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SOLVABILITY OF A (P, N-P)-TYPE MULTI-POINT BOUNDARY-VALUE PROBLEM FOR HIGHER-ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this article, we study the differential equation

$$(-1)^{n-p}x^{(n)}(t) = f(t, x(t), x'(t), \dots, x^{(n-1)}(t)), \quad 0 < t < 1,$$

subject to the multi-point boundary conditions

$$x^{(i)}(0) = 0 \quad \text{for } i = 0, 1, \dots, p-1,$$
$$x^{(i)}(1) = 0 \quad \text{for } i = p+1, \dots, n-1,$$
$$\sum_{i=1}^{m} \alpha_i x^{(p)}(\xi_i) = 0,$$

where $1 \le p \le n-1$. We establish sufficient conditions for the existence of at least one solution at resonance and another at non-resonance. The emphasis in this paper is that f depends on all higher-order derivatives. Examples are given to illustrate the main results of this article.

1. Introduction

In recent years, there have been many studies concerning the solvability of multi-point boundary-value problems for second order differential equations at resonance case; see for example [14, 15, 17, 20, 21, 22, 26] and the references therein. However, there has no publication concerning the solvability of multi-point boundary-value problems for higher order differential equations at resonance.

In this paper, we consider the differential equation

$$(-1)^{n-p}x^{(n)}(t) = f(t, x(t), x'(t), \dots, x^{(n-1)}(t)), \quad 0 < t < 1, \tag{1.1}$$

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subject to the boundary conditions

$$\begin{gathered} x^{(i)}(0)=0 \quad \text{for } i=0,1,\dots,p-1,\\ x^{(i)}(1)=0 \quad \text{for } i=p+1,\dots,n-1,\\ \sum_{i=1}^{m}\alpha_i x^{(p)}(\xi_i)=0, \end{gathered} \tag{1.2}$$

where $f:[0,1]\times\mathbb{R}^n\to\mathbb{R}$ is continuous, $m\ge 2$, $n\ge 2$ are integers, $1\le p\le n-1$ is a fixed value, $\alpha_i\in\mathbb{R}$ $(i=1,2,\dots,m)$ and $0\le\xi_1<\xi_2<\dots<\xi_m\le 1$ are fixed.

When $\sum_{i=1}^{m}\alpha_i\neq 0$, the linear operator $Lx(t)=(-1)^{n-p}x^{(n)}(t)$, defined in a suitable Banach space, is invertible. This is called the non-resonance case; otherwise, it is called the resonance case.

If $n=3$, $m=1$, $p=1$, $f(t,x,y)\equiv g(x)$ and $0<\xi_1<1$, the boundary-value problem (1.1)–(1.2) becomes

$$\begin{gathered} x'''(t)=g(x), \quad 0<t<1,\\ x(0)=0, \alpha_1 x'(\xi_1)=0, \quad x''(1)=0, \end{gathered} \tag{1.3}$$

where g is continuous. Anderson [8] studied the existence of multiple positive solutions of (1.3) when $\alpha_1\neq 0$.

The boundary-value problem

$$\begin{gathered} x^{(n)}(t)=f(t,x(t)), \quad 0<t<1,\\ x^{(i)}(0)=0, \quad \text{for } i=0,1,\dots,p-1,\\ x^{(i)}(1)=0 \quad \text{for } i=p,\dots,n-1, \end{gathered} \tag{1.4}$$

is called the $(p,n-p)$ right focal boundary-value problem [1, 3, 4, 5, 7, 13, 18], and is a special case of (1.1)-(1.2). Many authors studied (1.4) and its special cases; see for example [1, 13, 18, 29]. We remark that in the papers mentioned above, f depends only on t and $x(t)$, or on t and even order derivatives $x(t), x''(t),\dots$. Since (1.1)–(1.2) is a generalization of (1.4), we call this $(p,n-p)$-type boundary-value problem.

To the best of our knowledge, (1.1)–(1.2) has not been studied till now. Motivated and inspired by [10, 15, 19, 25], we establish sufficient conditions for the existence of at least one solution of (1.1)–(1.2) at resonance and another solution at non-resonance. The emphasis in this paper is that f depends on all higher-order derivatives. The method used is based on the coincidence degree method developed by Gaines and Mawhin [16] and on Shaeffer's theorem [27].

This paper can be placed in the existence theory of boundary-value problems for ordinary differential equations. The foundations and many important results on this theory were established by Agarwal, O'Regan and Wong, whose scientific output is summarized in the monographs [1, 6]. It is observed that this particular branch of differential equations has been developed and gained prominence since the early 1980s. In recent years, many authors have discussed the boundary-value problems at non-resonance or resonance for second-order differential equations [1, 16, 21, 26].

This paper is organized as follows. In Section 2, we establish existence results for solutions of (1.1)–(1.2) at resonance. In section 3, we show the existence of solutions of (1.1)–(1.2) at non-resonance. In section 4, we give some examples to illustrate the main results of this paper.

2. Solvability of (1.1)–(1.2) at resonance

In this section, we establish sufficient conditions for the existence of at least one solution of (1.1)–(1.2) in the resonance case, i.e. $\sum_{i=1}^m \alpha_i = 0$. In this case, the operator $Lx(t) = (-1)^{n-p}x^{(n)}(t)$ is not invertible. We assume that $\sum_{i=1}^m \alpha_i^2 \neq 0$. For convenience, we first introduce some notation and an abstract existence theorem proved by Gaines and Mawhin [16].

Let X and Y be Banach spaces, $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator of index zero, $P : X \to X$, $Q : Y \to Y$ be projectors such that

$$\operatorname{Im} P = \ker L, \quad \ker Q = \operatorname{Im} L, \quad X = \ker L \oplus \ker P, \quad Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

It follows that

$$L|_{\operatorname{dom} L \cap \ker P} : \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$$

is invertible, we denote the inverse of that map by K_p.

If Ω is an open bounded subset of X, $\operatorname{dom} L \cap \overline{\Omega} \neq \Phi$, the map $N : X \to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I - Q)N : \overline{\Omega} \to X$ is compact.

Theorem 2.1 ([16]). *Let L be a Fredholm operator of index zero and let N be L-compact on Ω. Assume that the following conditions are satisfied:*

- (i) *$Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(\operatorname{dom} L/ \ker L) \cap \partial\Omega] \times (0, 1)$*
- (ii) *$Nx \notin \operatorname{Im} L$ for every $x \in \ker L \cap \partial\Omega$;*
- (iii) *$\deg(\Lambda QN\big|_{\ker L}, \Omega \cap \ker L, 0) \neq 0$, where $\Lambda : Y/ \operatorname{Im} L \to \ker L$ is an isomorphism.*

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

We use the classical Banach spaces $C^k[0, 1]$. Let $X = C^{n-1}[0, 1]$ and $Y = c^0[0, 1]$. The space Y is endowed with the norm $\|y\|_\infty = \max_{t\in[0,1]} |y(t)|$. The space X is endowed with the norm $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty, \dots, \|x^{(n-1)}\|_\infty\}$. Define the linear operator L and the nonlinear operator N by

$$\begin{gathered} L : X \cap \operatorname{dom} L \to Y, \quad Lx(t) = (-1)^{n-p}x^{(n)}(t), \\ N : X \to Y, \quad Nx(t) = f(t, x(t), x'(t), \dots, x^{(n-1)}(t)), \end{gathered}$$

where

$$\begin{aligned} \operatorname{dom} L =& \big\{x \in C^n[0, 1] : x^{(i)}(0) = 0 \text{ for } i = 0, 1, \dots, p - 1, \\ & x^{(i)}(1) = 0 \text{ for } i = p + 1, \dots, n - 1, \ \sum_{i=1}^m \alpha_i x^{(p)}(\xi_i) = 0\big\}. \end{aligned}$$

Lemma 2.2. *The following results hold.*

- (i) $\ker L = \{ct^p, \ t \in [0, 1], \ c \in \mathbb{R}\}$
- (ii) $\operatorname{Im} L = \big\{y \in Y, \ \sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} y(s)ds = 0\big\}$
- (iii) *L is a Fredholm operator of index zero*
- (iv) *There are projectors $P : X \to X$ and $Q : Y \to Y$ such that* $\ker L = \operatorname{Im} P$ *and* $\ker Q = \operatorname{Im} L$. *Furthermore, let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \cap \operatorname{dom} L \neq \Phi$, then N is L-compact on $\overline{\Omega}$*
- (v) *$x(t)$ is a solution of* (1.1)–(1.2) *if and only if x is a solution of the operator equation $Lx = Nx$ in* $\operatorname{dom} L$.

Proof. (i) Let $x \in \ker L$, then $x^{(n)}(t) = 0$ and $x^{(i)}(0) = 0$ for $i = 0, 1, \dots, p-1$ and $x^{(i)}(1) = 0$ for $i = p+1, \dots, n-1$ and $\sum_{i=1}^m \alpha x^{(p)}(\xi_i) = 0$. It is easy to get $x(t) = ct^p$, thus $x \in \{ct^p : t \in [0,1],\ c \in \mathbb{R}\ \}$. On the other hand, if $x(t) = ct^p$, then we find that $x \in \ker L$. This completes the proof of (i).

(ii) For $y \in \operatorname{Im} L$, then there is $x \in \operatorname{dom} L$ such that $(-1)^{n-p} x^{(n)}(t) = y(t)$ and $x^{(i)}(0) = 0$ for $i = 0, 1, \dots, p-1$ and $x^{(i)}(1) = 0$ for $i = p+1, \dots, n-1$ and $\sum_{i=1}^m \alpha x^{(p)}(\xi_i) = 0$. Thus

$$x^{(p)}(t) = \int_t^1 \frac{(s-t)^{n-p-1}}{(n-p-1)!} y(s)ds + A.$$

Then

$$x^{(p)}(\xi_i) = \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} y(s)ds + A \ for\ i = 1, \dots, m.$$

Hence

$$\sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} y(s)ds = 0. \tag{2.1}$$

On the other hand, if (2.1) holds, we let

$$x(t) = \int_0^t \frac{(t-s)^{p-1}}{(p-1)!} \int_s^1 \frac{(u-s)^{n-p-1}}{(n-p-1)!} y(u)du\,ds + \frac{At^p}{p!}, \quad t \in [0,1].$$

Then $x \in \operatorname{dom} L \cap X$ and $Lx = y$. Thus the proof of (ii) is completed.

(iii) From (i), $\dim \ker L = 1$. On the other hand, we claim that there is $k \in \{0, 1, \dots, m-1\}$ such that

$$\sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} s^k ds \neq 0.$$

In fact, if for all $k \in \{0, 1, \dots, m-1\}$, we have

$$\sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} s^k ds = 0.$$

It is easy to see that the determinant of coefficients of above equations is

$$\begin{aligned}
&\Big| \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} s^k ds \Big|_{m\times m} \\
&= \begin{vmatrix} \int_{\xi_1}^1 \frac{(s-\xi_1)^{n-p-1}}{(n-p-1)!} ds & \dots & \int_{\xi_1}^1 \frac{(s-\xi_1)^{n-p-1}}{(n-p-1)!} s^{m-1} ds \\ \vdots & & \vdots \\ \int_{\xi_m}^1 \frac{(s-\xi_m)^{n-p-1}}{(n-p-1)!} ds & \dots & \int_{\xi_m}^1 \frac{(s-\xi_m)^{n-p-1}}{(n-p-1)!} s^{m-1} ds \end{vmatrix} \\
&= \Big| \frac{(1-\xi_i)^{n-p}}{(n-p)!} - k \frac{(1-\xi_i)^{n-p+1}}{(n-p+1)!} + k(k-1) \frac{(1-\xi_i)^{n-p+2}}{(n-p+2)!} - \dots \\
&\quad + (-1)^k k! \frac{(1-\xi_i)^{n-p+k}}{n-p+k)!} \Big|_{m\times m} \\
&= \begin{vmatrix} \frac{(1-\xi_1)^{n-p}}{(n-p)!} & -k\frac{(1-\xi_1)^{n-p+1}}{(n-p+1)!} & \dots & (-1)^{m-1}(m-1)!\frac{(1-\xi_1)^{n-p+m-1}}{(n-p+m-1)!} \\ \vdots & & & \vdots \\ \frac{(1-\xi_m)^{n-p}}{(n-p)!} & -k\frac{(1-\xi_m)^{n-p+1}}{(n-p+1)!} & \dots & (-1)^{m-1}(m-1)!\frac{(1-\xi_m)^{n-p+m-1}}{(n-p+m-1)!} \end{vmatrix} \neq 0
\end{aligned}$$

since it can be transformed into a Vandermon dominant and $0 \le \xi_1 < \xi_2 < \cdots < \xi_m \le 1$. Hence, we get $\alpha_1 = \cdots = \alpha_m = 0$, which contradicts $\sum_{i=1}^m \alpha_i^2 \neq 0$.

Now, for $y \in Y$, let

$$y_0 = y - \Big(\sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} y(s)ds\, t^k\Big)/\Big(\sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} s^k ds\Big).$$

It is easy to check that $y_0 \in \operatorname{Im} L$. Let $\overline{R} = \{ct^k : t \in [0,1],\ c \in \mathbb{R}\}$. Then $Y = \overline{R} + \operatorname{Im} L$. Again, $\overline{R} \cap \operatorname{Im} L = \{0\}$, so $Y = \overline{R} \oplus \operatorname{Im} L$. Hence $\dim Y/\operatorname{Im} L = 1$. On the other hand, $\operatorname{Im} L$ is closed. So L is a Fredholm operator of index zero.

(iv) Define the projectors $Q : Y \to Y$ and $P : X \to X$ by

$$Qy(t) = t^k \sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} y(s)ds \quad \text{for } y \in Y,$$
$$Px(t) = x^{(p)}(1)t^p \quad \text{for } x \in X.$$

It is easy to prove that $\ker L = \operatorname{Im} P$ and $\operatorname{Im} L = \ker Q$. Then the inverse $K_p : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ of the map $L : \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$ can be written by

$$K_p y(t) = \int_0^t \frac{(t-s)^{p-1}}{(p-1)!} \int_s^1 \frac{(u-s)^{n-p-1}}{(n-p-1)!} y(u)du\, ds \quad \text{for } y \in \operatorname{Im} L.$$

In fact, for $y \in \operatorname{Im} L$, we have

$$(LK_p)y(t) = L\Big(\int_0^t \frac{(t-s)^{p-1}}{(p-1)!} \int_s^1 \frac{(u-s)^{n-p-1}}{(n-p-1)!} y(u)du\, ds\Big) = y(t).$$

On the other hand, for $x \in \ker P \cap \operatorname{dom} L$, it follows that

$$\begin{aligned}
(K_pL)x(t) &= K_p((-1)^{n-p}x^{(n)}(t)) \\
&= \int_0^t \frac{(t-s)^{p-1}}{(p-1)!} \int_s^1 \frac{(u-s)^{n-p-1}}{(n-p-1)!} (-1)^{n-p}x^{(n)}(u)du\, ds \\
&= \int_0^t \frac{(t-s)^{p-1}}{(p-1)!} (-x^{(p)}(1) + x^{(p)}(s))ds \\
&= \int_0^t \frac{(t-s)^{p-1}}{(p-1)!} x^{(p)}(s)ds \\
&= x(t).
\end{aligned}$$

Furthermore, one has

$$\begin{aligned}
QNx(t) &= Qf(t, x(t), x'(t), \dots, x^{(n-1)}(t)) \\
&= \sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} f(s, x(s), x'(s), \dots, x^{(n-1)}(s))ds
\end{aligned}$$

and

$$\begin{aligned}
&K_p(I-Q)Nx(t) \\
&= K_p\Big[f(t, x(t), x'(t), \dots, x^{(n-1)}(t)) \\
&\quad - \sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} f(s, x(s), x'(s), \dots, x^{(n-1)}(s))ds\Big]
\end{aligned}$$

$$\begin{aligned}
&= \int_0^t \frac{(t-s)^{p-1}}{(p-1)!}\Big(\int_s^1 \frac{(u-s)^{n-p-1}}{(n-p-1)!} f(u,x(u),x'(u),\dots,x^{(n-1)}(u))du\Big)ds \\
&- \sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} f(s,x(s),x'(s),\dots,x^{(n-1)}(s))ds \\
&\times \int_0^t \frac{(t-s)^{p-1}}{(p-1)!}\Big(\int_s^1 \frac{(u-s)^{n-p-1}}{(n-p-1)!}\Big)ds.
\end{aligned}$$

Since f is continuous, using the Ascoli-Arzela theorem, we can prove that $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N : \overline{\Omega} \to X$ is compact, thus N is L-compact on $\overline{\Omega}$.
(v) The proof is simple and is omitted. $\square$

For the next theorem, we set the following asumptions:

(A1) There is $M > 0$ such that for any $x \in \operatorname{dom} L/\ker L$, if $|x^{(p)}(t)| > M$ for all $t \in (0, \frac{1}{2})$, then
$$\sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} f(s,x(s),x'(s),\dots,x^{(n-1)}(s))ds \neq 0$$

(A2) There is a function $a \in C^0[0,1]$ and positive numbers $a_i (i = 0,1,\dots,n-1)$ and $\beta_i \in [0,1)$ $(i = 0,1,\dots,n-1)$ such that
$$|f(t,x_0,x_1,\dots,x_{n-1})| \le a(t) + \sum_{i=0}^{n-1} a_i|x_i|^{\beta_i}$$
for $t \in [0,1]$ and $(x_0,x_1,\dots,x_{n-1}) \in \mathbb{R}^n$

(A3) There is $M^* > 0$ such that for any $c \in \mathbb{R}$ then either
$$c\sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} f(s,cs^p,cps^{p-1},\dots,cp!,0,\dots,0)ds < 0 \quad \forall |c| > M^*$$
or
$$c\sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} f(s,cs^p,cps^{p-1},\dots,cp!,0,\dots,0)ds > 0 \quad \forall |c| > M^*.$$

Theorem 2.3. *Under Assumptions (A1)–(A3), the boundary-value problem* (1.1)–(1.2) *has at least one solution.*

Proof. To apply Theorem 2.1, we define an open bounded subset Ω of X so that (i), (ii) and (iii) of Theorem 2.1 hold. To obtain Ω, we do three steps. The proof of this theorem is divide into four steps.
Step 1. Let
$$\Omega_1 = \{x \in \operatorname{dom} L/\ker L,\ Lx = \lambda Nx \text{ for } \textit{some } \lambda \in (0,1)\}.$$
For $x \in \Omega_1$, $x \notin \ker L, \lambda \neq 0$ and $Nx \in \operatorname{Im} L$, thus $QNx = 0$. Then
$$\sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} f(s,x(s),x'(s),\dots,x^{(n-1)}(s))ds = 0.$$
Hence by (A1), we know that there is $t_0 \in (0, \frac{1}{2})$ such that $|x^{(p)}(t_0)| \le M$. Thus
$$|x^{(p)}(t)| \le |x^{(p)}(t_0)| + \Big|\int_{t_0}^t x^{(p+1)}(s)ds\Big|$$

$$
\begin{aligned}
&\le M + \int_0^1 |x^{(p+1)}(s)|ds \\
&\le M + \|x^{(p+1)}\|_\infty,
\end{aligned}
$$

i.e. $\|x^{(p)}\|_\infty \le M + \|x^{(p+1)}\|_\infty$. On the other hand, it is easy to prove that

$$
\|x\|_\infty \le \|x'\|_\infty \le \dots \le \|x^{(p)}\|_\infty \ \textit{and}\ \|x^{(p+1)}\|_\infty \le \dots \le \|x^{(n-1)}\|_\infty.
$$

So $\|x\| = \max\{\|x^{(p)}\|_\infty,\ \|x^{(n-1)}\|_\infty\}$. Now, we prove that there is $t_1 \in [0,1]$ such that

$$
|x^{(n-1)}(t_1)| \le \frac{(n-p-1)!M}{(1-t_0)^{n-p-1}}. \tag{2.2}
$$

In fact, if

$$
|x^{(n-1)}(t)| > \frac{(n-p-1)!M}{(1-t_0)^{n-p-1}} \quad \text{for all } t \in [0,1],
$$

then either

$$
x^{(n-1)}(t) > \frac{(n-p-1)!M}{(1-t_0)^{n-p-1}} \quad \text{for all } t \in [0,1] \tag{2.3}
$$

or

$$
x^{(n-1)}(t) < -\frac{(n-p-1)!M}{(1-t_0)^{n-p-1}} \quad \text{for all } t \in [0,1], \tag{2.4}
$$

or

$$
\begin{gathered}
x^{(n-1)}(t) > \frac{(n-p-1)!M}{(1-t_0)^{n-p-1}} \quad \text{for some } t \in [0,1] \\
x^{(n-1)}(t) < -\frac{(n-p-1)!M}{(1-t_0)^{n-p-1}} \quad \text{for other } t \in [0,1].
\end{gathered} \tag{2.5}
$$

It is easy to see that if (2.5) holds, there exists $t_1 \in [0,1]$ such that $x^{(n-1)}(t_1) = (n-p-1)!M/(1-t_0)^{n-p-1}$, thus (2.2) holds, which is a contradiction. Therefore, for all $t \in [0,1]$, we have

$$
(-1)^{n-p-1}x^{(p)}(t) > \frac{(1-t)^{n-p-1}}{(n-p-1)!}\frac{(n-p-1)!M}{(1-t_0)^{n-p-1}}
$$

or

$$
(-1)^{n-p-1}x^{(p)}(t) < -\frac{(1-t)^{n-p-1}}{(n-p-1)!}\frac{(n-p-1)!M}{(1-t_0)^{n-p-1}}.
$$

Hence

$$
|x^{(p)}(t)| > \frac{(1-t)^{n-p-1}}{(n-p-1)!}\frac{(n-p-1)!M}{(1-t_0)^{n-p-1}}.
$$

Then we obtain

$$
|x^{(p)}(t_0)| > \frac{(1-t_0)^{n-p-1}}{(n-p-1)!}\frac{(n-p-1)!M}{(1-t_0)^{n-p-1}} = M,
$$

which contradicts $|x^{(p)}(t_0)| \le M$. Hence there is $t_1 \in [0,1]$ such that

$$
|x^{(n-1)}(t_1)| \le \frac{(n-p-1)!M}{(1-t_0)^{n-p-1}} \le 2^{n-p-1}(n-p-1)!M.
$$

Thus we get

$$
\begin{aligned}
|x^{(n-1)}(t)| &\le |x^{(n-1)}(t_1)| + |\int_{t_1}^t x^{(n)}(s)ds| \\
&\le 2^{n-p-1}(n-p-1)!M + \int_0^1 |f(s,x(s),x'(s),\dots,x^{(n-1)}(s))|ds
\end{aligned}
$$

$$
\begin{aligned}
&\le 2^{n-p-1}(n-p-1)!M + \int_0^1 a(s)ds + \sum_{i=0}^{n-1} a_i \int_0^1 |x^{(i)}(s)|^{\beta_i} ds \\
&\le 2^{n-p-1}(n-p-1)!M + \int_0^1 a(s)ds + \sum_{i=0}^{n-1} a_i \|x^{(i)}\|_\infty^{\beta_i} \\
&\le 2^{n-p-1}(n-p-1)!M + \int_0^1 a(s)ds + \Big(\sum_{i=0}^{p} a_i\Big)\|x^{(p)}\|_\infty^{\beta_i} \\
&\quad + \Big(\sum_{i=p+1}^{n-1} a_i\Big)\|x^{(n-1)}\|_\infty^{\beta_i}.
\end{aligned}
$$

and

$$
\begin{aligned}
|x^{(p)}(t)| &\le |x^{(p)}(t_0)| + \Big|\int_{t_0}^t x^{(p+1)}(s)ds\Big| \\
&\le M + \int_0^1 |x^{(p+1)}(s)|ds \\
&= M + \int_0^1 \int_s^1 \frac{(u-s)^{n-p-2}}{(n-p-2)!} |f(u,x(u),x'(u),\dots,x^{(n-1)}(u))|du\, ds \\
&\le M + \int_0^1 \frac{s^{n-p-2}}{(n-p-2)!} |f(s,x(s),x'(s),\dots,x^{(n-1)}(s))|ds \\
&\le M + \int_0^1 \frac{s^{n-p-2}}{(n-p-2)!} a(s)ds + \frac{1}{(n-p-1)!}\sum_{i=0}^{n-1} a_i \|x^{(i)}\|_\infty^{\alpha_i} \\
&\le M + \int_0^1 \frac{s^{n-p-2}}{(n-p-2)!} a(s)ds + \frac{1}{(n-p-1)!}\sum_{i=0}^{p} a_i \|x^{(p)}\|_\infty^{\beta_i} \\
&\quad + \frac{1}{(n-p-1)!}\sum_{i=p+1}^{n-1} a_i \|x^{(n-1)}\|_\infty^{\beta_i}.
\end{aligned}
$$

Without loss of generality, suppose that $\|x^{(n-1)}\|_\infty > 1$, then

$$
\begin{aligned}
&\|x^{(n-1)}\|_\infty \\
&\le 2^{n-p-1}(n-p-1)!M + \int_0^1 a(s)ds + \sum_{i=0}^{p} a_i \|x^{(p)}\|_\infty^{\beta_i} + \sum_{i=p+1}^{n-1} a_i \|x^{(n-1)}\|_\infty^{\beta_i} \\
&\le 2^{n-p-1}(n-p-1)!M + \int_0^1 a(s)ds + \sum_{i=0}^{p} a_i (M + \|x^{(p+1)}\|_\infty)^{\beta_i} \\
&\quad + \sum_{i=p+1}^{n-1} a_i \|x^{(n-1)}\|_\infty^{\beta_i} \\
&\le 2^{n-p-1}(n-p-1)!M + \int_0^1 a(s)ds + \sum_{i=0}^{p} a_i (M + \|x^{(n-1)}\|_\infty)^{\beta_i} \\
&\quad + \sum_{i=p+1}^{n-1} a_i \|x^{(n-1)}\|_\infty^{\beta_i}.
\end{aligned}
$$

It follows from $\beta_i \in [0,1)$ that there is $M_1 > 0$ such that $\|x^{(n-1)}\|_\infty \le M_1$. Hence

$$\|x^{(p)}\|_\infty \le M + \int_0^1 \frac{s^{n-p-2}}{(n-p-2)!} a(s)ds + \frac{1}{(n-p-1)!}\sum_{i=0}^{p} a_i \|x^{(p)}\|_\infty)^{\beta_i} + \frac{1}{(n-p-1)!}\sum_{i=p+1}^{n-1} a_i M^{\beta_i}.$$

We see from above inequality and $\beta_i \in [0,1)$ that there is $M_2 > 0$ such that $\|x^{(p)}\|_\infty \le M_2$. Hence we get $\|x\| \le \max\{M_1,\ M_2\} = M'$. It follows that Ω_1 is bounded.

Step 2. Let $\Omega_2 = \{x \in \ker L : Nx \in \operatorname{Im} L\}$. For $x \in \Omega_2$, then $x(t) = ct^p$ for some $c \in [0,1]$. It suffices to prove that there is $M'' > 0$ such that $|c| \le M''$. $Nx \in \operatorname{Im} L$ implies

$$\sum_{i=1}^{m} \alpha_i \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} f(s, cs^p, cps^{p-1}, \dots, cp!, 0, \dots, 0)ds = 0.$$

By (A3), we get $|c| \le M^*$. Thus Ω_2 is bounded.

Step 3. According to (A3), for any $c \in \mathbb{R}$ if $|c| > M^*$, then either

$$c\sum_{i=1}^{m} \alpha_i \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} f(s, cs^p, cps^{p-1}, \dots, cp!, 0, \dots, 0)ds < 0 \tag{2.6}$$

or

$$\sum_{i=1}^{m} \alpha_i \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} f(s, cs^p, cps^{p-1}, \dots, cp!, 0, \dots, 0)ds > 0. \tag{2.7}$$

If (2.6) holds, let

$$\Omega_3 = \{x \in \ker L : -\lambda \wedge x + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\},$$

where $\wedge$ is the isomorphism given by $\wedge(ct^p) = ct^k$ for all $c \in \mathbb{R}$. Now, we shall show that Ω_3 is bounded. Since for $ct^p \in \Omega_3$, we have

$$\lambda c = (1-\lambda)\sum_{i=1}^{m} \alpha_i \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} f(s, cs^p, cps^{p-1}, \dots, cp!, 0, \dots, 0)ds.$$

If $\lambda = 1$, it follows from above equality that $c = 0$. Otherwise, if $|c| > M^*$, in view of (2.2), one has

$$\lambda c^2 = (1-\lambda)c\sum_{i=1}^{m} \alpha_i \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} f(s, cs^p, cps^{p-1}, \dots, cp!, 0, \dots, 0)ds < 0,$$

which contradicts $\lambda c^2 \ge 0$. Thus Ω_3 is bounded.

If (2.7) holds, let

$$\Omega_3 = \{x \in \ker L : \lambda \wedge x + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\}.$$

Similarly to above argument, we can prove that Ω_3 is bounded.

Next, we show that all conditions of Theorem 2.1 are satisfied. Set Ω be a open bounded subset of X such that $\Omega \supset \cup_{i=1}^3 \overline{\Omega_i}$. By Lemma 2.2, L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. From the definition of Ω, we have the first two conditions for Theorem 2.1:

- $Lx \ne \lambda Nx$ for $x \in (\operatorname{dom} L/\ker L) \cap \partial\Omega$ and $\lambda \in (0,1)$
- $Nx \notin \operatorname{Im} L$ for $x \in \ker L \cap \partial\Omega$.

Step 4. We shall prove the third condition for applying Theorem 2.1:

- $\deg(QN|_{\ker L},\ \Omega\cap\ker L, 0)\neq 0$.

Let $H(x,\lambda)=\pm\lambda\wedge x+(1-\lambda)QNx$. According the definition of Ω, we know $H(x,\lambda)\neq 0$ for $x\in\partial\Omega\cap\ker L$, thus by homotopy property of degree,

$$\begin{aligned}\deg(QN|\ker L,\Omega\cap\ker L,0)&=\deg(H(\cdot,0),\Omega\cap\ker L,0)\\&=\deg(H(\cdot,1),\Omega\cap\ker L,0)\\&=\deg(\pm\wedge,\Omega\cap\ker L,0)\neq 0.\end{aligned}$$

Thus by Theorem 2.1, $Lx=Nx$ has at least one solution in $\operatorname{dom}L\cap\overline{\Omega}$, which is a solution of (1.1)–(1.2). $\square$

For the following theorem, we need the following assumptions:

(A4) There exists $M>0$ such that for all $x\in\operatorname{dom}L$ if $|x^{(p)}(t)|>M$ for all $t\in[0,1]$, then
$$\sum_{i=1}^m\alpha_i\int_{\xi_i}^1\frac{(s-\xi_i)^{n-1-p}}{(n-1-p)!}f(s,x(s),x'(s),\dots,x^{(n-1)}(s))ds\neq 0.$$

(A5) There exists $a_0\in C^0[0,1]$ and non-negative numbers a_i such that
$$|f(t,x_0,x_1,\dots,x_{n-1})|\leq a_0(t)+\sum_{i=0}^{n-1}a_i|x_i|$$
for all $t\in[0,1]$ and $(x_0,\dots,x_{n-1})\in\mathbb{R}^n$.

Theorem 2.4. *Under Assumptions (A3), (A4), (A5), the boundary-value problem* (1.1)–(1.2) *has at least one solution provided that*
$$\sum_{i=0}^p a_i<(n-1-p)!,\qquad\sum_{i=p+1}^{n-1}a_i<1,$$
$$\sum_{i=p+1}^{n-1}a_i+\frac{\Big(\sum_{i=0}^p a_i\Big)\Big(\sum_{i=p+1}^{n-1}a_i\Big)}{(n-1-p)!-\sum_{i=0}^p a_i}<1.$$

Proof. The proof is similar to that of Theorem 2.3. We need to do four steps. Let $\Omega_i(i=1,2,3)$ be defined in the proof of Theorem 2.3.

Step 1. Prove that Ω_1 is bounded. For $x\in\Omega_1$,
$$\sum_{i=1}^m\alpha_i\int_{\xi_i}^1\frac{(s-\xi_i)^{n-1-p}}{(n-1-p)!}f(s,x(s),x'(s),\dots,x^{(n-1)}(s))ds=0.$$
It follows from (A4) that there is $t_0\in[0,1]$ such that $|x^{(p)}(t_0)|\leq M$. On the other hand, $x\in\Omega_1$ implies
$$x^{(n)}(t)=\lambda f(t,x(t),x'(t),\dots,x^{(n-1)}(t)),\quad t\in(0,1).$$
Integrating it from 0 to t if $p\leq n-2$, or from t_0 to t if $p=n-1$, we get
$$\begin{aligned}|x^{(n-1)}(t)|&=\begin{cases}\big|x^{(n-1)}(0)+\lambda\int_0^t f(s,x(s),\dots,x^{(n-1)}(s))ds\big| & \text{for } p\leq n-2,\\ \big|x^{(n-1)}(t_0)+\lambda\int_{t_0}^t f(s,x(s),\dots,x^{(n-1)}(s))ds\big| & \text{for } p=n-1\end{cases}\\&\leq\begin{cases}\int_0^1|f(s,x(s),\dots,x^{(n-1)}(s))|ds,\\ M+\int_0^1|f(s,x(s),\dots,x^{(n-1)}(s))|ds\end{cases}\end{aligned}$$

$$
\begin{aligned}
&\le M + \int_0^1 (a_0(s) + \sum_{i=0}^{n-1} a_i |x^{(i)}(s)|)ds \\
&\le M + \int_0^1 a_0(s)ds + \sum_{i=0}^{n-1} a_i \int_0^1 |x^{(i)}(s)|ds.
\end{aligned}
$$

It is easy to see that $x^{(i)}(t)| \le \|x^{(p)}\|_\infty$ for $i = 0, 1, \dots, p$ and $|x^{(i)}(t)| \le \|x^{(n-1)}\|_\infty$ for all $i = p+1, \dots, n-1$ and $t \in [0, 1]$. Hence

$$
|x^{(n-1)}(t)| \le M + \int_0^1 a_0(s)ds + \Big(\sum_{i=0}^{p} a_i\Big)\|x^{(p)}\|_\infty + \Big(\sum_{i=p+1}^{n-1} a_i\Big)\|x^{(n-1)}\|_\infty.
$$

Thus

$$
\|x^{(n-1)}\|_\infty \le M + \int_0^1 a_0(s)ds + \Big(\sum_{i=0}^{p} a_i\Big)\|x^{(p)}\|_\infty + \Big(\sum_{i=p+1}^{n-1} a_i\Big)\|x^{(n-1)}\|_\infty.
$$

On the other hand, we have

$$
x^{(p+1)}(t) = \lambda \int_t^1 \frac{(s-t)^{n-1-p}}{(n-1-p)!} f(s, x(s), \dots, x^{(n-1)}(s))ds.
$$

Integrating from t_0 to t, we get

$$
\begin{aligned}
|x^{(p)}(t)| &= \Big|x^{(p)}(t_0) + \lambda \int_{t_0}^t f(s, x(s), \dots, x^{(n-1)}(s))ds\Big| \\
&\le M + \int_0^1 \int_s^1 \frac{(u-s)^{n-1-p}}{(n-1-p)!} f(u, x(u), \dots, x^{(n-1)}(u))du\, ds \\
&\le M + \frac{1}{(n-1-p)!} \int_0^1 |f(s, x(s), \dots, x^{(n-1)}(s))|ds \\
&\le M + \frac{1}{(n-1-p)!} \Big(\int_0^1 a_0(s)ds + \sum_{i=0}^{n-1} a_i |x^{(i)}(s)|ds\Big).
\end{aligned}
$$

Similarly, we get

$$
\|x^{(p)}\|_\infty \le M + \frac{1}{(n-1-p)!} \Big(\int_0^1 a_0(s)ds + \sum_{i=0}^{p} a_i \|x^{(p)}\|_\infty + \sum_{i=p+1}^{n-1} a_i \|x^{(n-1)}\|_\infty\Big).
$$

Hence

$$
\begin{gathered}
\Big(1 - \sum_{i=p+1}^{n-1} a_i\Big)\|x^{(n-1)}\|_\infty \le M + \int_0^1 a_0(s)ds + \Big(\sum_{i=0}^{p} a_i\Big)\|x^{(p)}\|_\infty, \\
\begin{aligned}
&\Big(1 - \frac{1}{(n-1-p)!} \sum_{i=0}^{p} a_i\Big)\|x^{(p)}\|_\infty \\
&\le M + \frac{1}{(n-1-p)!} \Big(\int_0^1 a_0(s)ds + \sum_{i=n-1-p}^{n-1} a_i \|x^{(n-1)}\|_\infty\Big).
\end{aligned}
\end{gathered}
$$

Thus we get from the assumptions of the Theorem 2.4

$$
\Big(1 - \sum_{i=p+1}^{n-1} a_i\Big)\|x^{(n-1)}\|_\infty \le M + \int_0^1 a_0(s)ds + \frac{\sum_{i=0}^{p} a_i}{1 - \frac{1}{(n-1-p)!}\sum_{i=0}^{p} a_i}\Big[M
$$

$$+\frac{1}{(n-1-p)!}\Big(\int_0^1 a_0(s)ds+\sum_{i=n-1-p}^{n-1} a_i\|x^{(n-1)}\|_\infty\Big)\Big].$$

i.e.,

$$\begin{aligned}&\Big(1-\sum_{i=p+1}^{n-1} a_i-\frac{\left(\sum_{i=0}^p a_i\right)\left(\sum_{i=p+1}^{n-1} a_i\right)}{(n-1-p)!-\sum_{i=0}^p a_i}\Big)\|x^{(n-1)}\|_\infty\\ &\le M+\int_0^1 a_0(s)ds+\frac{(n-1-p)!\sum_{i=0}^p a_i}{(n-1-p)!-\sum_{i=0}^p a_i}\Big[M+\frac{1}{(n-1-p)!}\int_0^1 a_0(s)ds\Big].\end{aligned}$$

It follows from the assumptions of Theorem 2.4 that there is $M_1>0$ such that $\|x^{(n-1)}\|_|infty\le M_1$. Thus there is $M_2>0$ such that $\|x^{(p)}\|_\infty\le M_2$. So $\|x\|\le\max\{M_1,M_2\}$. Thus Ω_1 is bounded.

Step2. Prove that Ω_2 is bounded. It similar to the Step 2 of the proof of Theorem 2.3 and is omitted.

Step 3. Prove thatΩ_3 is bounded. It is same to the Step 3 of the proof of Theorem 2.3 and is omitted.

Step 4. It is same to the Step 4 of the proof of Theorem 2.3 and is omitted.

Thus the proof is complete. □

3. Solvability of (1.1)–(1.2) at non-resonance

In this section, we obtain sufficient conditions for the existence of at least one solution of (1.1)–(1.2) at non-resonance, i.e. when $\sum_{i=1}^n\alpha_i\ne 0$. In this case, the operator $Lx(t)=(-1)^{n-p}x^{(n)}(t)$ is invertible. The method employed is based on Scheaffer's theorem, see for example [28, Theorem 4.3.2] or [[27].

Theorem 3.1 ([27, 28])**.** *Let $(X,\|*\|)$ be a Banach space. T is a continuous mapping of X into X which is compact on each bounded subset of X. Then either*

(i) *The equation $x=\lambda Tx$ has a solution for $\lambda=1$, or*

(ii) *The set of all such solutions x, for $\lambda\in(0,1)$, is unbounded.*

Combining the differential equation (1.1) with the boundary conditions (1.2), a solutions $x(t)$ satisfies

$$x^{(p)}(1)-x^{(p)}(t)=\int_t^1\frac{(s-t)^{n-p-1}}{(n-p-1)!}f(s,x(s),x'(s),\dots,x^{(n-1)}(s))ds.$$

Since $\sum_{i=1}^m\alpha_i x^{(p)}(\xi_i)=0$, we have

$$x^{(p)}(1)=\frac{1}{\sum_{i=1}^m\alpha_i}\sum_{i=1}^m\alpha_i\int_{\xi_i}^1\frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!}f(s,x(s),x'(s),\dots,x^{(n-1)}(s))ds.$$

Thus

$$\begin{aligned}x^{(p)}(t)=&\frac{1}{\sum_{i=1}^m\alpha_i}\sum_{i=1}^m\alpha_i\int_{\xi_i}^1\frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!}f(s,x(s),x'(s),\dots,x^{(n-1)}(s))ds\\ &-\int_t^1\frac{(s-t)^{n-p-1}}{(n-p-1)!}f(s,x(s),x'(s),\dots,x^{(n-1)}(s))ds.\end{aligned}$$

Integrating above equation, we have

$$x(t)=\frac{1}{\sum_{i=1}^m\alpha_i}\sum_{i=1}^m\alpha_i\int_{\xi_i}^1\frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!}f(s,x(s),x'(s),\dots,x^{(n-1)}(s))ds\frac{t^p}{p!}$$

$$
\begin{aligned}
&-\int_0^t \frac{(t-s)^{p-1}}{(p-1)!}\Big(\int_s^1 \frac{(u-s)^{n-p-1}}{(n-p-1)!} f(u,x(u),x'(u),\dots,x^{(n-1)}(u))du\Big)ds\\
&=\frac{1}{\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} f(s,x(s),x'(s),\dots,x^{(n-1)}(s))ds\frac{t^p}{p!}\\
&\quad+\sum_{j=0}^{n-p}\frac{(-1)^j t^{j+p}}{(j+p)!}\int_0^1 \frac{s^{n-p-1-j}}{(n-p-1-j)!} f(s,x(s),x'(s),\dots,x^{(n-1)}(s))ds\\
&\quad+(-1)^{n-p+1}\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,x(s),x'(s),\dots,x^{(n-1)}(s))ds.
\end{aligned}
$$

Define the Banach space

$$
\begin{aligned}
X=\big\{x\in C^{n-1}[0,1] : x^{(i)}(0)=0 \text{ for } i=0,1,\dots,p-1\\
\text{and } x^{(i)}(1)=0 \text{ for } i=p+1,\dots,n-1\big\},
\end{aligned}
$$

whose norm is $\|x\|=\max\{\|x\|_\infty,\dots,\|x^{(n-1)}\|_\infty\}$, where $\|x\|_\infty=\max_{t\in[0,1]}|x(t)|$. It is easy to show that

$$
\|x\|=\max\{\|x^{(p)}\|_\infty,\ \|x^{(n-1)}\|_\infty\}.
$$

Define the nonlinear operator $T: X\to X$ as

$$
\begin{aligned}
Tx(t)&=\frac{1}{\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} f(s,x(s),x'(s),\dots,x^{(n-1)}(s))ds\frac{t^p}{p!}\\
&\quad+\sum_{j=0}^{n-p}\frac{(-1)^j t^{j+p}}{(j+p)!}\int_0^1 \frac{s^{n-p-1-j}}{(n-p-1-j)!} f(s,x(s),x'(s),\dots,x^{(n-1)}(s))ds\\
&\quad+(-1)^{n-p+1}\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,x(s),x'(s),\dots,x^{(n-1)}(s))ds\,.
\end{aligned}
$$

Theorem 3.2. *Assume that the nonlinearity f is bounded. Then (1.1)–(1.2) has at least one solution.*

Proof. Let $M>0$ be such that $|f(t,x(t),x'(t),\dots,x^{(n-1)}(t))|\le M$ for $t\in[0,1]$, $(x_0,x_1,\dots,x_{n-1})\in\mathbb{R}^n$. For $\mu\in[0,1]$, consider the equation

$$
x=\mu Tx. \tag{3.1}
$$

If $x(t)$ is a solution of this equation, then:

$$
\begin{aligned}
x(t)&=\mu\Big[\frac{1}{\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} f(s,x(s),x'(s),\dots,x^{(n-1)}(s))ds\frac{t^p}{p!}\\
&\quad+\sum_{j=0}^{n-p}\frac{(-1)^j t^{j+p}}{(j+p)!}\int_0^1 \frac{s^{n-p-1-j}}{(n-p-1-j)!} f(s,x(s),x'(s),\dots,x^{(n-1)}(s))ds\\
&\quad+(-1)^{n-p+1}\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,x(s),x'(s),\dots,x^{(n-1)}(s))ds\Big],
\end{aligned}
$$

$$
x^{(p)}(t)=\mu\Big[\frac{1}{\sum_{i=1}^m \alpha_i}\sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} f(s,x(s),x'(s),\dots,x^{(n-1)}(s))ds
$$

$$- \int_t^1 \frac{(s-t)^{n-p-1}}{(n-p-1)!} f(s, x(s), x'(s), \dots, x^{(n-1)}(s)) ds \Big],$$

and

$$(-1)^{n-p-1} x^{(n-1)}(t) = \mu \int_t^1 f(s, x(s), x'(s), \dots, x^{(n-1)}(s)) ds.$$

So, we have

$$|x^{(p)}(t)| \le \mu M \Big[\frac{1}{\sum_{i=1}^m \alpha_i} \sum_{i=1}^m \alpha_i \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} ds + \int_0^1 \frac{s^{n-p-1}}{(n-p-1)!} ds \Big],$$
$$|x^{(n-1)}(t)| \le \mu M.$$

This shows that all solutions of (12) satisfy $\|x\| = \max\{\|x^{(p)}\|_\infty,\ \|x^{(n-1)}\|_\infty\}$ is bounded. Taking into account that T is continuous and compact on each bounded subset of X and using Schaeffer's theorem, we obtain that T has a fixed point, which is a solution of (1.1)–(1.2). □

We remark that the hypotheses in Theorem 3.2 are strong, but it is convenient to apply them. Next, we give another existence result.

Theorem 3.3. *Assume there exist $a_i \in [0, +\infty)$ $(i = 0, 1, \dots, n-1)$ and $a \in C[0,1]$ and $\beta_i \in [0,1] (i = 0, 1, \dots, n-1)$ such that*

$$|f(t, x_0, x_1, \dots, x_{n-1})| \le a(t) + a_0|x_0|^{\beta_0} + \dots + a_{n-1}|x_{n-1}|^{\beta_{n-1}} \tag{3.2}$$

for $t \in [0,1]$ and $(x_0, x_1, \dots, x_{n-1}) \in \mathbb{R}^n$ and $\sum_{i=p+1}^{n-1} a_i < 1$. Then (1.1)–(1.2) has at least one solution.

Proof. For $x \in X$, we have

$$|f(t, x(t), x'(t), \dots, x^{(n-1)}(t))| \le a(t) + \sum_{i=0}^{n-1} a_i |x^{(i)}(t)|^{\beta_i}.$$

If $x(t)$ is a solution of (3.1), then

$$\begin{aligned} |f(t, x(t), x'(t), \dots, x^{(n-1)}(t))| &= a(t) + \sum_{i=0}^{p-1} a_i t |x^{(i+1)}(\xi_i)|^{\beta_i} + a_p |x^{(p)}(t)|^{\beta_p} \\ &\quad + \sum_{i=p+1}^{n-2} a_i t |x^{(i+1)}(\xi_i)|^{\beta_i} + a_{n-1} |x^{(n-1)}(t)|^{\beta_{n-1}} \\ &\le a(t) + \sum_{i=0}^{p} a_i \|x^{(p)}\|_\infty^{\beta_i} + \sum_{i=p+1}^{n-1} a_i \|x^{(n-1)}\|_\infty^{\beta_i}. \end{aligned}$$

Thus

$$\begin{aligned} &|x^{(p)}(t)| \\ &\le \mu \Big[\frac{1}{|\sum_{i=1}^m \alpha_i|} \sum_{i=1}^m |\alpha_i| \int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!} \Big(a(s) + \sum_{i=0}^{p} a_i \|x^{(p)}\|_\infty^{\beta_i} \\ &\quad + \sum_{i=p+1}^{n-1} a_i \|x^{(n-1)}\|_\infty^{\beta_i} \Big) ds + \int_0^1 \frac{s^{n-p-1}}{(n-p-1)!} \Big(a(s) + \sum_{i=0}^{p} a_i \|x^{(p)}\|_\infty^{\beta_i} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=p+1}^{n-1} a_i\|x^{(n-1)}\|_\infty\Big)ds\Big]\\
&\le \mu\Big\{\frac{1}{|\sum_{i=1}^m \alpha_i|}\sum_{i=1}^m|\alpha_i|\Big[\int_{\xi_i}^1\frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!}a(s)ds+\int_{\xi_i}^1\frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!}ds\\
&\times\Big(\sum_{i=0}^p a_i\|x^{(p)}\|_\infty^{\beta_i}\Big)+\int_{\xi_i}^1\frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!}ds\Big(\sum_{i=p+1}^{n-1}a_i\|x^{(n-1)}\|_\infty^{\beta_i}\Big)\Big]\\
&+\int_0^1\frac{s^{n-p-1}}{(n-p-1)!}a(s)ds+\int_0^1\frac{s^{n-p-1}}{(n-p-1)!}a(s)ds\Big(\sum_{i=0}^p a_i\|x^{(p)}\|_\infty^{\beta_i}\Big)\\
&+\int_0^1\frac{s^{n-p-1}}{(n-p-1)!}a(s)ds\Big(\sum_{i=p+1}^{n-1}a_i\|x^{(n-1)}\|_\infty^{\beta_i}\Big)\Big\}\\
&=\mu\Big\{\frac{1}{|\sum_{i=1}^m \alpha_i|}\sum_{i=1}^m|\alpha_i|\Big[\int_{\xi_i}^1\frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!}a(s)ds\\
&+\frac{(1-\xi_i)^{n-p}}{(n-p)!}\Big(\sum_{i=0}^p a_i\|x^{(p)}\|_\infty^{\beta_i}\Big)+\frac{(1-\xi_i)^{n-p}}{(n-p)!}\Big(\sum_{i=p+1}^{n-1}a_i\|x^{(n-1)}\|_\infty^{\beta_i}\Big)\Big]\\
&+\int_0^1\frac{s^{n-p-1}}{(n-p-1)!}a(s)ds+\frac{1}{(n-p)!}\Big(\sum_{i=0}^p a_i\|x^{(p)}\|_\infty^{\beta_i}\Big)\\
&+\frac{1}{(n-p)!}\Big(\sum_{i=p+1}^{n-1}a_i\|x^{(n-1)}\|_\infty^{\beta_i}\Big)\Big\}\\
&=\mu\Big[\sum_{i=0}^p a_i\|x^{(p)}\|_\infty^{\beta_i}\int_{\xi_i}^1\frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!}a(s)ds+\int_0^1\frac{s^{n-p-1}}{(n-p-1)!}a(s)ds\\
&+\frac{1}{(n-p)!}\Big(\frac{\sum_{i=0}^m|\alpha_i|}{|\sum_{i=1}^m\alpha_i|}(1-\xi_i)^{n-p}+1\Big)\sum_{i=0}^p a_i\|x^{(p)}\|_\infty^{\beta_i}\\
&+\frac{1}{(n-p)!}\Big(\frac{\sum_{i=0}^m|\alpha_i|}{|\sum_{i=1}^m\alpha_i|}(1-\xi_i)^{n-p}+1\Big)\sum_{i=p+1}^{n-1}a_i\|x^{(n-1)}\|_\infty^{\beta_i}\Big].
\end{aligned}
$$

and

$$
|x^{(n-1)}(t)|\le\mu\Big[\int_0^1 a(s)ds+\sum_{i=0}^p a_i\|x^{(p)}\|_\infty^{\beta_i}+\sum_{i=p+1}^{n-1}a_i\|x^{(n-1)}\|_\infty^{\beta_i}\Big].
$$

Hence

$$
\|x^{(n-1)}\|_\infty\le\mu\Big[\int_0^1 a(s)ds+\sum_{i=0}^p a_i\|x^{(p)}\|_\infty^{\beta_i}+\sum_{i=p+1}^{n-1}a_i\|x^{(n-1)}\|_\infty^{\beta_i}\Big].
$$

Without loss of generality, suppose $\|x^{(n-1)}\|_\infty\ge 1$, then

$$
\|x^{(n-1)}\|_\infty\le\int_0^1 a(s)ds+\sum_{i=0}^p a_i\|x^{(p)}\|_\infty^{\beta_i}+\sum_{i=p+1}^{n-1}a_i\|x^{(n-1)}\|_\infty.
$$

Thus

$$\|x^{(n-1)}\|_\infty \le \Big(1-\sum_{i=p+1}^{n-1} a_i\Big)^{-1}\Big(\int_0^1 a(s)ds+\sum_{i=0}^{p} a_i\|x^{(p)}\|_\infty^{\beta_i}\Big).$$

Hence

$$\begin{aligned}
&\|x^{(p)}\|_\infty\\
&\le \mu\Big[|\sum_{i=1}^m \alpha_i|^{-1}\sum_{i=1}^m|\alpha_i|\int_{\xi_i}^1 \frac{(s-\xi_i)^{n-p-1}}{(n-p-1)!}a(s)ds+\int_0^1\frac{s^{n-p-1}}{(n-p-1)!}a(s)ds\\
&\quad+\frac{1}{(n-p)!}\Big(\frac{\sum_{i=0}^m|\alpha_i|}{|\sum_{i=1}^m\alpha_i|}(1-\xi_i)^{n-p}+1\Big)\Big(\sum_{i=0}^p a_i\|x^{(p)}\|_\infty^{\beta_i}\Big)\\
&\quad+\frac{1}{(n-p)!}\Big(\frac{\sum_{i=0}^m|\alpha_i|}{|\sum_{i=1}^m\alpha_i|}(1-\xi_i)^{n-p}+1\Big)\sum_{j=p+1}^{n-1}a_j\Big(1-\sum_{i=p+1}^{n-1}a_i\Big)^{-\beta_j}\\
&\quad\times\Big(\int_0^1 a(s)ds+\sum_{i=0}^p a_i\|x^{(p)}\|_\infty^{\beta_i}\Big)^{\beta_j}\Big].
\end{aligned}$$

Since $\beta_i\in[0,1)$, there exists $M_1>0$ sufficiently large and independent on μ such that $\|x^{(p)}\|_\infty\le M_1$, and

$$\|x^{(n-1)}\|_\infty\le\int_0^1 a(s)ds+\sum_{i=1}^p a_iM_1^{\beta_i}+\sum_{i=p+1}^{n-1}a_i\|x^{(n-1)}\|_\infty^{\beta_i}.$$

Similarly, it follows that there is $M_2>0$ sufficiently large and independent on μ such that $\|x^{(n-1)}\|_\infty\le M_2$. These show that $\|x\|=\max\{\|x^{(p)}\|_\infty,\ \|x^{(n-1)}\|_\infty\ \}$ is bounded. On the other hand, T is continuous and compact on each bounded subset of X. Therefore, by Schaeffer's theorem, we obtain the existence of at least one fixed point for the operator T, which is a solution of (1.1)–(1.2). The proof is complete. □

4. Examples

In this section, we present some examples to illustrate the main results.

Example 1. Consider the following boundary-value problem

$$\begin{gathered}
x''(t)=e(t)+\frac{1}{14}[x'(t)]^{2/3}+\frac{t}{7}\cos^2 t\ \sin[x(t)]^{2/3},\\
x(0)=0,\quad x'(1)=\frac12 x'(\frac12)+\frac12 x'(0).
\end{gathered}\tag{4.1}$$

Corresponding to (1.1) and (1.2), we find $n=2$, $\xi_1=0$, $\xi_2=\frac12$, $\xi_3=1$, and $\alpha_1=\frac12$, $\alpha_2=\frac12$, $\alpha_3=-1$. $f(t,x,y)=e(t)+\frac{1}{14}y^{2/3}+\frac{t}{7}\cos^2 t\ \sin x^{2/3}$. It is easy to see $|f(t,x,y)|\le|e(t)+\frac17|x|^{2/3}+\frac{1}{14}|y|^{2/3}$ with $\beta_0=\frac23$ and $\beta_1=\frac23$. So Assumption (A2) holds. Since

$$\begin{aligned}
&\frac12\int_0^1 f(s,x(s),x'(s))ds+\frac12\int_{1/2}^1 f(s,x(s),x'(s))ds-\int_1^1 f(s,x(s),x'(s))ds\\
&=\frac12\int_0^{1/2} f(s,x(s),x'(s))ds+\int_{1/2}^1 f(s,x(s),x'(s))ds,
\end{aligned}$$

it is easy to see that if $|x'(t)| > 14^{3/2}\left(\|e\|_\infty + \frac{1}{7}\right)^{3/2}$ for all $t \in [0, \frac{1}{2}]$, and $e(t) \geq \frac{t}{7}\cos^2 t$ for $t \in [\frac{1}{2}, 1]$, choosing $M = 14^{3/2}\left(\|e\|_\infty + \frac{1}{7}\right)^{3/2}$, Assumption (A1) holds. Furthermore,

$$
\begin{aligned}
&c\Big[\frac{1}{2}\int_0^1 f(s,x(s),x'(s))ds + \frac{1}{2}\int_{1/2}^1 f(s,x(s),x'(s))ds - \int_1^1 f(s,x(s),x'(s))ds\Big] \\
&= \frac{1}{2}\int_0^{1/2}\Big(ce(s) + \frac{1}{14}c^{5/3} + \frac{cs}{7}\cos^2 s\sin(cs)^{2/3}\Big)\,ds \\
&\quad + \int_{1/2}^1\Big(ce(s) + \frac{1}{14}c^{5/3} + \frac{cs}{7}\cos^2 s\sin(cs)^{2/3}\Big)\,ds > 0
\end{aligned}
$$

for sufficiently large $|c|$. So (A3) of Theorem 2.3 holds. From Theorem 2.3, (4.1) has at least one solution for every $e \in C^0[0,1]$ with $e(t) \geq \frac{t}{7}\cos^2 t$ for all $t \in [\frac{1}{2}, 1]$.

Example 2. Consider the boundary-value problem

$$
\begin{gathered}
x'''(t) = e(t) + \frac{1}{14}[x'(t)]^{2/3} + \frac{t}{7}\cos^2 t\ \sin[x(t)]^{2/3} + \frac{t^2}{8}\sin^2 t\cos[x''(t)]^{4/5}, \\
x(0) = 0, \quad x'(1) = \frac{1}{2}x'(\frac{1}{2}) + \frac{1}{2}x'(0),\ x''(0) = 0.
\end{gathered}
\tag{4.2}
$$

Corresponding to (1.1)–(1.2) we find $n = 3$, $\xi_1 = 0$, $\xi_2 = \frac{1}{2}$, $\xi_3 = 1$, and $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{2}$, $\alpha_3 = -1$. $f(t,x,y,z) = e(t) + \frac{1}{14}y^{2/3} + \frac{t}{7}\cos^2 t\ \sin x^{2/3} + \frac{t^2}{8}\sin^2 t\ \cos z^{4/5}$. It is easy to see $|f(t,x,y,z)| \leq |e(t) + \frac{1}{7}|x|^{2/3} + \frac{1}{14}|y|^{2/3} + \frac{1}{8}|z|^{4/5}$ with $\beta_0 = \frac{2}{3}$ and $\beta_1 = \frac{2}{3}$ and $\beta_2 = \frac{4}{5}$. So Assumption (A2) holds. Similarly, we can prove that (A1) and (A3) hold if $e(t) \geq \frac{t}{7}\cos^2 t + \frac{t^2}{8}\sin^2 t$ for all $t \in [\frac{1}{2}, 1]$. Hence from Theorem 2.3, (4.2) has at least one solution for every $e \in C^0[0,1]$ with $e(t) \geq \frac{t}{7}\cos^2 t + \frac{t^2}{8}\sin^2 t$ for all $t \in [\frac{1}{2}, 1]$.

Example 3. Consider the boundary-value problem

$$
\begin{gathered}
x^{(n)}(t) = \sum_{i=0, i\neq p}^{n-1} a_i \sin x^{(i)}(t) + a_p x^{(p)}(t) + e(t), \\
x^{(i)}(0) = 0, \quad \text{for } i = 0, 1, \dots, p-1, p+1, \dots, n-1, \\
x^{(p)}(1) = \sum_{i=1}^{m} \alpha_i x^{(p)}(\xi_i),
\end{gathered}
\tag{4.3}
$$

where $1 \leq p \leq n-1$, $0 < \xi_1 < \dots < \xi_m < 1$, $a_p > 0$, $\alpha_i \geq 0$ for all $i \neq p$ with $\sum_{i=1}^m \alpha_i = 1$. It is easy to see above problem is a special case of (1.1)–(1.2). Furthermore, $|f(t, x_0, \dots, x_{n-1})| \leq |e(t)| + \sum_{i=1}^{n-1} a_i|x_i|$. So (A5) holds. Since $|f(t, x_0, \dots, x_{n-1})| \geq a_p|x_p| - \sum_{i=1, i\neq p}^{n-1} |a_i||x_i| - \|e\|_\infty$, we find that there is $M > 0$ such that if $|x^{(p)}(t)| \geq M$ for all $t \in [0,1]$, then (A4) holds. As in Example 1, we find that there is $M^* > 0$ such that (A3) holds. Thus from Theorem 2.4, (4.3) has at least one solution provided that

$$
\sum_{i=0}^{p} |a_i| < (n-1-p)!, \qquad \sum_{i=p+1}^{n-1} |a_i| < 1,
$$

$$\sum_{i=p+1}^{n-1}|a_i|+\frac{\left(\sum_{i=0}^{p}|a_i|\right)\left(\sum_{i=p+1}^{n-1}|a_i|\right)}{(n-1-p)!-\sum_{i=0}^{p}|a_i|}<1.$$

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