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APPROXIMATIONS OF SOLUTIONS TO NONLINEAR SOBOLEV TYPE EVOLUTION EQUATIONS

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Abstract. In the present work we study the approximations of solutions to a class of nonlinear Sobolev type evolution equations in a Hilbert space. These equations arise in the analysis of the partial neutral functional differential equations with unbounded delay. We consider an associated integral equation and a sequence of approximate integral equations. We establish the existence and uniqueness of the solutions to every approximate integral equation using the fixed point arguments. We then prove the convergence of the solutions of the approximate integral equations to the solution of the associated integral equation. Next we consider the Faedo-Galerkin approximations of the solutions and prove some convergence results. Finally we demonstrate some of the applications of the results established.

1. Introduction

In the present work we are concerned with the approximation of solutions to the nonlinear Sobolev type evolution equation

$$\begin{gathered} \frac{d}{dt}(u(t) + g(t, u(t))) + Au(t) = f(t, u(t)), \quad t > 0, \\ u(0) = \phi, \end{gathered} \tag{1.1}$$

in a separable Hilbert space $(H, \|.\|, (.,.))$, where the linear operator A satisfies the assumption (H1) stated later in this section so that $-A$ generates an analytic semigroup. The functions f and g are the appropriate continuous functions of their arguments in H.

The case of (1.1) in which $g \equiv 0$ has been extensively studied in literature, see for instance, the books of Krein [11], Pazy [14], Goldstein [7] and the references cited in these books.

The study of (1.1) with linear g was initiated by Showalter [15, 16, 17, 18, 19] with the applications to the degenerate parabolic equations. Brill [3] has reformulated a class of pseudoparabolic partial differential equations as (1.1) with linear g and has considered the applications to a variety of physical problems, for example, in the thermodynamics [6], in the flow of fluid through fissured rocks [2], in the shear in second-order fluids [21] and in the soil mechanics [20].

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The nonlinear Sobolev type equations of the form (1.1) arise in the study of partial neutral functional differential equations with an unbounded delay which can be modelled in the form (cf. [9, 10])

$$\frac{d}{dt}(u(t)+G(t,u_t)) = Au(t)+F(t,u_t), \quad t>0, \tag{1.2}$$

in a Banach space X where A is the infinitesimal generator of an analytic semigroup in X, F and G are appropriate nonlinear functions from $[0,T]\times W$ into X and for any function $u \in C((-\infty,\infty),X)$ the history function $u_t \in C((-\infty,0],X)$ of u is given by $u_t(\theta)=u(t+\theta)$.

In the present work we are interested in the Faedo-Galerkin approximations of solutions to (1.1). The Faedo-Galerkin approximations of solutions to the particular case of (1.1) where $g\equiv 0$ and $f(t,u)=M(u)$ has been considered by Miletta [13]. The more general case has been dealt with by Bahuguna, Srivastava and Singh [1]. The existence and uniqueness of solutions to (1.1) has been studied by Hernández [8] under the assumptions that $-A$ is the infinitesimal generator of an analytic semigroup of bounded linear operators defined on a Banach space X and f and g are appropriate continuous functions on $[0,T]\times W$ into X where W is an open subset of X.

Now, we consider some assumptions on A, f and g. We assume that the operator A satisfies the following.

(H1) A is a closed, positive definite, self-adjoint, linear operator from the domain $D(A)\subset H$ of A into H such that $D(A)$ is dense in H, A has the pure point spectrum

$$0<\lambda_0\le\lambda_1\le\lambda_2\le\cdots$$

and a corresponding complete orthonormal system of eigenfunctions $\{u_i\}$, i.e., $Au_i=\lambda_i u_i$ and $(u_i,u_j)=\delta_{ij}$, where $\delta_{ij}=1$ if $i=j$ and zero otherwise.

These assumptions on A guarantee that $-A$ generates an analytic semigroup, denoted by e^{-tA}, $t\ge 0$.

We mention some notions and preliminaries essential for our purpose. It is well known that there exist constants $\tilde{M}\ge 1$ and $\omega\ge 0$ such that

$$\|e^{-tA}\|\le \tilde{M}e^{\omega t}, \quad t\ge 0.$$

Since $-A$ generates the analytic semigroup e^{-tA}, $t\ge 0$, we may add cI to $-A$ for some constant c, if necessary, and in what follows we may assume without loss of generality that $\|e^{-tA}\|$ is uniformly bounded by M, i.e., $\|e^{-tA}\|\le M$ and $0\in\rho(A)$. In this case it is possible to define the fractional power A^η for $0\le\eta\le 1$ as closed linear operator with domain $D(A^\eta)\subseteq H$ (cf. Pazy [14], pp. 69-75 and p. 195). Furthermore, $D(A^\eta)$ is dense in H and the expression

$$\|x\|_\eta = \|A^\eta x\|,$$

defines a norm on $D(A^\eta)$. Henceforth we represent by X_η the space $D(A^\eta)$ endowed with the norm $\|.\|_\eta$. In the view of the facts mentioned above we have the following result for an analytic semigroup e^{-tA}, $t\ge 0$ (cf. Pazy [14] pp. 195-196).

Lemma 1.1. *Suppose that $-A$ is the infinitesimal generator of an analytic semigroup e^{-tA}, $t\ge 0$ with $\|e^{-tA}\|\le M$ for $t\ge 0$ and $0\in\rho(-A)$. Then we have the following properties.*

(i) *X_η is a Banach space for $0\le\eta\le 1$.*

(ii) *For* $0 < \delta \le \eta < 1$, *the embedding* $X_\eta \hookrightarrow X_\delta$ *is continuous.*

(iii) A^η *commutes with* e^{-tA} *and there exists a constant* $C_\eta > 0$ *depending on* $0 \le \eta \le 1$ *such that*

$$\|A^\eta e^{-tA}\| \le C_\eta t^{-\eta}, \quad t > 0.$$

We assume the following assumptions on the nonlinear maps f and g.

(H2) There exist positive constants $0 < \alpha < \beta < 1$ and R such that the functions f and $A^\beta g$ are continuous for $(t,u) \in [0,\infty) \times B_R(X_\alpha, \phi)$, where $B_R(Z, z_0) = \{z \in Z \mid \|z - z_0\|_Z \le R\}$ for any Banach space Z with its norm $\|.\|_Z$ and there exist constants L, $0 < \gamma \le 1$ and a nondecreasing function F_R from $[0,\infty)$ into $[0,\infty)$ depending on $R > 0$ such that for every (t,u), (t,u_1) and (t,u_2) in $[0,\infty) \times B_R(X_\alpha, \phi)$,

$$\begin{gathered}
\|A^\beta g(t,u_1) - A^\beta g(s,u_2)\| \le L\{|t-s|^\gamma + \|u_1 - u_2\|_\alpha\},\\
\|f(t,u)\| \le F_R(t),\\
\|f(t,u_1) - f(t,u_2)\| \le F_R(t)\|u_1 - u_2\|_\alpha,\\
L\|A^{\alpha-\beta}\| < 1.
\end{gathered}$$

The plan of this paper is as follows. In the second section, we consider an integral equation associated with (1.1). We then consider a sequence of approximate integral equations and establish the existence and uniqueness of solutions to each of the approximate integral equations. In the third section we prove the convergence of the solutions of the approximate integral equations and show that the limiting function satisfies the associated integral equation. In the fourth section we consider the Faedo-Galerkin approximations of solutions and prove some convergence results for such approximations. Finally in the last section we demonstrate some of the applications of the results established in earlier sections.

2. Approximate Integral Equations

We continue to use the notions and notations of the earlier section. The existence of solutions to (1.1) is closely associated with the existence of solutions to the integral equation

$$\begin{aligned}
u(t) = &e^{-tA}(\phi + g(0,\phi)) - g(t,u(t)) + \int_0^t Ae^{-(t-s)A} g(s,u(s))ds\\
&+ \int_0^t e^{-(t-s)A} f(s,u(s))ds, \quad t \ge 0.
\end{aligned}$$

In this section we will consider an approximate integral equation associated with (2.1) and establish the existence and uniqueness of the solutions to the approximate integral equations. By a solution u to (2.1) on $[0,T]$, $0 < T < \infty$, we mean a function $u \in X_\alpha(T)$ satisfying (2.1) on $[0,T]$ where $X_\alpha(T)$ is the Banach space $C([0,T], X_\alpha)$ of all continuous functions from $[0,T]$ into X_α endowed with the supremum norm

$$\|u\|_{X_\alpha(T)} = \sup_{0 \le t \le T} \|u(t)\|_\alpha.$$

By a solution u to (2.1) on $[0,\tilde{T})$, $0 < \tilde{T} \le \infty$, we mean a function u such that $u \in X_\alpha(T)$ satisfying (2.1) on $[0,T]$ for every $0 < T < \tilde{T}$.

Let H_n denote the finite dimensional subspace of the Hilbert space H spanned by $\{u_0, u_1, \dots, u_n\}$ and let $P^n : H \to H_n$ for $n = 1, 2, \cdots$, be the corresponding projection operators.

Let $0 < T_0 < \infty$ be arbitrarily fixed and let

$$B = \max_{0 \le t \le T_0} \|A^\beta g(t, \phi)\|.$$

We choose $0 < T \le T_0$ such that

$$\|(e^{-tA} - I)A^\alpha(\phi + g(0, P^n\phi))\| \le (1-\mu)\frac{R}{3},$$

$$\|A^{\alpha-\beta}\|LT^\gamma + C_{1+\alpha-\beta}(L\tilde{R} + B)\frac{T^{\beta-\alpha}}{\beta-\alpha} + C_\alpha F_{\tilde{R}}(T_0)\frac{T^{1-\alpha}}{1-\alpha} < (1-\mu)\frac{R}{6},$$

$$C_{1+\alpha-\beta}L\frac{T^{\beta-\alpha}}{\beta-\alpha} + C_\alpha F_{\tilde{R}}(T_0)\frac{T^{1-\alpha}}{1-\alpha} < 1-\mu,$$

where $\mu = \|A^{\alpha-\beta}\|L$, $\tilde{R} = \sqrt{R^2 + \|\phi\|_\alpha^2}$ and C_α and $C_{1+\alpha-\beta}$ are the constants in Lemma 1.1.

For each n, we define

$$f_n : [0,T] \times X_\alpha(T) \to H \quad \text{by} \quad f_n(t,u) = f(t, P^n u(t)),$$
$$g_n : [0,T] \times X_\alpha(T) \to X_\beta(T) \quad \text{by} \quad g_n(t,u) = g(t, P^n u(t)).$$

We set $\tilde{\phi}(t) = \phi$ for $t \in [0,T]$ and define a map S_n on $B_R(X_\alpha(T), \tilde{\phi})$ by

$$\begin{aligned}(S_n u)(t) =& e^{-tA}(\phi + g_n(0, \tilde{\phi})) - g_n(t,u) + \int_0^t Ae^{-(t-s)A} g_n(s,u)ds \\ &+ \int_0^t e^{-(t-s)A} f_n(s,u)ds.\end{aligned} \tag{2.1}$$

Proposition 2.1. *Let (H1) and (H2) hold. Then there exists a unique function $u_n \in B_R(X_\alpha(T), \tilde{\phi})$ such that $S_n u_n = u_n$ for each $n = 0, 1, 2, \dots$; i.e., u_n satisfies the approximate integral equation*

$$\begin{aligned}u_n(t) =& e^{-tA}(\phi + g_n(0, \tilde{\phi})) - g_n(t,u_n) + \int_0^t Ae^{-(t-s)A} g_n(s,u_n)ds \\ &+ \int_0^t e^{-(t-s)A} f_n(s,u_n)ds.\end{aligned} \tag{2.2}$$

Proof. First we show that the map $t \mapsto (S_n u)(t)$ is continuous from $[0,T]$ into X_α with respect to norm $\|.\|_\alpha$. For $t \in [0,T]$ and sufficiently small $h > 0$, we have

$$\begin{aligned}&\|(S_n u)(t+h) - (S_n u)(t)\|_\alpha \\ &\le \|(e^{-hA} - I)A^\alpha e^{-tA}\|(\|\phi\| + \|g(0, P^n\phi)\|) \\ &\quad + \|A^{\alpha-\beta}\|\,\|A^\beta g_n(t+h,u) - A^\beta g_n(t,u)\| \\ &\quad + \int_0^t \|(e^{-hA} - I)A^{1+\alpha-\beta}e^{-(t-s)A}\|\,\|A^\beta g_n(s,u)\|ds \\ &\quad + \int_t^{t+h} \|e^{-(t+h-s)A}A^{1+\alpha-\beta}\|\,\|A^\beta g_n(s,u)\|ds \\ &\quad + \int_0^t \|(e^{-hA} - I)A^\alpha e^{-(t-s)A}\|\,\|f_n(s,u)\|ds\end{aligned} \tag{2.3}$$

$$+\int_t^{t+h} \|e^{-(t+h-s)A}A^\alpha\| \, \|f_n(s,u)\| ds.$$

Using (H2) we obtain

$$\begin{aligned} \|A^\beta g_n(t+h,u) - A^\beta g_n(t,u)\| &\le L(h^\gamma + \|P^n u(t+h) - P^n u(t)\|_\alpha) \\ &\le L(h^\gamma + \|u(t+h) - u(t)\|_\alpha) \end{aligned} \tag{2.4}$$

and

$$\int_t^{t+h} \|e^{-(t+h-s)A}A^{1+\alpha-\beta}\| \, \|A^\beta g_n(s,u)\| ds \le \frac{(L\tilde{R}+B)C_{1+\alpha-\beta}h^{\beta-\alpha}}{\beta-\alpha}, \tag{2.5}$$

since

$$\begin{aligned} \|A^\beta g_n(s,u)\| &\le \|A^\beta g_n(s,u) - A^\beta g(s,\phi)\| + \|A^\beta g(s,\phi)\| \\ &\le L\|P^n u(s) - \phi\|_\alpha + B \le L\tilde{R} + B \end{aligned} \tag{2.6}$$

and

$$\int_t^{t+h} \|e^{-(t+h-s)A}A^\alpha\| \, \|f_n(s,u)\| ds \le \frac{C_\alpha F_{\tilde{R}}(T_0)h^{1-\alpha}}{1-\alpha}. \tag{2.7}$$

Part (d) of Theorem 2.6.13 in Pazy [14] implies that for $0 < \vartheta \le 1$ and $x \in D(A^\vartheta)$,

$$\|(e^{-tA} - I)x\| \le C'_\vartheta t^\vartheta \|x\|_\vartheta. \tag{2.8}$$

Let ϑ be a real number with $0 < \vartheta < \min\{1-\alpha, \beta-\alpha\}$, then $A^\alpha y \in D(A^\vartheta)$ for any $y \in D(A^{\alpha+\vartheta})$. For all $t, s \in [0,T]$, $t \ge s$ and $0 < h < 1$, we get the following inequalities:

$$\|(e^{-hA} - I)A^\alpha e^{-tA}\| \le C'_\vartheta h^\vartheta \|A^{\alpha+\vartheta}e^{-tA}\| \le \frac{\tilde{C}h^\vartheta}{t^{\alpha+\vartheta}}, \tag{2.9}$$

$$\|(e^{-hA} - I)A^\alpha e^{-(t-s)A}\| \le \frac{\tilde{C}h^\vartheta}{(t-s)^{\alpha+\vartheta}}, \tag{2.10}$$

$$\|(e^{-hA} - I)A^{1+\alpha-\beta} e^{-(t-s)A}\| \le \frac{\tilde{C}h^\vartheta}{t^{1+\alpha+\vartheta-\beta}}, \tag{2.11}$$

where $\tilde{C} = C'_\vartheta \max\{C_{\alpha+\vartheta}, C_{1+\alpha+\vartheta-\beta}\}$. Using the estimates (2.6), (2.10) and (2.11), we get

$$\int_0^t \|(e^{-hA} - I)A^{1+\alpha-\beta} e^{-(t-s)A}\| \, \|A^\beta g_n(s,u)\| ds \le \tilde{C}h^\vartheta (L\tilde{R}+B)\frac{T_0^{\beta-(\alpha+\vartheta)}}{\beta-(\alpha+\vartheta)} \tag{2.12}$$

and

$$\int_0^t \|(e^{-hA} - I)A^\alpha e^{-(t-s)A}\| \, \|f_n(s,u)\| ds \le \tilde{C}h^\vartheta F_{\tilde{R}}(T_0)\frac{T_0^{1-(\alpha+\vartheta)}}{1-(\alpha+\vartheta)}. \tag{2.13}$$

From the inequalities (2.3), (2.4), (2.5), (2.7), (2.9), (2.12) and (2.13), it follows that $(S_n u)(t)$ is continuous from $[0,T]$ into X_α with respect to the norm $\|.\|_\alpha$. Now, we show $S_n u \in B_R(X_\alpha(T), \tilde{\phi})$. Consider

$$\begin{aligned} &\|(S_n u)(t) - \phi\|_\alpha \\ &\le \|(e^{-tA} - I)A^\alpha(\phi + g_n(0,\tilde{\phi}))\| + \|A^{\alpha-\beta}\| \, \|A^\beta g_n(0,\tilde{\phi}) - A^\beta g_n(t,u)\| \\ &\quad + \int_0^t \|A^{1+\alpha-\beta}e^{-(t-s)A}\| \, \|A^\beta g_n(s,u)\| ds + \int_0^t \|e^{-(t-s)A}A^\alpha\| \, \|f_n(s,u)\| ds \end{aligned}$$

$$\begin{aligned}
&\le (1-\mu)\frac{R}{3} + \|A^{\alpha-\beta}\| L\{T^\gamma + \|u(t)-\phi\|_\alpha\} \\
&\quad + C_{1+\alpha-\beta}(L\tilde{R}+B)\frac{T^{\beta-\alpha}}{\beta-\alpha} + C_\alpha F_{\tilde{R}}(T_0)\frac{T^{1-\alpha}}{1-\alpha} \\
&\le (1-\mu)\frac{R}{3} + (1-\mu)\frac{R}{6} + \mu R \le R.
\end{aligned}$$

Taking the supremum over $[0,T]$, we obtain

$$\|S_n u - \tilde{\phi}\|_{X_\alpha(T)} \le R.$$

Hence S_n maps $B_R(X_\alpha(T),\tilde{\phi})$ into $B_R(X_\alpha(T),\tilde{\phi})$. Now we show that S_n is a strict contraction on $B_R(X_\alpha(T),\tilde{\phi})$. For $u,v \in B_R(X_\alpha(T),\tilde{\phi})$, we have

$$\begin{aligned}
&\|(S_n u)(t) - (S_n v)(t)\|_\alpha \\
&\le \|A^{\alpha-\beta}\|\,\|A^\beta g_n(t,u) - A^\beta g_n(t,v)\|_\alpha \\
&\quad + \int_0^t \|A^{1+\alpha-\beta}e^{-(t-s)A}\|\,\|A^\beta g_n(s,u) - A^\beta g_n(s,v)\|ds \\
&\quad + \int_0^t \|e^{-(t-s)A}A^\alpha\|\,\|f_n(s,u) - f_n(s,v)\|ds.
\end{aligned} \tag{2.14}$$

Now,

$$\|A^\beta g_n(t,u) - A^\beta g_n(t,v)\| \le L\|u(t)-v(t)\|_\alpha \le L\|u-v\|_{X_\alpha(T)}. \tag{2.15}$$

Also, we have

$$\|f_n(s,u) - f_n(s,v)\| \le F_{\tilde{R}}(T_0)\|u(s)-v(s)\|_\alpha \le F_{\tilde{R}}(T_0)\|u-v\|_{X_\alpha(T)}. \tag{2.16}$$

Using (2.15) and (2.16) in (2.14) and taking supremum over $[0,T]$, we get

$$\|S_n u - S_n v\|_{X_\alpha(T)} \le \big(\|A^{\alpha-\beta}\|L + C_{1+\alpha-\beta}L\frac{T^{\beta-\alpha}}{\beta-\alpha} + C_\alpha F_{\tilde{R}}(T_0)\frac{T^{1-\alpha}}{1-\alpha}\big)\|u-v\|_{X_\alpha(T)}.$$

The above estimate and the definition of T imply that S_n is a strict contraction on $B_R(X_\alpha(T),\tilde{\phi})$. Hence there exists a unique $u_n \in B_R(X_\alpha(T),\tilde{\phi})$ such that $S_n u_n = u_n$. Clearly u_n satisfies (2.2). This completes the proof of the proposition. $\square$

Proposition 2.2. *Let (H1) and (H2) hold. If $\phi \in D(A^\alpha)$ then $u_n(t) \in D(A^\vartheta)$ for all $t \in (0,T]$ where $0 \le \vartheta \le \beta < 1$. Furthermore, if $\phi \in D(A)$ then $u_n(t) \in D(A^\vartheta)$ for all $t \in [0,T]$ where $0 \le \vartheta \le \beta < 1$.*

Proof. From Proposition 2.1, we have the existence of a unique $u_n \in B_R(X_\alpha(T),\tilde{\phi})$ satisfying (2.2). Part (a) of Theorem 2.6.13 in Pazy [14] implies that for $t > 0$ and $0 \le \vartheta < 1$, $e^{-tA} : H \to D(A^\vartheta)$ and for $0 \le \vartheta \le \beta < 1$, $D(A^\beta) \subseteq D(A^\vartheta)$. (H2) implies that the map $t \mapsto A^\beta g(t,u_n(t))$ is Hölder continuous on $[0,T]$ with the exponent $\rho = \min\{\gamma,\vartheta\}$ since the Hölder continuity of u_n can be easily established using the similar arguments from (2.3) to (2.13). It follows that (cf. Theorem 4.3.2 in [14])

$$\int_0^t e^{-(t-s)A}A^\beta g_n(s,u_n)ds \in D(A).$$

Also from Theorem 1.2.4 in Pazy [14], we have $e^{-tA}x \in D(A)$ if $x \in D(A)$. The required result follows from these facts and the fact that $D(A) \subseteq D(A^\vartheta)$ for $0 \le \vartheta \le 1$. $\square$

Proposition 2.3. *Let (H1) and (H2) hold. If $\phi \in D(A^\alpha)$ and $t_0 \in (0,T]$ then*

$$\|u_n(t)\|_\vartheta \le U_{t_0}, \quad \alpha < \vartheta < \beta, \quad t \in [t_0,T], \quad n = 1,2,\cdots,$$

for some constant U_{t_0}, dependent of t_0 and

$$\|u_n(t)\|_\vartheta \le U_0, \quad 0 < \vartheta \le \alpha, \quad t \in [0,T], \quad n = 1,2,\cdots,$$

for some constant U_0. Moreover, if $\phi \in D(A^\beta)$, then there exists a constant U_0, such that

$$\|u_n(t)\|_\vartheta \le U_0, \quad 0 < \vartheta < \beta, \quad t \in [0,T], \quad n = 1,2,\cdots.$$

Proof. First, we assume that $\phi \in D(A^\alpha)$. Applying A^ϑ on both the sides of (2.2) and using (iii) of Lemma 1.1, for $t \in [t_0,T]$ and $\alpha < \vartheta < \beta$, we have

$$\begin{aligned}
\|u_n(t)\|_\vartheta \le & \|A^\vartheta e^{-tA}(\phi + g_n(0,\tilde{\phi}))\| + \|A^{\vartheta-\beta}\|\,\|A^\beta g_n(t,u_n)\| \\
& + \int_0^t \|A^{1+\vartheta-\beta} e^{-(t-s)A}\|\,\|A^\beta g_n(s,u_n)\| ds \\
& + \int_0^t \|e^{-(t-s)A} A^\vartheta\|\,\|f_n(s,u_n)\| ds \\
\le & C_\vartheta t_0^{-\vartheta}(\|\phi\| + \|g_n(0,\tilde{\phi})\|) + \|A^{\vartheta-\beta}\|(L\tilde{R} + B) \\
& + \ C_{1+\vartheta-\beta}(L\tilde{R}+B)\frac{T^{\beta-\vartheta}}{\beta-\vartheta} + C_\vartheta F_{\tilde{R}}(T_0)\frac{T^{1-\vartheta}}{1-\vartheta} \le U_{t_0}.
\end{aligned}$$

Again, for $t \in [0,T]$ and $0 < \vartheta \le \alpha$, $\phi \in D(A^\vartheta)$ and

$$\begin{aligned}
\|u_n(t)\|_\vartheta \le & M(\|A^\vartheta \phi\| + \|g_n(0,\tilde{\phi}\|_\vartheta) + \|A^{\vartheta-\beta}\|(L\tilde{R}+B) \\
& + C_{1+\vartheta-\beta}(L\tilde{R}+B)\frac{T^{\beta-\vartheta}}{\beta-\vartheta} + C_\vartheta F_{\tilde{R}}(T_0)\frac{T^{1-\vartheta}}{1-\vartheta} \le U_0.
\end{aligned}$$

Furthermore, If $\phi \in D(A^\beta)$ then $\phi \in D(A^\vartheta)$ for $0 < \vartheta \le \beta$ and we can easily get the required estimate. This completes the proof of the proposition. □

3. Convergence of Solutions

In this section we establish the convergence of the solution $u_n \in X_\alpha(T)$ of the approximate integral equation (2.2). to a unique solution u of (2.1).

Proposition 3.1. *Let (H1) and (H2) hold. If $\phi \in D(A^\alpha)$, then for any $t_0 \in (0,T]$,*

$$\lim_{m\to\infty} \sup_{\{n\ge m,\ t_0\le t\le T\}} \|u_n(t) - u_m(t)\|_\alpha = 0.$$

Proof. Let $0 < \alpha < \vartheta < \beta$. For $n \ge m$, we have

$$\begin{aligned}
\|f_n(t,u_n) - f_m(t,u_m)\| &\le \|f_n(t,u_n) - f_n(t,u_m)\| + \|f_n(t,u_m) - f_m(t,u_m)\| \\
&\le F_{\tilde{R}}(T_0)[\|u_n(t) - u_m(t)\|_\alpha + \|(P^n - P^m)u_m(t)\|_\alpha].
\end{aligned}$$

Also,

$$\|(P^n - P^m)u_m(t)\|_\alpha \le \|A^{\alpha-\vartheta}(P^n - P^m)A^\vartheta u_m(t)\| \le \frac{1}{\lambda_m^{\vartheta-\alpha}}\|A^\vartheta u_m(t)\|.$$

Thus, we have

$$\|f_n(t,u_n) - f_m(t,u_m)\| \le F_{\tilde{R}}(T_0)[\|u_n(t) - u_m(t)\|_\alpha + \frac{1}{\lambda_m^{\vartheta-\alpha}}\|A^\vartheta u_m(t)\|].$$

Similarly

$$\begin{aligned}&\|A^\beta g_n(t,u_n) - A^\beta g_m(t,u_m)\|\\&\le \|A^\beta g_n(t,u_n) - A^\beta g_n(t,u_m)\| + \|A^\beta g_n(t,u_m) - A^\beta g_m(t,u_m)\|\\&\le L[\|u_n(t) - u_m(t)\|_\alpha + \frac{1}{\lambda_m^{\vartheta-\alpha}}\|A^\vartheta u_m(t)\|].\end{aligned}$$

Now, for $0 < t_0' < t_0$, we may write

$$\begin{aligned}&\|u_n(t) - u_m(t)\|_\alpha\\&\le \|e^{-tA}A^\alpha(g_n(0,\tilde{\phi}) - g_m(0,\tilde{\phi}))\| + \|A^{\alpha-\beta}\|\,\|A^\beta g_n(t,u_n) - A^\beta g_m(t,u_m)\|\\&\quad + \Big(\int_0^{t_0'} + \int_{t_0'}^t\Big)\|A^{1+\alpha-\beta}e^{-(t-s)A}\|\,\|A^\beta g_n(s,u_n) - A^\beta g_m(s,u_m)\|ds\\&\quad + \Big(\int_0^{t_0'} + \int_{t_0'}^t\Big)\|A^\alpha e^{-(t-s)A}\|\,\|f_n(s,u_n) - f_m(s,u_m)\|ds.\end{aligned}$$

We estimate the first term as

$$\begin{aligned}\|e^{-tA}A^\alpha(g_n(0,\tilde{\phi}) - g_m(0,\tilde{\phi}))\| &\le M\|A^{\alpha-\beta}\|\,\|A^\beta g(0,P^n\phi) - A^\beta g(0,P^m\phi)\|\\&\le M\|A^{\alpha-\beta}\|L\|(P^n - P^m)A^\alpha\phi\|.\end{aligned}$$

The first and the third integrals are estimated as

$$\begin{gathered}\int_0^{t_0'}\|A^{1+\alpha-\beta}e^{-(t-s)A}\|\,\|A^\beta g_n(s,u_n) - A^\beta g_m(s,u_m)\|ds\\\le 2C_{1+\alpha-\beta}(L\tilde{R} + B)(t_0 - t_0')^{-(1+\alpha-\beta)}t_0',\end{gathered}$$

$$\int_0^{t_0'}\|A^\alpha e^{-(t-s)A}\|\,\|f_n(s,u_n) - f_m(s,u_m)\|ds \le 2C_\alpha F_{\tilde{R}}(T_0)(t_0 - t_0')^{-\alpha}t_0'.$$

For the second and the fourth integrals, we have

$$\begin{aligned}&\int_{t_0'}^t\|A^{1+\alpha-\beta}e^{-(t-s)A}\|\,\|A^\beta g_n(s,u_n) - A^\beta g_m(s,u_m)\|ds\\&\le C_{1+\alpha-\beta}L\int_{t_0'}^t(t-s)^{-(1+\alpha-\beta)}[\|u_n(s) - u_m(s)\|_\alpha + \frac{1}{\lambda_m^{\vartheta-\alpha}}\|A^\vartheta u_m(s)\|]ds\\&\le C_{1+\alpha-\beta}L\Big(\frac{U_{t_0'}T^{\beta-\alpha}}{\lambda_m^{\vartheta-\alpha}(\beta-\alpha)} + \int_{t_0'}^t(t-s)^{-(1+\alpha-\beta)}\|u_n(s) - u_m(s)\|_\alpha ds\Big),\end{aligned}$$

$$\begin{aligned}&\int_{t_0'}^t\|A^\alpha e^{-(t-s)A}\|\,\|f_n(s,u_n) - f_m(s,u_m)\|ds\\&\le C_\alpha F_{\tilde{R}}(T_0)\int_{t_0'}^t(t-s)^{-\alpha}[\|u_n(s) - u_m(s)\|_\alpha + \frac{1}{\lambda_m^{\vartheta-\alpha}}\|A^\vartheta u_m(s)\|]ds\\&\le C_\alpha F_{\tilde{R}}(T_0)\Big(\frac{U_{t_0'}T^{1-\alpha}}{\lambda_m^{\vartheta-\alpha}(1-\alpha)} + \int_{t_0'}^t(t-s)^{-\alpha}\|u_n(s) - u_m(s)\|_\alpha ds\Big).\end{aligned}$$

Therefore,

$$\|u_n(t) - u_m(t)\|_\alpha \le M\|A^{\alpha-\beta}\|L\|(P^n - P^m)A^\alpha\phi\|$$

$$
\begin{aligned}
&+ \|A^{\alpha-\beta}\| L\Big(\|u_n(t) - u_m(t)\|_\alpha + \frac{U_{t_0'}}{\lambda_m^{\vartheta-\alpha}}\Big)\\
&+ 2\Big(\frac{C_{1+\alpha-\beta}(L\tilde{R} + B)}{(t_0 - t_0')^{1+\alpha-\beta}} + \frac{C_\alpha F_{\tilde{R}}(T_0)}{(t_0 - t_0')^\alpha}\Big)t_0' + C_{\alpha,\beta}\frac{U_{t_0'}}{\lambda_m^{\vartheta-\alpha}}\\
&+ \int_{t_0'}^t \Big(\frac{C_\alpha F_{\tilde{R}}(T_0)}{(t-s)^\alpha} + \frac{C_{1+\alpha-\beta}L}{(t-s)^{1+\alpha-\beta}}\Big)\|u_n(s) - u_m(s)\|_\alpha ds,
\end{aligned}
$$

where

$$
C_{\alpha,\beta} = C_\alpha F_{\tilde{R}}(T_0)\frac{T^{1-\alpha}}{1-\alpha} + C_{1+\alpha-\beta}L\frac{T^{\beta-\alpha}}{\beta-\alpha}.
$$

Since $\|A^{\alpha-\beta}\|L < 1$, we have

$$
\begin{aligned}
\|u_n(t) - u_m(t)\|_\alpha \le &\frac{1}{(1 - \|A^{\alpha-\beta}\|L)}\Big\{M\|(P^n - P^m)A^\alpha\phi\| + \|A^{\alpha-\beta}\|L\frac{U_{t_0'}}{\lambda_m^{\vartheta-\alpha}}\\
&+ 2\Big(\frac{C_{1+\alpha-\beta}(L\tilde{R} + B)}{(t_0 - t_0')^{1+\alpha-\beta}} + \frac{C_\alpha F_{\tilde{R}}(T_0)}{(t_0 - t_0')^\alpha}\Big)t_0' + C_{\alpha,\beta}\frac{U_{t_0'}}{\lambda_m^{\vartheta-\alpha}}\\
&+ \int_{t_0'}^t \Big(\frac{C_\alpha F_{\tilde{R}}(T_0)}{(t-s)^\alpha} + \frac{C_{1+\alpha-\beta}L}{(t-s)^{1+\alpha-\beta}}\Big)\|u_n(s) - u_m(s)\|_\alpha ds\Big\}.
\end{aligned}
$$

Lemma 5.6.7 in [14] implies that there exists a constant C such that

$$
\begin{aligned}
&\|u_n(t) - u_m(t)\|_\alpha\\
&\le \frac{1}{(1 - \|A^{\alpha-\beta}\|L)}\Big\{M\|(P^n - P^m)A^\alpha\phi\| + (\|A^{\alpha-\beta}\|L + C_{\alpha,\beta})\frac{U_{t_0'}}{\lambda_m^{\vartheta-\alpha}}\\
&+ 2\Big(\frac{C_{1+\alpha-\beta}(L\tilde{R} + B)}{(t_0 - t_0')^{1+\alpha-\beta}} + \frac{C_\alpha F_{\tilde{R}}(T_0)}{(t_0 - t_0')^\alpha}\Big)t_0'\Big\}C.
\end{aligned}
$$

Taking supremum over $[t_0, T]$ and letting $m \to \infty$, we obtain

$$
\begin{aligned}
&\lim_{m\to\infty} \sup_{\{n\ge m, t\in[t_0,T]\}} \|u_n(t) - u_m(t)\|_\alpha\\
&\le \frac{2}{(1 - \|A^{\alpha-\beta}\|L)}\Big(\frac{C_{1+\alpha-\beta}(L\tilde{R} + B)}{(t_0 - t_0')^{1+\alpha-\beta}} + \frac{C_\alpha F_{\tilde{R}}(T_0)}{(t_0 - t_0')^\alpha}\Big)C.
\end{aligned}
$$

As t_0' is arbitrary, the right hand side may be made as small as desired by taking t_0' sufficiently small. This completes the proof of the proposition. □

Corollary 3.2. *If $\phi \in D(A^\beta)$ then*

$$
\lim_{m\to\infty} \sup_{\{n\ge m,\ 0\le t\le T\}} \|u_n(t) - u_m(t)\|_\alpha = 0.
$$

Proof. Propositions 2.2 and 2.3 imply that in the proof of Proposition 3.1 we may take $t_0 = 0$. □

For the convergence of the solution $u_n(t)$ of the approximate integral equation (2.2) we have the following result.

Theorem 3.3. *Let (H1) and (H2) hold and let $\phi \in D(A^\alpha)$. Then there exists a unique function $u \in X_\alpha(T)$ such that $u_n \to u$ as $n \to \infty$ in $X_\alpha(T)$ and u satisfies (2.1) on $[0, T]$. Furthermore u can be extended to the maximal interval of existence $[0, t_{\max})$, $0 < t_{\max} \le \infty$ satisfying (2.1) on $[0, t_{\max})$ and u is a unique solution to (2.1) on $[0, t_{\max})$.*

Proof. Let us assume that $\phi \in D(A^\alpha)$. Since, for $0 < t \le T$, $A^\alpha u_n(t)$ converges to $A^\alpha u(t)$ as $n \to \infty$ and $u_n(0) = u(0) = \phi$ for all n, we have, for $0 \le t \le T$, $A^\alpha u_n(t)$ converges to $A^\alpha u(t)$ in H as $n \to \infty$. Since $u_n \in B_R(X_\alpha(T), \tilde{\phi})$, it follows that $u \in B_R(X_\alpha(T), \tilde{\phi})$ and for any $0 < t_0 \le T$,

$$\lim_{n\to\infty} \sup_{\{t_0 \le t \le T\}} \|u_n(t) - u(t)\|_\alpha = 0.$$

Also,

$$\sup_{t_0 \le t \le T} \|f_n(t, u_n) - f(t, u(t))\| \le F_{\tilde{R}}(T_0)(\|u_n - u\|_{X_\alpha(T)} + \|(P^n - I)u\|_{X_\alpha(T)}) \to 0$$

as $n \to \infty$ and

$$\sup_{t_0 \le t \le T} \|A^\beta g_n(t, u_n) - A^\beta g(t, u(t))\| \le L(\|u_n - u\|_{X_\alpha(T)} + \|(P^n - I)u\|_{X_\alpha(T)}) \to 0$$

as $n \to \infty$. Now, for $0 < t_0 < t$, we may rewrite (2.2) as

$$\begin{aligned} u_n(t) = e^{-tA}(\phi + g_n(0, \tilde{\phi})) - g_n(t, u_n) + \Big(\int_0^{t_0} + \int_{t_0}^t\Big) Ae^{-(t-s)A} g_n(s, u_n) ds \\ + \Big(\int_0^{t_0} + \int_{t_0}^t\Big) e^{-(t-s)A} f_n(s, u_n) ds \end{aligned}$$

The first and third integrals are estimated as

$$\begin{aligned} \|\int_0^{t_0} Ae^{-(t-s)A} g_n(s, u) ds\| &\le \int_0^{t_0} \|A^{1-\beta} e^{-(t-s)A}\| \, \|A^\beta g_n(s, u_n)\| ds \\ &\le C_{1-\beta}(L\tilde{R} + B) T^{1-\beta} t_0, \end{aligned}$$

$$\|\int_0^{t_0} e^{-(t-s)A} f_n(s, u_n) ds\| \le M F_{\tilde{R}}(T_0) t_0.$$

Thus, we have

$$\begin{aligned} &\Big\| u_n(t) - e^{-tA}(\phi + g_n(0, \tilde{\phi})) + g_n(t, u_n) \\ &- \int_{t_0}^t Ae^{-(t-s)A} g_n(s, u_n) ds - \int_{t_0}^t e^{-(t-s)A} f_n(s, u_n) ds \Big\| \\ &\le (C_{1-\beta}(L\tilde{R} + B) T^{1-\beta} + M F_{\tilde{R}}(T_0)) t_0. \end{aligned}$$

Letting $n \to \infty$ in the above inequality, we get

$$\begin{aligned} &\Big\| u(t) - e^{-tA}(\phi + g(0, \phi)) + g(t, u(t)) \\ &- \int_{t_0}^t Ae^{-(t-s)A} g(s, u(s)) ds - \int_{t_0}^t e^{-(t-s)A} f(s, u(s)) ds \Big\| \\ &\le (C_{1-\beta}(L\tilde{R} + B) T^{1-\beta} + M F_{\tilde{R}}(T_0)) t_0. \end{aligned}$$

Since $0 < t_0 \le T$ is arbitrary, we obtain that u satisfies the integral equation (2.1).

If u satisfies (2.1) on $[0, T_1]$ for some $0 < T_1 \le T_0$, then we show that, u can be extended further. Since $0 < T_0 < \infty$, was arbitrary, we assume that $0 < T_1 < T_0$. We consider the equation

$$\begin{gathered} \frac{d}{dt}(w(t) + G(t, w(t))) + Aw(t) = F(t, w(t)), \quad 0 \le t \le T_0 < \infty, \\ w(0) = u(T_1), \end{gathered}$$

where, $F, G : [0, T_0 - T_1] \times D(A^\alpha) \to H$ are defined by

$$F(t,x) = f(t + T_1, x), \quad G(t,x) = g(t + T_1, x),$$

for $(t,x) \in [0, T_0 - T_1] \times D(A^\alpha)$. We note that F and G satisfy (H2), where T_0 is replaced by $T_0 - T_1$. Hence, there exists a unique $w \in C([0,T_2], D(A^\alpha))$ for some $0 < T_2 < T_0 - T_1$ satisfying the integral equation

$$\begin{aligned} w(t) &= e^{-tA}(u(T_1) + G(0, u(T_1)) - G(t, w(t)) \\ &\quad + \int_0^t Ae^{-(t-s)A} G(s, w(s)) ds + \int_0^t e^{-(t-s)A} F(s, w(s)) ds, \quad 0 \le t \le T_2. \end{aligned}$$

We define

$$\tilde{u}(t) = \begin{cases} u(t), & 0 \le t \le T_1, \\ w(t - T_1), & T_1 \le t \le T_1 + T_2. \end{cases}$$

Then $\tilde{u}$ satisfies the integral equation

$$\begin{aligned} \tilde{u}(t) &= e^{-tA}(\phi + g(0,\phi)) - g(t, \tilde{u}(t)) + \int_0^t Ae^{-(t-s)A} g(s, \tilde{u}(s)) ds \\ &\quad + \int_0^t e^{-(t-s)A} f(s, \tilde{u}(s)) ds, \quad 0 \le t \le T_1 + T_2. \end{aligned} \tag{3.1}$$

To see this, we need to verify (3.1) only on $[T_1, T_1 + T_2]$. For $t \in [T_1, T_1 + T_2]$,

$$\begin{aligned} \tilde{u}(t) &= w(t - T_1) \\ &= e^{-(t-T_1)A}(u(T_1) + G(0, u(T_1))) - G(t - T_1, w(t - T_1)) \\ &\quad + \int_0^{t-T_1} Ae^{-(t-T_1-s)A} G(s, w(s)) ds + \int_0^{t-T_1} e^{-(t-T_1-s)A} F(s, w(s)) ds. \end{aligned}$$

Putting $T_1 + s = \eta$, we get

$$\begin{aligned} \tilde{u}(t) &= e^{-(t-T_1)A}(\{e^{-T_1 A}(\phi + g(0,\phi)) - g(T_1, u(T_1)) \\ &\quad + \int_0^{T_1} Ae^{-(T_1-s)A} g(s, u(s)) ds + \int_0^{T_1} e^{-(T_1-s)A} f(s, u(s)) ds\} \\ &\quad + G(0, u(T_1))) - G(t - T_1, w(t - T_1)) \\ &\quad + \int_{T_1}^t Ae^{-(t-\eta)A} G(\eta - T_1, w(\eta - T_1)) d\eta \\ &\quad + \int_{T_1}^t e^{-(t-\eta)A} F(\eta - T_1, w(\eta - T_1)) ds \\ &= e^{-tA}(\phi + g(0,\phi)) - g(t, w(t - T_1)) + \int_0^{T_1} Ae^{-(t-s)A} g(s, u(s)) ds \\ &\quad + \int_{T_1}^t Ae^{-(t-s)A} g(s, w(s - T_1)) ds + \int_0^{T_1} e^{-(t-s)A} f(s, u(s)) ds \\ &\quad + \int_{T_1}^t e^{-(t-s)A} f(s, w(s - T_1)) ds, \end{aligned}$$

as $G(0,u(T_1)) = g(T_1,u(T_1))$, $G(t-T_1,w(t-T_1)) = g(t,w(t-T_1))$ and $F(t-T_1,w(t-T_1)) = f(t,w(t-T_1))$. Hence, we have

$$\begin{aligned}\tilde{u}(t) = e^{-tA}(\phi + g(0,\phi)) - g(t,\tilde{u}(t)) + \int_0^t Ae^{-(t-s)A} g(s,\tilde{u}(s))ds \\ + \int_0^t e^{-(t-s)A} f(s,\tilde{u}(s))ds,\end{aligned}$$

for $t \in [0, T_1 + T_2]$. Thus, we see $\tilde{u}(t)$ satisfy (3.1) on $[0, T_1 + T_2]$. hence, we may extend $u(t)$ to maximal interval $[0, t_{\max})$ satisfying (3.1) on $[0, t_{\max})$ with $0 < t_{\max} \le \infty$.

Now, we show the uniqueness of solutions to (2.1). Let u_1 and u_2 be two solutions to (2.1) on some interval $[0, T_3]$, where T_3 be any number such that $0 < T_3 < t_{\max}$. Then, for $0 \le t \le T_3$, we have

$$\begin{aligned}\|u_1(t) - u_2(t)\|_\alpha &\le \|A^{\alpha-\beta}\| \, \|A^\beta g(t,u_1(t)) - A^\beta g(t,u_2(t))\| \\ &\quad + \int_0^t \|A^{1+\alpha-\beta} e^{-(t-s)A}\| \, \|A^\beta g(s,u_1(s)) - A^\beta g(s,u_2(s))\| ds \\ &\quad + \int_0^t \|e^{-(t-s)A} A^\alpha\| \, \|f(s,u_1(s)) - f(s,u_2(s))\| ds \\ &\le \|A^{\alpha-\beta}\| L \|u_1(t) - u_2(t)\|_\alpha \\ &\quad + C_{1+\alpha-\beta} L \int_0^t (t-s)^{-(1+\alpha-\beta)} \|u_1(s) - u_2(s)\|_\alpha ds \\ &\quad + C_\alpha F_{\tilde{R}}(T_3) \int_0^t (t-s)^{-\alpha} \|u_1(s) - u_2(s)\|_\alpha ds.\end{aligned}$$

Since, $\|A^{\alpha-\beta}\| L < 1$, we have

$$\begin{aligned}&\|u_1(t) - u_2(t)\|_\alpha \\ &\le \frac{1}{(1 - \|A^{\alpha-\beta}\| L)} \int_0^t \Big(\frac{C_{1+\alpha-\beta} L}{(t-s)^{1+\alpha-\beta}} + \frac{C_\alpha F_{\tilde{R}}(T_3)}{(t-s)^\alpha} \Big) \|u_1(s) - u_2(s)\|_\alpha ds.\end{aligned}$$

Using Lemma 5.6.7 in Pazy [14], we get

$$\|u_1(t) - u_2(t)\|_\alpha = 0$$

for all $0 \le t \le T_3$. From the fact that

$$\|u_1(t) - u_2(t)\| \le \frac{1}{\lambda_0^\alpha} \|u_1(t) - u_2(t)\|_\alpha,$$

it follows that $u_1 = u_2$ on $[0, T_3]$. Since $0 < T_3 < t_{\max}$ was arbitrary, we have $u_1 = u_2$ on $[0, t_{\max})$. This completes the proof of the theorem. □

4. Faedo-Galerkin Approximations

For any $0 < T < t_{\max}$, we have a unique $u \in X_\alpha(T)$ satisfying the integral equation

$$\begin{aligned}u(t) = e^{-tA}(\phi + g(0,\phi)) - g(t,u(t)) + \int_0^t Ae^{-(t-s)A} g(s,u(s))ds \\ + \int_0^t e^{-(t-s)A} f(s,u(s))ds.\end{aligned}$$

Also, we have a unique solution $u_n \in X_\alpha(T)$ of the approximate integral equation

$$u_n(t) = e^{-tA}(\phi + g_n(0,\tilde{\phi})) - g_n(t,u_n) + \int_0^t Ae^{-(t-s)A} g_n(s,u_n)ds + \int_0^t e^{-(t-s)A} f_n(s,u_n)ds.$$

If we project (4.1) onto H_n, we get the Faedo-Galerkin approximation $\hat{u}_n(t) = P^n u_n(t)$ satisfying

$$\begin{aligned} \hat{u}_n(t) &= e^{-tA}(P^n\phi + P^n g(0, P^n\phi)) - P^n g(t, \hat{u}_n(t)) \\ &\quad + \int_0^t Ae^{-(t-s)A} P^n g(s,\hat{u}_n(s))ds + \int_0^t e^{-(t-s)A} P^n f(s,\hat{u}_n(s))ds \end{aligned} \tag{4.1}$$

The solution u of (4.1) and $\hat{u}_n$ of (4.1), have the representation

$$u(t) = \sum_{i=0}^{\infty} \alpha_i(t) u_i, \quad \alpha_i(t) = (u(t), u_i), \quad i = 0,1,\dots; \tag{4.2}$$

$$\hat{u}_n(t) = \sum_{i=0}^{n} \alpha_i^n(t) u_i, \quad \alpha_i^n(t) = (\hat{u}_n(t), u_i), \quad i = 0,1,\dots; \tag{4.3}$$

Using (4.3) in (4.1), we get the following system of first order ordinary differential equations

$$\begin{gathered} \frac{d}{dt}\left(\alpha_i^n(t) + G_i^n(t, \alpha_0^n(t), \dots, \alpha_n^n(t))\right) + \lambda_i \alpha_i^n(t) = F_i^n(t, \alpha_0^n(t), \dots, \alpha_n^n(t)), \\ \alpha_i^n(0) = \phi_i, \end{gathered} \tag{4.4}$$

where

$$G_i^n(t, \alpha_0^n(t), \dots, \alpha_n^n(t)) = \big(g(t, \sum_{i=0}^{n} \alpha_i^n(t) u_i), u_i\big),$$
$$F_i^n(t, \alpha_0^n(t), \dots, \alpha_n^n(t)) = \big(f(t, \sum_{i=0}^{n} \alpha_i^n(t) u_i), u_i\big),$$

and $\phi_i = (\phi, u_i)$ for $i = 1, 2, \dots n$.

The system (4.4) determines the $\alpha_i^n(t)$'s. Now, we shall show the convergence of $\alpha_i^n(t) \to \alpha_i(t)$. It can easily be checked that

$$A^\alpha[u(t) - \hat{u}(t)] = A^\alpha\Big[\sum_{i=0}^{\infty}(\alpha_i(t) - \alpha_i^n(t))u_i\Big] = \sum_{i=0}^{\infty} \lambda_i^\alpha(\alpha_i(t) - \alpha_i^n(t))u_i.$$

Thus, we have

$$\|A^\alpha[u(t) - \hat{u}(t)]\|^2 \geq \sum_{i=0}^{n} \lambda_i^{2\alpha}(\alpha_i(t) - \alpha_i^n(t))^2.$$

We have the following convergence theorem.

Theorem 4.1. *Let (H1) and (H2) hold. Then we have the following.*

(a) *If $\phi \in D(A^\alpha)$, then for any $0 < t_0 \leq T$,*

$$\lim_{n\to\infty} \sup_{t_0 \leq t \leq T} \Big[\sum_{i=0}^{n} \lambda_i^{2\alpha}(\alpha_i(t) - \alpha_i^n(t))^2\Big] = 0.$$

(b) *If* $\phi \in D(A^\beta)$*, then*

$$\lim_{n\to\infty} \sup_{0\le t\le T} \Big[\sum_{i=0}^{n} \lambda_i^{2\alpha} (\alpha_i(t) - \alpha_i^n(t))^2 \Big] = 0.$$

The assertion of this theorem follows from the facts mentioned above and the following result.

Proposition 4.2. *Let (H1) and (H2) hold and let* T *be any number such that* $0 < T < t_{\max}$*, then we have the following.*

(a) *If* $\phi \in D(A^\alpha)$*, then for any* $0 < t_0 \le T$*,*

$$\lim_{n\to\infty} \sup_{\{n\ge m, t_0\le t\le T\}} \|A^\alpha[\hat{u}_n(t) - \hat{u}_m(t)]\| = 0.$$

(b) *If* $\phi \in D(A^\beta)$*, then*

$$\lim_{n\to\infty} \sup_{\{n\ge m, 0\le t\le T\}} \|A^\alpha[\hat{u}_n(t) - \hat{u}_m(t)]\| = 0.$$

Proof. For $n \ge m$, we have

$$\begin{aligned} \|A^\alpha[\hat{u}_n(t) - \hat{u}_m(t)]\| &= \|A^\alpha[P^n u_n(t) - P^m u_m(t)]\| \\ &\le \|P^n[u_n(t) - u_m(t)]\|_\alpha + \|(P^n - P^m)u_m\|_\alpha \\ &\le \|u_n(t) - u_m(t)\|_\alpha + \frac{1}{\lambda_m^{\vartheta-\alpha}} \|A^\vartheta u_m\|. \end{aligned}$$

If $\phi \in D(A^\alpha)$ then the result in (a) follows from Proposition 3.1. If $\phi \in D(A^\beta)$, (b) follows from Corollary 3.2. $\square$

5. Applications

In this section we give some applications of the results established in the earlier sections. Consider the initial boundary value problem

$$\begin{gathered} \frac{\partial}{\partial t}(w(x,t) - \Delta w(x,t)) + \Delta^2 w(x,t) = h(x,t,w(x,t)), \\ w(x,0) = w_0(x), \quad x \in \Omega, \end{gathered} \tag{5.1}$$

with the homogeneous boundary conditions where Ω is a bounded domain in the $\mathbb{R}^N$ with the sufficiently smooth boundary $\partial\Omega$ and Δ is N-dimensional Laplacian. The nonlinear function h is sufficiently smooth in all its arguments.

Let $X = L^2(\Omega)$ and define the operator A by

$$D(A) = H_0^1(\Omega) \cap H^2(\Omega), \quad Au = -\Delta u, \quad u \in D(A),$$

then we can reformulate (5.1) in the abstract form

$$\begin{gathered} \frac{d}{dt}(u(t) + Au(t)) + A^2 u(t) = h(t, u(t)), \\ u(0) = w_0. \end{gathered} \tag{5.2}$$

The operator A is not invertible but for $c > 0$ large enough $(A + cI)$ is invertible and $\|(A + cI)^{-1}\| \le C$. Therefore, we can write (5.2) as a Sobolev type evolution equation of the form (1.1) where

$$g(t, u) = (1 - c)(A + cI)^{-1} u$$

and

$$f(t,u) = cA(A+cI)^{-1}u + h(t,(A+cI)^{-1}u).$$

We see that the operator A satisfies (H1). Also we can easily check that g and f satisfy (H2). Thus, we may apply the results of the earlier sections to guarantee the existence of Faedo-Galerkin approximations and their convergence to the unique solution of (5.1).

A particular example of (5.1) is the meta-parabolic (cf. Carroll and Showalter [5], Showalter [19] and Brown [4]) problem

$$\begin{gathered} \frac{\partial}{\partial t}\Big(u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2}\Big) + \frac{\partial^4 u(x,t)}{\partial x^4} = f(x,t,u(x,t)), \quad 0<x<1, \\ u(0,t) = u(1,t) = \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(1,t) = 0, \quad t>0, \\ u(x,0) = u_0(x), \quad 0<x<1. \end{gathered} \tag{5.3}$$

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