$. Actually, this branch can be extended slightly beyond λ_k, i.e. to $\lambda \in (\lambda_k - \delta, \lambda_k]$, and the corresponding solutions are clearly bounded away from the negative solution. Since in λ_k starts a bifurcation branch (λ, u) emanating from the negative solution and bending to the left (as shown by de Figueiredo - Jianfu [6]), we conclude that for λ to the left of and close to λ_k there exist at least three solutions.

2 Statement of theorems

In this section we give the precise statements of the theorems. Furthermore, the notation and basic properties are introduced. We consider problem (1.1) under the following conditions on the nonlinearity g:

(g_1) $g : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}^+$ is continuous;

(g_2) $g(x, s) \equiv 0$ for $s \leq 0$, i.e. $g(x, s) = g(x, s_+)$ with $g(x, 0) = 0$;

(g_3) There exist constants $c_1 > 0$ and $1 < p < (N + 2)/(N - 2)$ such that $g(x, s) \leq c_1|s|^p$, for all $s \in \mathbb{R}$;

For $N = 3, 4, 5$ we assume in addition:

(g_4) There exist $c > 0$ and q with $\max\left\{\frac{N}{2(N-1)}, \frac{2}{3}\right\} < \frac{q+1}{2^\star} < 1$, such that $g(x, s_+) \geq c(s_+)^q$, for all $s \in \mathbb{R}$.

We prove the following results:

Theorem 2.1 $N \geq 6$*: Let* $h \in L^r(\Omega), r > N$*, be given, and let* $T_0 = T_0(h)$ *the number given by Theorem 0. Assume that* $\lambda > \lambda_1$*, and that* g *satisfies* $(g_1), (g_2), (g_3)$*. If* λ *is an eigenvalue, say* $\lambda = \lambda_k$*, assume in addition that* $h \perp ker(-\Delta - \lambda_k)$*. Then problem (1.1) has a second solution for* $f = h + te_1$*, with* $t > T_0$*.*

Theorem 2.2 $3 \leq N \leq 5$*: Let* $h \in L^r, r > N$*, be given, and suppose that all other assumptions of Theorem 2.1 are satisfied. If* g *satisfies also* (g_4)*, then problem (1.1) has a second solution for* $f = h + te_1$*, with* $t > T_0$*.*

Theorem 2.3 *Assume the hypotheses of Theorems 2.1 and 2.2, and assume in addition that* $h \perp \ker(-\Delta - \lambda_k)$*, for some* $k \geq 2$*. Then there exists* $\delta > 0$ *such that for* $\lambda \in (\lambda_k - \delta, \lambda_k)$ *problem (1.1) has at least three solutions for* $f = h + te_1$*, with* $t > T_0$*.*

The first solution of equation (1.1) is a negative solution, and its existence, for t sufficiently large, is not difficult to prove (see [4], [6]): first note that a negative solution satisfies the *linear equation*

$$\begin{gathered} -\Delta y - \lambda y = h + te_1 \quad \text{in } \Omega \\ y = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{2.1}$$

The solution of this equation is unique, and we denote it, in dependence of te_1, by ϕ_t (we remark that if $\lambda = \lambda_k$ then $h \perp ker(-\Delta - \lambda_k)$ is required, and the solution is unique in $(ker(-\Delta - \lambda_k))^\perp$). Note that we may assume that $\int_\Omega he_1 dx = 0$, and then the solution ϕ_t can be written as $\phi_t = w + s_t e_1$, with $\int_\Omega we_1 dx = 0$ and $s_t = t/(\lambda_1 - \lambda)$ (and with $w \perp ker(-\Delta - \lambda_k)$ if $\lambda = \lambda_k$). Since $h \in L^r, r > N$, we have $w \in C^{1,1-N/r}(\overline{\Omega})$, and it is known (see [9]) that on $\partial\Omega$ the (interior) normal derivative $\frac{\partial}{\partial n} e_1(x)$ is positive; hence $\phi_t < 0$ for t sufficiently large.

To find a second solution of equation (1.1), we set $u = v + \phi_t$; then v solves

$$\begin{gathered} -\Delta v = \lambda v + (v + \phi_t)_+^{2^\star - 1} + g(x, (v + \phi_t)_+) \quad \text{in } \Omega \\ v = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{2.2}$$

Clearly $v = 0$ is a solution of this equation, corresponding to the negative solution ϕ_t for equation (1.1). To find a second solution of equation (2.2) one can look for non trivial critical points of the functional

$$J(v) = \frac{1}{2}\int_\Omega (|\nabla v|^2 - \lambda v^2)dx - \frac{1}{2^\star}\int_\Omega (v + \phi_t)_+^{2^\star} dx - \int_\Omega G(x, (v + \phi_t)_+)dx,$$

where $G(x, s) := \int_0^s g(x, \xi)\, d\xi$.

This was the approach of de Figueiredo-Jianfu in [6]. For applying the Generalized Mountain Pass theorem of Rabinowitz [10] one needs to prove some geometric estimates. Furthermore, since the nonlinearities have critical growth, one needs to show that the minimax level avoids the non-compactness levels given by the "concentrating sequences" u_ϵ (see [3] and below).

In these estimates, the terms of the form $v + \phi_t + u_\epsilon$ are not easy to handle. In this paper we apply a method introduced in [8] to make such estimates easier. The idea consists in separating the supports of $v + \phi_t$ and u_ϵ by concentrating the support of the functions u_ϵ in small balls, and "cutting small holes" into the functions $v + \phi_t$ such that the respective supports are disjoint. These manipulations create some errors, but these are easier to handle than to estimate the "mixed terms" arising in expressions like $(v + \phi_t + u_\epsilon)_+^p$. Moving these "small holes" near $\partial\Omega$ where the first solution ϕ_t is small allows to further improve the estimates.

3 Variational setting and preliminary properties

We begin by replacing equation (2.2) by an approximate equation. We denote by $B_r(x_0) \subset \Omega$ a ball of radius r and center $x_0 \in \Omega$. Choose $m \in \mathbb{N}$ so large that $B_{2/m}(x_0) \subseteq \Omega$, and let $\eta_m \in C_0^\infty(\Omega)$ such that $0 \le \eta_m(x) \le 1$, $|\nabla \eta_m(x)| \le 2m$ and

$$\eta_m(x) = \begin{cases} 0 & \text{in } B_{1/m}(x_0) \\ 1 & \text{in } \Omega \setminus B_{2/m}(x_0) \end{cases}$$

Define the functions $\phi_t^m = \eta_m \phi_t$ and set

$$f_m = -\Delta\phi_t^m - \lambda\phi_t^m .$$

Setting as before $u = v + \phi_t^m$ in equation (1.1), we see that then $v = \phi_t - \phi_t^m$ solves the equation

$$\begin{gathered} -\Delta v = \lambda v + (v + \phi_t^m)_+^{2^\star - 1} + g(x, (v + \phi_t^m)_+) + (f - f_m) \text{ in } \Omega \\ v = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{3.1}$$

Clearly $v = \phi_t - \phi_t^m$ corresponds to the trivial solution of this equation; for finding other solutions of (3.1) we look for critical points of the functional, $J : H \to \mathbb{R}$,

$$\begin{aligned} J(v) =& \frac{1}{2} \int_\Omega (|\nabla v|^2 - \lambda v^2) - \frac{1}{2^\star} \int_\Omega (v + \phi_t^m)_+^{2^\star} \\ & - \int_\Omega G(x, (v + \phi_t^m)_+) - \int_\Omega (f - f_m) v \,, \end{aligned}$$

where H denotes the Sobolev space $H = H_0^1(\Omega)$, equipped with the Dirichlet norm $\|u\| = (\int_\Omega |\nabla u|^2 \, dx)^{1/2}$.

We begin by estimating the "error" given by the term $f - f_m$.

Lemma 3.1 *For $N \geq 3$, as $m \to +\infty$ we have:*

$$\|\phi_t - \phi_t^m\| \leq cm^{-N/2}; \tag{3.2}$$

$$\Big| \int_\Omega (f - f_m)\psi \, dx \Big| \leq c\|\psi\| m^{-N/2}, \quad \textit{for all } \psi \in H. \tag{3.3}$$

Proof. Note first that by the regularity assumption $h \in L^r, r > N$, it follows that $\phi_t \in C^{1,1-N/r}(\overline{\Omega})$, and hence in particular that there exists $c > 0$ such that for any point $\bar{x} \in \partial\Omega$

$$|\phi_t(x)| \leq c|x - \bar{x}|.$$

Furthermore we may choose for every large $m \in \mathbb{N}$ a point x_0 at distance $4/m$ from the boundary point $\bar{x}$, such that

$$|\phi_t(x)| \leq \frac{c_1}{m}, \quad \forall x \in B_{4/m}(x_0). \tag{3.4}$$

We may assume that $x_0 = 0$ for every choice of m; from now on we write $B_r = B_r(0)$. Thus, we can estimate

$$\begin{aligned}
&\int_\Omega |\nabla(\phi_t - \phi_t^m)|^2 \\
&= \int_\Omega |\nabla\phi_t(1 - \eta_m) - \phi_t \nabla\eta_m|^2 \\
&= \int_{B_{\frac{2}{m}}} |\nabla\phi_t|^2 |1 - \eta_m|^2 - 2\int_{B_{\frac{2}{m}} \setminus B_{\frac{1}{m}}} |\nabla\phi_t|(1 - \eta_m)|\phi_t| \, |\nabla\eta_m| \\
&\quad + \int_{B_{\frac{2}{m}} \setminus B_{\frac{1}{m}}} |\phi_t|^2 |\nabla\eta_m|^2 \\
&\leq c_1 m^{-N} + c_2 m^{-N} + c_3 m^{-N} = cm^{-N},
\end{aligned}$$

hence (3.2). The estimate (3.3) is now obtained as follows:

$$\begin{aligned}
\Big| \int_\Omega (f - f_m)\psi \, dx \Big| &= \Big| \int_\Omega |\nabla(\phi_t - \phi_t^m)\nabla\psi - \lambda(\phi_t - \phi_t^m)\psi dx \Big| \\
&\leq c\|\phi_t - \phi_t^m\| \, \|\psi\| \leq c\|\psi\| m^{-N/2},
\end{aligned}$$

for all $\psi \in H$.

Let $\lambda_1 < \lambda_2 \leq \dots$ the eigenvalues of $-\Delta$ and $e_1, e_2 \dots$, the corresponding eigenfunctions. Take m as before and let $\zeta_m : \Omega \to \mathbb{R}$ be smooth functions such that $0 \leq \zeta_m \leq 1, |\nabla\zeta_m| \leq 4m$ and

$$\zeta_m(x) = \begin{cases} 0 & \text{if } x \in B_{2/m} \\ 1 & \text{if } x \in \Omega \setminus B_{3/m} \end{cases}$$

We define "approximate eigenfunctions" by $e_i^m = \zeta_m e_i$. Then the following estimates hold.

Lemma 3.2 *As $m \to \infty$, we have $e_i^m \to e_i$ in H. Moreover, in the space $H_{j,m}^- = \operatorname{span}\{e_1^m, \dots, e_j^m\}$, we have*

$$\max\Big\{\|u\|^2 : u \in H_{j,m}^-, \int_\Omega u^2 = 1\Big\} \le \lambda_j + c_j m^{-N}$$

and

$$\int_\Omega \nabla e_i^m \nabla e_j^m dx = \delta_{ij} + O(m^{-N}).$$

Proof. See [8] and observe that, since $\partial\Omega$ is of class C^1, also for the eigenfunctions e_i an estimate as (3.4) holds.

Consider the family of functions

$$u_\varepsilon^\star(x) = \Big[\frac{\sqrt{N(N-2)}\varepsilon}{\varepsilon^2 + |x|^2}\Big]^{(N-2)/2}$$

which are solutions to the equation

$$\begin{gathered} -\Delta u = |u|^{2^\star - 2}u \quad \text{in } \mathbb{R}^N \\ u(x) \to 0 \quad \text{as } |x| \to \infty \end{gathered}$$

and which realize the best Sobolev embedding constant $H^1(\mathbb{R}^N) \subset L^{2^*}(\mathbb{R}^N)$, i.e. the value

$$S = S_N = \inf_{u \ne 0} \frac{\|u\|_H}{\|u\|_{L^{2^*}}}.$$

Let $\xi \in C_0^1(B_{1/m})$ be a cut–off function such that $\xi(x) = 1$ on $B_{1/2m}$, $0 \le \xi(x) \le 1$ in $B_{1/m}$ and $\|\nabla\xi\|_\infty \le 4m$.

Let $u_\varepsilon(x) := \xi(x) u_\varepsilon^\star(x) \in H$. For $\varepsilon \to 0$ we have the following estimates due to Brezis and Nirenberg

Lemma 3.3 (Brezis-Nirenberg, [3]) *For fixed m we have*

(a) $\|u_\varepsilon\|^2 = S^{N/2} + O(\varepsilon^{N-2})$

(b) $\|u_\varepsilon\|_{2^\star}^{2^\star} = S^{N/2} + O(\varepsilon^N)$

(c) $\|u_\varepsilon\|_2^2 \ge K_1 \varepsilon^2 + O(\varepsilon^{N-2})$

(d) $\|u_\varepsilon\|_s^s \ge K_2 \varepsilon^{N - \frac{N-2}{2}s}$.

Moreover, for $m \to \infty$ and $\varepsilon = o(1/m)$, we have (see [8])

(e) $\|u_\varepsilon\|^2 = S^{N/2} + O((\varepsilon m)^{N-2})$

(f) $\|u_\varepsilon\|_{2^\star}^{2^\star} = S^{N/2} + O((\varepsilon m)^N)$

while (c) and (d) hold independently of m.

Proof. We only prove (d); for the other estimates, see [3, 8, 12].

$$\begin{aligned}\int_\Omega u_\varepsilon^s\,dx \geq& c\int_{B_{\frac{1}{2m}}} \big(\frac{\varepsilon}{\varepsilon^2+|x|^2}\big)^{\frac{N-2}{2}s}\,dx\\ \geq& c\int_0^\varepsilon \big(\frac{\varepsilon}{\varepsilon^2+\rho^2}\big)^{\frac{N-2}{2}s}\rho^{N-1}d\rho \geq c\varepsilon^{-\frac{N-2}{2}s}\varepsilon^N.\end{aligned}$$

which completes the proof.

4 The linking structure

In this section we prove that the functional J has a "linking structure" as required by the *Generalized Mountain Pass Theorem* by P. Rabinowitz [10]. For the rest of this article, we assume $\lambda \in [\lambda_k, \lambda_{k+1})$. Let $H^+ = [\mathrm{span}\{e_1, \dots, e_k\}]^\perp$, $S_r = \partial B_r \cap H^+$, $H_m^- = span\{e_1^m, \dots, e_k^m\}$ and $Q_m^\varepsilon = (B_R \cap H_m^-) \oplus [0, R]\{u_\varepsilon\}$, where $m \in \mathbb{N}$ is fixed. Define the family of maps $\mathcal{H} = \{h : Q_m^\varepsilon \to H$ continuous : $h\big|_{\partial Q_m^\varepsilon} = id\}$, and set

$$\bar{c} = \inf_{h\in\mathcal{H}} \sup_{u\in h(Q_m^\varepsilon)} J(u) \tag{4.1}$$

Then the Generalized Mountain Pass theorem of P. Rabinowitz states that if

1) $J : H \to \mathbb{R}$ satisfies the Palais-Smale condition (PS)
2) there exist numbers $0 < r < R$ and $\alpha_1 > \alpha_0$ such that

$$J(v) \geq \alpha_1, \quad \text{for all } v \in S_r \tag{4.2}$$
$$J(v) \leq \alpha_0, \quad \text{for all } v \in \partial Q_m^\varepsilon\,, \tag{4.3}$$

then the value $\bar{c}$ defined by (4.1) satisfies $\bar{c} \geq \alpha_1$, and it is a critical value for J.

First note that for $v \in H_m^- \oplus \mathbb{R}\{u_\varepsilon\}$, $v = w + su_\varepsilon$, we have by definition $\mathrm{supp}(u_\varepsilon) \cap \mathrm{supp}(w) = \emptyset$. It is easy to prove that this implies that

$$J(v) \equiv J(w + su_\varepsilon) = J(w) + J(su_\varepsilon)\,.$$

We begin by showing that the functional J satisfies condition (4.2).

Lemma 4.1 *There exist numbers $r > 0$ and $\alpha_1 > 0$ such that*

$$J(v) \geq \alpha_1 \quad \textit{for all } v \in S_r = \partial B_r \cap H^+$$

Proof. Let $v \in H^+$. From the variational characterization of λ_{k+1} and the Sobolev embedding theorem, we have, using (g_3) and Lemma 3.1,

$$\begin{aligned} J(v) \;&\geq\; \frac{1}{2}\big(1 - \frac{\lambda}{\lambda_{k+1}}\big)\int_\Omega |\nabla v|^2 - \frac{1}{2^\star}\int_\Omega v_+^{2^\star} - c\int_\Omega v_+^{p+1} - \int_\Omega (f - f_m)v \\ &\geq\; c_1\|v\|^2 - c_2\|v\|^{2^\star} - c_3\|v\|^{p+1} - c_4 m^{-N/2}\|v\| \end{aligned}$$

Let $k_m(s) = c_1|s|^2 - c_2|s|^{2^\star} - c_3|s|^{p+1} - c_4 m^{-N/2}|s|$. Clearly, there exists m_0 such that $\max_{\mathbb{R}} k_m(s) = M_m \geq M_{m_0} > 0$ for all $m \geq m_0$. Thus there exist $\alpha_1 > 0$ and $r > 0$ such that

$$J(v) \geq \alpha_1 > 0 \quad \text{for } \|v\| = r.$$

which completes the proof.

Next we prove condition (4.3).

Lemma 4.2 *There exist $R > r$ and $\alpha_0 < \alpha_1$ such that for ε sufficiently small*

$$J\big|_{\partial Q_m^\varepsilon} < \alpha_0 \,.$$

Proof. Let $v = w + su_\varepsilon \in (H_m^- \cap \bar{B}_R) \oplus [0,R]\{u_\varepsilon\}$. Since $J(v) = J(w) + J(su_\varepsilon)$ we can estimate $J(w)$ and $J(su_\varepsilon)$ separately.

$$J(w) \leq \frac{1}{2}\int_\Omega |\nabla w|^2\,dx - \frac{\lambda}{2}\int_\Omega |w|^2\,dx - \frac{1}{2^\star}\int_\Omega (w+\phi_t^m)_+^{2^\star}\,dx\,,$$

since $\int_\Omega (f - f_m)w\,dx = \int_\Omega \nabla(\phi_t - \phi_t^m)\nabla w\,dx - \lambda\int_\Omega(\phi_t - \phi_t^m)w\,dx = 0$.

$$\begin{aligned} J(su_\varepsilon) \;\leq\;& \frac{s^2}{2}\int_{B_{\frac{1}{m}}} |\nabla u_\varepsilon|^2\,dx - \frac{\lambda s^2}{2}\int_{B_{\frac{1}{m}}} |u_\varepsilon|^2\,dx \\ -\;& \frac{s^{2^\star}}{2^\star}\int_{B_{\frac{1}{m}}} |u_\varepsilon|^{2^\star}\,dx - \int_{B_{\frac{1}{m}}} f\, su_\varepsilon\,dx. \end{aligned}$$

Let $\partial Q_m^\varepsilon = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\begin{aligned} \Gamma_1 &= \{v \in H : v = w + su_\varepsilon,\ w \in H_m^-,\ \|w\| = R,\ 0 \leq s \leq R\}, \\ \Gamma_2 &= \{v \in H : v = w + Ru_\varepsilon,\ w \in H_m^- \cap \bar{B}_R\}, \\ \Gamma_3 &= H_m^- \cap \bar{B}_R\,. \end{aligned}$$

Note that it follows by Lemma 3.2 that

$$\int_\Omega |\nabla w|^2\,dx \leq (\lambda_k + c_k m^{-N})\int_\Omega |w|^2\,dx, \quad \text{for all } w \in H_m^-$$

1. Suppose $v \in \Gamma_1$; then $v = w + su_\varepsilon$ with $\|w\| = R$ and $0 \leq s \leq R$.
(i) if $\lambda \in (\lambda_k, \lambda_{k+1})$, we choose m_0 such that $c_k m^{-n} < \frac{\lambda - \lambda_k}{2}$, for $m \geq m_0$. Then, using Lemma 3.3

$$\begin{aligned} J(v) \leq& \frac{1}{2}\big(1 - \frac{\lambda}{\lambda_k + c_k m^{-N}}\big)\int_\Omega |\nabla w|^2\,dx - \frac{1}{2^\star}\int_\Omega (w+\phi_t^m)_+^{2^\star}\,dx \\ &+ \frac{s^2}{2}\int_{B_{\frac{1}{m}}} |\nabla u_\varepsilon|^2\,dx - \frac{\lambda s^2}{2}\int_{B_{\frac{1}{m}}} u_\varepsilon^2\,dx - \frac{s^{2^\star}}{2^\star}\int_{B_{\frac{1}{m}}} u_\varepsilon^{2^\star}\,dx + s\|f\|_2\|u_\varepsilon\|_2 \\ \leq& -cR^2 + S^{N/2}\big[\frac{s^2}{2} - \frac{s^{2^\star}}{2^\star}\big] + R^{2^\star}O(\varepsilon^N) + R^2 O(\varepsilon^{N-2}) + cs \\ \leq& -cR^2 + c_1 + c_2 R + c_3 R^{2^\star}\varepsilon^{N-2}\,. \end{aligned}$$

Thus $J(v) \le 0$ for $R \ge R_0$ and $\varepsilon > 0$ sufficiently small.
(ii) if $\lambda = \lambda_k$, for $w = R\bar{w} \in H_m^-$ with $\|\bar{w}\| = 1$ we write $\bar{w} = \alpha y + \beta e_k^m$, with $y \in \operatorname{span}\{e_1^m, \dots, e_{k-1}^m\}$ and $\|y\| = 1$. Then

$$J(w) = \frac{R^2}{2}\int_\Omega (|\alpha\nabla y + \beta\nabla e_k^m|^2 - \lambda_k|\alpha y + \beta e_k^m|^2)\,dx - \frac{R^{2^\star}}{2^\star}\int_\Omega \big(\bar{w} + \frac{\phi_t}{R}\big)_+^{2^\star}\,dx.$$

Using Lemma 3.2, we can estimate the first integral as follows

$$\begin{aligned}
&\int_\Omega (|\alpha\nabla y + \beta\nabla e_k^m|^2 - \lambda_k|\alpha y + \beta e_k^m|^2)\,dx\\
&\le \alpha^2\big(1 - \frac{\lambda_k}{\lambda_{k-1} + cm^{-N}}\big) + \beta^2\big(1 - \frac{\lambda_k}{\lambda_k + cm^{-N}}\big)\\
&\quad +2\alpha\beta\int_\Omega (\nabla y\nabla e_k^m - \lambda_k y e_k^m)\,dx\\
&\le -c\alpha^2 + c_1(\beta^2 + 2\alpha\beta)m^{-N}.
\end{aligned}$$

Note now that if $|\alpha| \ge \delta > 0$, for some $\delta > 0$, then

$$J(w) \le -\frac{c\delta^2R^2}{2} + c_1m^{-N}$$

and hence $J(v) \le 0$ for $R \ge R_1(\delta)$.

We show now that there exists $\delta > 0$ such that if $|\alpha| \le \delta$, then there exist constants $c_2 > 0$ and $R_2 > 0$ such that

$$\int_\Omega \big(\bar{w} + \frac{\phi_t}{R}\big)_+^{2^\star} \ge c_2 > 0$$

for all $R \ge R_2$ and for all $w \in H_m^- \cap \partial B_R$. To this aim we prove that there exist $\delta > 0$ and $\eta > 0$ such that

$$\max_{\bar\Omega}(\alpha y + \beta e_k^m) \ge \eta > 0 \quad \text{for all } y \in H_{k-1,m}^-,\ \|y\| = 1,\ |\alpha| \le \delta\,.$$

By contradiction assume that there exist sequences $|\alpha_n| \le 1/n$, $y_n \in H_{k-1,m}^-$ with $\|y_n\| = 1$ such that

$$\max_{\bar\Omega}\{\alpha_n y_n + \beta_n e_k^m\} \to 0 \quad \text{as} \quad n \to +\infty.$$

Then $\alpha_n y_n \to 0$, $\beta_n^2 = 1 - \alpha_n^2 + O(m^{-N}) \to \beta^2 = 1 + O(m^{-N}) \ge 1/2$, for $m \ge m_0$. Therefore, we conclude that

$$\max_{\bar\Omega}(\beta e_k^m) = 0\ ,\ \text{with } \beta^2 \ge \frac{1}{2} \text{ for } m \ge m_0 \text{ i.e. } \ (e_k^m)^+ = 0\,.$$

This is a contradiction since $e_k^m \to e_k$ in H implies that e_k^m must change sign, for m large. Therefore, there exist $\delta > 0$, $\eta > 0$ such that

$$\max_{\bar\Omega}\{\bar{w}, |\alpha| \le \delta\} \ge \eta > 0\ ,\ \forall \bar{w} \in H_{k,m}^-, \|\bar{w}\| = 1, m \ge m_0\,.$$

Denoting $\Omega_{\bar w} = \{x \in \Omega : \ \bar w(x) \geq \ \eta/2\}$, then $|\Omega_{\bar w}| \geq \nu > 0$, $\forall \bar w \in H^-_{k,m}$, with $\|\bar w\| = 1$ and $|\alpha| \leq \delta$, $m \geq m_0$, since the functions $\bar w \in H^-_{k,m}$ are equicontinuous. Moreover

$$\frac{\phi_t}{R} > -\frac{\eta}{4} \quad \text{for } R \text{ sufficiently large}\,.$$

Then

$$\int_\Omega (\bar w + \frac{\phi_t}{R})_+^{2^\star} \geq (\frac{\eta}{4})^{2^\star} |\Omega_{\bar w}|.$$

Thus, we can conclude that there exists $R_2 > 0$ such that

$$J(w) \leq cR^2 - R^{2^\star} \int_\Omega (\bar w + \frac{\phi_t}{R})_+^{2^*} \leq cR^2 - R^{2^\star} (\frac{\eta}{4})^{2^*}\ \nu \leq 0\ ,$$

for all $R \geq R_2$. In particular $J(w) \to -\infty$ as $R \to +\infty$.

2. Let $v \in \Gamma_2$, i.e. $v = w + Ru_\varepsilon$ with $\|w\| \leq R$. Then

$$\begin{aligned} J(v) =& J(w) + J(Ru_\varepsilon) \\ \leq& cm^{-N}\|w\|^2 + \frac{R^2}{2}\int_{B_{\frac{1}{m}}} |\nabla u_\varepsilon|^2\,dx - \frac{R^{2^\star}}{2^\star}\int_\Omega u_\varepsilon^{2^\star}\,dx + R\|f\|_2\|u_\varepsilon\|_2 < 0\,, \end{aligned}$$

for R sufficiently large. Now fix $R > 0$ such that the previous estimates hold.

3. Let $v \in \Gamma_3$, i.e. $v = w \in H^-_m \cap B_R$. Hence

$$J(v) \leq c_1 m^{-N}\|w\|^2 - \frac{1}{2^\star}\int_\Omega (w + \phi_t)_+^{2^\star} \leq \alpha_0$$

if m is sufficiently large.

5 Existence of a second solution

In this section we prove Theorems 2.1 and 2.2. By the Linking Theorem we construct a Palais–Smale sequence $\{v_n\} \subset H$ at the minimax level $\bar c$; the sequence $\{v_n\}$ satisfies

$$\begin{aligned} J(v_n) =& \frac{1}{2}\int_\Omega (|\nabla v_n|^2 - \lambda v_n^2)\,dx - \frac{1}{2^\star}\int_\Omega (v_n + \phi_t^m)_+^{2^\star}\,dx \\ &- \int_\Omega G(x, (v_n + \phi_t^m)_+)\,dx + \int_\Omega (f - f_m)v_n\,dx = \bar c + o(1) \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} \langle J'(v_n), z\rangle =& \int_\Omega (\nabla v_n \nabla z - \lambda v_n z)\,dx - \int_\Omega (v_n + \phi_t^m)_+^{2^\star - 1} z\,dx \\ &- \int_\Omega g(x, (v_n + \phi_t^m)_+)z\,dx + \int_\Omega (f - f_m)z\,dx = o(1)\|z\| \end{aligned} \tag{5.2}$$

for all $z \in H$.

Lemma 5.1 *Under the hypotheses of Theorem 2.1 or Theorem 2.2, the sequence $\{v_n\}$ is bounded in H.*

Proof. ¿From (5.1) and (5.2), it follows that

$$\begin{aligned}
&J(v_n) - \frac{1}{2}\langle J'(v_n), v_n\rangle \\
&= \frac{1}{N}\int_\Omega (v_n+\phi_t^m)_+^{2^\star} - \frac{1}{2}\int_\Omega (v_n+\phi_t^m)_+^{2^\star-1}\phi_t^m \\
&\quad - \int_\Omega G(x,(v_n+\phi_t^m)_+) + \frac{1}{2}\int_\Omega g(x,(v_n+\phi_t^m)_+)v_n - \frac{1}{2}\int_\Omega (f-f_m)v_n \\
&= \bar{c} + o(1) + o(1)\|v_n\|
\end{aligned}$$

Therefore, using that $\phi_t^m \le 0$ and Lemma 4,

$$\begin{aligned}
\frac{1}{N}\int_\Omega (v_n+\phi_t^m)_+^{2^\star}\,dx \le\ & \int_\Omega G(x,(v_n+\phi_t^m)_+)\,dx \\
& -\frac{1}{2}\int_\Omega g(x,(v_n+\phi_t^m)_+)(v_n+\phi_t^m)\,dx \\
& + c + (o(1)+dm^{-N})\|v_n\|
\end{aligned}$$

Then by (g_3), we get

$$\begin{aligned}
&\int_\Omega (v_n+\phi_t^m)_+^{2^\star}\,dx \\
&\le c_1\int_\Omega (v_n+\phi_t^m)_+^{p+1}\,dx + c + (o(1)+dm^{-N})\|v_n\| \\
&\le c_1\Big(\int_\Omega (v_n+\phi_t^m)_+^{2^\star}\,dx\Big)^{\frac{p+1}{2^\star}} + c + (o(1)+dm^{-N})\|v_n\|
\end{aligned}$$

Since $p+1<2^\star$, we obtain

$$\int_\Omega (v_n+\phi_t)_+^{2^\star}\,dx \le c + (o(1)+dm^{-N})\|v_n\| \le c_1 + c_2\|v_n\| \tag{5.3}$$

(i) First we consider the case $\lambda\in(\lambda_k,\lambda_{k+1})$. Let $v_n = v_n^+ + v_n^-$ (as in [6]), with $v_n^-\in H_k^- = \operatorname{span}\{e_1\dots e_k\}$ and $v_n^+\in(H_k^-)^\perp$. We obtain

$$\begin{aligned}
\langle J'(v_n), v_n^+\rangle =\ & \int_\Omega (|\nabla v_n^+|^2 - \lambda(v_n^+)^2)\,dx - \int_\Omega (v_n+\phi_t^m)_+^{2^\star-1}v_n^+\,dx \\
& - \int_\Omega g(x,(v_n+\phi_t^m)_+)v_n^+\,dx - \int_\Omega (f-f_m)v_m^+ dx = o(1)\|v_n^+\|
\end{aligned}$$

From the variational characterization of λ_{k+1} we get, using the Hölder and Young inequalities and (5.3),

$$
\begin{aligned}
&\big(1-\frac{\lambda}{\lambda_{k+1}}\big)\|v_n^+\|^2\\
&\le \int_\Omega (v_n+\phi_t^m)_+^{2^\star-1}v_n^+ + c\int_\Omega (v_n+\phi_t^m)_+^p|v_n^+| + o(1)\|v_n^+\| + dm^{-N}\|v_n^+\|\\
&\le \varepsilon\Big(\int_\Omega |v_n^+|^{2^\star}\Big)^{2/2^\star} + c_\varepsilon\big(\int_\Omega (v_n+\phi_t^m)_+^{2^\star}\big)^{\frac{2(2^\star-1)}{2^\star}}\\
&\quad +c\Big(\int_\Omega (v_n+\phi_t^m)_+^{2^\star}\Big)^{p/2^\star}\Big(\int_\Omega |v_n^+|^{\frac{2^\star}{2^\star-p}}\Big)^{\frac{2^\star-p}{2^\star}} + c\|v_n^+\|\\
&\le \varepsilon\|v_n^+\|^2 + c_\varepsilon\Big(\int_\Omega (v_n+\phi_t^m)_+^{2^\star}\Big)^{\frac{2(2^\star-1)}{2^\star}} + \varepsilon\Big(\int_\Omega |v_n^+|^{\frac{2^\star}{2^\star-p}}\Big)^{2\frac{2^\star-p}{2^\star}}\\
&\quad +c_\varepsilon\big(\int_\Omega (v_n+\phi_t^m)_+^{2^\star}\Big)^{2p/2^\star} + c\|v_n^+\|
\end{aligned}
$$

By (5.3) and by the Sobolev embedding theorems, we obtain

$$\|v_n^+\|^2 \le c + o(1)\big(\|v_n\|^{\frac{N+2}{N}} + \|v_n\|^{\frac{2p}{2^\star}}\big) + c\|v_n^+\| \tag{5.4}$$

For $v_n^- \in H^-$, we have

$$
\begin{aligned}
&(\frac{\lambda}{\lambda_k}-1)\int_\Omega |\nabla v_n^-|^2\\
&\le \int_\Omega (v_n+\phi_t^m)_+^{2^\star-1}|v_n^-| + \int_\Omega g(x,(v_n+\phi_t^m)_+)|v_n^-| + \int_\Omega |f-f_m||v_n^-|
\end{aligned}
$$

In the same way we obtain

$$\|v_n^-\|_{H^1}^2 \le \bar{c} + o(1)\big(\|v_n\|^{\frac{N+2}{N}} + \|v_n\|^{\frac{2p}{2^\star}}\big) + c\|v_n^-\| \tag{5.5}$$

Joining (5.4) and (5.5), we find

$$\|v_n\|^2 \le c + c(\|v_n\|^{\frac{N+2}{N}} + \|v_n\|^{\frac{2p}{2^\star}}) + c\|v_n\|,$$

so v_n is bounded in H.

(ii) If $\lambda=\lambda_k$ we write $v_n = v_n^- + v_n^+ + \beta_n e_k = w_n + \beta_n e_k$, where we denote with v_n^- and v_n^+ the projections of v_n onto the subspace $H_{k-1}^- = \operatorname{span}\{e_1,\dots,e_{k-1}\}$ and $H_k^+ = (H_k^-)^\perp$ respectively. With a similar argument as above we obtain the following estimate

$$\|w_n\|^2 \le c + c\big(\|v_n\|^{\frac{N+2}{N}} + \|v_n\|^{\frac{2p}{2^\star}}\big) + c\|w_n\| \tag{5.6}$$

We can assume $\|v_n\|\ge 1$. Then, from (5.6), we have

$$\|w_n\|^2 \le c + c\|v_n\|^{\frac{2p}{2^\star}} \le c + c(\|w_n\| + |\beta_n|)^{\frac{2p}{2^\star}} \tag{5.7}$$

If β_n is bounded we conclude as above. If not, we may assume $\beta_n \to +\infty$ and $\|w_n\| \to +\infty$ (we neglect the case $\|w_n\| \le c$ which is much easier). Therefore, from (5.7),

$$\|w_n\|^2 \le \big[\frac{1}{2}(\|w_n\| + \beta_n^2)\big]^{p/2^\star}.$$

Then $\|w_n\| \le c\beta_n^{p/2^\star}$ and $\|\frac{w_n}{\beta_n}\| \to 0$ since $\frac{p}{2^\star} < 1$. Therefore, possibly up to a subsequence, $w_n/\beta_n \to 0$ a.e. and strongly in L^q, $2 \le q < 2^\star$. Therefore, for all $q \in (2, 2^\star)$

$$\int_\Omega (\frac{w_n + \phi_t^m}{\beta_n} + e_k)_+^q e_k \, dx \quad \to \quad \int_\Omega (e_k)_+^{q+1} \, dx. \tag{5.8}$$

Moreover, since

$$\begin{aligned} o(1) &= \langle J'(v_n), e_k \rangle \\ &= -\int_\Omega (v_n + \phi_t^m)_+^{2^\star-1} e_k - \int_\Omega g(x, (v_n + \phi_t^m)_+) e_k + \int_\Omega (f - f_m) e_k \end{aligned}$$

we get, using (g_3)

$$\int_\Omega \big(\frac{w_n + \phi_t^m}{\beta_n} + e_k\big)_+^{2^\star-1} e_k \le o(1) + \frac{c}{\beta_n^{2^\star-1-p}} \int_\Omega \big(\frac{w_n + \phi_t^m}{\beta_n} + e_k\big)_+^p e_k.$$

Finally, by (5.8) we get

$$\int_\Omega (e_k)_+^{2^\star} \le 0$$

which is a contradiction. Thus (v_n) is bounded.

Returning to relation (5.2), we may therefore assume, as $n \to +\infty$: $v_n \rightharpoonup v$ weakly in H_0^1, $v_n \to v$ in L^q $\;2 \le q < 2^\star$ and $v_n \to v$ a. e. in Ω. In particular, it follows that v is a weak solution of

$$\begin{gathered} -\Delta v = \lambda v + (v + \phi_t^m)_+^{2^\star-1} + g(x, (v + \phi_t^m)_+) + f - f_m \quad \text{in } \Omega \\ v = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{5.9}$$

To conclude the proof, It remains to show that $v \ne \phi_t - \phi_t^m$, the "trivial" solution of (5.9). First, we estimate

$$\begin{aligned} &J(\phi_t - \phi_t^m) \\ &= \frac{1}{2}\int_\Omega |\nabla(\phi_t - \phi_t^m)|^2 - \frac{\lambda}{2}\int_\Omega |\phi_t - \phi_t^m|^2 - \int_\Omega (f - f_m)(\phi_t - \phi_t^m) \\ &= -\frac{1}{2}\int_\Omega \big[|\nabla(\phi_t - \phi_t^m)|^2 - \lambda|\phi_t - \phi_t^m|^2\big]. \end{aligned}$$

From Lemma 3.1 we get, taking $\varepsilon^\beta = \frac{1}{m}$ $\quad 0 < \beta < 1$,

$$|J(\phi_t - \phi_t^m)| \le cm^{-N} := c\varepsilon^{\beta N}. \tag{5.10}$$

Since v is a weak solution of (5.9), we have

$$\begin{aligned}\int_\Omega (|\nabla v|^2 - \lambda v^2)\,dx - \int_\Omega (v+\phi_t^m)_+^{2^\star}\,dx + \int_\Omega (v+\phi_t^m)_+^{2^\star-1}\phi_t^m\,dx \\ - \int_\Omega g(x,(v+\phi_t^m)_+)v\,dx - \int_\Omega (f-f_m)v\,dx &= 0\end{aligned}$$

By the Brezis–Lieb Lemma ([2])

$$\int_\Omega (v_n+\phi_t^m)_+^{2^\star}\,dx = \int_\Omega (v_n-v)_+^{2^\star}\,dx + \int_\Omega (v+\phi_t^m)_+^{2^\star}\,dx + o(1) \tag{5.11}$$

Since $v_n \to v$ in $L^q \quad 2 \le q < 2^\star$, we have

$$\int_\Omega G(x,(v_n+\phi_t^m)_+)\,dx - \int_\Omega G(x,(v+\phi_t^m)_+)\,dx = o(1). \tag{5.12}$$

Since $v_n \rightharpoonup v$ in H,

$$\int |\nabla v_n|^2 = \int |\nabla v|^2 + \int |\nabla (v_n - v)|^2 + o(1). \tag{5.13}$$

Then, by (5.1), (5.11) and (5.13), we find

$$\begin{aligned}\bar c + o(1) &= J(v_n) \qquad (5.14)\\ &= J(v) + \frac{1}{2}\int_\Omega |\nabla(v_n - v)|^2\,dx - \frac{1}{2^\star}\int_\Omega (v_n-v)_+^{2^\star}\,dx + o(1)\end{aligned}$$

Similarly, since $J'(v)=0$, we obtain

$$\begin{aligned}\langle J'(v_n), v_n\rangle &= \int_\Omega |\nabla(v_n-v)|^2 - \int_\Omega (v_n-v)_+^{2^\star} - \int_\Omega (v_n-v)_+^{2^\star-1}\phi_t^m \\ &\quad - \int_\Omega g(x,(v_n+\phi_t^m)_+)v_n + \int_\Omega g(x,(v+\phi_t^m)_+)v + o(1).\end{aligned}$$

Since

$$\int_\Omega (v_n-v)_+^{2^\star-1}\phi_t^m\,dx = o(1)$$

and

$$\int_\Omega g(x,(v_n+\phi_t^m)_+)v_n\,dx - \int_\Omega g(x,(v+\phi_t^m)_+)v\,dx = o(1),$$

we get

$$\int_\Omega |\nabla(v_n-v)|^2\,dx = \int_\Omega (v_n-v)_+^{2^\star}\,dx + o(1). \tag{5.15}$$

Now, let

$$K = \lim_{n\to+\infty}\int_\Omega |\nabla(v_n-v)|^2\,dx.$$

If $K=0$, then $v_n \to v$ strongly in H and in $L^{2^\star}$; then by (5.14) and (5.10)

$$J(v)=\bar{c}\geq \alpha_1 > c\varepsilon^{\beta N} \geq J(\phi_t-\phi_t^m)$$

so that $v\neq \phi_t-\phi_t^m$. If $K>0$, using the Sobolev inequality and (5.15), we have (as in [6])

$$\begin{aligned}\|v_n-v\|^2 &\geq S\Big(\int_\Omega |v_n-v|^{2^\star}\,dx\Big)^{2/2^\star} \geq S\Big(\int_\Omega (v_n-v)_+^{2^\star}\,dx\Big)^{2/2^\star}\\ &\geq S\Big(\int_\Omega |\nabla(v_n-v)|^2\,dx+o(1)\Big)^{2/2^\star}\end{aligned}$$

This implies that $K\geq SK^{\frac{N-2}{N}}$, that is $K\geq S^{N/2}$.

To complete the proof we use the following Lemmas which will be proved below.

Lemma 5.2 *Under the hypotheses of Theorem 2.1 one has, for $\varepsilon^\beta=1/m$, with $\alpha/N<\beta<(N-4)/(N-2)$,*

$$\bar{c}<\frac{1}{N}S^{N/2}-c\varepsilon^2. \tag{5.16}$$

Lemma 5.3 *Suppose that the hypotheses of Theorem 2.22 are satisfied. Then for $\varepsilon^\beta=1/m$, with*

$$\max\big\{1-\frac{N-2}{2N}(q+1),\frac{2(N-1)}{N-2}-(q+1)\big\}<\beta<\frac{1}{2}(q+1)-\frac{2}{N-2},$$

we have

$$\bar{c}<\frac{1}{N}S^{N/2}-c\varepsilon^{N-\frac{N-2}{2}(q+1)}.$$

By (5.14) and (5.15) we get

$$J(v)+\frac{K}{N}=\bar{c}\leq\frac{1}{N}S^{N/2}-\begin{cases}c\varepsilon^2 & \text{(Theorem 2.1)}\\ c\varepsilon^{N-\frac{N-2}{2}(q+1)} & \text{(Theorem 2.2)}\end{cases}$$

Assume now by contradiction that $v\equiv\phi_t-\phi_t^m$. Then we get by (5.10)

$$\frac{K}{N}+J(v)\geq\frac{1}{N}S^{N/2}-c\varepsilon^{\beta N}$$

which is impossible, due to the choice of β.

Proof of Lemma 5.2. Let $\varepsilon>0$ and $m\in\mathbb{N}$ be fixed such that the hypotheses of the Linking Theorem are satisfied. For $v=w+su_\varepsilon$, we have

$$\begin{aligned}J(v) &= J(w)+J(su_\varepsilon)\\ &\leq J(w)+\frac{s^2}{2}\int_{B_{\frac{1}{m}}}|\nabla u_\varepsilon|^2-\frac{s^{2^\star}}{2^\star}\int_{B_{\frac{1}{m}}}u_\varepsilon^{2^\star}-\frac{\lambda s^2}{2}\int_{B_{\frac{1}{m}}}u_\varepsilon^2\\ &\quad +s\int_{B_{\frac{1}{m}}}|f-f_m|\,|u_\varepsilon|\end{aligned}$$

As above, we have $J(w) \le cm^{-N}$.

To estimate the last inequality we use the argument developed in [8] (Lemma 3.1). We have from (e) and (f) in Lemma 3.3

$$\begin{aligned} \frac{s^2}{2}\int_{B_{1/m}} |\nabla u_\varepsilon|^2 - \frac{s^{2^\star}}{2^\star}\int_{B_{1/m}} u_\varepsilon^{2^\star} &\le (\frac{s^2}{2} - \frac{s^{2^\star}}{2^\star})\big[S^{N/2} + O((\varepsilon m)^{N-2})\big] \\ &\le (\frac{1}{2} - \frac{1}{2^\star})\big[S^{N/2} + O((\varepsilon m)^{N-2})\big] \\ &\le \frac{S^{N/2}}{N} + c(\varepsilon m)^{N-2}. \end{aligned}$$

(since $\frac{s^2}{2} - \frac{s^{2^\star}}{2^\star}$ attains its maximum at $s = 1$)

We make use of the Hölder inequality to estimate the last term. Let α be such that $\frac{1}{\alpha} + \frac{1}{r} + \frac{1}{2} = 1$, i.e. $\frac{1}{\alpha} = \frac{1}{2} - \frac{1}{r} > \frac{1}{2} - \frac{1}{N} = \frac{1}{2^\star}$, in particular $2^\star > \alpha$. Since $\operatorname{supp} f_m \subseteq \Omega \backslash B_{1/m}$ we have

$$\begin{aligned} \int_{B_{1/m}} |f - f_m|\,|u_\varepsilon| &\le \Big(\int_{B_{1/m}} |f|^r\Big)^{1/r}\Big(\int_{B_{1/m}} |u_\varepsilon|^2\Big)^{1/2}(\mu(B_{1/m}))^{1/\alpha} \\ &\le c\varepsilon m^{-N/\alpha}. \end{aligned}$$

If $\varepsilon^\beta = 1/m$ with $0 < \beta < 1$ we have, by (c) in Lemma 3.3,

$$\begin{aligned} J(v) \le& \frac{1}{N}S^{N/2} + c_1 m^{-N} + c_2(\varepsilon m)^{N-2} - c_3\varepsilon^2 + c_4\varepsilon m^{N/\alpha} \\ =& \frac{1}{N}S^{N/2} + c_1\varepsilon^{N\beta} + c_2\varepsilon^{(1-\beta)(N-2)} - c_3\varepsilon^2 + \varepsilon^{1+\beta N/\alpha} \end{aligned} \tag{5.17}$$

We choose β such that

$$\frac{\alpha}{N} < \beta < 1 - \frac{2}{N-2}\,.$$

Such a choice is possible only for $N \ge 6$. This then implies that for $\varepsilon > 0$ sufficiently small

$$J(v) < \frac{1}{N}S^{N/2} - c\varepsilon^2,$$

in particular $\bar{c} < \frac{1}{N}S^{N/2} - c\varepsilon^2$.

Proof of Lemma 5.3. With a similar argument as above and using (g_4) and Lemma 3.3(d), we obtain

$$\begin{aligned} J(v) &= J(w) + J(su_\varepsilon) \\ &\le J(w) + \frac{s^2}{2}\int_\Omega |\nabla u_\varepsilon|^2\,dx - \frac{\lambda s^2}{2}\int_\Omega u_\varepsilon^2\,dx - \frac{s^{2^\star}}{2^\star}\int_\Omega u_\varepsilon^{2^\star}\,dx \\ &\quad - \int_\Omega G(x,(su_\varepsilon + \phi_t^m)_+)\,dx - s\int_\Omega (f - f_m)u_\varepsilon\,dx \end{aligned}$$

Reasoning as in Lemma 5.2 from (d) in Lemma 3.3 we obtain as in (5.17)

$$\begin{aligned} J(v) &\leq \frac{S^{N/2}}{N} + c_1\varepsilon^{N\beta} + c_2\varepsilon^{(1-\beta)(N-2)} - c_3 s^{q+1}\int_\Omega u_\varepsilon^{q+1}\,dx + c_4\varepsilon^{1+\beta N/\alpha} \\ &\leq \frac{S^{N/2}}{N} + c_1\varepsilon^{N\beta} + c_2\varepsilon^{(1-\beta)(N-2)} - c_3\varepsilon^{-\frac{N-2}{2}(q+1)+N} + c_4\varepsilon^{1+\beta N/\alpha} \end{aligned}$$

Using that $\alpha < 2^\star$, we get the following condition on $\beta \in (0,1)$

$$N - \frac{N-2}{2}(q+1) < \begin{cases} \beta N \\ 1 + \frac{\beta N}{2^\star} \\ (1-\beta)(N-2) \end{cases}$$

which is equivalent to the system

$$\begin{aligned} 1 - \frac{N-2}{2N}(q+1) < \beta < -\frac{2}{N-2} + \frac{1}{2}(q+1) \\ \frac{2(N-1)}{N-2} - (q+1) < \beta < -\frac{2}{N-2} + \frac{1}{2}(q+1) \end{aligned}$$

This choice is possible if

$$q+1 > \begin{cases} \frac{N^2}{(N-1)(N-2)} \\ \frac{2}{3}\cdot\frac{2N}{N-2} \end{cases}$$

Therefore the result follows.

6 Existence of a third solution

We consider only the case $N \geq 6$ and $g \equiv 0$; the other cases follow with small changes.

Let $\phi_t(\lambda)$ denote the negative solution of (1) for $\lambda \in (\lambda_k - \delta, \lambda_k)$. Since $h \in \ker(-\Delta - \lambda_k)^\perp$, $\phi_t(\lambda)$ is uniformly bounded when $\lambda \to \lambda_k$. Consider the functional

$$J_\lambda(v) = \frac{1}{2}\int_\Omega (|\nabla v|^2 - \lambda v^2)\,dx - \frac{1}{2^\star}\int_\Omega (v + \phi_t(\lambda))_+^{2^\star}\,dx - \int_\Omega (f - f_m)v\,dx.$$

We prove the main geometrical properties of the functional J_λ.

Lemma 6.1 *Let $H^+ = \operatorname{span}\{e_1, \dots, e_k\}^\perp$. There exist $\bar\alpha > 0$, $\bar r > 0$ such that*

$$J_\lambda(v) \geq \bar\alpha > 0 \quad \textit{for all } v \in S_{\bar r} = \partial B_{\bar r} \cap H^+,\ \lambda \in (\lambda_{k-1}, \lambda_k).$$

Proof. With a similar argument as in Lemma 4.1, since $\lambda < \lambda_k$, we have

$$\begin{aligned} J_\lambda(v) &\geq \frac{1}{2}\big(1-\frac{\lambda}{\lambda_{k+1}}\big)\|v\|^2 - c_2\|v\|^{2^\star} - c_3 m^{-N/2}\|v\| \\ &\geq \frac{1}{2}\big(1-\frac{\lambda_k}{\lambda_{k+1}}\big)\|v\|^2 - c_2\|v\|^{2^\star} - c_3 m^{-N/2}\|v\| \end{aligned}$$

Then there exist m_0, $\bar\alpha$ and $\bar r$ such that for $m \geq m_0$,

$$J_\lambda(v) \geq \bar\alpha > 0$$

for $\|v\| = \bar r$ and all $\lambda \in (\lambda_{k-1}, \lambda_k)$.

Now let $H_m^- = \operatorname{span}\{e_1^m, \dots, e_k^m\}$ as in the proof of Theorems 2.1 and 2.2.

Lemma 6.2 *Let $Q_m = (H_m^- \cap B_R(0)) \oplus [0,R]u_\varepsilon$, where $\varepsilon^\beta = 1/m$. Then, if R is large enough, there exists $\bar m \geq m_0$ such that:*
(i) $J_{\partial Q_{\bar m}} < \bar\alpha$
(ii) $\sup_{Q_{\bar m}} J < \frac{S^{N/2}}{N} - c\bar\varepsilon^2$.

Proof. (i) Let $\partial Q_{\bar m} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ as in Lemma 4.2.
(1) For $v = w + su_\varepsilon \in \Gamma_1$ we have $J(v) = J(w) + J(su_\varepsilon)$. Arguing as in Lemma 4.2 (ii) we have

$$J(su_\varepsilon) \leq c_3 R^{2^\star}\varepsilon^{N-2} + c_1 + cR \tag{6.1}$$

Writing $w = R\bar w \in H_m^-$ with $\|\bar w\| = 1$, $\bar w = \alpha y + \beta e_k^m$, $y \in \operatorname{span}\{e_1^m, \dots, e_{k-1}^m\}$, $\|y\| = 1$, it was shown in Lemma 4.2 (ii) that there exists a number $\sigma > 0$ such that for $|\alpha|^2 \leq \sigma$,

$$J(w) \leq c_1 R^2 - c_2 R^{2^\star}. \tag{6.2}$$

On the other hand, if $|\alpha|^2 > \sigma$, one deduces as in Lemma 4.2 (ii) (using that $\delta < \lambda_k - \lambda_{k-1}$) that

$$\begin{aligned} J(w) &\leq \frac{R^2}{2}\int (\alpha\nabla y + \beta\nabla e_k^m)^2 - \lambda|\alpha y + \beta e_k^m|^2 \\ &\leq \frac{R^2}{2}\Big[\alpha^2\big(1-\frac{\lambda}{\lambda_{k-1}+cm^{-N}}\big) + \beta^2\big(1-\frac{\lambda}{\lambda_k + cm^{-N}}\big) \\ &\quad +2\alpha\beta\int(\nabla y\nabla e_k^m - \lambda y e_k^m)\Big] \\ &\leq \frac{R^2}{2}\Big[\sigma\frac{\lambda_{k-1}-(\lambda_k-\delta)+cm^{-N}}{\lambda_{k-1}+cm^{-N}} + \beta^2\frac{\delta + cm^{-N}}{\lambda_k + cm^{-N}} + 2\alpha\beta cm^{-N}\Big] \\ &\leq R^2[-c\sigma + c_1\delta + c_2 m^{-N}]. \end{aligned}$$

Thus, we get in this case, for δ sufficiently small and m sufficiently large

$$J(w) \leq -\frac{c}{2}\sigma R^2. \tag{6.3}$$

Joining (6.1), (6.2) and (6.3) we have

$$J(v) = J(w) + J(su_\varepsilon) \le 0,$$

for δ small, R sufficiently large and ε sufficiently small.
(2) Let $v \in \Gamma_2$, i.e. $v = w + Ru_\varepsilon$ with $\|w\| \le R$; then

$$\begin{aligned} J(v) &= J(w) + J(Ru_\varepsilon) \le c(m^{-N} + \delta)R^2 + \frac{R^2}{2}\int_{B_{\frac{1}{m}}} |\nabla u_\varepsilon|^2\,dx \\ &\quad -\frac{R^{2^\star}}{2^\star}\int_{B_{\frac{1}{m}}} u_\varepsilon^{2^\star}\,dx + R\|f\|\,\|u_\varepsilon\| < 0 \end{aligned}$$

for R sufficiently large.
(3) Let $v \in \Gamma_3 = \bar{B}_R \cap H_m^-$. Then

$$\begin{aligned} J(v) &\le c_1(m^{-N} + \delta)\|w\|^2 - \frac{1}{2^\star}\int_{B_{\frac{1}{m}}} (w + \phi_t(\lambda))_+^{2^\star}\,dx \\ &\le c_1(m^{-N} + \delta)\|w\|^2 < \alpha_0 \end{aligned}$$

if $m \ge m_1$ and $\delta > 0$ sufficiently small.
ii) For $v \in Q_m$ we have as in Lemma 5.2 inequality (5.17),

$$J(v) \le \frac{1}{N}S^{N/2} + c_1(m^{-N} + \delta)R^2 + c_2(\varepsilon m)^{N-2} - c_3\varepsilon^2 - c_4\varepsilon m^{-N/2}$$

If $\varepsilon^\beta = 1/m$, with β as in Lemma 5.2, then

$$J(v) \le \frac{S^{N/2}}{N} - c\varepsilon^2 + c_4\delta R^2$$

Then there exists δ_1 such that if $0 < \delta < \delta_1$, then

$$J(v) \le \frac{S^{N/2}}{N} - \frac{c}{2}\varepsilon^2.$$

Arguing as in Theorem 2.1, we find a critical point v_λ of the functional J_λ at a level $c_\lambda \ge \bar{\alpha} > 0$ with $\bar{\alpha}$ independent of $\lambda \in (\lambda_k - \delta, \lambda_k)$.

It was shown in [6, Theorem 1.3] that in every $\lambda_k, k \ge 1$, starts a bifurcation branch (λ, k) emanating from the negative solution and bending to the left (Prop. 4.2) (and thus “corresponding” to our second solution). We conclude that for λ to the left and close to λ_k there exist at least three solutions for equation (1.1), for $t > T_0$ with $T_0 = T_0(h)$ sufficiently large.

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