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On the result of He concerning the smoothness of solutions to the Navier-Stokes equations *

Milan Pokorný

Abstract

We improve the regularity criterion for the Navier-Stokes equations proved by He [4]. We show that for the Cauchy problem the Leray-Hopf weak solution is smooth provided $\nabla u_3 \in L^t(0,T;L^s)$, $\frac{2}{t}+\frac{3}{s}\le\frac{3}{2}$.

1 Introduction and Main Theorem

We consider the Cauchy problem for the Navier-Stokes equations in three space dimensions, i.e. the system of PDE's

$$\left.\begin{array}{c}\varrho\dfrac{\partial \mathbf{u}}{\partial t}+\varrho(\mathbf{u}\cdot\nabla)\mathbf{u}-\nu\Delta\mathbf{u}+\nabla p=\varrho\mathbf{f}\\ \nabla\cdot\mathbf{u}=0\end{array}\right\}\quad \text{in } (0,T)\times\mathbb{R}^3 \\ \mathbf{u}(0,\mathbf{x})=\mathbf{u}_0(\mathbf{x})\qquad \text{in } \mathbb{R}^3\,. \tag{1.1}$$

Here, $\mathbf{u}:(0,T)\times\mathbb{R}^3\mapsto\mathbb{R}^3$ is the velocity field, $p:(0,T)\times\mathbb{R}^3\mapsto\mathbb{R}^3$ is the pressure, $\mathbf{f}:(0,T)\times\mathbb{R}^3\mapsto\mathbb{R}^3$ denotes the volume force, $0<T<\infty$. For our purpose, the values of the constant density ϱ and the constant viscosity ν do not play any role; we therefore assume without loss of generality $\varrho=\nu=1$. Moreover, in order to simplify the presentation of the result, we take $\mathbf{f}=\mathbf{0}$.

As is well known, the existence of globally in time smooth solution to system (1.1) is proved only for small data [6]; for large data we only have the existence of a weak solution [8], which is locally in time smooth provided the data are smooth enough [5].

On the other hand, if we assume that our weak solution is "slightly" smoother than it follows from the definition then such a solution is as smooth as the data of the problem allow (provided the data are smooth enough). We call $\mathbf{u}\in L^\infty(0,T;L^2)\cap L^2(0,T;W^{1,2})$ with $\nabla\cdot\mathbf{u}=\mathbf{0}$ a weak solution to (1.1) with

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$\mathbf{f} = \mathbf{0}$, if $\langle \mathbf{u}', \mathbf{v}\rangle + \int \nabla \mathbf{u} : \nabla \mathbf{v} + \int ((\mathbf{u}\cdot\nabla)\mathbf{u})\cdot \mathbf{v} = 0$ for a.a. $t \in (0,T)$ and all $\mathbf{v} \in W^{1,2}$ with $\nabla \cdot \mathbf{v} = 0$, and $\lim_{t\to 0+} \mathbf{u}(t) = \mathbf{u}_0$ in the weak L^2 sense.

Let us mention some of these regularity criteria

(I) $\mathbf{u} \in L^t(I; L^s)$, $\frac{2}{t} + \frac{3}{s} \le 1$, $2 \le t \le \infty$, $3 \le s \le \infty$ (see [13], for the case $s = 3$ see [12], [3])

(II) $\nabla \mathbf{u} \in L^t(I; L^s)$, $\frac{2}{t} + \frac{3}{s} \le 2$, $1 \le t \le \infty$, $\frac{3}{2} < s \le \infty$ (see [1])

(III) $p \in L^t(I; L^s)$, $\frac{2}{t} + \frac{3}{s} \le 2$, $1 \le t \le \infty$, $\frac{3}{2} < s \le \infty$ (see [2])

On the other hand, in two space dimensions the weak solution is known to be unique and as regular as the data of the problem allow (see [7]). Therefore several authors tried to find regularity criteria which depend only on one velocity component and/or on the derivatives of one velocity component or derivatives only in the x_3 direction

(IV) $u_3 \in L^t(I; L^s)$, $\frac{2}{t} + \frac{3}{s} \le \frac{1}{2}$, $4 \le t \le \infty$, $6 < s \le \infty$ (see [9])

(V) $u_3 \in L^t(I; L^s)$, $\frac{2}{t} + \frac{3}{s} \le 1$, $2 \le t \le \infty$, $3 < s \le \infty$ and $\frac{\partial u_1}{\partial x_3}, \frac{\partial u_2}{\partial x_3} \in L^t(I; L^s)$, $\frac{2}{t} + \frac{3}{s} \le 2$, $1 \le t \le \infty$, $\frac{3}{2} < s \le \infty$

(VI) $\frac{\partial u_3}{\partial x_3} \in L^\infty(I; L^\infty)$

(VII) $\frac{\partial \mathbf{u}}{\partial x_3} \in L^t(I; L^s)$, $\frac{2}{t} + \frac{3}{s} \le \frac{3}{2}$, $\frac{4}{3} \le t \le \infty$, $2 \le s \le \infty$

(VIII) $\frac{\partial u_3}{\partial x_3} \in L^t(I; L^s)$, $\frac{2}{t} + \frac{3}{s} \le 1$, $2 \le t \le \infty$, $3 \le s \le \infty$ and $\frac{\partial u_1}{\partial x_3}, \frac{\partial u_2}{\partial x_3} \in L^t(I; L^s)$, $\frac{2}{t} + \frac{3}{s} \le 2$, $1 \le t \le \infty$, $\frac{3}{2} < s \le \infty$ (For the results (V)–(VIII) see [11].)

In the recent paper [4], He followed similar aim and obtained the regularity of the Navier-Stokes system provided $\nabla \mathbf{u_3} \in L^t(I; L^s)$, $\frac{2}{t} + \frac{3}{s} \le 1$, $2 \le t \le \infty$, $3 \le s \le \infty$. This result, in comparison to the result of Neustupa, Novotný and Penel [9], does not seem to be optimal. One would rather expect in this case $\frac{2}{t} + \frac{3}{s} \le \frac{3}{2}$. The aim of this note is to show that this is indeed true. More precisely

Theorem 1.1 *Let* $\mathbf{u}_0 \in W^{1,2}(\mathbb{R}^3)$ *with* $\operatorname{div} \mathbf{u}_0 = 0$, $\mathbf{f} = \mathbf{0}$ *and let* $\mathbf{u}$ *be a weak solution to the Navier-Stokes equations (1.1) which satisfies the energy inequality*[1]*. Assume moreover that* $\nabla u_3 \in L^t(0,T; L^s)$ *with* $\frac{2}{t} + \frac{3}{s} \le \frac{3}{2}$, $\frac{4}{3} \le t \le \infty$, $2 \le s \le \infty$*. Then* $\mathbf{u}$ *and the corresponding pressure* p *is the smooth solution*

[1] We say that a weak solution to the Navier-Stokes equations (1.1) satisfies the energy inequality if for almost all $t \in (0,T)$ it holds

$$\frac{1}{2}\frac{d}{dt}\Big(\int |\mathbf{u}|^2(t)\Big) + \int |\nabla \mathbf{u}|^2(t) \le 0\,.$$

It is not difficult to show that such weak solutions exist; on the other hand, it is not known whether any weak solution in the sense above satisfies the energy inequality. Weak solutions satisfying the energy inequality are usually called Leray-Hopf weak solutions *to the Navier-Stokes equations, i.e.* $\mathbf{u} \in L^\infty(0,T;W^{1,2}(\mathbb{R}^3) \cap L^2(0,T;W^{2,2}(\mathbb{R}^3))$, $\nabla p \in L^2(0,T;W^{1,2}(\mathbb{R}^3))$. *Moreover* $\mathbf{u} \in C^\infty([\delta,T) \times \mathbb{R}^3)$ *and* $p \in C^\infty([\delta,T) \times \mathbb{R}^3)$ *for delta any small positive number.*

Remark 1.2 Assuming $\mathbf{f} \neq \mathbf{0}$ we would get the regularity of the solution in dependence on the regularity $\mathbf{f}$. Since these calculations are relatively standard, we omit them here.

Remark 1.3 At the first sight Theorem 1.1 seems to be a direct consequence of the result from [9]. But this is true only for $2 \leq s < 3$, i.e. for the case when $W^{1,s} \hookrightarrow L^{\frac{3s}{3-s}}$. Nevertheless, we will prove Theorem 1 also in this case.

2 Proof of Theorem 1.1

In what follows, we use standard notation for the Sobolev and Lebesgue spaces ($W^{k,p}$ and L^p, respectively) as well as for the corresponding norms ($\|\cdot\|_{k,p}$ and $\|\cdot\|_p$, respectively) without specifying the domain (always $\mathbb{R}^3$). Morerover, the Bochner spaces $L^p(I;X)$ will be in the case of $X = L^q$ denoted shortly $L^{p,q}$. In order to simplify the notation, we will not distinguish between $(L^p)^m$ and L^p.

Any generic constant will denoted by C; its value may vary, even on the same line or in the same formula. We also use the summation convention.

The proof will be a modification of the procedure used by Neustupa, Novotný and Penel (see [9] and also [10]), where regularity criteria only for suitable weak solutions were studied. This proof can be also regarded as a way how to transform the results from the above mentioned papers to the Cauchy problem.

First, as $\mathbf{u}_0 \in W^{1,2}$ with $\operatorname{div} \mathbf{u}_0 = 0$, we know that there exists exactly one strong solution to the Navier-Stokes equations with the initial condition $\mathbf{u}_0$ (on a possibly short time interval). Denote

$$\tau_0 = \sup_{\tau > 0} \Big\{ \text{there exists a strong solution to (1.1) on } (0,\tau) \Big\}.$$

It is well known that $\tau_0 > 0$. As our weak solution from Theorem 1.1 satisfies the energy inequality, it coincides with the strong solution on its interval of existence (see e.g. [15]). We will show that the assumption $\tau_0 < T$ leads to a contradiction. Note that the solution is smooth on the open interval $(0,\tau_0)$ and thus the equations are satisfied pointwise here.

Denote by $Y = L^\infty(0,\tau;L^2) \cap L^2(0,\tau;W^{1,2})$ with $0 < \tau < \tau_0$. We will not specify the length of the time interval $(0,\tau)$ in the notation for Y. Our aim will be to show that under the assumptions of Theorem 1.1, $\nabla \mathbf{u}$ remains bounded in Y independently of τ, provided $\tau_0 < T$. Thus, using standard extension argument, we get a contradiction with the maximality of τ_0.

To this aim, we first show that for any $\tau < \tau_0$

$$\|\omega_3\|_Y^2 \equiv \|\omega_3\|^2_{L^\infty(0,\tau;L^2)} + \|\nabla \omega_3\|^2_{L^2(0,\tau;L^2)} \leq C_1 + C_2 \|\omega\|_Y \tag{2.1}$$

with $C_i = C_i(\mathbf{u}_0, \|\nabla u_3\|_{L^{t,s}})$, $i = 1, 2$. In particular, the constants are independent of τ. (Here, by ω we denote the vorticity, i.e. $\omega = \operatorname{curl}\mathbf{u}$.) Using (2.1) it will be relatively easy to show that

$$\|\nabla \mathbf{u}\|_Y \le C(\mathbf{u}_0, \|\nabla u_3\|_{L^{s,t}})$$

with the constant independent of τ. This finishes our proof as our weak solution cannot blow up at τ_0.

Let us first prove (2.1).

Lemma 2.1 *Under the assumptions of Theorem 1.1, there exist positive constants* $C_1(\mathbf{u}_0, \|\nabla \mathbf{u}\|_{L^{t,s}})$ *and* $C_2(\mathbf{u}_0, \|\nabla \mathbf{u}\|_{L^{t,s}})$ *such that (2.1) holds true.*

Proof: As explained above, it is enough to show inequality (2.1) for smooth solutions to (1.1). To this aim, let us look at the equation for the vorticity. We have

$$\frac{\partial \omega}{\partial t} - \Delta\omega + (\mathbf{u}\cdot\nabla)\omega - (\omega\cdot\nabla)\mathbf{u} = \mathbf{0}\,. \tag{2.2}$$

Multiplying the equation for ω_3 by ω_3 and integrating over $\mathbb{R}^3$ we get (note that all integrals are finite)

$$\frac{1}{2}\frac{d}{dt}\|\omega_3\|_2^2 + \|\nabla\omega_3\|_2^2 = \int (\omega\cdot\nabla)u_3\omega_3 \equiv I_1\,. \tag{2.3}$$

We will now estimate I_1. Using Hölder's inequality and standard interpolation inequalities we have ($\frac{1}{p} + \frac{1}{q} + \frac{1}{s} = 1$, $2 \le s \le \infty$, $2 \le p \le 6$, $2 \le q \le 3$)

$$\begin{aligned}|I_1| &\le \|\nabla u_3\|_s \|\omega_3\|_p \|\omega\|_q \\ &\le \|\nabla u_3\|_s \|\omega_3\|_2^{\frac{6-p}{2p}} \|\omega_3\|_6^{\frac{3p-6}{2p}} \|\omega\|_2^{\frac{6-q}{2q}} \|\omega\|_6^{\frac{3q-6}{2q}} \\ &\le \frac{1}{2}\|\nabla\omega_3\|_2^2 + C\|\nabla u_3\|_s^{\frac{4p}{6+p}} \|\omega\|_2^{2\frac{p}{q}\frac{6-q}{6+p}} \|\omega\|_6^{2\frac{p}{q}\frac{3q-6}{6+p}} \|\omega_3\|_2^{2\frac{6-p}{6+p}}\,.\end{aligned}$$

Thus

$$\frac{d}{dt}\|\omega_3\|_2^{\frac{4p}{6+p}} \le C\|\nabla u_3\|_s^{\frac{4p}{6+p}} \|\omega\|_2^{2\frac{p}{q}\frac{6-q}{6+p}} \|\omega\|_6^{2\frac{p}{q}\frac{3q-6}{6+p}}\,,$$

i.e.

$$\|\omega_3\|_2^{\frac{4p}{6+p}}(\tau) \le \|\omega_3\|_2^{\frac{4p}{6+p}}(0) + C\|\omega\|_{L^{\infty,2}}^{4\frac{p}{q}(\frac{3-q}{6+p})} \int_0^\tau \|\nabla u_3\|_s^{\frac{4p}{6+p}} \|\omega\|_2^{\frac{2p}{6+p}} \|\omega\|_6^{2\frac{p}{q}\frac{3q-6}{6+p}}\, ds\,.$$

Now, $\frac{3s-4}{6s} + \frac{p}{6+p} + \frac{3q-6}{q}\frac{p}{6+p} = 1$ (recall that $\frac{1}{p} + \frac{1}{q} + \frac{1}{s} = 1$) and we get

$$\|\omega_3\|_{L^\infty(0,\tau;L^2)}^{\frac{4p}{6+p}} \le C(\mathbf{u}_0) + C\|\nabla u_3\|_{L^{t,s}}^{\frac{4p}{6+p}} \|\omega\|_{L^{2,2}}^{\frac{2p}{6+p}} \|\omega\|_Y^{\frac{2p}{6+p}}\,,$$

i.e.

$$\|\omega_3\|_{L^{\infty,2}}^2 \le C_1 + C_2\|\omega\|_Y\,. \tag{2.4}$$

Returning to (2.3), repeating calculations above and using (2.4) we get the desired inequality (2.1). □

Proof of Theorem 1.1: We rewrite equation $(1.1)_1$ in the form

$$\frac{\partial \mathbf{u}}{\partial t} - \Delta \mathbf{u} + (\omega \times \mathbf{u}) + \nabla (p + \frac{1}{2}|\mathbf{u}|^2) = \mathbf{0}\,. \tag{2.5}$$

Multiplying equation (2.5) by $-\Delta \mathbf{u}$ and integrating over $\mathbb{R}^3$ we easily see that for $0 < \tau < \tau_0$

$$\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{u}\|_2^2(\tau) + \int \|\nabla^2 \mathbf{u}\|_2^2(\tau) = \int (\omega \times \mathbf{u}) \cdot \Delta \mathbf{u} \equiv I_2 \tag{2.6}$$

Using

$$(\omega \times \mathbf{u}) \cdot \Delta \mathbf{u} = (\omega_2 u_3 - \omega_3 u_2)\Delta u_1 + (\omega_3 u_1 - \omega_1 u_3)\Delta u_2 + (\omega_1 u_2 - \omega_2 u_1)\Delta u_3$$

we get

$$\begin{aligned} |I_2| &\le C\Big(\int |\nabla \mathbf{u}|^2 |\nabla u_3| + |\mathbf{u}||\nabla^2 \mathbf{u}||\nabla u_3| + \int |\mathbf{u}||\nabla^2 \mathbf{u}||\omega_3| \Big) \\ &\equiv C(I_{21} + I_{22} + I_{23})\,. \end{aligned}$$

We estimate each term separately; I_{21} and I_{22} using better regularity properties of ∇u_3, I_{23} using Lemma 2.1. If $s < \infty$,

$$I_{21} \le \|\nabla u_3\|_s \|\nabla \mathbf{u}\|^2_{\frac{2s}{s-1}} \le \varepsilon \|\nabla^2 \mathbf{u}\|_2^2 + C(\varepsilon)\|\nabla \mathbf{u}_3\|_s^{\frac{2s}{2s-3}} \|\nabla \mathbf{u}\|_2^2\,. \tag{2.7}$$

Similarly we proceed for $s = \infty$.

$$I_{22} \le \varepsilon \|\nabla^2 \mathbf{u}\|_2^2 + C(\varepsilon) \int |\mathbf{u}|^2 |\nabla u_3|^2\,. \tag{2.8}$$

If $2 \le s \le 3$ we estimate the integral on the right-hand side by $\|\nabla u_3\|_s^2 \|\mathbf{u}\|^2_{\frac{2s}{s-2}}$ and using the interpolation inequality

$$\|\mathbf{u}\|_{\frac{2s}{s-2}} \le C\|\nabla \mathbf{u}\|_2^{\frac{2s-3}{s}} \|\nabla^2 \mathbf{u}\|_2^{\frac{3-s}{s}}$$

we get

$$I_{22} \le 2\varepsilon \|\nabla^2 \mathbf{u}\|_2^2 + C(\varepsilon)\|\nabla u_3\|_s^{\frac{2s}{2s-3}} \|\nabla \mathbf{u}\|_2^2\,. \tag{2.9}$$

For $s > 3$ we estimate

$$\int |\mathbf{u}|^2 |\nabla u_3|^2 \le \|\nabla u_3\|_s^{2(1-\alpha)} \|\nabla u_3\|_2^{2\alpha} \|\mathbf{u}\|^2_{\frac{2s}{(s-2)(1-\alpha)}}\,,$$

where for our purpose the optimal choice of α is $\frac{2s-6}{5s-6}$ (for $s < \infty$) and $s = \frac{2}{5}$ (for $s = \infty$), respectively. Note that $0 \le \alpha \le \frac{2}{5}$. Now, as $\frac{10}{3} \le \frac{2s}{(s-2)(1-\alpha)} \le 6$, we use the interpolation inequality

$$\|\mathbf{u}\|^2_{\frac{2}{3}\frac{5s-6}{s-2}} \le C\|\mathbf{u}\|_2^{4\frac{s-3}{5s-6}} \|\nabla \mathbf{u}\|_2^{\frac{6s}{5s-6}}$$

and thus
$$\int |\mathbf{u}|^2|\nabla u_3|^2 \le C\|\nabla u_3\|_s^{\frac{6s}{5s-6}}\|\nabla \mathbf{u}\|_2^2\|\mathbf{u}\|_2^{4\frac{s-3}{5s-6}}\,.$$
Taking (2.8) into account we end up with
$$I_{22} \le \varepsilon\|\nabla^2\mathbf{u}\|_2^2 + C(\varepsilon,\mathbf{u}_0)\|\nabla \mathbf{u}_3\|_s^{\frac{6s}{5s-6}}\|\nabla\mathbf{u}\|_2^2\,.$$
Note that, even though $\frac{6s}{5s-6} \ge \frac{2s}{2s-3}$ for $s \ge 3$, we still have with $t = \frac{6s}{5s-6}$ that $\frac{2}{t} + \frac{3}{s} = \frac{5}{3} + \frac{1}{s}$ for $3 \le s \le \infty$.

Finally we consider I_{23}. Here we apply Lemma 2.1 and
$$I_{23} \le \|\nabla^2\mathbf{u}\|_2\|\omega_3\|_3\|\mathbf{u}\|_6 \le \varepsilon\|\nabla^2\mathbf{u}\|_2^2 + \varepsilon\|\omega_3\|_3^4 + C(\varepsilon)\|\nabla\mathbf{u}\|_2^4\,. \tag{2.10}$$
Inequalities (2.6)–(2.10), after integrating over $(0,\tau)$, $\tau < \tau_0$ read as follows
$$\begin{aligned} &\frac{1}{2}\|\nabla\mathbf{u}\|_2^2(\tau) + \int_0^\tau \|\nabla^2\mathbf{u}\|_2^2 \\ &\le K\varepsilon\int_0^\tau \|\nabla^2\mathbf{u}\|_2^2 + \varepsilon\int_0^\tau \|\omega_3\|_3^4 \\ &\quad + C(\varepsilon,\mathbf{u}_0)\int_0^\tau (\|\nabla u_3\|_s^{\frac{2s}{2s-3}} + g(s)\|\nabla u_3\|_s^{\frac{6s}{5s-6}} + \|\nabla\mathbf{u}\|_2^2)\|\nabla\mathbf{u}\|_2^2\,, \end{aligned} \tag{2.11}$$
where $g(s) = 0$ for $2 \le s \le 3$ and $g(s) = 1$ for $s > 3$. Lemma 2.1 yields
$$\int_0^\tau \|\omega_3\|_3^4 \le C_1(\mathbf{u}_0, \|\nabla u_3\|_{L^{t,s}}) + C_2(\mathbf{u}_0, \|\nabla u_3\|_{L^{t,s}})\|\nabla\mathbf{u}\|_Y^2\,.$$
(Note that the larger $\|\nabla u_3\|_{L^{t,s}}$ is, the smaller ε must be.) Thus from (2.11), taking ε sufficiently small, it follows that
$$\begin{aligned} &\|\nabla\mathbf{u}\|_2^2(\tau) + \int_0^\tau \|\nabla^2\mathbf{u}\|_2^2 \\ &\le \sup_{\sigma\in(0,\tau)} \|\nabla\mathbf{u}\|_2^2(\sigma) + \int_0^\tau \|\nabla^2\mathbf{u}\|_2^2 \\ &\le C_1(\mathbf{u}_0, \|\nabla u_3\|_{L^{t,s}}) \\ &\quad + C_2(\mathbf{u}_0, \|\nabla u_3\|_{L^{t,s}})\int_0^\tau \big(\|\nabla u_3\|_s^{\frac{2s}{2s-3}} + g(s)\|\nabla u_3\|_s^{\frac{6s}{5s-6}} + \|\nabla\mathbf{u}\|_2^2\big)\|\nabla\mathbf{u}\|_2^2 \end{aligned}$$
and, applying the Gronwall inequality, we obtain
$$\|\nabla\mathbf{u}\|_{L^\infty(0,\tau;L^2)}^2 + \|\nabla^2\mathbf{u}\|_{L^2(0,\tau;L^2)}^2 \le C(\mathbf{u}_0, \|\nabla u_3\|_{L^{t,s}})\,,$$
where the constant C is in particular independent of τ as $\tau \to \tau_0$. Theorem 1.1 is proved. □

Remark 2.2 I would like to thank the referee who kindly informed me that a similar result as presented in Theorem 1.1 was recently obtained by Y. Zhou [16]. The main idea of the proof (i.e. the estimate of ω_3 in Lemma 2.1 and consequently of ω (proof of Theorem 1.1)) is basically the same. On the other hand, the too papers differ in the way how the quantities on the right-hand side are estimated as well as in the argument how the formally obtained a priori estimates are verified for only weak solutions to the Navier-Stokes equations. I was also kindly informed by the authors below that a similar problem, for $s = 3$ and suitable weak solutions in bounded domains, was also considered by Z. Skalák and P. Kučera in [14].

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Milan Pokorný
Math. Institute of Charles University
Sokolovská 83, 186 75 Praha 8, Czech Republic
e-mail: pokorny@karlin.mff.cuni.cz