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EXPONENTIAL STABILITY OF LINEAR AND ALMOST PERIODIC SYSTEMS ON BANACH SPACES

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Abstract. Let $v_f(\cdot,0)$ the mild solution of the well-posed inhomogeneous Cauchy problem

$$\dot v(t) = A(t)v(t) + f(t), \quad v(0) = 0 \quad t \geq 0$$

on a complex Banach space X, where $A(\cdot)$ is an almost periodic (possible unbounded) operator-valued function. We prove that $v_f(\cdot,0)$ belongs to a suitable subspace of bounded and uniformly continuous functions if and only if for each $x \in X$ the solution of the homogeneous Cauchy problem

$$\dot u(t) = A(t)u(t), \quad u(0) = x \quad t \geq 0$$

is uniformly exponentially stable. Our approach is based on the spectral theory of evolution semigroups.

1. Introduction

Let X be a complex Banach space and $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on X. The norms on X and $\mathcal{L}(X)$ will be denoted by $\|\cdot\|$. We recall that a family $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ of bounded linear operators acting on X, is a *strongly continuous and exponentially bounded evolution family* (which we will call simply an evolution family), if $U(t,t) = \mathrm{Id}$ (Id is the identity operator on X), $U(t,s)U(s,r) = U(t,r)$ for all $t \geq s \geq r$, for each $x \in X$ the map $(t,s) \mapsto U(t,s)x$ is continuous and there exist $\omega \in \mathbb{R}$ and $M_\omega \geq 1$ such that

$$\|U(t,s)\| \leq M_\omega e^{\omega(t-s)} \quad \text{for all } t \geq s. \tag{1.1}$$

If $\mathcal{F}(\mathbb{R},X)$ is a suitable Banach function space, then for each $t \geq 0$ the operator $\mathcal{T}(t)$ defined by

$$(\mathcal{T}(t)f)(s) = U(s,s-t)f(s-t), \quad s \in \mathbb{R} \tag{1.2}$$

acts on $\mathcal{F}(\mathbb{R},X)$ and the family $\{\mathcal{T}(t)\}_{t\geq 0}$ is a strongly continuous semigroup which is called the *evolution semigroup* associated with the family $\mathcal{U}$ on the space $\mathcal{F}(\mathbb{R},X)$. For example, $\mathcal{F}(\mathbb{R},X) = C_{00}(\mathbb{R},X)$ the Banach space of all continuous functions that vanish at infinities and $\mathcal{F}(\mathbb{R},X) = L^p(\mathbb{R},X)$ with $1 \leq p < \infty$, the usual Lebesgue-Bochner space, are suitable. Similar results were obtained when $\mathcal{F}(\mathbb{R},X)$ are certain subspaces of $BUC(\mathbb{R},X)$ the Banach space of all X-valued, bounded

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and uniformly continuous functions on $\mathbb{R}$, endowed with the sup-norm. Let $\mathbb{R}_+ := [0, \infty)$. The space $BUC(\mathbb{R}_+, X)$ can be defined in a similar way.

We will use the following closed subspaces of $BUC(\mathbb{R}, X)$, see [9, 14, 18]:

$AP(\mathbb{R}, X)$ is the smallest closed subspace of $BUC(\mathbb{R}, X)$ which contains all functions of the form:

$$t \mapsto e^{i\mu t}x : \mathbb{R} \to X, \quad \mu \in \mathbb{R}, \quad x \in X;$$

$C_0^+(\mathbb{R}, X)$ is the subspace of $BUC(\mathbb{R}, X)$ consisting by all functions vanishing at ∞;

$AAP_r^+(\mathbb{R}, X)$ is the space consisting by all functions f with relatively compact range for which there exist $g \in AP(\mathbb{R}, X)$ and $h \in C_0^+(\mathbb{R}, X)$ such that $f = g + h$. $P_q(\mathbb{R}, X)$, with strictly positive fixed q, is the space consisting by all continuous and q-periodic functions.

The evolution family $\mathcal{U}$ is called q-periodic if the function $U(t + \cdot, s + \cdot)$ is q-periodic for every pair (t, s) with $t \geq s$. Also we say that the family $\mathcal{U}$ is *asymptotically almost periodic with relatively compact range* (a.a.p.r.) if for each $x \in X$ and each pair (t, s) with $t \geq s$, the map $U(t + \cdot, s + \cdot)x$ lies in the space $AAP_r^+(\mathbb{R}, X)$. If the evolution family $\mathcal{U}$ is q-periodic and $\mathcal{F}(\mathbb{R}, X) = P_q(\mathbb{R}, X)$ or $\mathcal{F}(\mathbb{R}, X) = AP(\mathbb{R}, X)$ then the semigroup $\mathcal{T} = \{\mathcal{T}(t)\}_{t\geq 0}$ defined in (1.2) acts on $P_q(\mathbb{R}, X)$ or $AP(\mathbb{R}, X)$ and it is strongly continuous. Moreover, if $\mathcal{U}$ is a.a.p.r. and for each $x \in X$, $\lim_{t\to 0+} U(s, s - t)x = x$, uniformly for $s \in \mathbb{R}$, then the evolution semigroup $\mathcal{T}$ is defined on $AAP_r^+(\mathbb{R}, X)$ and is strongly continuous. More details related to these results can be found in [1, 2, 10, 11, 12, 13, 15, 16]. Interesting results on this subject in the general framework of dynamical systems have been obtained by D. N. Cheban [6, 7].

2. Almost periodic evolution families and evolution semigroups

An X-valued function f defined on $\mathbb{R}$ is called almost periodic (a.p.) if it belongs to the space $AP(\mathbb{R}, X)$. Let $\mathcal{U}$ be a strongly continuous and exponentially bounded evolution family on the Banach space X and let f be a X-valued function on $\mathbb{R}$. We will consider the following hypotheses about $\mathcal{U}$ and f.

- (H1) The function $U(\cdot, \cdot - t)x$ is a.p. for every $t \geq 0$ and any $x \in X$.
- (H2) The function $U(\cdot, \cdot - t)x$ has relatively compact range for every $t \geq 0$ and any $x \in X$.
- (H3) For each $x \in X$ $\lim_{t\to 0} U(s, s - t)x = x$ uniformly for $s \in \mathbb{R}$.
- (H4) The function f is a.p.

It is well-known that (H1) implies (H2).

Theorem 2.1. (i) *If the evolution family $\mathcal{U}$ satisfies (H1) and f satisfies (H4) then for each $t \geq 0$, the function $\mathcal{T}(t)f$ is a.p.*

(ii) *If $\mathcal{U}$ satisfies (H2) and f satisfies (H4) then for each $t \geq 0$, the map $\mathcal{T}(t)f$ has relatively compact range.*

(iii) *If $\mathcal{U}$ satisfies (H1) and (H3) then the semigroup $\mathcal{T}$ acts on $AP(\mathbb{R}, X)$ and is strongly continuous.*

(iv) *If $\mathcal{U}$ satisfies (H1) and (H3) then the evolution semigroup $\mathcal{T}$ is defined on $AAP_r^+(\mathbb{R}, X)$ and is strongly continuous.*

Proof. (i) Let $p_n(t) := \sum_{k=0}^{n} c_k e^{i\mu_k t}x_k$ with $c_k \in \mathbb{C}$, $\mu_k \in \mathbb{R}$, $t \in \mathbb{R}$ and $x_k \in X$ such that $p_n(s)$ converges uniformly at $f(s)$ for $s \in \mathbb{R}$. Then $U(s, s - t)p_n(s - t)$

converges uniformly at $U(s, s-t)f(s-t)$ for $s \in \mathbb{R}$. Since the map:

$$s \mapsto U(s, s-t)p_n(s-t) = \sum_{k=0}^{n} c_k e^{i\mu_k(s-t)} U(s, s-t)x_k$$

is a. p. its limit $U(\cdot, \cdot - t)f(\cdot - t)$ is a.p. as well.
(ii) Let $t \geq 0$ be fixed. First we prove that for each $x \in X$ and each $\mu \in \mathbb{R}$ the function $s \mapsto U(s, s-t)e^{i\mu(s-t)}x$ has relatively compact range. Let (s_n) be a sequence of real numbers such that $(U(s_n, s_n - t)x)$ converges in X. Since the sequence $(e^{i\mu(s_n-t)})$, is bounded in $\mathbb{C}$, we can suppose that the sequence $(e^{i\mu(s_n-t)}U(s_n, s_n - t)x))$ converges in X. Let $p_N(s-t) = \sum_{k=0}^{N} c_k e^{i\mu_k(s-t)} x_k$, as above, be such that $p_N(s-t) \to f(s-t)$ uniformly for $s \in \mathbb{R}$. Let $\varepsilon > 0$ and $N_0 \in \mathbb{N}$ be such that the inequality

$$Me^{\omega t}\|f(s_n - t) - p_{N_0}(s_n - t)\| < \frac{\varepsilon}{2}$$

holds for n sufficiently large. We denote by y_t the limit in X of the sequence $(U(s_n, s_n - t)p_{N_0}(s_n - t))$. Then, for n sufficiently large, we have

$$\begin{aligned}
&\|U(s_n, s_n - t)f(s_n - t) - y_t\| \\
&\leq \|U(s_n, s_n - t)f(s_n - t) - U(s_n, s_n - t)p_{N_0}(s_n - t)\| \\
&\quad + \|U(s_n, s_n - t)p_{N_0}(s_n - t)\| \\
&\leq Me^{\omega t}\|f(s_n - t) - p_{N_0}(s_n - t)\| + \|U(s_n, s_n - t)p_{N_0}(s_n - t) - y_t\| < \varepsilon.
\end{aligned}$$

Hence the map $U(\cdot, \cdot - t)f(\cdot - t)$ has relatively compact range.
(iii) Let $f \in AP(\mathbb{R}, X)$ and $\varepsilon > 0$. We can choose $N_0 \in \mathbb{N}$ and $\delta > 0$ such that the following three inequalities

$$\sup_{s\in\mathbb{R}} \|U(s, s-t)p_{N_0}(s-t) - p_{N_0}(s-t)\| \leq \sum_{k=0}^{N_0} |c_k| \|U(s, s-t)x_k - x_k\| < \frac{\varepsilon}{3},$$
$$\sup_{s\in\mathbb{R}} \|p_{N_0}(s-t) - f(s-t)\| < \frac{\varepsilon}{3},$$
$$\sup_{s\in\mathbb{R}} \|f(s-t) - f(s)\| < \frac{\varepsilon}{3}$$

hold for all $0 \leq t < \delta$. Now it is clear that $\lim_{t\to 0} \|\mathcal{T}(t)f - f\|_\infty = 0$, hence the semigroup $\mathcal{T}$ is strongly continuous.
(iv) Finally we show that the semigroup $\mathcal{T}$ given in (1.2) on $AAP_r^+(\mathbb{R}, X)$ is strongly continuous. Let $\varepsilon > 0$ be fixed. We can choose $\delta_1 > 0$ such that the inequality

$$\sup_{s\in\mathbb{R}} \|f(s-t) - f(s)\| < \frac{\varepsilon}{2}$$

holds for $0 \leq t < \delta_1$. Since f has relatively compact range there exist $s_1, s_2, \dots, s_\nu$ in $\mathbb{R}$ such that:

$$\overline{\text{range}(f)} \subset \cup_{k=1}^{\nu} B\big(f(s_k), \frac{\varepsilon}{6Me^{\omega t}}\big), \quad \omega > 0,\ t \geq 0.$$

Let $s \in \mathbb{R}, t \geq 0$ and $k \in \{1, \dots, \nu\}$ such that $f(s-t) \in B\left(f(s_k), \frac{\varepsilon}{6Me^{\omega t}}\right)$. From hypothesis it follows that there exists $\delta_2 > 0$ such that the inequality

$$\|U(s, s-t)f(s_k) - f(s_k)\| < \varepsilon/6$$

holds for $0 \le t < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then for every t in $[0, \delta)$, we have

$$\begin{aligned}
&\|U(s, s-t)f(s-t) - f(s)\| \\
&\le \|U(s, s-t)f(s-t) - U(s, s-t)f(s_k)\| + \|U(s, s-t)f(s_k) - f(s_k)\| \\
&\quad + \|f(s_k) - f(s-t)\| + \|f(s-t) - f(s)\| \\
&\le Me^{\omega t}\|f(s-t) - f(s_k)\| + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{2} < \varepsilon;
\end{aligned}$$

therefore, $\lim_{t\to 0} \|\mathcal{T}(t)f - f\|_\infty = 0$. In the above considerations we supposed that $\mathcal{T}$ acts on $AAP_r^+(\mathbb{R}, X)$. Next, we show that this is true. Let $f \in AAP_r^+(\mathbb{R}, X)$ and $t \ge 0$ be fixed. From the hypothesis it results that there exist a sequence (s_n) of real numbers and y_t, z_t in X such that

$$f(s_n - t) \to y_t \quad \text{and} \quad U(s_n, s_n - t)y_t \to z_t \quad \text{as} n \to \infty.$$

Then $U(s_n, s_n - t)f(s_n - t) \to z_t$ as $n \to \infty$. Indeed, we have

$$\|U(s_n, s_n-t)f(s_n-t)-z_t\| \le \|U(s_n, s_n-t)[f(s_n-t)-y_t]\|+\|U(s_n, s_n-t)y_t-z_t\| \to 0$$

as $n \to \infty$. $\square$

3. Evolution semigroups and exponential stability

Let $\mathcal{F}_q(\mathbb{R}, X) := P_q(\mathbb{R}, X) \oplus C_0^+(\mathbb{R}, X)$ and $\mathcal{U}$ be a q-periodic evolution family of bounded linear operators on the Banach space X. It is easy to see that the evolution semigroup $\mathcal{T}$ defined in (1.2) acts on $\mathcal{F}_q(\mathbb{R}, X)$ and it is strongly continuous. By $\mathcal{F}_q^0(\mathbb{R}_+, X)$ we will denote the subspace of $BUC(\mathbb{R}_+, X)$ consisting of all functions f on $\mathbb{R}_+$ for which $f(0) = 0$ and there exists F_f in $\mathcal{F}_q(\mathbb{R}, X)$ such that $F_f(t) = f(t)$ for all $t \ge 0$. For such f we consider the map:

$$(\mathcal{S}(t)f)(s) := \begin{cases} U(s, s-t)f(s-t) & \text{if } s \ge t \\ 0 & \text{if } 0 \le s < t. \end{cases} \tag{3.1}$$

Proposition 3.1. *With the previous notation we have that $\mathcal{S}(t)$ acts on $\mathcal{F}_q^0(\mathbb{R}_+, X)$ for each $t \ge 0$ and the evolution semigroup $\mathcal{S} = \{\mathcal{S}(t)\}_{t\ge 0}$ is strongly continuous.*

Proof. Let $t \ge 0$ be fixed, $f \in \mathcal{F}_q^0(\mathbb{R}_+, X)$ and $\tilde{f} := \mathcal{S}(t)f$. Then $F_f = G_f + H_f$ with $G_f \in P_q(\mathbb{R}, X)$, $H_f \in C_0^+(\mathbb{R}, X)$ and $f = G_f + H_f$ on $\mathbb{R}_+$. Let us consider the maps $\tilde{G}_f \in P_q(\mathbb{R}, X)$ and $\tilde{H}_f \in C_0^+(\mathbb{R}, X)$ defined by

$$\tilde{G}_f(s) = (\mathcal{T}(t)G_f)(s), \quad s \in \mathbb{R},$$

$$\tilde{H}_f(s) = \begin{cases} (\mathcal{T}(t)H_f)(s) & \text{if } s \ge t \\ -(\mathcal{T}(t)G_f)(s) & \text{if } s < t. \end{cases}$$

If $t > 0$ then $\tilde{G}_f(0) + \tilde{H}_f(0) = 0$, and if $t = 0$ then

$$\tilde{G}_f(0) + \tilde{H}_f(0) = (\mathcal{T}(0)G_f)(0) + (\mathcal{T}(0)H_f)(0) = U(0,0)G_f(0) + U(0,0)H_f(0) = 0.$$

On the other hand it is clear that $\tilde{f} = \tilde{G}_f + \tilde{H}_f$ on $\mathbb{R}_+$, hence $\tilde{f}$ belongs to $\mathcal{F}_q^0(\mathbb{R}_+, X)$. Using the strong continuity of $\mathcal{T}$ and the uniform continuity of f, it follows that

$$\begin{aligned}
\|\mathcal{S}(t)f - f\|_\infty &\le \sup_{s\ge t} \|(\mathcal{T}(t)F_f)(s) - F_f(s)\| + \sup_{s\in[0,t]} \|f(s)\| \\
&\le \|\mathcal{T}(t)F_f - F_f\|_{\mathcal{F}_q(\mathbb{R},X)} + \sup_{s\in[0,t]} \|f(s)\|.
\end{aligned}$$

The last term tends to 0 when t tends to 0. Therefore, the semigroup $\mathcal{S}$ is strongly continuous. □

The following theorem seems to be a new characterization of the exponential stability for evolution families.

Theorem 3.2. *Let $\mathcal{U}$ be a q-periodic evolution family of bounded linear operators on the Banach space X. The following two statements are equivalent.*

1. *The family $\mathcal{U}$ is exponentially stable, that is, we can choose a negative ω such that (1.1) holds.*
2. *For each f in $\mathcal{F}_q^0(\mathbb{R}_+, X)$ the map $t \mapsto \int_0^t U(t,\tau)f(\tau)d\tau : \mathbb{R}_+ \to X$ is an element of $\mathcal{F}_q^0(\mathbb{R}_+, X)$.*

Proof. (2) $\Rightarrow$ (1) It is clear that $\mathcal{F}_q^0(\mathbb{R}_+, X)$ contains $C_{00}(\mathbb{R}_+, X)$. Then we can apply [3, Theorem 3] which works with $C_{00}(\mathbb{R}_+, X)$ instead of $C_0(\mathbb{R}_+, X)$. Here $C_{00}(\mathbb{R}_+, X)$ denotes the subspace of $BUC(\mathbb{R}_+, X)$ consisting by all functions that vanish at 0 and ∞.

(1) $\Rightarrow$ (2) $\mathcal{U}$ is exponentially stable so the semigroup $\mathcal{S}$ defined in (3.1) is exponentially stable as well. Then the generator

$$G : D(G) \subset \mathcal{F}_q^0(\mathbb{R}_+, X) \to \mathcal{F}_q^0(\mathbb{R}_+, X)$$

of $\mathcal{S}$ is an invertible operator. The proof of Theorem 3.2 will be complete using the following lemma. □

Lemma 3.3. *Let $\{u, f\}$ belong to $\mathcal{F}_q^0(\mathbb{R}_+, X)$. The following statements are equivalent.*

1. *$u \in D(G)$ and $Gu = -f$.*
2. *$u(t) = \int_0^t U(t,s)f(s)ds$ for all $t \geq 0$.*

This Lemma is well-known for certain spaces instead of $\mathcal{F}_q^0(\mathbb{R}_+, X)$.

Let $\mathcal{A}_0(\mathbb{R}_+, X)$ be the set of all X-valued functions f on $\mathbb{R}_+$ for which there exist $t_f \geq 0$ and $F_f \in AP(\mathbb{R}, X)$ such that $F_f(t_f) = 0$ and

$$f(t) = \begin{cases} 0 & \text{if } t \in [0, t_f] \\ F_f(t) & \text{if } t > t_f. \end{cases}$$

The smallest closed subspaces of $BUC(\mathbb{R}_+, X)$ which contains $\mathcal{A}_0(\mathbb{R}_+, X)$ will be denoted by $\mathcal{AP}_0(\mathbb{R}_+, X)$. By $AAP_{r0}^+(\mathbb{R}_+, X)$ we will denote the space consisting by all functions f for which there exists $F_f \in AAP_r^+(\mathbb{R}, X)$ such that $F_f(0) = 0$ and $F_f = f$ on $\mathbb{R}_+$.

Proposition 3.4.

1. *If the evolution family $\mathcal{U}$ satisfies the hypothesis (H1) and (H3) then the semigroup $\mathcal{S}$, given in (3.1) acts on $\mathcal{AP}_0(\mathbb{R}, X)$. Moreover the semigroup $\mathcal{S}$ is strongly continuous.*
2. *If the family $\mathcal{U}$ satisfies $\mathbf{h_1}, \mathbf{h_2}$ and (H3) then the semigroup $\mathcal{S}$ acts on $AAP_{r0}^+(\mathbb{R}, X)$ and is strongly continuous.*

The proof of (1) can be obtained as in [4, Lemma 2.2], and the proof on (2) as in [5, Lemma 2.2]. Thus we omit their proof.

For every real fixed T we consider the spaces $BUC([T,\infty), X)$ and $AP([T,\infty), X)$ Recall that $AP([T,\infty))$ is bounded locally dense in $BUC([T,\infty), X)$; that is, for every $\varepsilon > 0$, every bounded and closed interval $I \subset [T,\infty)$ and every $f \in C(I, X)$

there exist a function $f_{\varepsilon,I} \in AP([T,\infty),X)$ and a positive constant L, independent of ε and I such that

$$\sup_{s\in I} \|f(s) - f_{\varepsilon,I}(s)\| \le \varepsilon$$

and $\|f_{\varepsilon,I}\|_{BUC([T,\infty),X)} \le L\|f\|_{C(I,X)}$ (see [17], page 335).

Let $BUC_0(\mathbb{R}_+,X)$ be the space of functions in $BUC(\mathbb{R}_+,X)$ for which $f(0) = 0$. It is clear that $\mathcal{A}_0(\mathbb{R}_+,X)$ is bounded locally dense in $BUC_0(\mathbb{R}_+,X)$ hence $\mathcal{AP}_0(\mathbb{R}_+,X)$ is bounded locally dense in $BUC_0(\mathbb{R}_+,X)$ as well.

Theorem 3.5. *Suppose that $\mathcal{U}$ is an evolution family that satisfies hypotheses (H1) and (H3). The following statements are equivalent.*

1. *The family $\mathcal{U}$ is exponentially stable.*
2. *For each $f \in \mathcal{AP}_0(\mathbb{R}_+,X)$ the map $t \mapsto \int_0^t U(t,s)f(s)ds : \mathbb{R}_+ \to X$ is in $\mathcal{AP}_0(\mathbb{R}_+,X)$.*

Proof. The implication (1) $\Rightarrow$ (2) follows as in [4, Theorem 2.3]. Now we shoe that (2) $\Rightarrow$ (1). By the uniform boundedness theorem there is a constant $K > 0$ such that for every $g \in \mathcal{AP}_0(\mathbb{R}_+,X)$,

$$\sup_{t>0} \Big\| \int_0^t U(t,s)g(s)ds \Big\| \le K\|g\|_\infty\,.$$

For a given $f \in C_0(\mathbb{R}_+,X)$ and $t > 0$, let $M_t = \sup_{0\le r\le s\le t} \|U(s,r)\|$ and let $f_t \in \mathcal{AP}_0(\mathbb{R}_+,X)$ be a mapping such that

$$\sup_{0\le s\le t} \|f(s) - f_t(s)\| \le \frac{1}{tM_t}\|f\|_{C_0(\mathbb{R}_+,X)},$$
$$\|f_t\|_{BUC_0(\mathbb{R}_+,X)} \le L\|f\|_{C_0(\mathbb{R}_+,X)}.$$

It follows that

$$\begin{aligned}\Big\| \int_0^t U(t,s)f(s)ds \Big\| &\le \Big\| \int_0^t U(t,s)[f(s) - f_t(s)]ds \Big\| + \Big\| \int_0^t U(t,s)f_t(s)ds \Big\| \\ &\le (1 + KL)\cdot \|f\|_{C_0(\mathbb{R}_+,X)}\,.\end{aligned}$$

Then by [3, Theorem 3], $\mathcal{U}$ is exponentially stable. $\square$

Now we can write the spectral mapping theorem for the evolution semigroup $\mathcal{S}$ on $\mathcal{AP}_0(\mathbb{R}_+,X)$ corresponding to an evolution family $\mathcal{U}$. Of course similar results hold for the spaces $\mathcal{F}_q^0(\mathbb{R}_+,X)$ and $AAP_{r0}^+(\mathbb{R}_+,X)$. With $(G, D(G))$ we will denote the generator of $\mathcal{S}$ with its maximal domain. By $\sigma(G)$ we denote the spectrum of G. The spectral bound $s(G)$ is defined by

$$s(G) = \sup\{\mathrm{Re}(\lambda) : \lambda \in \sigma(G)\},$$

and the spectral radius of $\mathcal{S}(t)$ is defined by

$$r(\mathcal{S}(t)) = \sup\{|\lambda| : \lambda \in \sigma(\mathcal{S}(t))\}.$$

Theorem 3.6. *If $\mathcal{U}$ is an evolution family that satisfies the hypothesis (H1) and (H3) then the evolution semigroup $\mathcal{S}$ associated with $\mathcal{U}$, defined on $\mathcal{AP}_0(\mathbb{R}_+,X)$, satisfies the spectral mapping theorem; that is,*

$$\sigma(\mathcal{S}(t)) \setminus \{0\} = e^{t\sigma(G)}, \quad t \ge 0.$$

Moreover, $\sigma(G) = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \le s(G)\}$, and for every $t > 0$,

$$\sigma(\mathcal{S}(t)) = \{\lambda \in \mathbb{C} : |\lambda| \le r(\mathcal{S}(t))\,.$$

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