Generalized quasilinearization method for a second order three point boundary-value problem with nonlinear boundary conditions *

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Abstract

The generalized quasilinearization technique is applied to obtain a monotone sequence of iterates converging uniformly and quadratically to a solution of three point boundary value problem for second order differential equations with nonlinear boundary conditions. Also, we improve the convergence of the sequence of iterates by establishing a convergence of order k.

1 Introduction

The method of quasilinearization pioneered by Bellman and Kalaba [1] and generalized by Lakshmikantham [8, 9] has been applied to a variety of problems [2, 10, 11, 12, 13, 16].

Multipoint boundary value problems for second order differential equations have also been receiving considerable attention recently. Kiguradze and Lomtatidze [7] and Lomtatidze [14, 15] have studied closely related problems. Gupta et.al. [4, 5, 6] have studied problems related to three point boundary value problems. More recently, Paul Eloe and Yang Gao [3] discussed the method of quasilinearization for a three point boundary value problem. In this paper, we develop the method of generalized quasilinearization for a three point boundary value problem involving nonlinear boundary conditions and obtain a monotone sequence of approximate solutions converging uniformly and quadratically to a solution of the problem. Also, we have discussed the convergence of order k.

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Basic Results

Consider the three point boundary value problem with nonlinear boundary conditions
\[ x'' = f(t, x(t)), \quad t \in [0, 1] = J \]
\[ x(0) = a, \quad x(1) = g(x(\frac{1}{2})), \]
where \( f \in C[J \times \mathbb{R}, \mathbb{R}] \) and \( g : \mathbb{R} \to \mathbb{R} \) is continuous. Let \( G(t, s) \) denote the Green’s function for the conjugate or Dirichlet boundary value problem and is given by
\[
G(t, s) = \begin{cases} 
  t(s - 1), & 0 \leq t < s \leq 1 \\
  s(t - 1), & 0 \leq s < t \leq 1.
\end{cases}
\]

We note that \( G(t, s) < 0 \) on \((0, 1) \times (0, 1)\). If \( x(t) \) is the solution of (1.1) and (1.2), then
\[
x(t) = a(1-t) + g(x(\frac{1}{2}))t + \int_0^1 G(t, s)f(s, x(s))ds.
\]

Let \( \alpha, \beta \in C^2[0, 1] \). We say that \( \alpha \) is a lower solution of the BVP (1.1), if
\[
\alpha'' \geq f(t, \alpha), \quad t \in [0, 1] \\
\alpha(0) \leq a, \quad \alpha(1) \leq g(\alpha(\frac{1}{2})),
\]
and \( \beta \) be an upper solution of the BVP (1.1), if
\[
\beta'' \leq f(t, \beta), \quad t \in [0, 1] \\
\beta(0) \geq a, \quad \beta(1) \geq g(\beta(\frac{1}{2})).
\]

Now, we state the following theorems without proof [3].

**Theorem 1.1** Assume that \( f \) is continuous with \( f_x > 0 \) on \([0, 1] \times \mathbb{R} \) and \( g \) is continuous with \( 0 \leq g' < 1 \) on \( \mathbb{R} \). Let \( \beta \) and \( \alpha \) be the upper and lower solutions of the BVP (1.1) respectively. Then \( \alpha(t) \leq \beta(t), \ t \in [0, 1] \).

**Theorem 1.2 (Method of upper and lower solutions)** Assume that \( f \) is continuous on \([0, 1] \times \mathbb{R} \) and \( g \) is continuous on \( \mathbb{R} \) satisfying \( 0 \leq g' < 1 \). Further, we assume that there exists an upper solution \( \beta \) and a lower solution \( \alpha \) of the BVP (1.1) such that \( \alpha(t) \leq \beta(t), \ t \in [0, 1] \). Then there exists a solution \( x \) of the BVP (1.1) such that
\[
\alpha(t) \leq x \leq \beta(t), \quad t \in [0, 1].
\]
2 Main Result

Theorem 2.1 (Generalized quasilinearization method)

\((A_1)\) \(f, f_x\) are continuous on \([0, 1] \times \mathbb{R}\) and \(f_{xx}\) exists on \([0, 1] \times \mathbb{R}\). Further, \(f_x > 0\) and \(f_{xx} + \phi_{xx} \leq 0\), where \(\phi, \phi_x\) are continuous on \([0, 1] \times \mathbb{R}\) and \(\phi_{xx} \leq 0\).

\((A_2)\) \(g, g'\) are continuous on \(\mathbb{R}\) and \(g''\) exists and \(0 \leq g' < 1\), \(g''(x) \geq 0, x \in \mathbb{R}\).

\((A_3)\) \(\alpha\) and \(\beta\) are lower and upper solutions of the BVP (1.1) respectively.

Then there exists a monotone sequence \(\{w_n\}\) of solutions converging quadratically to the unique solution \(x\) of the BVP (1.1).

Proof. Define \(F: [0, 1] \times \mathbb{R} \to \mathbb{R}\) as

\[ F(t, x) = f(t, x) + \phi(t, x). \]

Then, in view of \((A_1)\), we note that \(F, F_x\) are continuous on \([0, 1] \times \mathbb{R}\), and \(F_{xx}\) exists such that

\[ F_{xx}(t, x) \leq 0. \] (2.1)

Using the mean value theorem and the assumptions \((A_1)\) and \((A_2)\), we obtain

\[ f(t, x) \leq F(t, y) + F_x(t, y)(x - y) - \phi(t, x), \] (2.2)
\[ g(x) \geq g(y) + g'(y)(x - y), \] (2.3)

where \(x, y \in \mathbb{R}\) such that \(x \geq y\) and \(t \in [0, 1]\). Here, we remark that (2.2) and (2.3) are also valid independent of the requirement \(x \geq y\). Define the functions \(\hat{F}(t, x, y)\) and \(h(x, y)\) as

\[ \hat{F}(t, x, y) = F(t, y) + F_x(t, y)(x - y) - \phi(t, x), \]
\[ h(x, y) = g(y) + g'(y)(x - y). \]

We observe that

\[ f(t, x) = \min_y F^*(t, x, y). \] (2.4)

Further

\[ \hat{F}_x(t, x, y) = F_x(t, y) - \phi_x(t, x) \geq F_x(t, x) - \phi_x(t, x) \]
\[ = f_x(t, x) > 0, \] (2.5)

implies that \(\hat{F}(t, x, y)\) is increasing in \(x\) for each fixed \((t, y) \in [0, 1] \times \mathbb{R}\). Similarly

\[ g(x) = \max_y h(x, y), \] (2.6)
\[ 0 \leq h'(x, y) < 1. \] (2.7)
Now, set $\alpha = w_0$, and consider the three point BVP
\[ x'' = \tilde{F}(t, x(t), w_0(t)), \quad t \in [0, 1] = J \]
\[ x(0) = a, \quad x(1) = h(x(\frac{1}{2}), w_0(\frac{1}{2})). \]
\[ (2.8) \]
Using (A3) together with (2.4) and (2.6), we have
\[ w''_0 \geq f(t, w_0) = \tilde{F}(t, w, w_0), \quad t \in [0, 1] \]
\[ w_0(0) \leq a, \quad w_0(1) = g(w_0(\frac{1}{2}), w_0(\frac{1}{2})) = h(w_0(\frac{1}{2}), w_0(\frac{1}{2})), \]
and
\[ \beta'' \leq f(t, \beta) = \tilde{F}(t, \beta, w_0), \quad t \in [0, 1] \]
\[ \beta(0) \geq a, \quad \beta(1) = g(\beta(\frac{1}{2}), \beta(\frac{1}{2})) = h(\beta(\frac{1}{2}), w_0(\frac{1}{2})), \]
which imply that $w_0$ and $\beta$ are lower and upper solutions of the BVP (2.8) respectively. In view of (2.5) (2.7) and the fact that $w_0$ and $\beta$ are lower and upper solutions of the BVP (2.8) respectively, it follows by Theorems 1.1 and 1.2 that there exists a unique solution $w_1$ of the BVP (2.8) such that
\[ w_0(t) \leq w_1(t) \leq \beta(t), \quad t \in [0, 1]. \]
Now, consider the BVP
\[ x'' = \tilde{F}(t, x(t), w_1(t)), \quad t \in [0, 1] = J \]
\[ x(0) = a, \quad x(1) = h(x(\frac{1}{2}), w_1(\frac{1}{2})). \]
\[ (2.9) \]
Again, using (A3), (2.4) and (2.6), we find that $w_1$ and $\beta$ are lower and upper solutions of (2.9) respectively, that is,
\[ w''_1 = \tilde{F}(t, w_1, w_0) \geq \tilde{F}(t, w_1, w_1), \quad t \in [0, 1] \]
\[ w_1(0) = a, \quad w_1(1) = h(w_1(\frac{1}{2}), w_0(\frac{1}{2})) \leq h(w_1(\frac{1}{2}), w_1(\frac{1}{2})), \]
and
\[ \beta'' \leq f(t, \beta) = \tilde{F}(t, \beta, w_1), \quad t \in [0, 1] \]
\[ \beta(0) \geq a, \quad \beta(1) = g(\beta(\frac{1}{2}), \beta(\frac{1}{2})) = h(\beta(\frac{1}{2}), w_1(\frac{1}{2})), \]
Hence, by Theorems 1.1 and 1.2, there exists a unique solution $w_2$ of (2.9) such that
\[ w_1(t) \leq w_2(t) \leq \beta(t), \quad t \in [0, 1]. \]
Continuing this process successively, we obtain a monotone sequence \( \{w_n\} \) of solutions satisfying

\[
\begin{align*}
    w_0(t) &\leq w_1(t) \leq \cdots \leq w_n(t) \leq \beta(t), \quad t \in [0, 1],
\end{align*}
\]

where each element \( w_n \) of the sequence is a solution of the BVP

\[
    x'' = F(t, x(t), w_{n-1}(t)), \quad t \in [0, 1] = J
\]

\[
    x(0) = a, \quad x(1) = h(x(\frac{1}{2}), w_{n-1}(\frac{1}{2})),
\]

and

\[
    w_n(t) = a(1 - t) + h(w_n(\frac{1}{2}), w_{n-1}(\frac{1}{2})) t + \int_0^1 G(t, s) \hat{F}(s, w_n, w_{n-1}) ds. \tag{2.10}
\]

Employing the fact that \([0, 1]\) is compact and the monotone convergence is pointwise, it follows that the convergence of the sequence is uniform. If \( x(t) \) is the limit point of the sequence, then passing onto the limit \( n \to \infty \), (2.10) gives

\[
    x(t) = a(1 - t) + h(x(\frac{1}{2}), x(\frac{1}{2})) t + \int_0^1 G(t, s) \hat{F}(s, x(s), x(s)) ds
\]

\[
    = a(1 - t) + g(x(\frac{1}{2})) t + \int_0^1 G(t, s) f(s, x(s)) ds.
\]

Thus, \( x(t) \) is the solution of the BVP (1.1). Now, we show that the convergence of the sequence is quadratic. For that, set

\[
    e_n(t) = x(t) - w_n(t), \quad t \in [0, 1].
\]

Observe that

\[
    e_n(t) \geq 0, \quad e_n(0) = 0,
\]

\[
    e_n(1) = g(x(\frac{1}{2})) - h(w_n(\frac{1}{2}), w_{n-1}(\frac{1}{2})).
\]

Using the mean value theorem repeatedly, \((A_1)\) and the nonincreasing property of \( F_2 \), we have

\[
    e_{n+1}''(t) = x''(t) - w_{n+1}''(t)
\]

\[
    = f(t, x) - [F(t, w_n) + F(t, w_n)(w_{n+1} - w_n) - \phi(t, w_{n+1})]
\]

\[
    = F_2(t, c_1)(x - w_n) - F_2(t, c_1)(x - w_n) + F_2(t, w_n)(x - w_{n+1}) - \phi_x(t, c_2)(x - w_{n+1})
\]

\[
    = (F_2(t, c_1)(c_1 - w_n) - (F_2(t, w_n) - \phi_x(t, c_2))(x - w_{n+1})\n\]

\[
    \geq F_{xx}(t, c_2)(x - w_n)^2 + (F_2(t, c_2) - \phi_x(t, c_2))(x - w_{n+1})
\]

\[
    = F_{xx}(t, c_2)(e_n)^2 + f_x(t, c_2)e_{n+1}
\]

\[
    \geq F_{xx}(t, c_2)(e_n)^2 \geq -M \| e_n \|^2,
\]
where $M$ is a bound on $F_{xx}(t,x)$ for $t \in [0,1]$, $w_n < c_3 < c_1 < x(t)$, $w_{n+1} < c_2 < x(t)$, and $||\cdot||$ denotes the supremum norm on $C[0,1]$. Thus, we have

\[
e_{n+1}(t) = \left[g(x(\frac{1}{2}))-h(w_{n+1}(\frac{1}{2}),w_n(\frac{1}{2}))\right]t + \int_0^1 G(t,s)e''_{n+1}(s)ds
\]

\[
\leq \left[g(x(\frac{1}{2}))-g(w_n(\frac{1}{2})) - g'(w_n(\frac{1}{2}))(w_{n+1}(\frac{1}{2}) - w_n(\frac{1}{2}))\right]t
\]

\[
+ \int_0^1 G(t,s)M||e_n||^2ds
\]

\[
\leq \left[g'(c_o)(x(\frac{1}{2})-w_n(\frac{1}{2})) - g'(w_n(\frac{1}{2}))(w_{n+1}(\frac{1}{2}) - w_n(\frac{1}{2}))\right]t
\]

\[
+ M||e_n||^2\int_0^1 |G(t,s)|ds
\]

\[
= \left[g''(c_1)c_o - w_n(\frac{1}{2}))(x(\frac{1}{2}) - w_n(\frac{1}{2})) + g'(w_n(\frac{1}{2}))e_{n+1}\right]t
\]

\[
+ M_1||e_n||^2
\]

\[
\leq \left[g''(c_1)e_n^2(t) + g'(w_n(\frac{1}{2}))e_{n+1}\right]t + M_1 ||e_n||^2
\]

where $w_n(\frac{1}{2}) < c_1 < c_o < x(\frac{1}{2})$. Taking the maximum over the interval $[0,1]$, we get

\[
||e_{n+1}|| \leq M_2||e_n||^2 + \lambda||e_{n+1}|| + M_1||e_n||^2
\]

Solving algebraically, we get

\[
||e_{n+1}|| \leq \frac{M_3}{1-\lambda}||e_n||^2
\]

where, $|g'| \leq \lambda < 1$, $M_1$ provides a bound on $M \int_0^1 |G(t,s)|ds$, $M_2$ provides a bound for $|g''|$ on $[w_n(\frac{1}{2}), x(\frac{1}{2})]$ and $M_3 = M_1 + M_2$. This establishes the quadratic convergence.

### 3 Rapid Convergence

**Theorem 3.1** Assume that

\[(B_1) \quad \frac{\partial^i}{\partial x^i} f(t,x) (i = 0, 1, 2, \ldots k) \text{ are continuous on } [0,1] \times \mathbb{R} \text{ satisfying}
\]

\[
\frac{\partial^i}{\partial x^i} f(t,x) \geq 0, \quad (i = 0, 1, 2, \ldots k - 1)
\]

\[
\frac{\partial^k}{\partial x^k} (f(t,x) + \phi(t,x)) \leq 0,
\]

where $\frac{\partial^i}{\partial x^i} \phi(t,x) (i = 0, 1, 2, \ldots k)$ are continuous and $\frac{\partial^i}{\partial x^i} \phi(t,x) < 0$ for some function $\phi(t,x)$.
(B\textsubscript{2}) $\alpha, \beta \in C^2[J, \mathbb{R}]$ are lower and upper solutions of the BVP (1.1).

(B\textsubscript{3}) $\frac{d^i}{dx^i} g(x)$ ($i = 0, 1, 2, \ldots k$) are continuous on $\mathbb{R}$ satisfying
\[ 0 \leq \frac{d^i}{dx^i} g(x) < \frac{M}{(\beta - \alpha)^{i-1}}, \]
with $0 < M < \frac{1}{3}$ and $\frac{d^k}{dx^k} g(x) \geq 0$.

Then there exists a monotone sequence of solutions $\{w_n\}$ that converge to the unique solution $x$, of the BVP (1.1) with the order of convergence $k \geq 2$.

Proof. Define $F: [0, 1] \times \mathbb{R} \to \mathbb{R}$ as
\[ F(t, x) = f(t, x) + \phi(t, x). \]
Using (B\textsubscript{1}), (B\textsubscript{3}) and the generalized mean value theorem, we obtain
\[ f(t, x) \leq \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} F(t, y) \frac{(x-y)^i}{i!} - \phi(t, x), \]
\[ g(x) \geq \sum_{i=0}^{k-1} \frac{d^i}{dx^i} g(y) \frac{(x-y)^i}{i!}. \]
Define
\[ \tilde{\Phi}^*(t, x, y) = \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} F(t, y) \frac{(x-y)^i}{i!} - \phi(t, x), \] (3.1)
and
\[ \tilde{h}^*(x, y) = \sum_{i=0}^{k-1} \frac{d^i}{dx^i} g(y) \frac{(x-y)^i}{i!}. \] (3.2)
Observe that $\tilde{\Phi}^*(t, x, y)$ and $\tilde{h}^*(x, y)$ are continuous and further
\[ f(t, x) = \min_y \tilde{\Phi}^*(t, x, y), \] (3.3)
\[ g(x) = \max_y \tilde{h}^*(x, y). \] (3.4)
Using generalized mean value theorem, (3.1) can be written as
\[ \tilde{\Phi}^*(t, x, y) = \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t, y) \frac{(x-y)^i}{i!} - \frac{\partial^k}{\partial x^k} \phi(t, \xi) \frac{(x-y)^k}{k!}. \] (3.5)
Differentiating (3.5) and using (B\textsubscript{1}), we get
\[ \tilde{\Phi}^*_{xx}(t, x, y) > \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t, y) \frac{(x-y)^i}{(i-1)!} \geq 0, \] (3.6)
which implies that $\hat{F}(t,x,y)$ is increasing in $x$ for each $(t,y) \in [0,1] \times \mathbb{R}$.

Similarly, differentiation of (3.2), in view of (B3), yields

$$
\hat{h}'(x,y) = \sum_{i=0}^{k-1} \frac{d^i}{dx^i} g(y) \frac{(x-y)^{i-1}}{(i-1)!}.
$$

Clearly $\hat{h}'(x,y) \geq 0$ and

$$
\hat{h}'(x,y) \leq \sum_{i=0}^{k-1} \frac{M}{(i-1)!} \leq M(1 + \sum_{i=0}^{k-2} \frac{1}{2^{i-1}}) = M(3 - \frac{1}{2^{k-1}}) < M < 3.
$$

Now, set $\alpha = w_0$, and consider the linear BVP

$$
x'' = F(t,x(t),w_0(t)), \quad t \in [0,1] = J
x(0) = a, \quad x(1) = h(x(\frac{1}{2}),w_0(\frac{1}{2})).
$$

(3.7)

Using (B2), (3.3) and (3.4), we find that

$$
w_0'' \geq f(t,w_0) = F(t,w_0,w_0), \quad t \in [0,1]
w_0(0) \leq a, \quad w_0(1) \leq g(w_0(\frac{1}{2})) = h(w_0(\frac{1}{2}),w_0(\frac{1}{2})),
$$

and

$$
\beta'' \leq f(t,\beta) \leq F(t,\beta,w_0), \quad t \in [0,1]
\beta(0) \geq a, \quad \beta(1) \geq g(\beta(\frac{1}{2})) \geq h(\beta(\frac{1}{2}),w_0(\frac{1}{2})),
$$

imply that $w_0$ and $\beta$ are lower and upper solutions of the BVP (3.7) respectively. It follows by Theorems 1.1 and 1.2 that there exists a unique solution $w_1$ of the BVP (3.7) such that

$$
w_0(t) \leq w_1(t) \leq \beta(t), \quad t \in [0,1].
$$

Continuing this process successively, we obtain a monotone sequence $\{w_n\}$ of solutions satisfying

$$
w_0(t) \leq w_1(t) \leq w_2(t) \leq \cdots \leq w_n(t) \leq \beta(t), \quad t \in [0,1],
$$

where each element $w_n$ of the sequence is a solution of the BVP

$$
x'' = F(t,x(t),w_{n-1}(t)), \quad t \in [0,1] = J
x(0) = a, \quad x(1) = h(x(\frac{1}{2}),w_{n-1}(\frac{1}{2})).
$$
and is given by

\[ w_n(t) = a(1 - t) + \hat{h}(w_n(\frac{1}{2}), w_{n-1}(\frac{1}{2}))t + \int_0^1 G(t, s) * F(s, w_n, w_{n-1})ds. \] \tag{3.8} 

Again, using the standard arguments employed in the last section, it follows that

\[ x(t) = a(1 - t) + \hat{h}(x(\frac{1}{2}), x(\frac{1}{2}))t + \int_0^1 G(t, s) * F(s, x(s), x(s))ds, \]

Hence \( x(t) \) is the solution of the BVP (1.1). Now, we show that the convergence of the sequence of iterates is of order \( k \geq 2 \). For that, we set

\[ e_n(t) = x(t) - w_n(t), \quad a_n(t) = w_{n+1}(t) - w_n(t), \quad t \in [0, 1]. \]

Note that \( e_n(t) \geq 0, a_n(t) \geq 0, e_n(t) - a_n(t) = e_{n+1}(t) \), and

\[ e_n(0) = 0, \quad e_n(1) = g(x(\frac{1}{2})) - h(w_n(\frac{1}{2}), w_{n-1}(\frac{1}{2})). \]

Also, \( e_n(t) \geq a_n(t) \) and hence by induction \( e_n^k(t) \geq a_n^k(t) \). Using the generalized mean value theorem, we have

\[ e_{n+1}''(t) \]

\[ = x'' - w_{n+1}'' \]

\[ = \left[ \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t, w_n) \left( x - w_n \right)^i \right] + \frac{\partial^k}{\partial x^k} f(t, \xi) \left( x - w_n \right)^k \]

\[ - \left[ \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t, w_n) \left( w_{n+1} - w_n \right)^i \right] - \frac{\partial^k}{\partial x^k} f(t, \xi) \left( w_{n+1} - w_n \right)^k \]

\[ = \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t, w_n) \left( e_n^i - a_n^i \right) + \frac{\partial^k}{\partial x^k} f(t, \xi) \left( e_n^k \right) + \frac{\partial^k}{\partial x^k} \phi(t, \xi) \left( a_n^k \right) \]

\[ \geq \left( \sum_{i=0}^{k-1} \frac{\partial^i}{\partial x^i} f(t, w_n) \frac{1}{i!} \sum_{i=0}^{k-1} \epsilon_n^i a_n^{i-1-j} \right) e_n^{i+1} + \left( \frac{\partial^k}{\partial x^k} f(t, \xi) + \frac{\partial^k}{\partial x^k} \phi(t, \xi) \right) \left( e_n^k \right) \]

\[ \geq \frac{\partial^k}{\partial x^k} F(t, \xi) \left( e_n^k \right) \geq -M \|e_n\|^k, \] \tag{3.9}
where $M$ is a bound on $\frac{d^n}{dt^n}F(t, \xi)$ for $t \in [0,1]$. Thus, in view of (3.9), we have

$$e_{n+1}(t)$$

\[ = \left(g(x_1) - h(w_{n+1}(\frac{1}{2}), w_{n}(\frac{1}{2}))\right) t + \int_0^1 G(t, s)e_{n+1}(t)ds \]

\[ \leq \left(g(x_1) - h(w_{n+1}(\frac{1}{2}), w_{n}(\frac{1}{2}))\right) t + M\|e_n\|^k \int_0^1 |G(t, s)| ds \]

\[ = \left[ \sum_{i=0}^{k-1} \frac{d^i}{dx^i} g(w_n(\frac{1}{2})) \left( x(\frac{1}{2}) - w_n(\frac{1}{2}) \right)^i \right] t + M\|e_n\|^k \]

\[ \leq \left[ \sum_{i=0}^{k-1} \frac{d^i}{dx^i} g(w_n(\frac{1}{2})) \left( \frac{1}{2} \right)^i \right] t + M\|e_n\|^k \]

\[ = \left[ \sum_{i=0}^{k-1} \frac{d^i}{dx^i} g(w_n(\frac{1}{2})) \left( \frac{1}{2} \right)^i \sum_{j=0}^{i-1} \frac{1}{2} a_{n}^{i-1-j} e_{n+1}(\frac{1}{2}) \right] t + M\|e_n\|^k \]

\[ \leq \left[ \sum_{i=1}^{k-1} \frac{M}{(\beta - \alpha)^{i-1}} \frac{1}{i!} \sum_{j=0}^{i-1} \frac{1}{2} a_{n}^{i-1-j} e_{n+1}(\frac{1}{2}) \right] t + M\|e_n\|^k + M_1\|e_n\|^k. \]

Letting

\[ P_n(t) = \left[ \sum_{i=1}^{k-1} \frac{M}{(\beta - \alpha)^{i-1}} \frac{1}{i!} \sum_{j=0}^{i-1} \frac{1}{2} a_{n}^{i-1-j} e_{n+1}(\frac{1}{2}) \right], \]

we observe that

\[ \lim_{n \to \infty} P_n(t) = \lim_{n \to \infty} \sum_{i=1}^{k-1} \frac{M}{(\beta - \alpha)^{i-1}} \frac{1}{i!} \sum_{j=0}^{i-1} \frac{1}{2} a_{n}^{i-1-j} e_{n+1}(\frac{1}{2}) = M < \frac{1}{3}. \]

Therefore, we can choose $\lambda < 1/3$ and $n_0 \in N$ such that for $n \geq n_0$, we have $P_n(t) < \lambda$ and consequently (3.10) becomes

$$\|e_{n+1}\| < \lambda\|e_{n+1}\| + M_3\|e_n\|^k.$$ 

Solving algebraically, we obtain

$$\|e_{n+1}\| \leq \frac{M_3}{1 - \lambda} \|e_n\|^k,$$

where $M_3 = M_1 + M_2$, $M_1$ provides bound for $M \int_0^1 |G(t, s)| ds$, and $M_2$ provides bound for

$$\frac{d^k}{dx^k} g(\frac{1}{2}) \frac{1}{k!}.$$
References


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