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Nonexistence of solutions to systems of higher-order semilinear inequalities in cone-like domains *

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Abstract

In this paper, we obtain nonexistence results for global solutions to the system of higher-order semilinear partial differential inequalities

$$\frac{\partial^k u_i}{\partial t^k} - \Delta(a_i(x,t)u_i(x,t)) \geq t^{\gamma_{i+1}}|x|^{\sigma_{i+1}}|u_{i+1}(x,t)|^{p_{i+1}}, \quad 1 \leq i \leq n,$$
$$u_{n+1} = u_1,$$

in cones and cone-like domains in $\mathbb{R}^N$, $t > 0$. Our results apply to nonnegative solutions and to solutions which change sign. Moreover, we provide a general formula of the critical exponent corresponding to this system. Our proofs are based on the test function method, developed by Mitidieri and Pohozaev.

1 Introduction

This paper is devoted to the study of nonexistence results for global solutions to systems of semilinear higher-order evolution differential inequalities in unbounded cone-like domains. Nonexistence results concerning nonnegative solutions of parabolic equations in cones were obtained by Bandle & Levine [1] and Levine & Meier [18]. Recently, new nonexistence results dealing with solutions with arbitrary sign were established by Laptev [11, 13, 14] and by El Hamidi & Laptev [6] when the domains are cones or product of cones. On the other hand, for cone-like domains, only nonexistence results of nonnegative solutions to semilinear evolution differential inequalities, were obtained in [1, 6, 11, 13, 14]. Recently, Laptev [15] obtained a nonexistence result for the semilinear parabolic inequality

$$u_t - \Delta(|u|^{m-1}u) \geq |x|^\sigma |u|^q$$

with $1 \leq m < q$ and $\sigma > -2$, in cone-like domains. Which is the first result, to our knowledge, dealing with solutions of evolution problems which are not necessarily nonnegative in cone-like domains.

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In this paper, we obtain nonexistence results for systems of semilinear higher-order evolution differential inequalities in unbounded cones and cone-like domains. More precisely, for $n \geq 2$, we study the problem

$$\begin{gathered} \frac{\partial^k u_i}{\partial t^k} - \Delta(a_i u_i) \geq t^{\gamma_{i+1}} |x|^{\sigma_{i+1}} |u_{i+1}|^{p_{i+1}}, \quad 1 \leq i \leq n, \\ u_{n+1} = u_1, \end{gathered} \tag{1.1}$$

where x belongs to a cone (or a *cone-like domain*), $t \in]0, +\infty[$, $k \geq 1$, $p_{n+1} = p_1$, $\gamma_{n+1} = \gamma_1$ and $\sigma_{n+1} = \sigma_1$.

For $n = 1$, we study the problem

$$\frac{\partial^k u}{\partial t^k} - \Delta(au) \geq t^{\gamma} |x|^{\sigma} \, |u|^p, \tag{1.2}$$

where x belongs to a cone (or a *cone-like domain*) and $t \in]0, +\infty[$. Such systems were studied, in the whole space, by Renclawowicz [28], Guedda & Kirane [7], Igbida & Kirane [8] and Kirane, Nabana & Pohozaev [10].

Our results concern all weak solutions, specially nonnegative weak solutions. We obtain general formulas of the critical exponents corresponding to the systems considered. These formulas are also valid in the scalar case of one inequality ($n = 1$).

Our approach is based on the *test function* method developed by Mitidieri & Pohozaev [19], Pohozaev & Tesei [25], Pohozaev & Véron [27] and Laptev [11, 13, 14].

Let $\Omega \subset S^{N-1}$ be a connected submanifold of the unit sphere S^{N-1} in $\mathbb{R}^N$ with smooth boundary $\partial\Omega \subset S^{N-1}$ and having positive $N-2$ dimensional measure. By a *cone* in $\mathbb{R}^N$ with *cross section* Ω with vertex at the origin, we mean a set

$$K = \{(r, \omega) \in \mathbb{R}^N ;\ 0 < r < +\infty \text{ and } \omega \in \Omega\},$$

where $r = |x|$, $x \in \mathbb{R}^N$. The boundary of K is

$$\partial K = \{(r, \omega); \quad r = 0 \text{ or } \omega \in \partial\Omega\}.$$

For $\varepsilon > 0$ fixed, the *cone-like domain* K_ε is defined as

$$K_\varepsilon = \{x \in K;\ |x| > \varepsilon\}$$

and its boundary as

$$\partial K_\varepsilon = \{(r, \omega);\ r = \varepsilon \text{ or } \omega \in \partial\Omega\}.$$

The outward normal vector to the boundary $\partial\Omega$ (resp. ∂K) will be denoted by ν_ω (resp. ν). The restriction of the laplacian operator Δ to the unit sphere S^{N-1} will be denoted by Δ_ω, which is the Laplace-Beltrami operator. It is well-known that the laplacian operator in $\mathbb{R}^N$ can be written, in polar coordinates (r, ω), as

$$\Delta = \frac{1}{r^{N-1}} \frac{\partial}{\partial r} \big(r^{N-1} \frac{\partial}{\partial r} \big) + \frac{1}{r^2} \Delta_\omega = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_\omega.$$

for the rest of this paper, λ denotes the first Dirichlet eigenvalue, and Φ the corresponding eigenfunction, for the Laplace-Beltrami operator; namely,

$$-\Delta_\omega \Phi = \lambda\Phi \quad \text{in } \Omega,$$
$$\Phi = 0 \text{ on } \partial\Omega.$$

Recall that $\lambda > 0$ and $\Phi(\omega) > 0$, for any $\omega \in \Omega$. We shall assume Φ is normalized so that

$$0 < \Phi(\omega) \leq 1, \quad \forall\omega \in \Omega.$$

The space of the C^2 functions with respect to the first variable and C^j, $j \in \mathbb{N}^*$, with respect to the second variable, on $K\times]0,+\infty[$, will be denoted by $\mathcal{C}^{2,j}(K\times]0,+\infty[)$.

This article is organized as follows. In Section 2, we introduce notation and establish estimates which we shall use in the sequel. Section 3 is devoted to nonexistence results to the inequality (1.2), where the parameter $\gamma = 0$. In Section 4, we generalize the results of the Section 3 for $n \geq 2$ and the parameters $\gamma_i = 0$, $i \in \{1, 2, \dots, n\}$. Section 5 concerns the general system (1.1), with $\gamma_i \leq 0$, $i \in \{1, 2, \dots, n\}$.

2 Preliminary results

Throughout this paper, the letter C denotes a constant which may vary from line to line but is independent of the terms which will take part in any limit process. For a real number $p > 1$, we define p' such that $1/p + 1/p' = 1$.

We define now the weak solutions of the problems that we will consider in the sequel. Let us consider the higher order inequality

$$\frac{\partial^k u}{\partial t^k} - \Delta(au) \geq |x|^\sigma\, |u|^p, \quad x \in K_\varepsilon,\ t \in]0, +\infty[, \tag{2.1}$$

where $p > 1$, $\sigma > -2$, with the initial data

$$u(x, 0) = u_0(x), \quad \text{in } K_\varepsilon,$$
$$\frac{\partial^i u}{\partial t^i}(x, 0) = u_i(x), \quad i \in \{1, 2, \dots, k-1\}, \quad \text{in } K_\varepsilon.$$

Definition 2.1 Let a be in $L^\infty\left(\overline{K}_\varepsilon\times]0,+\infty[\right)$. A weak solution u of the inequality (2.1) on $K_\varepsilon\times]0,+\infty[$ is continuous function on $\overline{K}_\varepsilon \times [0,+\infty[$ such that the traces $\frac{\partial^j u}{\partial t^j}(x, 0)$, $j \in \{1, .., k-1\}$, are well defined and locally integrable on K_ε and

$$\begin{aligned}\int_0^\infty \int_{K_\varepsilon} \Big(a\, u\Delta\varphi - u(-1)^k \frac{\partial^k \varphi}{\partial t^k} + |x|^\sigma |u|^p \varphi\Big)\, dx\, dt \\ - \int_0^\infty \int_{\partial K_\varepsilon} a\, u \frac{\partial\varphi}{\partial\nu}\, dx\, dt + \sum_{j=0}^{k-1} (-1)^j \int_{K_\varepsilon} \frac{\partial^{k-1-j} u}{\partial t^{k-1-j}}(x, 0) \frac{\partial^j \varphi}{\partial t^j}(x, 0)\, dx \leq 0,\end{aligned} \tag{2.2}$$

holds for any nonnegative test function $\varphi \in \mathcal{C}^{2,k}(K_\varepsilon \times]0,+\infty[)$ with compact support, such that $\varphi|_{\partial K_\varepsilon \times]0,+\infty[} = 0$.

Similarly, we define the weak solutions of the system

$$\begin{gathered} \frac{\partial^k u_i}{\partial t^k} - \Delta(a_i u_i) \geq |x|^{\sigma_{i+1}} |u_{i+1}|^{p_{i+1}}, \quad x \in K_\varepsilon,\ t \in]0,+\infty[,\ 1 \leq i \leq n, \\ u_{n+1} = u_1, \end{gathered} \tag{2.3}$$

where $p_i > 1$, $\sigma_i > -2$, for $1 \leq i \leq n$, $p_{n+1} = p_1$, $\sigma_{n+1} = \sigma_1$, and the initial data $\left(u_i^{(0)}, u_i^{(1)}, \dots, u_i^{(k-1)}\right) \in \left[L^1_{\mathrm{loc}}(K_\varepsilon)\right]^k$, $1 \leq i \leq n$.

Definition 2.2 Let $a_i \in L^\infty\left(\overline{K}_\varepsilon \times]0,+\infty[\right)$, $i \in \{1,2,\dots,n\}$. A weak solution $(u_1,\dots,u_n)$ of the system (2.3) on $K_\varepsilon \times]0,+\infty[$ is a vector of continuous functions $(u_1,\dots,u_n)$ on $\overline{K}_\varepsilon \times [0,+\infty[$ such that the traces $\frac{\partial^j u_i}{\partial t^j}(x,0)$, $(i,j) \in \{1,..,n\} \times \{1,..,k-1\}$, are well defined and locally integrable on K_ε and the n estimates

$$\begin{aligned} &\int_0^\infty \int_{K_\varepsilon} \Big(a_i\, u_i \Delta\varphi - u_i (-1)^k \frac{\partial^k \varphi}{\partial t^k} + |x|^{\sigma_{i+1}} |u_{i+1}|^{p_{i+1}} \varphi \Big)\, dx\, dt \\ &- \int_0^\infty \int_{\partial K_\varepsilon} a_i\, u_i \frac{\partial \varphi}{\partial \nu}\, dx\, dt + \sum_{j=0}^{k-1} (-1)^j \int_{K_\varepsilon} \frac{\partial^{k-1-j} u_i}{\partial t^{k-1-j}}(x,0) \frac{\partial^j \varphi}{\partial t^j}(x,0)\, dx \leq 0, \end{aligned} \tag{2.4}$$

for any $i \in \{1,2,\dots,n-1\}$, and

$$\begin{aligned} &\int_0^\infty \int_{K_\varepsilon} \Big(a_n\, u_n \Delta\varphi - u_n (-1)^k \frac{\partial^k \varphi}{\partial t^k} + |x|^{\sigma_1} |u_1|^{p_1} \varphi \Big)\, dx\, dt \\ &- \int_0^\infty \int_{\partial K_\varepsilon} a_n\, u_n \frac{\partial \varphi}{\partial \nu}\, dx\, dt + \sum_{j=0}^{k-1} (-1)^j \int_{K_\varepsilon} \frac{\partial^{k-1-j} u_n}{\partial t^{k-1-j}}(x,0) \frac{\partial^j \varphi}{\partial t^j}(x,0)\, dx \leq 0, \end{aligned} \tag{2.5}$$

hold, for any nonnegative test function $\varphi \in \mathcal{C}^{2,k}(K_\varepsilon \times]0,+\infty[)$ with compact support, such that $\varphi|_{\partial K_\varepsilon \times]0,+\infty[} = 0$.

We shall construct the test functions which will be used in our proofs. Let ζ be the standard *cut-off function*

$$\zeta(y) = \begin{cases} 1 & \text{if } 0 \leq y \leq 1, \\ 0 & \text{if } y \geq 2. \end{cases}$$

Let $p_0 \geq k+1$ and $\eta(y) = [\zeta(y)]^{p_0}$.

Explicit computation shows that there is a positive constant $C(\eta) > 0$ such that, for $y \geq 0$ and $1 < p \leq p_0$,

$$|\eta^{(k)}(y)|^p \leq C(\eta) \eta^{p-1}(y). \tag{2.6}$$

We introduce the term of the test function which depends on the variable t. Let the parameter $\rho > \varepsilon$, the exponent $\theta > 0$ and the function $t \mapsto \eta(t/\rho^\theta)$. Remark that

$$\operatorname{supp}|\eta(t/\rho^\theta)| = \{t \in \mathbb{R}^+,\ \ 0 \le t \le 2\rho^\theta\}$$

and

$$\operatorname{supp}\Big|\frac{d^k\eta}{dt^k}(t/\rho^\theta)\Big| = \{t \in \mathbb{R}^+,\ \ \rho^\theta \le t \le 2\rho^\theta\},$$

where "supp" denotes the support. It follows that

$$\int_{\operatorname{supp}|\frac{d^k\eta}{dt^k}(t/\rho^\theta)|} \frac{|\frac{d^k\eta}{dt^k}(t/\rho^\theta)|^p}{\eta^{p-1}(t/\rho^\theta)}\,dt \ \le\ c_\eta \rho^{-\theta(kp-1)}. \tag{2.7}$$

Now, we construct the part of the test function in the space variable $x \equiv (r,\omega)$. Let $s \neq 0$ a real number, then

$$\Delta\left(r^s\Phi(\omega)\right) = r^{s-2}\Phi(\omega)\left(s^2 + (N-2)s - \lambda\right).$$

Denote by s_+ and s_- the two roots of the equation

$$s^2 + (N-2)s - \lambda = 0.$$

These roots are given by

$$s_+ = -\frac{N-2}{2} + \sqrt{\Big(\frac{N-2}{2}\Big)^2 + \lambda} \quad \text{and} \quad s_- = -\frac{N-2}{2} - \sqrt{\Big(\frac{N-2}{2}\Big)^2 + \lambda}.$$

Consider the function ζ defined on K_ε by

$$\zeta(x) \equiv \zeta(r,\omega) = \Big(\big(\frac{r}{\varepsilon}\big)^{s_+} - \big(\frac{r}{\varepsilon}\big)^{s_-}\Big)\,\Phi(\omega).$$

It is interesting to note that the function ζ is harmonic in K_ε and vanishes on the boundary ∂K_ε. Moreover,

$$\frac{\partial\zeta}{\partial\nu}\Big|_{\partial K_\varepsilon} \le 0,$$

where ν is the outer normal to the full surface ∂K_ε. Indeed, thanks to the Hopf lemma,

$$\frac{\partial\zeta}{\partial\nu_\omega} = \Big(\big(\frac{r}{\varepsilon}\big)^{s_+} - \big(\frac{r}{\varepsilon}\big)^{s_-}\Big)\,\frac{\partial\Phi(\omega)}{\partial\nu_\omega} \le 0.$$

Moreover, since $s_+ > 0$ and $s_- < 0$, then

$$-\frac{\partial\zeta}{\partial r}\Big|_{r=\varepsilon} = \frac{s_- - s_+}{\varepsilon}\Phi(\omega) \le 0.$$

Let us consider now the function of r, for $r \ge \varepsilon$,

$$\xi(r) = \Big(\big(\frac{r}{\varepsilon}\big)^{s_+} - \big(\frac{r}{\varepsilon}\big)^{s_-}\Big)\,\eta\big(\frac{r}{\rho}\big).$$

Now, we give estimates of $\partial\xi/\partial r$ and $\partial^2\xi/\partial r^2$. First, we have

$$\frac{\partial\xi}{\partial r}=\Big(\frac{s_+}{\varepsilon}\big(\frac{r}{\varepsilon}\big)^{s_+-1}-\frac{s_-}{\varepsilon}\big(\frac{r}{\varepsilon}\big)^{s_--1}\Big)\,\eta\big(\frac{r}{\rho}\big)+\frac{1}{\rho}\Big(\big(\frac{r}{\varepsilon}\big)^{s_+}-\big(\frac{r}{\varepsilon}\big)^{s_-}\Big)\,\eta'\big(\frac{r}{\rho}\big).$$

Whence, there is a positive constant C, independent of ρ and r, such that

$$\begin{aligned}&\Big|\frac{\partial\xi}{\partial r}\Big|^p\\ &\le C\Big\{\Big[\frac{s_+}{\varepsilon}\big(\frac{r}{\varepsilon}\big)^{s_+-1}-\frac{s_-}{\varepsilon}\big(\frac{r}{\varepsilon}\big)^{s_--1}\Big]^p\eta^p\big(\frac{r}{\rho}\big)+\frac{1}{\rho^p}\Big[\big(\frac{r}{\varepsilon}\big)^{s_+}-\big(\frac{r}{\varepsilon}\big)^{s_-}\Big]^p\Big|\eta'\big(\frac{r}{\rho}\big)\Big|^p\Big\}.\end{aligned}$$

Consequently,

$$\Big|\frac{\partial\xi}{\partial r}\Big|^p\le C\,r^{p(s_+-1)}\eta^{p-1}\big(\frac{r}{\rho}\big)\big(1+\frac{r^p}{\rho^p}\big).\tag{2.8}$$

Moreover,

$$\begin{aligned}\frac{\partial^2\xi}{\partial r^2}=&\Big(\frac{s_+(s_+-1)}{\varepsilon^2}\big(\frac{r}{\varepsilon}\big)^{s_+-2}-\frac{s_-(s_--1)}{\varepsilon^2}\big(\frac{r}{\varepsilon}\big)^{s_--2}\Big)\eta\big(\frac{r}{\rho}\big)\\ &+\frac{2}{\rho}\Big(\frac{s_+}{\varepsilon}\big(\frac{r}{\varepsilon}\big)^{s_+-1}-\frac{s_-}{\varepsilon}\big(\frac{r}{\varepsilon}\big)^{s_--1}\Big)\eta'\big(\frac{r}{\rho}\big)+\frac{1}{\rho^2}\Big(\big(\frac{r}{\varepsilon}\big)^{s_+}-\big(\frac{r}{\varepsilon}\big)^{s_-}\Big)\eta''\big(\frac{r}{\rho}\big).\end{aligned}$$

Similarly, there is $C>0$, independent on ρ and r, such that

$$\Big|\frac{\partial^2\xi}{\partial r^2}\Big|^p\le C\,r^{p(s_+-2)}\eta^{p-1}\big(\frac{r}{\rho}\big)\Big(1+\frac{r^p}{\rho^p}+\frac{r^{2p}}{\rho^{2p}}\Big).\tag{2.9}$$

We introduce now the final test function of the space variable

$$\psi_\rho(x)=\zeta(x)\eta\big(\frac{|x|}{\rho}\big)=\Big(\big(\frac{r}{\varepsilon}\big)^{s_+}-\big(\frac{r}{\varepsilon}\big)^{s_-}\Big)\,\eta\big(\frac{r}{\rho}\big)\Phi(\omega).$$

Then

$$\Delta\psi_\rho(x)=\frac{\partial^2\psi_\rho}{\partial r^2}(x)+\frac{N-1}{r}\frac{\partial\psi_\rho}{\partial r}(x)+\frac{1}{r^2}\Delta_\omega\psi_\rho(x),$$

where

$$\Delta_\omega\psi_\rho=\Big(\big(\frac{r}{\varepsilon}\big)^{s_+}-\big(\frac{r}{\varepsilon}\big)^{s_-}\Big)\,\eta\big(\frac{r}{\rho}\big)(-\lambda\Phi)=-\lambda\psi_\rho.$$

Whence

$$\begin{aligned}|\Delta\psi_\rho(x)|^p&=\Phi^p(\omega)\Big|\Big\{\frac{\partial^2}{\partial r^2}+\frac{N-1}{r}\frac{\partial}{\partial r}-\frac{\lambda}{r^2}\Big\}(\xi(r))\Big|^p,\\ &\le C\,\Phi^p(\omega)\eta^{p-1}\big(\frac{r}{\rho}\big)r^{p(s_+-2)}\Big(1+\frac{r^p}{\rho^p}+\frac{r^{2p}}{\rho^{2p}}\Big),\\ &\le C\,\psi_\rho^{p-1}(x)r^{s_+-2p}\Big(\frac{(r/\varepsilon)^{s_+}}{(r/\varepsilon)^{s_+}-(r/\varepsilon)^{s_-}}\Big)^{p-1}\Big(1+\frac{r^p}{\rho^p}+\frac{r^{2p}}{\rho^{2p}}\Big),\end{aligned}\tag{2.10}$$

where C is a positive constant, independent of ρ and r. Let us denote $\mathcal{N} = \operatorname{supp}(\Delta\psi_\rho)$. Since $\eta(r/\rho) = 1$ for $r \le \rho$ and $\eta(r/\rho) = 0$ for $r \ge 2\rho$, then $\mathcal{N} \subset \{x \in K_\varepsilon;\ \rho \le r \le 2\rho\}$. Moreover, since $\rho > \varepsilon$, then the expressions

$$1 + \frac{r^p}{\rho^p} + \frac{r^{2p}}{\rho^{2p}} \quad \text{and} \quad \frac{(r/\varepsilon)^{s_+}}{(r/\varepsilon)^{s_+} - (r/\varepsilon)^{s_-}}$$

are bounded on $\{x \in K_\varepsilon;\ \rho \le r \le 2\rho\}$. We conclude that there is a positive constant C such that

$$|\Delta\psi_\rho(x)|^p \le C\ \psi_\rho^{p-1}(x)\frac{r^{s_+}}{\rho^{2p}} \quad \forall x \in \mathcal{N}.$$

Finally, for ρ sufficiently large, we have the estimate

$$\begin{aligned} &\int_{\mathcal{N}} \frac{|\Delta(\psi_\rho)(x)|^p}{\psi_\rho^{p-1}(x)|x|^{\sigma(p-1)}}\,dx \\ &\le \frac{C}{\rho^{2p}}\int_\rho^{2\rho}\int_\Omega \frac{r^{s_++N-1}}{r^{\sigma(p-1)}}\,d\theta\,dr \\ &\le C\begin{cases} \rho^{s_++N-\sigma(p-1)-2p} & \text{if } s_+ + N - \sigma(p-1) > 0, \\ \rho^{-2p}\ln(\rho) & \text{if } s_+ + N - \sigma(p-1) = 0, \\ \rho^{-2p} & \text{if } s_+ + N - \sigma(p-1) < 0. \end{cases} \end{aligned} \tag{2.11}$$

Consider the final test function of the variables x and t:

$$\varphi_\rho(x,t) = \eta\Big(\frac{t}{\rho^\theta}\Big)\psi_\rho(x). \tag{2.12}$$

On one hand, the same arguments used in (2.11) give the estimate

$$\begin{aligned} &\int_0^{+\infty}\int_{\mathcal{N}} \frac{|\Delta(\varphi_\rho)(x,t)|^p}{\varphi_\rho^{p-1}(x,t)|x|^{\sigma(p-1)}}dxdt \\ &\le \frac{C}{\rho^{2p}}\int_0^{2\rho^\theta}\int_\rho^{2\rho}\int_\Omega \frac{r^{s_++N-1}}{r^{\sigma(p-1)}}\,d\theta\,dr\,dt \\ &\le C\begin{cases} \rho^{s_++N-\sigma(p-1)+\theta-2p} & \text{if } s_+ + N - \sigma(p-1) > 0, \\ \rho^{\theta-2p}\ln(\rho) & \text{if } s_+ + N - \sigma(p-1) = 0, \\ \rho^{\theta-2p} & \text{if } s_+ + N - \sigma(p-1) < 0. \end{cases} \end{aligned} \tag{2.13}$$

On the other hand, if we denote $\mathcal{C}(\varepsilon,\rho) := \{x \in K_\varepsilon;\ \varepsilon \le |x| \le 2\rho\}$, then

$$\begin{aligned} &\int\int_{\operatorname{supp}\left|\frac{\partial^k\varphi_\rho}{\partial t^k}\right|} \frac{\left|\frac{\partial^k\varphi_\rho}{\partial t^k}\right|^p}{\varphi_\rho^{p-1}|x|^{(p-1)\sigma}}\,dx\,dt \\ &\le \int_{\mathcal{C}(\varepsilon,\rho)} \frac{\psi_\rho(x)}{|x|^{(p-1)\sigma}}\,dx\int_{\operatorname{supp}\left|\frac{d^k\eta}{dt^k}(t/\rho^\theta)\right|} \frac{\left|\frac{d^k\eta}{dt^k}(t/\rho^\theta)\right|^p}{\eta^{p-1}(t/\rho^\theta)}\,dt. \end{aligned} \tag{2.14}$$

Furthermore,

$$\begin{aligned}\int_{\mathcal{C}(\varepsilon,\rho)} \frac{\tilde{\psi}_\rho(x)}{|x|^{(p-1)\sigma}}\,dx &= \int_\Omega \Phi(\omega)\,d\theta \int_\varepsilon^{2\rho} \Big(\big(\frac{r}{\varepsilon}\big)^{s_+} - \big(\frac{r}{\varepsilon}\big)^{s_-}\Big)\,\eta\big(\frac{r}{\rho}\big)\,\frac{r^{N-1}}{r^{(p-1)\sigma}}\,dr \\ &\leq \frac{|\Omega|}{\varepsilon^{s_+}} \int_\varepsilon^{2\rho} r^{s_+ + N - 1 - (p-1)\sigma}\,dr \\ &\leq C \begin{cases} \rho^{s_+ + N - \sigma(p-1)} & \text{if } s_+ + N - \sigma(p-1) > 0, \\ \ln(\rho) & \text{if } s_+ + N - \sigma(p-1) = 0, \\ 1 & \text{if } s_+ + N - \sigma(p-1) < 0. \end{cases}\end{aligned} \tag{2.15}$$

Combining the estimates (2.7), (2.14) and (2.15), we obtain

$$\begin{aligned}&\int\int_{\operatorname{supp}\left|\frac{\partial^k\varphi_\rho}{\partial t^k}\right|} \frac{\left|\frac{\partial^k \varphi_\rho}{\partial t^k}\right|^p}{\varphi_\rho^{p-1}|x|^{(p-1)\sigma}}\,dx\,dt \\ &\leq C \begin{cases} \rho^{s_+ + N - \sigma(p-1) - \theta(kp-1)} & \text{if } s_+ + N - \sigma(p-1) > 0, \\ \rho^{-\theta(kp-1)}\ln(\rho) & \text{if } s_+ + N - \sigma(p-1) = 0, \\ \rho^{-\theta(kp-1)} & \text{if } s_+ + N - \sigma(p-1) < 0. \end{cases}\end{aligned} \tag{2.16}$$

In the following section, we consider the case $n = 1$.

3 Higher-Order Evolution Semilinear Inequalities

In this section, we establish nonexistence results for global solutions to the semilinear problem (2.1). The weak solutions of (2.1) are defined in Definition 2.1.

Theorem 3.1 *Assume that for all $(x,t) \in \partial K_\varepsilon \times [0,+\infty[$, $a(x,t) \geq 0$ and $u(x,t) \geq 0$. Also assume that*

$$\forall x \in K_\varepsilon;\quad \frac{\partial^{k-1} u}{\partial t^{k-1}}(x,0) \geq 0.$$

Let

$$\frac{\sigma+2}{p-1} \geq s_+ + N - 2\big(1 - \frac{1}{k}\big),$$

where $p > 1$ and $\sigma > -2$. Then there is no weak nontrivial solution u of the inequality (2.1).

Proof. Assume that (2.1) admits a nontrivial global weak solution u with

$$\frac{\sigma+2}{p-1} \geq s_+ + N - 2\big(1 - \frac{1}{k}\big).$$

In definition 2.1, let us choose the test function $\varphi(x,t) = \varphi_\rho(x,t)$ defined in (2.12). Thanks to the Hopf lemma, we have

$$\int_0^\infty \int_{\partial K_\varepsilon} a\, u \frac{\partial \varphi_\rho}{\partial \nu}\, dx\, dt \leq 0.$$

Moreover, the test function φ_ρ satisfies the equalities

$$\frac{\partial^j \varphi_\rho}{\partial t^j}(x,0) = 0, \quad \text{for} \quad j \in \{1,2,\dots,k-1\}.$$

Finally, we have

$$\int_{K_\varepsilon} \frac{\partial^{k-1} u}{\partial t^{k-1}}(x,0)\varphi_\rho(x,0)\, dx \geq 0.$$

Then, inequality (2.2) implies that

$$\int_0^\infty \int_{K_\varepsilon} |x|^\sigma |u|^p \varphi_\rho\, dx\, dt \leq \int_0^\infty \int_{K_\varepsilon} u \left(-a\Delta + (-1)^k \frac{\partial^k}{\partial t^k}\right) \varphi_\rho dx dt. \tag{3.1}$$

Let us introduce the notation

$$\begin{aligned}
I(\rho) &:= \int_0^\infty \int_{K_\varepsilon} |x|^\sigma |u|^p \varphi_\rho\, dx\, dt, \\
\mathcal{A}(\rho) &= \int_0^\infty \int_{K_\varepsilon} \frac{|\Delta\varphi_\rho|^{p'}}{(|x|^\sigma \varphi_\rho)^{p'-1}}\, dx\, dt \\
\mathcal{B}(\rho) &= \int_0^\infty \int_{K_\varepsilon} \frac{|\frac{\partial^k \varphi_\rho}{\partial t^k}|^{p'}}{(|x|^\sigma \varphi_\rho)^{p'-1}}\, dx\, dt.
\end{aligned}$$

Applying Hölder's inequality to (3.1), we obtain

$$I(\rho) \leq \max\left(||a||_\infty, 1\right) I(\rho)^{\frac{1}{p}} \left(\mathcal{A}(\rho)^{\frac{1}{p'}} + \mathcal{B}(\rho)^{\frac{1}{p'}}\right), \tag{3.2}$$

or equivalently

$$I(\rho)^{1-\frac{1}{p}} \leq \max\left(||a||_\infty, 1\right) \left(\mathcal{A}(\rho)^{\frac{1}{p'}} + \mathcal{B}(\rho)^{\frac{1}{p'}}\right).$$

At this stage, we choose the real parameter $\theta = 2/k$ and obtain

$$\mathcal{A}(\rho) \leq C\,\Theta(\rho) \quad \text{and} \quad \mathcal{B}(\rho) \leq C\,\Theta(\rho),$$

where

$$\Theta(\rho) = \begin{cases} \rho^{s_+ + N - \sigma(p'-1) - 2(p'-1/k)} & \text{if} \quad s_+ + N - \sigma(p'-1) > 0, \\ \rho^{-2(p'-1/k)} \ln(\rho) & \text{if} \quad s_+ + N - \sigma(p'-1) = 0, \\ \rho^{-2(p'-1/k)} & \text{if} \quad s_+ + N - \sigma(p'-1) < 0. \end{cases}$$

If $s_+ + N - \sigma(p'-1) > 0$, then explicit computation gives , for ρ sufficiently large,

$$I^{1-\frac{1}{p}} \leq C\rho^{\alpha},$$

where

$$\alpha = \frac{p-1}{p}\Big(s_+ + N - 2\big(1-\frac{1}{k}\big) - \frac{\sigma+2}{p-1}\Big).$$

Now, we require that $\alpha \leq 0$, which is equivalent to

$$\frac{\sigma+2}{p-1} \geq s_+ + N - 2\big(1-\frac{1}{k}\big).$$

In this case, $I(\rho)$ is bounded uniformly with respect to the variable ρ. Moreover, the function $I(\rho)$ is increasing in ρ. Consequently, the monotone convergence theorem implies that the function

$$(x,t) \equiv (r,\omega,t) \longmapsto |u(x,t)|^p|x|^{\sigma}\left(\big(\frac{r}{\varepsilon}\big)^{s_+} - \big(\frac{r}{\varepsilon}\big)^{s_-}\right)\Phi(\omega)$$

is in $L^1(K_\varepsilon \times]0,+\infty[)$. Furthermore, note that

$$\operatorname{supp}(\Delta\varphi_\rho) \subset \{t \in \mathbb{R}^+, \quad 0 \leq t \leq 2\rho^{2/k}\} \times \{x \in K_\varepsilon, \quad \rho \leq |x| \leq 2\rho\}$$

and

$$\operatorname{supp}\Big(\frac{\partial^k \varphi_\rho}{\partial t^k}\Big) \subset \{t \in \mathbb{R}^+, \quad \rho^{2/k} \leq t \leq 2\rho^{2/k}\} \times \{x \in K_\varepsilon, \quad \varepsilon \leq |x| \leq 2\rho\}.$$

Whence, instead of (3.2) we have more precisely

$$I(\rho) \leq \max\left(||a||_\infty, 1\right) \tilde{I}(\rho)^{\frac{1}{p}}\Big(\mathcal{A}(\rho)^{\frac{1}{p'}} + \mathcal{B}(\rho)^{\frac{1}{p'}}\Big), \tag{3.3}$$

where $\tilde{I}(\rho) = \int_{\mathcal{C}_\rho} |x|^{\sigma}|u|^p\varphi_\rho \, dx\, dt$ and $\mathcal{C}_\rho = \operatorname{supp}(\Delta\varphi_\rho) \cup \operatorname{supp}\big(\frac{\partial^k\varphi_\rho}{\partial t^k}\big)$. Finally, using the dominated convergence theorem, we obtain

$$\lim_{\rho\to+\infty} I(\rho) = 0.$$

This implies $u \equiv 0$, which contradicts the fact that u is assumed to be nontrivial weak solution.

Now, if $s_+ + N - \sigma(p'-1) \leq 0$, then

$$\lim_{\rho\to+\infty} \rho^{-2(p'-1/k)}\ln(\rho) = 0 \quad \text{and} \quad \lim_{\rho\to+\infty} \rho^{-2(p'-1/k)} = 0.$$

Therefore the integral $I(\rho)$ is bounded uniformly with respect to the variable ρ. The same arguments used previously complete the proof. □

The previous result is also valid for cones instead of cone-like domains. Indeed, let us consider the higher order inequality

$$\frac{\partial^k u}{\partial t^k} - \Delta(au) \geq |x|^{\sigma}\,|u|^p, \quad x \in K,\ t \in]0,+\infty[, \tag{3.4}$$

where $p > 1$, $\sigma > -2$, with the initial data

$$u(x,0) = u_0(x), \quad \text{in } K,$$
$$\frac{\partial^i u}{\partial t^i}(x,0) = u_i(x), \quad i \in \{1,2,\dots,k-1\}, \quad \text{in} \quad K.$$

Then we have the following statement.

Theorem 3.2 *Assume that for all $(x,t) \in \partial K \times [0,+\infty[$, $a(x,t) \geq 0$ and $u(x,t) \geq 0$. Also assume that*

$$\forall x \in K; \ \frac{\partial^{k-1} u}{\partial t^{k-1}}(x,0) \geq 0.$$

Let

$$\frac{\sigma+2}{p-1} \geq s_+ + N - 2\big(1 - \frac{1}{k}\big),$$

where $p > 1$ and $\sigma > -2$. Then there is no weak nontrivial solution u of the system (3.4).

Proof. Note that the cone K coincides with K_ε for $\varepsilon = 0$. In this case, the test function φ_ρ given by (2.12) is not well defined. We choose the new test function

$$\tilde{\varphi}_\rho(x,t) \equiv \tilde{\varphi}_\rho(r,\omega,t) = r^{s_+}\,\Phi(\omega)\,\eta\big(\frac{r}{\rho^\theta}\big)\,\eta\big(\frac{t}{\rho}\big).$$

The function

$$\begin{array}{ccc} K & \longrightarrow & [0,+\infty[\\ (r,\omega) & \longmapsto & r^{s_+}\,\Phi(\omega), \end{array}$$

is also harmonic. Following the different steps of the last proof with $\tilde{\varphi}_\rho$ (resp. K) instead of φ_ρ (resp. K_ε), we obtain the result. $\square$

4 Higher Order Systems of Evolution Semilinear Inequalities

We establish here nonexistence results of global solutions to the system (S_k^n). The weak solutions of (S_k^n) are defined in Definition 2.2. In this section, the initial conditions $\frac{\partial^j u_i}{\partial t^j}(x,0)$ will be denoted by $u_i^{(j)}(x,0)$, for $(i,j) \in \{1,..,n\} \times \{0,..,k-1\}$, and the vector $(X_1, X_2, \dots, X_n)$ will denote the solution of the linear system

$$\begin{pmatrix} -1 & p_1 & 0 & \dots & 0 \\ 0 & -1 & p_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & p_{n-1} \\ p_n & 0 & \dots & 0 & -1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{n-1} \\ X_n \end{pmatrix} = \begin{pmatrix} \sigma_1+2 \\ \sigma_2+2 \\ \vdots \\ \sigma_{n-1}+2 \\ \sigma_n+2 \end{pmatrix} \tag{4.1}$$

We have the following non existence result.

Theorem 4.1 *Assume that for all* $(x,t) \in \partial K_\varepsilon \times [0,+\infty[$ *and* $i \in \{1,2,\dots,n\}$, $u_i(x,t) \geq 0$ *and* $a_i(x,t) \geq 0$. *Also assume that*

$$\forall x \in K_\varepsilon, \ \ \forall i \in \{1,2,\dots,n\}; \quad \frac{\partial^{k-1} u_i}{\partial t^{k-1}}(x,0) \geq 0.$$

Let

$$\max\{X_1, X_2, \dots, X_n\} \geq s_+ + N - 2\big(1 - \frac{1}{k}\big),$$

where $p_i > 1$ *and* $\sigma_i > -2$, *for* $1 \leq i \leq n$. *Then the problem* (2.3) *has no nontrivial global weak solution.*

Proof. By contradiction, assume that (2.3) admits a nontrivial global weak solution $(u_1, u_2, \dots, u_n)$ with $\max\{X_1, X_2, \dots, X_n\} \geq s_+ + N - 2(1 - 1/k)$. In definition 2.2, let us choose the test function $\varphi(x,t) = \varphi_\rho(x,t)$ defined in (2.12). Thanks to the Hopf lemma, we have

$$\int_0^\infty \int_{\partial K_\varepsilon} a_i\, u_i \frac{\partial \varphi_\rho}{\partial \nu}\, dx\, dt \leq 0, \quad \text{for} \quad i \in \{1,2,\dots,n\}.$$

Moreover, the test function φ_ρ satisfies

$$\frac{\partial^j \varphi_\rho}{\partial t^j}(x,0) = 0, \quad \text{for} \quad j \in \{1,2,\dots,k-1\}.$$

Finally, we have

$$\int_{K_\varepsilon} \frac{\partial^{k-1} u_i}{\partial t^{k-1}}(x,0) \varphi_\rho(x,0)\, dx \geq 0, \ \ i \in \{1,2,\dots,n\}.$$

Then, inequalities (2.4) and (2.5) imply

$$\begin{gathered}\int_0^\infty \int_{K_\varepsilon} |x|^{\sigma_1} |u_1|^{p_1} \varphi_\rho \leq \int_0^\infty \int_{K_\varepsilon} u_n \left(-a_n \Delta + (-1)^k \frac{\partial^k}{\partial t^k} \right) \varphi_\rho, \\ \int_0^\infty \int_{K_\varepsilon} |x|^{\sigma_i} |u_i|^{p_i} \varphi_\rho \leq \int_0^\infty \int_{K_\varepsilon} u_{i-1} \left(-a_{i-1} \Delta + (-1)^k \frac{\partial^k}{\partial t^k} \right) \varphi_\rho, \ \ 2 \leq i \leq n.\end{gathered} \tag{4.2}$$

For $i \in \{1,2,\dots,n\}$, we use the notation

$$\begin{aligned} I_i(\rho) &:= \int_0^\infty \int_{K_\varepsilon} |x|^{\sigma_i} |u_i|^{p_i} \varphi_\rho\, dx\, dt, \\ \mathcal{A}_i(\rho) &= \int_0^\infty \int_{K_\varepsilon} \frac{|\Delta \varphi_\rho|^{p_i'}}{(|x|^{\sigma_i} \varphi_\rho)^{p_i'-1}}\, dx\, dt \\ \mathcal{B}_i(\rho) &= \int_0^\infty \int_{K_\varepsilon} \frac{|\frac{\partial^k \varphi_\rho}{\partial t^k}|^{p_i'}}{(|x|^{\sigma_i} \varphi_\rho)^{p_i'-1}}\, dx\, dt. \end{aligned}$$

Using the Hölder's inequality for system (4.2), we obtain

$$
\begin{aligned}
I_1(\rho) &\le \max(\|a_n\|_\infty, 1) I_n(\rho)^{\frac{1}{p_n}} \big(\mathcal{A}_n(\rho)^{\frac{1}{p'_n}} + \mathcal{B}_n(\rho)^{\frac{1}{p'_n}}\big),\\
I_i(\rho) &\le \max\left(\|a_{i-1}\|_\infty, 1\right) I_{i-1}(\rho)^{\frac{1}{p_{i-1}}} \big(\mathcal{A}_{i-1}(\rho)^{\frac{1}{p'_{i-1}}} + \mathcal{B}_{i-1}(\rho)^{\frac{1}{p'_{i-1}}}\big), \quad 2 \le i \le n.
\end{aligned} \tag{4.3}
$$

Whence, there is a positive constant C, independent on ρ and r such that

$$
I_i^{1-\frac{1}{p_1 p_2 \cdots p_n}} \le C \prod_{j=1}^{n} \Big(\mathcal{A}_j^{1/p'_j} + \mathcal{B}_j^{1/p'_j}\Big)^{\frac{1}{\mu_{ij}}}, \quad 1 \le i \le n,
$$

where

$$
\mu_{ij} = \begin{cases} \prod_{k=j+1}^{i-1} p_k & \text{for } i > j,\\ \prod_{k=1}^{i-1} p_k \prod_{k=j+1}^{n} p_k & \text{for } i \le j. \end{cases}
$$

At this stage, we choose the real parameter $\theta = 2/k$ to obtain

$$
\mathcal{A}_i(\rho) \le C\,\Theta_i(\rho) \quad \text{and} \quad \mathcal{B}_i(\rho) \le C\,\Theta_i(\rho), \quad \text{for } i \in \{1, 2, \dots, n\},
$$

where

$$
\Theta_i(\rho) = \begin{cases} \rho^{s_+ + N - \sigma_i(p'_i - 1) - 2(p'_i - 1/k)} & \text{if } s_+ + N - \sigma_i(p'_i - 1) > 0,\\ \rho^{-2(p'_i - 1/k)} \ln(\rho) & \text{if } s_+ + N - \sigma_i(p'_i - 1) = 0,\\ \rho^{-2(p'_i - 1/k)} & \text{if } s_+ + N - \sigma_i(p'_i - 1) < 0. \end{cases}
$$

To estimate the expressions $I_i(\rho)$, $i \in \{1, 2, \dots, n\}$, we consider two cases.

Case 1: $s_+ + N - \sigma_i(p'_i - 1) > 0$, for any $i \in \{1, 2, \dots, n\}$. Explicit computation gives , for ρ sufficiently large,

$$
I_i^{1-\frac{1}{p_1 p_2 \cdots p_n}} \le C\rho^{\alpha_i}, \quad 1 \le i \le n,
$$

where

$$
\alpha_i = \Big(1 - \frac{1}{p_1 p_2 \dots p_n}\Big)\Big(s_+ + N - 2\big(1 - \frac{1}{k}\big) - \frac{\sum_{j=1}^{n} (2 + \sigma_j)\tau_{ij}}{p_1 p_2 \dots p_n - 1}\Big),
$$

and

$$
\tau_{ij} = \begin{cases} \prod_{k=i}^{n} p_k \prod_{k=1}^{j-1} p_k & \text{for } i > j,\\ \tau_{ij} = \prod_{k=i}^{j-1} p_k & \text{for } i \le j. \end{cases}
$$

Now, we require that, at least, one of α_i, $i \in \{1, 2, \dots, n\}$, be less than zero, which is equivalent to

$$
\max\{X_1, X_2, \dots, X_n\} \ge s_+ + N - 2\big(1 - \frac{1}{k}\big),
$$

where the vector $(X_1, X_2, \dots, X_n)^T$ is the solution of (4.1). In this case, there is $i_0 \in \{1, 2, \dots, n\}$ such that $I_{i_0}(\rho)$ is bounded uniformly with respect to the variable ρ. Using the systems (4.2) and (4.3), we obtain that both $I_i(\rho)$, $1 \le i \le n$, are bounded uniformly with respect to the variable ρ. Moreover, the functions $I_i(\rho)$, $i \in \{1,2,\dots,n\}$, are increasing in ρ. Consequently, the monotone convergence theorem implies that the functions

$$(x,t) \equiv (r,\omega,t) \longmapsto |u_i(x,t)|^{p_i}|x|^{\sigma_i}\left(\big(\frac{r}{\varepsilon}\big)^{s_+} - \big(\frac{r}{\varepsilon}\big)^{s_-}\right)\Phi(\omega).$$

is in $L^1(K_\varepsilon \times]0,+\infty[)$. Precise that these functions correspond to

$$\lim_{\rho\to+\infty} |u_i(x,t)|^{p_i}|x|^{\sigma_i}\varphi_\rho(x,t).$$

Furthermore, note that

$$\operatorname{supp}(\Delta\varphi_\rho) \subset \{t \in \mathbb{R}^+,\ 0 \le t \le 2\rho^{2/k}\} \times \{x \in K_\varepsilon,\ \rho \le |x| \le 2\rho\}$$

and

$$\operatorname{supp}\big(\frac{\partial^k\varphi_\rho}{\partial t^k}\big) \subset \{t \in \mathbb{R}^+,\ \rho^{2/k} \le t \le 2\rho^{2/k}\} \times \{x \in K_\varepsilon,\ \varepsilon \le |x| \le 2\rho\}.$$

Whence, instead of (4.3) we have more precisely

$$\begin{aligned} I_1(\rho) &\le \max\left(||a_n||_\infty, 1\right) \tilde{I}_n(\rho)^{\frac{1}{p_n}}\left(\mathcal{A}_n(\rho)^{\frac{1}{p'_n}} + \mathcal{B}_n(\rho)^{\frac{1}{p'_n}}\right),\\ I_i(\rho) &\le \max\left(||a_{i-1}||_\infty, 1\right) \tilde{I}_{i-1}(\rho)^{\frac{1}{p_{i-1}}}\left(\mathcal{A}_{i-1}(\rho)^{\frac{1}{p'_{i-1}}} + \mathcal{B}_{i-1}(\rho)^{\frac{1}{p'_{i-1}}}\right),\quad 2 \le i \le n. \end{aligned} \tag{4.4}$$

where

$$\tilde{I}_i(\rho) = \int_{\mathcal{C}_\rho} |x|^{\sigma_i}|u_i|^{p_i}\varphi_\rho\,dx\,dt,$$

and $\mathcal{C}_\rho = \operatorname{supp}(\Delta\varphi_\rho) \cup \operatorname{supp}\big(\frac{\partial^k\varphi_\rho}{\partial t^k}\big)$. Finally, using the dominated convergence theorem, we obtain

$$\lim_{\rho\to+\infty} I_i(\rho) = 0, \quad i \in \{1,2,\dots,n\}.$$

This implies that $(u_1,u_2,\dots,u_n) \equiv (0,0,\dots,0)$, which contradicts the fact that $(u_1,u_2,\dots,u_n)$ is assumed to be nontrivial weak solution. We complete the proof by treating the case

Case 2: There is $i_0 \in \{1,2,\dots,n\}$, such that $s_+ + N - \sigma_{i_0}(p'_{i_0} - 1) \le 0$. The same arguments used in Case 1 give, for ρ sufficiently large,

$$I_i^{1-\frac{1}{p_1p_2\cdots p_n}} \le C\rho^{\tilde{\alpha}_i}, \quad 1 \le i \le n,$$

where $\tilde{\alpha}_i \le \alpha_i$, for $i \in \{1,2,\dots,n\}$. Then, if there is $i_1 \in \{1,2,\dots,n\}$ such that $\alpha_{i_1} \le 0$ then $\tilde{\alpha}_{i_1} \le 0$. This implies that $I_{i_1}(\rho)$ is bounded uniformly with respect to the variable ρ, which leads to the same conclusion as the Case 1. This completes the proof. $\square$

When problem (2.3) is posed on the cone K instead of the cone-like domain K_ε, our result is also valid and the proof changes very slightly. Indeed, consider

$$\begin{gathered}\frac{\partial^k u_i}{\partial t^k} - \Delta(a_i u_i) \geq |x|^{\sigma_{i+1}} |u_{i+1}|^{p_{i+1}}, \quad x \in K,\ t \in]0,+\infty[,\ 1 \leq i \leq n, \\ u_{n+1} = u_1,\end{gathered} \tag{4.5}$$

where $p_i > 1$, $\sigma_i > -2$, for $1 \leq i \leq n$, $p_{n+1} = p_1$, $\sigma_{n+1} = \sigma_1$, and the initial data $(u_i^{(0)}, u_i^{(1)}, \dots, u_i^{(k-1)}) \in [L^1_{\rm loc}(K)]^k$, $1 \leq i \leq n$.

The weak solutions of (4.5) are defined similarly as in Definition 2.1 with K instead of K_ε. Then we have the following result.

Theorem 4.2 *Assume that for all $(x,t) \in \partial K \times [0,+\infty[$ and $i \in \{1,2,\dots,n\}$, $u_i(x,t) \geq 0$ and $a_i(x,t) \geq 0$. Also assume that*

$$\forall x \in K,, \quad \forall i \in \{1,2,\dots,n\}; \quad \frac{\partial^{k-1} u_i}{\partial t^{k-1}}(x,0) \geq 0.$$

Let $\max\{X_1, X_2, \dots, X_n\} \geq s_+ + N - 2\big(1 - \frac{1}{k}\big)$, where $p_i > 1$ and $\sigma_i > -2$, for $1 \leq i \leq n$. Then problem $(\tilde{S}_k^n)$ has no nontrivial global weak solution.

For the Proof of this theorem, it suffices to use the same arguments of the last proof with $\tilde{\varphi}_\rho$ (resp K) instead of φ_ρ (resp K_ε) to obtain the result. □

5 General case

We consider the problem

$$\begin{aligned}&\tfrac{\partial^k u_i}{\partial t^k} - \Delta(a_i u_i) \geq t^{\gamma_{i+1}}\, |x|^{\sigma_{i+1}}\, |u_{i+1}|^{p_{i+1}}, \quad x \in K_\varepsilon,\ t \in]0,+\infty[,\ 1 \leq i \leq n, \\ &u_{n+1} = u_1,\end{aligned} \tag{5.1}$$

where $p_i > 1$, $\sigma_i > -2$, for $1 \leq i \leq n$, $p_{n+1} = p_1$, $\gamma_{n+1} = \gamma_1$, $\sigma_{n+1} = \sigma_1$, and the initial data $(u_i^{(0)}, u_i^{(1)}, \dots, u_i^{(k-1)}) \in [L^1_{\rm loc}(K_\varepsilon)]^k$, $1 \leq i \leq n$. We will assume that $\gamma_i \leq 0$ for $i \in \{1,2,\dots,n\}$.

We start by giving the new estimates corresponding to (2.13) and (2.16), (for given $p > 1$ and $\gamma \leq 0$)

$$\begin{aligned}&\int_0^{+\infty} \int_{\mathcal{N}} \frac{|\Delta(\varphi_\rho)(x,t)|^p}{(t^\gamma |x|^\sigma \varphi_\rho)^{p-1}} dx dt \\ &\leq \frac{C}{\rho^{2p}} \int_0^{2\rho^\theta} t^{-\gamma(p-1)}\, dt \int_\rho^{2\rho} \int_\Omega \frac{r^{s_+ + N - 1}}{r^{\sigma(p-1)}}\, d\theta\, dr \\ &\leq C \begin{cases} \rho^{s_+ + N - \sigma(p-1) + \theta - 2p - \gamma\theta(p-1)} & \text{if } s_+ + N - \sigma(p-1) > 0, \\ \rho^{\theta - 2p - \gamma\theta(p-1)} \ln(\rho) & \text{if } s_+ + N - \sigma(p-1) = 0, \\ \rho^{\theta - 2p - \gamma\theta(p-1)} & \text{if } s_+ + N - \sigma(p-1) < 0. \end{cases}\end{aligned}$$

$$\leq \quad C \begin{cases} \rho^{s_+ + N - (\sigma+\gamma\theta)(p-1)+\theta-2p} & \text{if } s_+ + N - \sigma(p-1) > 0, \\ \rho^{\theta-2p-\gamma\theta(p-1)} \ln(\rho) & \text{if } s_+ + N - \sigma(p-1) = 0, \\ \rho^{\theta-2p-\gamma\theta(p-1)} & \text{if } s_+ + N - \sigma(p-1) < 0. \end{cases} \tag{5.2}$$

and

$$\begin{aligned} &\int\int_{\mathrm{supp}\left|\frac{\partial^k \varphi_\rho}{\partial t^k}\right|} \frac{\left|\frac{\partial^k \varphi_\rho}{\partial t^k}\right|^p}{(t^\gamma |x|^\sigma \varphi_\rho)^{p-1}}\, dx\, dt \\ &\leq \quad C \begin{cases} \rho^{s_+ + N - \sigma(p-1) - \theta(kp-1) - \gamma\theta(p-1)} & \text{if } s_+ + N - \sigma(p-1) > 0, \\ \rho^{-\theta(kp-1)-\gamma\theta(p-1)} \ln(\rho) & \text{if } s_+ + N - \sigma(p-1) = 0, \\ \rho^{-\theta(kp-1)-\gamma\theta(p-1)} & \text{if } s_+ + N - \sigma(p-1) < 0. \end{cases} \\ &\leq \quad C \begin{cases} \rho^{s_+ + N - (\sigma+\gamma\theta)(p-1) - \theta(kp-1)} & \text{if } s_+ + N - \sigma(p-1) > 0, \\ \rho^{-\theta(kp-1)-\gamma\theta(p-1)} \ln(\rho) & \text{if } s_+ + N - \sigma(p-1) = 0, \\ \rho^{-\theta(kp-1)-\gamma\theta(p-1)} & \text{if } s_+ + N - \sigma(p-1) < 0. \end{cases} \end{aligned} \tag{5.3}$$

Let the vector $(Y_1, Y_2, \dots, Y_n)$ be the solution of the linear system

$$\begin{pmatrix} -1 & p_1 & 0 & \dots & 0 \\ 0 & -1 & p_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & p_{n-1} \\ p_n & 0 & \dots & 0 & -1 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{n-1} \\ Y_n \end{pmatrix} \begin{pmatrix} \sigma_1 + 2 + 2\gamma_1/k \\ \sigma_2 + 2 + 2\gamma_2/k \\ \vdots \\ \sigma_{n-1} + 2 + 2\gamma_{n-1}/k \\ \sigma_n + 2 + 2\gamma_n/k \end{pmatrix}$$

The we have the following non existence result.

Theorem 5.1 *Assume that for all $(x,t) \in \partial K_\varepsilon \times [0,+\infty[$ and $i \in \{1,2,\dots,n\}$, $u_i(x,t) \geq 0$ and $a_i(x,t) \geq 0$. Also assume that*

$$\forall x \in K_\varepsilon, \ \ \forall i \in \{1,2,\dots,n\}; \ \ \frac{\partial^{k-1} u_i}{\partial t^{k-1}}(x,0) \geq 0.$$

Let

$$\max\{Y_1, Y_2, \dots, Y_n\} \geq s_+ + N - 2\big(1 - \frac{1}{k}\big),$$

where $p_i > 1$ and $\sigma_i + 2\gamma_i/k > -2$, for $1 \leq i \leq n$. Then the problem 1.1 has no nontrivial global weak solution.

Proof. We follow the previous proof and replace the expressions $I_i(\rho)$, $\mathcal{A}_i(\rho)$ and $\mathcal{B}_i(\rho)$ by

$$\overline{I}_i(\rho) := \int_0^\infty \int_{K_\varepsilon} |x|^{\sigma_i} |u_i|^{p_i} \varphi_\rho \, dx\, dt,$$

$$\overline{\mathcal{A}}_i(\rho) = \int_0^\infty \int_{K_\varepsilon} \frac{|\Delta\varphi_\rho|^{p_i'}}{(t^{\gamma_i}|x|^{\sigma_i}\varphi_\rho)^{p_i'-1}}\,dx\,dt$$
$$\overline{\mathcal{B}}_i(\rho) = \int_0^\infty \int_{K_\varepsilon} \frac{|\frac{\partial^k\varphi_\rho}{\partial t^k}|^{p_i'}}{(t^{\gamma_i}|x|^{\sigma_i}\varphi_\rho)^{p_i'-1}}\,dx\,dt,$$

respectively. By setting the parameter $\theta = 2/k$, we conclude that

$$\overline{\mathcal{A}}_i(\rho) \le C\,\overline{\Theta}_i(\rho) \quad \text{and} \quad \overline{\mathcal{B}}_i(\rho) \le C\,\overline{\Theta}_i(\rho),$$

where

$$\overline{\Theta}_i(\rho) = \begin{cases} \rho^{s_+ + N - (\sigma_i + 2\gamma_i/k)(p_i'-1) - 2(p_i'-1/k)} & \text{if } s_+ + N - \sigma_i(p_i'-1) > 0, \\ \rho^{-2(p_i'-1/k) - 2\gamma_i(p_i'-1)/k}\ln(\rho) & \text{if } s_+ + N - \sigma_i(p_i'-1) = 0, \\ \rho^{-2(p_i'-1/k) - 2\gamma_i(p_i'-1)/k} & \text{if } s_+ + N - \sigma(p_i'-1) < 0, \end{cases}$$

for any $i \in \{1, 2, \dots, N\}$. As in the previous proof, note that the leading exponent in the previous estimate is

$$s_+ + N - (\sigma_i + 2\gamma_i/k)(p_i'-1) - 2(p_i'-1/k).$$

In other words, the only difference with the last proof is that the parameter σ_i must be replaced by $\sigma_i + 2\gamma_i/k$. Which achieves the proof. □

Remark. If we consider the semilinear problem (1.2) instead of (2.1) with $\gamma \le 0$, then σ has to be replaced by $\sigma + 2\gamma/k$ in Theorem 3.1 and Theorem 3.2.

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