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WEAK SOLUTIONS FOR THE p-LAPLACIAN WITH A NONLINEAR BOUNDARY CONDITION AT RESONANCE

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Abstract. We study the existence of weak solutions to the equation

$$\Delta_p u = |u|^{p-2}u + f(x,u)$$

with the nonlinear boundary condition

$$|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u - h(x,u)\,.$$

We assume Landesman-Lazer type conditions and use variational arguments to prove the existence of solutions.

1. Introduction

This paper shows conditions for the existence of weak solutions to the problem

$$\begin{gathered}\Delta_p u = |u|^{p-2}u + f(x,u) \quad \text{in } \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u - h(x,u) \quad \text{on } \partial\Omega.\end{gathered} \tag{1.1}$$

Here Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian with $p > 1$, and $\frac{\partial}{\partial \nu}$ is the outer normal derivative. We assume that the perturbations $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $h : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are bounded Caratheodory functions. For a variational approach, the functional associated to the problem is

$$J_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p + \frac{1}{p}\int_\Omega |u|^p - \frac{\lambda}{p}\int_{\partial\Omega} |u|^p + \int_\Omega F(x,u) + \int_{\partial\Omega} H(x,u),$$

where F and H are primitives of f and h with respect to u respectively. Weak solutions of (1.1) are critical points of J_λ in $W^{1,p}(\Omega)$. In fact if $u \in W^{1,p}(\Omega)$ is a critical point of J_λ, we have

$$\begin{aligned}J'_\lambda(u)\cdot v = &\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v + \int_\Omega |u|^{p-2}uv - \lambda\int_{\partial\Omega} |u|^{p-2}uv\\ &+ \int_\Omega f(x,u)v + \int_{\partial\Omega} h(x,u)v = 0, \quad \forall\, v \in W^{1,p}(\Omega).\end{aligned}$$

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Let us introduce some notation. We say that λ is an eigenvalue for the p-Laplacian with a nonlinear boundary condition if the problem

$$\begin{gathered} \Delta_p u = |u|^{p-2}u \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u \quad \text{on } \partial\Omega. \end{gathered} \tag{1.2}$$

has non trivial solutions. The set of solutions (called eigenfunctions) for a given λ will be denoted by A_λ. Problems of the form (1.2) appear in a natural way when one considers the Sobolev trace inequality. In fact, the immersion $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$ is compact, hence there exits a constant λ_1 such that

$$\lambda_1^{1/p}\|u\|_{L^p(\partial\Omega)} \le \|u\|_{W^{1,p}(\Omega)}.$$

The Sobolev trace constant λ_1 can be characterized as

$$\lambda_1 = \inf_{u\in W^{1,p}(\Omega)} \Big\{ \int_\Omega |\nabla u|^p + |u|^p \, dx \text{ such that } \int_{\partial\Omega} |u|^p = 1 \Big\}, \tag{1.3}$$

and is the first eigenvalue of (1.2) in the sense that $\lambda_1 \le \lambda$ for any other eigenvalue λ. The extremals (functions where the constant is attained) are solutions of (1.2). The first eigenvalue is simple and isolated with a first eigenfunction that is $C^\alpha(\overline{\Omega})$ and strictly positive in $\overline{\Omega}$, see [17]. In [11] it is proved that there exists a sequence of eigenvalues λ_n of (1.2) such that $\lambda_n \to +\infty$ as $n \to +\infty$.

The study of the eigenvalue problem when the nonlinear term is placed in the equation, that is when one considers a quasilinear problem of the form $-\Delta_p u = \lambda|u|^{p-2}u$ with Dirichlet boundary conditions, has received considerable attention, see for example [1, 2, 13, 14, 16], etc.

Resonance problems are well known in the literature. For example, for the resonance problem for the p-lapacian with Dirichlet boundary conditions see [3, 4, 9] and references therein.

In problem (1.1) we have a perturbation of the eigenvalue problem (1.2) given by the two nonlinear terms $f(x,u)$, $h(x,u)$. Following ideas from [9], we prove the following result, that establishes Landesman-Lazer type conditions on the nonlinear perturbation terms in order to have existence of weak solutions for (1.1).

Theorem 1.1. *Let $f^\pm := \lim_{t\to\pm\infty} f(x,t)$, $h^\pm := \lim_{t\to\pm\infty} h(x,t)$. Assume that there exists $\bar f \in L^q(\Omega)$ and $\bar h \in L^q(\partial\Omega)$, such that $|f(x,t)| \le \overline{f}\ \forall (x,t) \in \Omega\times\mathbb{R}$ and $|h(x,t)| \le \overline{h}\ \forall (x,t) \in \partial\Omega\times\mathbb{R}$ (where $q = p/p-1$). Also assume that either*

$$\int_{\{v>0\cap\Omega\}} f^+ v + \int_{\{v>0\cap\partial\Omega\}} h^+ v + \int_{\{v<0\cap\Omega\}} f^- v + \int_{\{v<0\cap\partial\Omega\}} h^- v > 0 \tag{1.4}$$

for all $v \in A_\lambda\backslash\{0\}$, or

$$\int_{\{v>0\cap\Omega\}} f^+ v + \int_{\{v>0\cap\partial\Omega\}} h^+ v + \int_{\{v<0\cap\Omega\}} f^- v + \int_{\{v<0\cap\partial\Omega\}} h^- v < 0 \tag{1.5}$$

for all $v \in A_\lambda\backslash\{0\}$, then (1.1) *has a weak solution.*

Note that when λ is not an eigenvalue the hypotheses trivially hold.

The integral conditions (of Landesman-Lazer type) that we impose for f and h will be used to prove a Palais-Smale condition for the functional J_λ associated to the problem (1.1). Observe that these conditions involve an integral balance (with the eingenfunctions v as weights) between f and h. Hence we allow perturbations both in the equation and in the boundary condition.

Let us have a close look at the conditions for the first eigenvalue. As the first eigenvalue is isolated and simple with an eigenfunction that do not change sign in Ω (we call it ϕ_1 and assume $\phi_1 > 0$ in $\bar{\Omega}$), [17], the conditions involved in Theorem 1.1 for λ_1 read as

$$\int_\Omega f^+\phi_1 + \int_{\partial\Omega} h^+\phi_1 > 0 \quad \text{and} \quad \int_\Omega f^-\phi_1 + \int_{\partial\Omega} h^-\phi_1 < 0 \tag{1.6}$$

or

$$\int_\Omega f^+\phi_1 + \int_{\partial\Omega} h^+\phi_1 < 0 \quad \text{and} \quad \int_\Omega f^-\phi_1 + \int_{\partial\Omega} h^-\phi_1 > 0. \tag{1.7}$$

For this case, $\lambda = \lambda_1$, we will prove a general result which improve the conditions on f and h. In [3] the resonance problem for the Dirichlet problem was analyzed using bifurcation theory. If we adapt the arguments of [3] to our situation, using bifurcation techniques to deal with (1.1), we can improve the previous result by measuring the speed and the form at which f and h approaches the limits $f^\pm$ and $h^\pm$. To this end, let us suppose that there exists α and β such that

$$\begin{gathered} \lim_{s\to+\infty}(f(x,s) - f^+(x))s^\alpha = A_\alpha(x), \\ \lim_{s\to-\infty}(f(x,s) - f^-(x))s^\beta = B_\beta(x), \quad \text{a.e. } x \in \Omega, \\ \lim_{s\to+\infty}(h(x,s) - h^+(x))s^\alpha = \overline{A}_\alpha(x), \\ \lim_{s\to-\infty}(h(x,s) - h^-(x))s^\beta = \overline{B}_\beta(x), \quad \text{a.e, } x \in \partial\Omega. \end{gathered}$$

The limits $A_\alpha, \overline{A}_\alpha, B_\beta$ and $\overline{B}_\beta$ are taken in a pointwise sense and dominated by functions in $L^1(\Omega)$ and $L^1(\partial\Omega)$.

We consider the conditions:

(G_α^+) $\int_\Omega f^+\phi_1 + \int_{\partial\Omega} h^+\phi_1 > 0$ or
$\int_\Omega f^+\phi_1 + \int_{\partial\Omega} h^+\phi_1 = 0$ and $\int_\Omega A_\alpha(x)\phi_1^{1-\alpha} + \int_{\partial\Omega} \overline{A}_\alpha(x)\phi_1^{1-\alpha} > 0$

(G_β^-) $\int_\Omega f^-\phi_1 + \int_{\partial\Omega} h^-\phi_1 < 0$ or
$\int_\Omega f^-\phi_1 + \int_{\partial\Omega} h^-\phi_1 = 0$ and $\int_\Omega B_\beta(x)\phi_1^{1-\beta} + \int_{\partial\Omega} \overline{B}_\beta(x)\phi_1^{1-\beta} < 0$

(G_β^+) $\int_\Omega f^-\phi_1 + \int_{\partial\Omega} h^-\phi_1 > 0$ or
$\int_\Omega f^-\phi_1 + \int_{\partial\Omega} h^-\phi_1 = 0$ and $\int_\Omega B_\beta(x)\phi_1^{1-\beta} + \int_{\partial\Omega} \overline{B}_\beta(x)\phi_1^{1-\beta} > 0$

(G_α^-) $\int_\Omega f^+\phi_1 + \int_{\partial\Omega} h^+\phi_1 < 0$ or
$\int_\Omega f^+\phi_1 + \int_{\partial\Omega} h^+\phi_1 = 0$ and $\int_\Omega A_\alpha(x)\phi_1^{1-\alpha} + \int_{\partial\Omega} \overline{A}_\alpha(x)\phi_1^{1-\alpha} < 0.$

Where $f^\pm := \lim_{t\to\pm\infty} f(x,t)$ and $h^\pm := \lim_{t\to\pm\infty} h(x,t)$. We remark that this set of conditions extend the hypothesis of Theorem 1.1.

Theorem 1.2. *Let f and h be such that there exists f in $L^q(\Omega)$ and h in $L^q(\partial\Omega)$, with $|f(x,t)| \le \overline{f}$ for all $(x,t) \in \Omega \times \mathbb{R}$ and $|h(x,t)| \le \overline{h}$ for all (x,t) in $\partial\Omega \times \mathbb{R}$ (where $q = p/p-1$). If (G_α^+) and (G_β^-) or (G_α^-) and (G_β^+) hold then (1.1) with $\lambda = \lambda_1$ has at least one solution.*

We can continue with this procedure and obtain even more general conditions considering the rate of convergence to zero of $(f(x,s) - f^+(x))s^\alpha - A_\alpha(x)$, for example. We leave the details to the reader. Also it is possible to consider different rates of convergence, in this case the conditions involve signs of integrals of A_α and B_α separately.

In the case $p = 2$, we have a linear operator in the Hilbert space $H^1(\Omega)$, so using the Spectral Theorem for compact self-adjoint linear operators and the Fredholm alternative, we have that when λ is not an eigenvalue we do not need any additional condition to have solutions for (1.1), and if λ is an eigenvalue, we need an orthogonality condition. However when dealing with $p \neq 2$ we have to consider the problem in $W^{1,p}(\Omega)$ (which is not Hilbert) and the results is not straightforward.

Note that nonlinear boundary conditions have only been considered in recent years. For reference purposes, we cite previous works. For the Laplace operator with nonlinear boundary conditions see for example [7, 8, 12]. For previous work for the p-Laplacian with nonlinear boundary conditions of different types see [6], [11], [18] and [17]. Also, one is lead to nonlinear boundary conditions in the study of conformal deformations on Riemannian manifolds with boundary, see for example [10].

2. Proof of the results

In this section we prove theorems 1.1 and 1.2 that provide existence of solutions to (1.1). First, let us prove Theorem 1.1. We will divide the proof in two steps. Following [9], we first prove a Palais-Smale condition for the functional J_λ using the conditions of Theorem 1.1. Then we split the proof of the theorem in two cases, first we deal with $\lambda_k < \lambda < \lambda_{k+1}$, where λ_k are the variational eigenvalues of (1.2) this allows us to obtain some geometric structure on J_λ (see [11]), and finally we treat the case where $\lambda = \lambda_k$. In this case we obtain solutions as limit of solutions for a sequence $\lambda_n \to \lambda_k$. We will see that if there is any bifurcation from infinity in $\lambda = \lambda_k$ then the bifurcation is subcritical. This fact provides a priori bounds that allow us to pass to the limit in a sequence of solutions as $\lambda_n \to \lambda_k$.

To prove these results we will need some preliminary lemmas (the proofs are straightforward, see [11]).

Lemma 2.1. *Let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be given by*
$$A(u) \cdot v := \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v + \int_\Omega |u|^{p-2} uv,$$
then A is a continuous, odd, $(p-1)$-homogeneous and continuously invertible.

Lemma 2.2. *Let $B : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be given by*
$$B(u) \cdot v := \int_{\partial\Omega} |u|^{p-2} uv.$$
Then B is a continuous, odd, $(p-1)$-homogeneous and compact.

Lemma 2.3. *Let $C : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be given by*
$$C(u) \cdot v := \int_\Omega f(x,u)v + \int_{\partial\Omega} h(x,u)v.$$
Then C is continuous and compact and $\|C(u)\|_{W^{1,p}(\Omega)^*} \le \|\overline{f}\|_{L^q(\Omega)} + K\|\overline{h}\|_{L^q(\partial\Omega)}$. where K is the best constant for the Sobolev trace inequality $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$.

With these lemmas we can prove the following theorem.

Theorem 2.4. *Suppose that the hypotheses of Theorem 1.1 are satisfied, then J_λ satisfies the Palais-Smale condition, that is, for any sequence $\{u_n\} \subset W^{1,p}(\Omega)$ such that $\|J_\lambda(u_n)\|_{W^{1,p}(\Omega)} \leq c$ and $J'_\lambda(u_n) \to 0$ there exists $u \in W^{1,p}(\Omega)$ such that $u_n \to u$ strongly in $W^{1,p}(\Omega)$.*

Proof. Let $\{u_n\}$ be a Palais-Smale sequence. If u_n is bounded then we have that there exists $u \in W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$. Using that

$$A(u_n) - \lambda B(u_n) + C(u_n) = J'_\lambda(u_n) \to 0,$$

the compactness of B and C, and the continuity of A^{-1} we have that

$$u_n \to A^{-1}(\lambda B(u) - C(u))$$

strongly in $W^{1,p}(\Omega)$. Hence if we prove that Palais-Smale sequences are bounded, the result follows. To see this, let us argue by contradiction. Assume that u_n is a Palais-Smale sequence and that $\|u_n\|_{W^{1,p}(\Omega)} \to \infty$. Let

$$v_n := \frac{u_n}{\|u_n\|_{W^{1,p}(\Omega)}}$$

then there exists v such that $v_n \rightharpoonup v$ in $W^{1,p}(\Omega)$ and $v_n \to v$ in $L^p(\partial\Omega)$. We have,

$$\frac{J'_\lambda(u_n)}{\|u_n\|^{p-1}_{W^{1,p}(\Omega)}} = A(v_n) - \lambda B(v_n) + \frac{C(u_n)}{\|u_n\|^{p-1}_{W^{1,p}(\Omega)}}. \tag{2.1}$$

Using compactness of B, continuity of A^{-1} and the fact that

$$\frac{C(u_n)}{\|u_n\|^{p-1}_{W^{1,p}(\Omega)}} \to 0$$

we have that $v_n \to A^{-1}(\lambda B(v))$ in $W^{1,p}(\Omega)$. Hence $v_n \to v$ in $W^{1,p}(\Omega)$ and then $A(v) - \lambda B(v) = 0$ with $\|v\|_{W^{1,p}(\Omega)} = 1$. That means that $v \in A_\lambda \backslash \{0\}$.

Observe that, for a.e. $x \in \{v(x) > 0\}$, we have $u_n(x) \to +\infty$ so,

$$\lim_{n\to\infty} f(x, u_n(x))v_n(x) + h(x, u_n(x))v_n(x) = f^+(x)v(x) + h^+(x)v(x),$$

and

$$\begin{aligned}
&\lim_{n\to\infty} \frac{F(x, u_n(x))}{\|u_n\|_{W^{1,p}(\Omega)}} + \frac{H(x, u_n(x))}{\|u_n\|_{W^{1,p}(\Omega)}} \\
&= \lim_{n\to\infty} v_n(x)\frac{1}{u_n(x)}\int_0^{u_n(x)} f(t, u_n(t)) + v_n(x)\frac{1}{u_n(x)}\int_0^{u_n(x)} h(t, u_n(t)) \\
&= v(x)f^+(x) + v(x)h^+(x).
\end{aligned}$$

In a similar way we obtain that, for a.e. $x \in \{x : v(x) < 0\}$, we have

$$\lim_{n\to\infty} f(x, u_n(x))v_n(x) + h(x, u_n(x))v_n(x) = f^-(x)v(x) + h^-(x)v(x),$$

and therefore

$$\lim_{n\to\infty} \frac{F(x, u_n(x))}{\|u_n\|_{W^{1,p}(\Omega)}} + \frac{H(x, u_n(x))}{\|u_n\|_{W^{1,p}(\Omega)}} = v(x)f^-(x) + v(x)h^-(x).$$

On the other hand, we have

$$pJ_\lambda(u_n) - J'_\lambda(u_n) \cdot u_n$$

$$= p\int_\Omega F(x,u_n(x)) + p\int_{\partial\Omega} H(x,u_n(x)) - \int_\Omega f(x,u_n(x))u_n - \int_{\partial\Omega} h(x,u_n(x))u_n.$$

Then

$$\begin{aligned} &p\frac{J_\lambda(u_n)}{\|u_n\|_{W^{1,p}(\Omega)}} - J'_\lambda(u_n)\cdot v_n \\ &= p\int_\Omega \frac{F(x,u_n(x))}{\|u_n\|_{W^{1,p}(\Omega)}} + p\int_{\partial\Omega} \frac{H(x,u_n(x))}{\|u_n\|_{W^{1,p}(\Omega)}} - \int_\Omega f(x,u_n(x))v_n - \int_{\partial\Omega} h(x,u_n(x))v_n. \end{aligned}$$

The left hand side approaches 0 as $n\to\infty$. Hence

$$0 = (p-1)\Big[\int_{\{v>0\cap\Omega\}} f^+v + \int_{\{v>0\cap\partial\Omega\}} h^+v + \int_{\{v<0\cap\Omega\}} f^-v + \int_{\{v<0\cap\partial\Omega\}} h^-v\Big]$$

which contradicts the hypothesis on f and h in Theorem 1.1. □

Now that we have proved the Palais-Smale condition, we can state a deformation theorem that will be used later to show that J_λ has critical points (see [19]).

Theorem 2.5. *Suppose that J_λ satisfies the Palais-Smale condition. Let $\beta\in\mathbb{R}$ be a regular value of J_λ and let $\bar\epsilon>0$. Then there exists $\epsilon\in(0,\bar\epsilon)$ and a continuous one-parameter family of homeomorphisms, $\Phi : W^{1,p}(\Omega)\times[0,1]\to W^{1,p}(\Omega)$ with the following properties:*

(1) *$\Phi(u,t)=u$ if $t=0$ or if $|J_\lambda-\beta|\ge\bar\epsilon$.*
(2) *$J_\lambda(\Phi(u,t))$ is non decreasing in t for any $u\in W^{1,p}(\Omega)$.*
(3) *If $J_\lambda(u)\le\beta+\epsilon$ then $J_\lambda(\Phi(u,1))\le\beta-\epsilon$.*

We now use a variational characterization for a sequence of eigenvalues for the problem (1.2). Indeed, solutions of (1.2) we can understood as critical points of the associated energy functional

$$I(u) = \int_\Omega |\nabla u|^p + \int_\Omega |u|^p,$$

under the constraint $u\in M$, where $M=\{u\in W^{1,p}(\Omega) : \|u\|_{L^p(\partial\Omega)}=1\}$. We can find a sequence of variational eigenvalues with the characterization,

$$\lambda_k := \inf_{A\in C_k}\sup_{u\in A} I(u),$$

where

$$C_k := \{A\subset M : \text{there exists } h : S^{k-1}\to A \text{ continuous, odd and surjective }\}.$$

To prove that these λ_k are critical values one first proves a Palais-Smale condition for the functional. Next, using a deformation argument, we prove that λ_k is an eigenvalue (see [11] for the details), but it is not known if this sequence contains all the eigenvalues.

As we mentioned before, we divide the proof in two cases, $\lambda_k<\lambda<\lambda_{k+1}$ and $\lambda=\lambda_k$.

Case $\lambda_k<\lambda<\lambda_{k+1}$. Let $A\in C_k$ such that $\sup_{u\in A} I(u)=m\in(\lambda_k,\lambda)$ (here we are using the definition of λ_k). Then we have, for $u\in A,\ t>0$, that

$$\begin{aligned} J_\lambda(tu) &= \frac{t^p}{p}[\|u\|^p_{W^{1,p}(\Omega)}-\lambda] + \int_\Omega F(x,tu) + \int_{\partial\Omega} H(x,tu) \\ &\le \frac{t^p}{p}(m-\lambda) + \Big|\int_\Omega F(x,tu)\Big| + \Big|\int_{\partial\Omega} H(x,tu)\Big| \end{aligned}$$

$$\begin{aligned}
&\le \frac{t^p}{p}(m-\lambda)t\Big(\int_\Omega |u|^p\Big)^{1/p}\Big(\int_\Omega |\overline{f}|^q\Big)^{1/q} + t\Big(\int_{\partial\Omega} |u|^p\Big)^{1/p}\Big(\int_{\partial\Omega} |\overline{h}|^q\Big)^{1/q} \\
&\le \frac{t^p}{p}(m-\lambda) + t\left(m\|\overline{f}\|_{L^q(\Omega)} + \|\overline{h}\|_{L^q(\partial\Omega)}\right).
\end{aligned}$$

Let

$$\xi_{k+1} = \Big\{u \in\ W^{1,p}(\Omega) : \int_\Omega |\nabla u|^p + \int_\Omega |u|^p \ge \lambda_{k+1}\int_{\partial\Omega} |u|^p\Big\}.$$

If $u \in \xi_{k+1}$ then,

$$\begin{aligned}
J_\lambda(u) &= \frac{1}{p}\Big[\int_\Omega |\nabla u|^p + \int_\Omega |u|^p\Big] - \frac{\lambda}{p}\int_{\partial\Omega} |u|^p + \int_\Omega F(x,u) + \int_{\partial\Omega} H(x,u) \\
&\ge \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)}\big[1 - \frac{\lambda}{\lambda_{k+1}}\big] + \int_\Omega F(x,u) + \int_{\partial\Omega} H(x,u) \\
&\ge \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)}\big[1 - \frac{\lambda}{\lambda_{k+1}}\big] - \|u\|_{W^{1,p}(\Omega)}\|\overline{f}\|_{L^q(\Omega)} \\
&\quad - K\|u\|_{W^{1,p}(\Omega))}\|\overline{h}\|_{L^q(\partial\Omega)}.
\end{aligned}$$

This proves the coercitivity of J_λ in ξ_{k+1}, then there exists α such that,

$$\alpha := \inf_{u\in\xi_{k+1}} J_\lambda(u).$$

On the other hand we have, for $u \in A$,

$$J_\lambda(tu) \le \frac{t^p}{p}(m-\lambda) + t\left(m\|\overline{f}\|_{L^q(\Omega)} + \|\overline{h}\|_{L^q(\partial\Omega)}\right),$$

where $m - \lambda < 0$. Then for all $u \in\ A$, as $t \to +\infty$ $J_\lambda(tu) \to -\infty$. Hence there exists $T > 0$ such that

$$\max_{u\in A, t\ge T} J_\lambda(tu) = \gamma < \alpha. \tag{2.2}$$

Let $TA := \{tu :\ u \in\ A,\ t \ge T\}$ and

$$\chi := \{h \in\ C(B_k(0,1), W^{1,p}(\Omega)) :\ h|_{S^{k-1}} \text{is odd into } TA\}.$$

Let us show that χ is nonempty. By the definition of C_k, there exists continuous function $h : S^{k-1} \to A$ odd and surjective. Let us define $\overline{h} : B_k \to W^{1,p}(\Omega)$ as $\overline{h}(ts) = tTh(s)$ $s \in\ S^{k-1}$, $t \in\ [0,1]$. Clearly $\overline{h} \in \chi$.

Next, let we prove that if $h \in \chi$ then $h(B_k) \cap \xi_{k+1} \ne \emptyset$. If there exists any $u \in h(B_k)$ such that $\int_{\partial\Omega} |u|^p = 0$ then $u \in \xi_{k+1}$. Suppose now that $\int_{\partial\Omega} |u|^p \ne 0$ for all $u \in h(B_k)$, and let us consider

$$\widetilde{h}(x_1,\dots,x_{k+1}) = \begin{cases} \pi h(x_1,\dots,x_k) & x_{k+1} \ge 0 \\ -\pi h(-x_1,\dots,-x_k) & x_{k+1} < 0, \end{cases}$$

where $\pi u = u/\|u\|_{L^p(\partial\Omega)}$. Then, if $x_{k+1} \ge 0$,

$$\widetilde{h}(x_1,\dots,x_{k+1}) = \pi(-h(-x_1,\dots,-x_k)) = -\pi h(-x_1,\dots,-x_k)$$

and hence

$$\widetilde{h}(-x_1,\dots,-x_{k+1}) = -\pi h(x_1,\dots,x_k) = -\widetilde{h}(x_1,\dots,x_{k+1}).$$

In an analogous way for $x_{k+1} < 0$, we have

$$\widetilde{h}(x_1,\dots,x_{k+1}) = -\widetilde{h}(-x_1,\dots,-x_{k+1}),$$

then $\widetilde{h}$ is odd. Hence $\widetilde{h}(S^k) \in C^{k+1}$. On the other hand, we have,

$$\lambda_{k+1} = \inf_{A \in C^{k+1}} \sup_{u \in A} I(u),$$

then

$$\lambda_{k+1} \le \sup_{u \in \widetilde{h}(S^k)} I(u).$$

Hence, for some $u \in \widetilde{h}(S^k)$, that is, for some $x \in S^k$ such that $u = \widetilde{h}(x)$ we have $\lambda_{k+1} \le I(u)$. This implies that $\widetilde{h}(x) \in \xi_{k+1}$. Using the definition of $\widetilde{h}$ we obtain that $h(x) \in \xi_{k+1}$. Then $h(B_k) \cap \xi_{k+1} \neq \emptyset$.

Theorem 2.6. *The value*

$$c := \inf_{h \in \chi} \sup_{x \in B_k} J_\lambda h(x),$$

is a critical value for J_λ, with $c \ge \alpha$.

Proof. For each $h \in \chi$, there exists $x \in B_k$ such that $h(x) \in \xi_{k+1}$, then $J_\lambda(h(x)) \ge \alpha$. Hence

$$\sup_{x \in B_k} J_\lambda(h(x)) \ge \alpha \quad \forall\, h \in \chi.$$

Therefore, $c \ge \alpha > \gamma$, where γ is given by (2.2).

Let us argue by contradiction. Suppose that c is a regular value, then using the deformation Theorem 2.5, with $\beta = c$ and $\bar{\epsilon} < c - \gamma$, we have that there exists a deformation $\Phi(u,t)$ that verifies the usual properties. If $u \in TA$ then,

$$J_\lambda(u) \le \gamma < \beta - \bar{\epsilon},$$

then by one of the properties of the deformation lemma we have $\Phi(u,t) = u$. By the definition of c, there exists $h \in \chi$ such that,

$$\sup_{x \in B_k} J_\lambda(h(x)) \le c + \epsilon. \tag{2.3}$$

Let $\tilde{h}(\cdot) := \Phi(h(\cdot),1)$, if $x \in S^{k-1}$ we have that $h(x) \in TA$, then $\tilde{h}(x) = \Phi(h(x),1) = h(x)$ and hence $\tilde{h}|_{S^{k-1}} = h|_{S^{k-1}}$. We also have $\tilde{h}(-x) = \Phi(h(-x),1) = \Phi(-h(x),1) = -\tilde{h}(x)$. We obtain that $\tilde{h} \in \chi$. Using (2.3) and the deformation theorem we have

$$\sup_{x \in B_k} J_\lambda(\tilde{h}(x)) = \sup_{x \in B_k} J_\lambda(\Phi(h(x),1)) \le c - \epsilon,$$

a contradiction that proves that c is a critical value. □

Case $\lambda = \lambda_k$. We will prove the result under condition $(LL)^+_{\lambda_k}$, the case where $(LL)^-_{\lambda_k}$ holds is completely analogous.

Lemma 2.7. *If $(LL)^+_{\lambda_k}$ is satisfied, then there exists $\delta > 0$ such that $(LL)^+_\mu$ is satisfied for all $\mu \in (\lambda_k - \delta, \lambda_k + \delta)$.*

Proof. Arguing by contradiction, let us assume that there exists $\mu_n \to \lambda_k$ and corresponding eigenfunctions $\{v_n\}$, $\|v_n\|_{W^{1,p}(\Omega)} = 1$, such that

$$\int_\Omega |\nabla v_n|^{p-2} \nabla v_n \nabla w + \int_\Omega |v_n|^{p-2} v_n w = \mu_n \int_{\partial\Omega} |v_n|^{p-2} v_n \quad \forall w \in W^{1,p}(\Omega) \tag{2.4}$$

and

$$\int_{\{v_n>0\cap\Omega\}} f^+ v_n + \int_{\{v_n>0\cap\partial\Omega\}} h^+ v_n + \int_{\{v_n<0\cap\Omega\}} f^- v_n + \int_{\{v_n<0\cap\partial\Omega\}} h^- v_n \le 0, \tag{2.5}$$

for all n. Then, since $\{v_n\}$ is bounded, there exists $v \in W^{1,p}(\Omega)$ such that $v_n \to v$ in $L^p(\partial\Omega)$. Taking

$$\phi_n(w) = \mu_n \int_{\partial\Omega} |v_n|^{p-2} v_n w \quad \text{and} \quad \phi(w) = \lambda_k \int_{\partial\Omega} |v|^{p-2} v w,$$

we have that $\phi_n \to \phi$ in $(W^{1,p}(\Omega))^*$. Using the continuity of A^{-1}, we have that $v_n \to v$ in $W^{1,p}(\Omega)$. Then, taking limits in (2.4) and (2.5) we have

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla w + \int_\Omega |v|^{p-2} v w = \lambda_k \int_{\partial\Omega} |v|^{p-2} v, \quad \forall w \in W^{1,p}(\Omega),$$

and

$$\int_{\{v>0\cap\Omega\}} f^+ v + \int_{\{v>0\cap\partial\Omega\}} h^+ v + \int_{\{v<0\cap\Omega\}} f^- v + \int_{\{v<0\cap\partial\Omega\}} h^- v \le 0.$$

Which contradicts the fact that $(LL)^+_{\lambda_k}$ is satisfied. □

Now we assume that $\lambda_{k-1} \le \lambda_k - \delta$ and let $\{\mu_n\} \subset (\lambda_k - \delta, \lambda_k)$ be an increasing sequence such that $\mu_n \to \lambda_k$. We will construct a decreasing sequence $\{c_n\}$ of critical values corresponding to J_{μ_n}, and then we will see that the sequence corresponding to the critical points $\{u_n\}$ is bounded and converges to a certain u that is a critical point for J_{λ_k}.

Lemma 2.8. *There exists a decreasing sequence of critical values, $\{c_n\}$ associated with the functional J_{μ_n}.*

Proof. Let $A \in C^{k-1}$, $T_1 > 0, \xi_k$ and χ_1 as in the first part ($\lambda_k < \lambda < \lambda_{k+1}$) such that,

$$c_1 := \inf_{h\in\chi_1} \sup_{x\in B_{k-1}} J_{\mu_1}(h(x))$$

is a critical value for J_{μ_1}. To define c_2, let us chose the same A and ξ_k, but we take $T_2 > T_1$ that provides the correspondent χ_2. Then $T_2 A \subset T_1 A$, $\chi_2 \subset \chi_1$ and,

$$\inf_{h\in\chi_2} \sup_{x\in B_{k-1}} J_{\mu_1}(h(x)) \ge \inf_{h\in\chi_1} \sup_{x\in B_{k-1}} J_{\mu_1}(h(x)) = c_1.$$

Let

$$h_2(x) := \begin{cases} h_1(2x) & |x| \le \frac{1}{2}, \\ h_1\big(\frac{x}{|x|}\big)[1 + 2(|x| - \frac{1}{2})T_2] & |x| > \frac{1}{2}. \end{cases}$$

For $|x| \ge 1/2$, $h_2(x) \in T_1 A$; therefore,

$$J_{\mu_1}(h_2(x)) \le \gamma < \alpha \le J_{\mu_1}(u), \quad \forall u \in \xi_{k+1}.$$

Then there exists $y \in B_k$ such that $h_2(y) \in \xi_{k+1}$ and

$$J_{\mu_1}(h_2(x)) \le \gamma < \alpha \le J_{\mu_1}(h_2(y)).$$

That is, for all x with $|x| \ge 1/2$ there exists $y \in B_k$ such that $J_{\mu_1}(h_2(x)) < J_{\mu_1}(h_2(y))$. Then

$$\sup_{x\in B_{k-1}} J_{\mu_1}(h_2(x)) = \sup_{|x|\le 1/2} J_{\mu_1}(h_2(x)) = \sup_{|x|\le 1/2} J_{\mu_1}(h_1(2x)) = \sup_{x\in B_{k-1}} J_{\mu_1}(h_1(x)).$$

Hence

$$c_1 := \inf_{h\in\chi_1} \sup_{x\in B_{k-1}} J_{\mu_1}(h(x)) = \inf_{h\in\chi_2} \sup_{x\in B_{k-1}} J_{\mu_1}(h(x)).$$

On the other hand we have,

$$J_{\mu_2}(u) = J_{\mu_1}(u) + \frac{1}{p}(\mu_1 - \mu_2)\int_{\partial\Omega} |u|^p \le J_{\mu_1}(u) \quad \forall u \in W^{1,p}(\Omega),$$

then

$$\inf_{h\in\chi_2} \sup_{x\in B_{k-1}} J_{\mu_1}(h(x)) \ge \inf_{h\in\chi_2} \sup_{x\in B_{k-1}} J_{\mu_2}(h(x)) := c_2.$$

We conclude that $c_1 \ge c_2$. Continuing with this procedure we find a sequence c_n with the desired properties. $\square$

Let $\{u_n\}$ be the sequence of critical points associated with $\{c_n\}$ then

$$J'_{\mu_n}(u_n) = A(u_n) - \mu_n B(u_n) + C(u_n) = 0.$$

If $\{u_n\}$ is bounded then there exists $u \in W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$, then $u_n \to A^{-1}(\lambda_k B(u) - C(u))$ in $W^{1,p}(\Omega)$. Hence u is a critical point for J_{λ_k} and we have proved our result.

Next, we show that $\{u_n\}$ must be bounded. This means that if there exists (μ_n, u_n) solutions of (1.1) with $\mu_n \to \lambda_k$ such that $\|u_n\|_{W^{1,p}(\Omega)} \to \infty$ then the sequence μ_n verifies $\mu_n > \lambda_k$, that is the only possible bifurcation from infinity at $\lambda = \lambda_k$ is subcritical.

Lemma 2.9. *If $\|u_n\|_{W^{1,p}(\Omega)} \to \infty$, then there exists $v \in A_{\lambda_k} \setminus \{0\}$ such that*

$$\frac{u_n}{\|u_n\|_{W^{1,p}(\Omega)}} \to v.$$

Proof. Let $v_n := u_n/\|u_n\|_{W^{1,p}(\Omega)}$. Then $v_n \rightharpoonup v$. Using that

$$A(v_n) - \mu_n B(v_n) - \frac{C(u_n)}{\|u_n\|^{p-1}} = 0, \tag{2.6}$$

the compactness of B and the continuity of A^{-1}, we have $v_n \to A^{-1}(\lambda_k B(v))$. Then $v_n \to v$, with $\|v\|_{W^{1,p}(\Omega)} = 1$. Taking limits in (2.6) we have $A(v) = \lambda_k B(v)$, then $v \in A_{\lambda_k} \setminus \{0\}$. $\square$

Making similar calculations to those in the proof of Theorem 2.1, we get

$$\begin{aligned} pc_n &= pJ_{\mu_n}(u_n) - J'_{\mu_n}(u_n)\cdot u_n \\ &= p\int_\Omega F(x,u_n) + p\int_{\partial\Omega} H(x,u_n) - \int_\Omega f(x,u_n)u_n - \int_{\partial\Omega} h(x,u_n)u_n. \end{aligned}$$

Then

$$\begin{aligned} &\lim_{n\to\infty} p\int_\Omega \frac{F(x,u_n)}{\|u_n\|_{W^{1,p}(\Omega)}} + p\int_{\partial\Omega} \frac{H(x,u_n)}{\|u_n\|_{W^{1,p}(\Omega)}} - \int_\Omega f(x,u_n)v_n - \int_{\partial\Omega} h(x,u_n)v_n \\ &= (p-1)\Big(\int_{\{v>0\cap\Omega\}} f^+v + \int_{\{v>0\cap\partial\Omega\}} h^+v + \int_{\{v<0\cap\Omega\}} f^-v + \int_{\{v<0\cap\partial\Omega\}} h^-v\Big) \\ &> 0. \end{aligned}$$

Then,

$$\lim_{n\to\infty} \frac{pc_n}{\|u_n\|_{W^{1,p}(\Omega)}} > 0,$$

which contradicts the fact that $\{c_n\}$ is bounded from above.

Then we have that $\{u_n\}$ is bounded. Hence there exists $u \in W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ weak in $W^{1,p}(\Omega)$, using the compactness of B and C and the continuity of A^{-1} we have $u_n \to u$ strong in $W^{1,p}(\Omega)$.

Case $\lambda = \lambda_1$ This corresponds to Theorem 1.2. In this theorem we improve the conditions on f and h for the case where $\lambda = \lambda_1$. We use ideas from [3], but first we find some estimates.

Lemma 2.10. *Let $u \in C^{\alpha}(\Omega)$ be a solution of* (1.1) *strictly positive in $\overline{\Omega}$. Then*

$$-\frac{\int_{\partial\Omega} h(x,u)\frac{\phi_1^p}{|u|^{p-2}u} + \int_{\Omega} f(x,u)\frac{\phi_1^p}{|u|^{p-2}u}}{\int_{\partial\Omega}\phi_1^p} \le \lambda_1 - \lambda \le -\frac{\int_{\partial\Omega} h(x,u)u + \int_{\Omega} f(x,u)u}{\int_{\partial\Omega}|u|^p}.$$

Proof. In the weak form with $v = u$, we have

$$\begin{aligned}
-\int_{\partial\Omega} g(x,u)u - \int_{\Omega} f(x,u)u &= \int_{\Omega}|\nabla u|^p + \int_{\Omega}|u|^p - \lambda\int_{\partial\Omega}|u|^p \\
&\ge (\lambda_1 - \lambda)\int_{\partial\Omega}|u|^p,
\end{aligned}$$

then we get the second inequality. If we take $v = \phi_1^p/(|u|^{p-2}u)$ we have,

$$\begin{aligned}
&-\int_{\partial\Omega} h(x,u)\frac{{\phi_1}^p}{|u|^{p-2}u} - \int_{\Omega} f(x,u)\frac{{\phi_1}^p}{|u|^{p-2}u} - (\lambda_1-\lambda)\int_{\partial\Omega}{\phi_1}^p \\
&= \int_{\Omega}|\nabla u|^{p-2}\nabla u\nabla\big(\frac{\phi_1^p}{|u|^{p-2}u}\big) + \int_{\Omega}|u|^{p-2}u\frac{\phi_1^p}{|u|^{p-2}u} - \int_{\Omega}|\nabla\phi_1|^p - \int_{\partial\Omega}|\phi_1|^p \\
&= \int_{\Omega} p|\nabla u|^{p-2}\frac{{\phi_1}^{p-1}}{|u|^{p-2}u}\nabla u\nabla\phi_1 - \int_{\Omega}(p-1)\frac{{\phi_1}^p}{|u|^p}|\nabla u|^p - \int_{\Omega}|\nabla\phi_1|^p \\
&\le \int_{\Omega} p\frac{{\phi_1}^{p-1}}{|u|^{p-1}}|\nabla u|^{p-1}|\nabla\phi_1| - \int_{\Omega}(p-1)\frac{{\phi_1}^p}{|u|^p}|\nabla u|^p - \int_{\Omega}|\nabla\phi_1|^p.
\end{aligned}$$

Using that

$$pt^{p-1}s - (p-1)t^p - s^p \le 0, \quad \forall t, s \ge 0$$

with $t = \frac{\phi_1}{|u|}|\nabla u|$ and $s = |\nabla\phi_1|$ we have that

$$-\int_{\partial\Omega} h(x,u)\frac{\phi_1}{|u|^{p-2}u} - \int_{\Omega} f(x,u)\frac{\phi_1}{|u|^{p-2}u} - (\lambda_1-\lambda)\int_{\partial\Omega}{\phi_1}^p \le 0,$$

the result follows. □

Now, let us proceed with the proof of the main theorem.

Proof of Theorem 1.2. Let us suppose that f and h satisfy conditions (G_{α}^{-}) and (G_{β}^{+}). We will prove that there exists (λ_n, u_n) solutions of problem (1.1) with $\lambda_n \to \lambda_1$ such that $\|u_n\|_{W^{1,p}(\Omega)} \le K$. This will follows from the fact that any possible bifurcation from infinity must be subcritical.

Let $\lambda_n \searrow \lambda_1$, and u_n be the solutions of (1.1). Remark that Theorem 1.1 shows the existence of u_n for every λ_n close but not equal to λ_1 (as λ_1 is isolated the conditions on f and h of Theorem 1.1 are trivially verified for any λ_n close to λ_1).

Suppose that $\|u_n\|_{W^{1,p}(\Omega)} \to \infty$. If $u_n/\|u_n\|_{W^{1,p}(\Omega)} \to \phi_1$ and

$$\int_{\Omega} f^+\phi_1 + \int_{\partial\Omega} h^+\phi_1 < 0$$

then we arrive to a contradiction. Otherwise, if $u_n/\|u_n\|_{W^{1,p}(\Omega)} \to -\phi_1$ and

$$\int_\Omega f^- \phi_1 + \int_{\partial\Omega} h^- \phi_1 < 0$$

we also arrive to a contradiction. Hence in both cases any bifurcation from infinity must be subcritical. Hence $\{u_n\}$ is bounded (see [3] for the details).

We have to consider only the case where

$$\begin{aligned} \int_\Omega f^+ \phi_1 + \int_{\partial\Omega} h^+ \phi_1 = 0, \\ \int_\Omega f^- \phi_1 + \int_{\partial\Omega} h^- \phi_1 = 0, \end{aligned} \tag{2.7}$$

and

$$\begin{gathered} \int_{\partial\Omega} \overline{A}_\alpha \phi_1^{1-\alpha} + \int_\Omega A_\alpha \phi_1^{1-\alpha} < 0, \\ \int_{\partial\Omega} \overline{B}_\alpha \phi_1^{1-\alpha} + \int_\Omega B_\alpha \phi_1^{1-\alpha} > 0. \end{gathered}$$

Let us assume by contradiction that $\|u_n\|_{W^{1,p}(\Omega)} \to \infty$. Then by Lemma 2.9,

$$\frac{u_n}{\|u_n\|_{W^{1,p}(\Omega)}} \to \pm\phi_1.$$

The convergence is uniform by regularity results that show that $u_n \in C^\alpha(\Omega)$, see [15]. Using the previous lemma,

$$0 > (\lambda_1 - \lambda_n) \int_{\partial\Omega} \phi_1^p \geq - \int_{\partial\Omega} h(x, u_n) \frac{\phi_1^p}{|u_n|^{p-2} u_n} - \int_\Omega f(x, u_n) \frac{\phi_1^p}{|u_n|^{p-2} u_n}.$$

Using (2.7),

$$\begin{aligned} 0 &< \int_{\partial\Omega} (h(x, u_n) \phi_1^{p-1} \frac{\|u_n\|^{p-1}}{|u_n|^{p-2} u_n} - h^+(x)) \phi_1 \\ &\quad + \int_\Omega (f(x, u_n) \phi_1^{p-1} \frac{\|u_n\|^{p-1}}{|u_n|^{p-2} u_n} - f^+(x)) \phi_1 \\ &= \int_{\partial\Omega} (h(x, u_n) - h^+(x)) \phi_1^{p-1} \frac{\|u_n\|^{p-1}}{|u_n|^{p-2} u_n} \phi_1 \\ &\quad - \int_{\partial\Omega} h^+(x) \phi_1 (1 - \phi_1^{p-1} \frac{\|u_n\|^{p-1}}{|u_n|^{p-2} u_n}) \\ &\quad + \int_\Omega (f(x, u_n) - f^+(x)) \phi_1^{p-1} \frac{\|u_n\|^{p-1}}{|u_n|^{p-2} u_n} \phi_1 \\ &\quad - \int_\Omega f^+(x) \phi_1 (1 - \phi_1^{p-1} \frac{\|u_n\|^{p-1}}{|u_n|^{p-2} u_n}). \end{aligned} \tag{2.8}$$

If $u_n/\|u_n\|_{W^{1,p}(\Omega)} \to \phi_1$, using our hypothesis on the dominated convergence of $(h(x, u_n) - h^+(x)) u_n^\alpha$ by a function in $L^1(\partial\Omega)$ and the uniform convergence of $u_n/\|u_n\|_{W^{1,p}(\Omega)}$ to ϕ_1, we have the hypotheses of the Lebesgue's Dominated Convergence Theorem. The second term also verifies these hypotheses. Then using our hypothesis over f and h, and taking the limit we have

$$\lim_{n\to\infty} \int_{\partial\Omega} (h(x, u_n) - h^+(x)) \|u_n\|^\alpha \phi_1^p \frac{\|u_n\|^{p-1}}{|u_n|^{p-2}}$$

$$
\begin{aligned}
&+\int_\Omega (f(x,u_n)-f^+(x))\|u_n\|^\alpha \phi_1^p \frac{\|u_n\|^{p-1}}{|u_n|^{p-2}u_n}\\
&=\int_{\partial\Omega} \overline{A}_\alpha \phi_1^{1-\alpha}+\int_\Omega A_\alpha \phi_1^{1-\alpha}<0.
\end{aligned}
$$

Therefore, for n large enough

$$
\begin{aligned}
&\int_{\partial\Omega} (h(x,u_n)-h^+(x))\|u_n\|^\alpha \phi_1^p \frac{\|u_n\|^{p-1}}{|u_n|^{p-2}}\\
&+\int_\Omega (f(x,u_n)-f^+(x))\|u_n\|^\alpha \phi_1^p \frac{\|u_n\|^{p-1}}{|u_n|^{p-2}u_n}<C<0.
\end{aligned}
$$

Using that the two negative terms of (2.8) go to zero (by the Lebesgue's Dominated Convergence Theorem), we have for n large enough that

$$
\begin{aligned}
&\int_{\partial\Omega} (h(x,u_n)\phi_1^{p-1} \frac{\|u_n\|^{p-1}}{|u_n|^{p-2}u_n}-h^+(x))\phi_1\\
&+\int_\Omega (f(x,u_n)\phi_1^{p-1} \frac{\|u_n\|^{p-1}}{|u_n|^{p-2}u_n}-f^+(x))\phi_1<0,
\end{aligned}
$$

which contradicts inequality (2.8). On the other hand if $u_n/\|u_n\|_{W^{1,p}(\Omega)} \to -\phi_1$, using

$$
\int_{\partial\Omega} \overline{B}_\beta \phi_1^{1-\beta}+\int_\Omega B_\beta \phi_1^{1-\beta}>0,
$$

and proceeding as before we arrive to a contradiction. Hence $\{u_n\}$ must be bounded. If f and h satisfy condition (G_α^+) and (G_β^-), using the other inequality we prove that if we take (λ_n, u_n) solutions of (1.1) with $\lambda_n \nearrow \lambda_1$ then $\{u_n\}$ must be bounded. Using the same argument as in the previous theorem we see that there exists $u \in W^{1,p}(\Omega)$ such that $u_n \to u$ and u is a solution for (1.1) with $\lambda=\lambda_1$. This completes the proof. $\square$

We can observe that in the proof of the previous theorem we prove that if f and h satisfy the condition (G_α^-) and (G_β^+) then any bifurcation from infinity must be subcritical, and in the second case any bifurcation from infinity must be supercritical.

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