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SYNCHRONIZATION OF NONAUTONOMOUS DYNAMICAL SYSTEMS

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Abstract. The synchronization of two nonautonomous dynamical systems is considered, where the systems are described in terms of a skew-product formalism, i. e., in which an inputed autonomous driving system governs the evolution of the vector field of a differential equation with the passage of time. It is shown that the coupled trajectories converge to each other as time increases for sufficiently large coupling coefficient and also that the component sets of the pullback attractor of the coupled system converges upper semi continuously as the coupling parameter increases to the diagonal of the product of the corresponding component sets of the pullback attractor of a system generated by the average of the vector fields of the original uncoupled systems.

1. Introduction

Synchronization of coupled dissipative systems is a well known phenomenon in biology and physics. It has been investigated mathematically in the case of autonomous systems by Rodrigues and his coauthors [1, 2, 6], who not only show that the coupled trajectories converge to each other as time increases for sufficiently large coupling coefficient but also that the global attractor of the coupled system converges upper semi continuously as the coupling parameter increases to the diagonal of the product of the global attractor of a system generated by the average of the vector fields of the original uncoupled systems. An important property of the systems here is their ultimate boundedness or dissipativity, which can be characterized through Lyapunov functions.

Afraimovich and Rodrigues [1] also considered the synchronization of nonautonomous systems. They could show that the coupled trajectories converge to each other with increasing time, but did not say anything about the attractors of the systems under consideration. Here we use a new concept of pullback attractor for nonautonomous systems [3] to make analogous statements about the attractors of the coupled system and limiting averaged system in the nonautonomous case. For this we use the skew-product formalism of the differential equations, that is, with an inputed autonomous driving system governing the evolution of the vector field of the system with the passage of time. Such a formalism is typical, for example, for

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almost periodic differential equations, for which the driving system is the shift operator on a set of admissible vector fields called the hull of the differential equation [4, 7, 8, 9]. In order to focus on the structure of the nonautonomous attractors we assume a simple uniform global dissipativity condition of the differential equations. Technical generalizations as in [1, 2, 6] are possible.

The paper is structured as follows. In the next section we recall the basic ideas on nonautonomous dynamical systems and their attractors. Then in Section 3 we formulate our main results on the synchronization of nonautonomous dynamical systems generated by ordinary differential equations. We present a simple example in terms of scalar differential equations in Section 4, with some additional remarks on generalizations to infinite dimensional reaction diffusion equations. Our proofs of the theorems formulated in section 3 are then given in the remaining sections of the paper.

We need some notation. Let H^*_Z denote the Hausdorff distance (semi-metric) between two nonempty sets of a complete metric space (Z, d_Z), that is

$$H^*_Z(A,B) := \sup_{a\in A} \operatorname{dist}_Z(a,B),$$

where $\operatorname{dist}_Z(a,B) = \inf_{b\in B} d_Z(a,b)$, and let

$$H_Z(A,B) = \max\{H^*_Z(A,B), H^*_Z(B,A)\}$$

be the Hausdorff metric on the space $\mathcal{K}(Z)$ of nonempty compact subsets of (Z, d_Z). Finally, let $B_Z[\bar{z}, R] := \{z \in Z : d_Z(z,\bar{z}) \le R\}$ be the closed ball in Z centered on $\bar{z}$ with radius R. When $Z = \mathbb{R}^d$ we will write $B_Z[\bar{z}, R]$ as $B_d[\bar{z}, R]$.

2. Attractors of nonautonomous dynamical systems

Following [3] and the papers cited therein, we define a nonautonomous dynamical system (θ, ϕ) in terms of a cocycle mapping ϕ on a state space X which is driven by an autonomous dynamical system θ acting on a base or parameter space P. Here we assume that (X, d_X) and (P, d_P) are complete metric spaces.

Specifically, $\theta = \{\theta_t : t \in \mathbb{R}\}$ is a *dynamical system* on P, i.e., a *group* of homeomorphisms under composition on P with the properties that

(1) $\theta_0(p) = p$ for all $p \in P$;
(2) $\theta_{s+t} = \theta_s(\theta_t(p))$ for all s, t in $\mathbb{R}$;
(3) The mapping $(t,p) \mapsto \theta_t(p)$ is continuous,

The *cocycle* mapping $\phi : \mathbb{R}^+ \times P \times X \to X$ satisfies

(1) $\phi(0,p,x) = x$ for all $(p,x) \in P \times X$;
(2) $\phi(s+t,p,x) = \phi(s, \theta_t(p), \phi(t,p,x))$ for all s, t, in $\mathbb{R}^+$, $(p,x) \in P \times X$;
(3) The mapping $(t,p,x) \mapsto \phi(t,p,x)$ is continuous.

A family $\widehat{A} = \{A_p : p \in P\}$ of nonempty compact subsets A_p of X, which is invariant under the the cocycle mapping in the sense that $\phi(t,p,A_p) = A_{\theta_t(p)}$ for all $t \ge 0$ and which is pullback attracting in the sense that

$$\lim_{t\to\infty} H^*_X\left(\phi\left(t, \theta_{-t}(p), D\right), A_p\right) = 0 \tag{2.1}$$

for any nonempty bounded subset D of X and $p \in P$ is called a *pullback attractor* of (θ, ϕ). It is called a *forward attractor* if the forward convergence

$$\lim_{t\to\infty} H^*_X(\phi(t,p,D), A_{\theta_t(p)})) = 0 \tag{2.2}$$

holds instead of the pullback convergence (2.1). Obviously, any uniform pullback attractor is also a uniform forward attractor, and vice versa, where uniformity is with respect to $p \in P$. (See [3] for a detailed discussion on such attractors and the relationship of the subset $\mathcal{A} := \cup_{p\in P}\{p\} \times A_p$ of $P \times X$ with a possible global attractor of the autonomous skew–product system π on the product space $Y = P \times X$, i.e., the mapping $\pi : \mathbb{R}^+ \times Y \to Y$ defined by $\pi(t,(p,x)) := (\theta_t(p), \phi(t,p,x))$; the name skew–product is due to the fact that the driving system acts independently of the state space dynamics).

The existence of a uniform pullback attractor follows from the assumed asymptotic compactness of the cocycle mapping and the existence of a uniform absorbing set B, which is a nonempty compact subset of X and uniformly absorbs nonempty bounded subsets D of X, i.e., there exists a $T_D \geq 0$ independent of $p \in P$, such that

$$\phi(t,p,D) \subset B \qquad \text{for all } t \geq T_D.$$

If, in addition, B is *ϕ–positively invariant* i.e., with $\phi(t,p,B) \subset B$ for all $t \geq 0$ and $p \in P$, then the nonautonomous dynamical system (θ, ϕ) has a uniform pullback attractor $\widehat{A} = \{A_p : p \in P\}$ with component sets given by

$$A_p = \bigcap_{t\geq 0} \phi\left(t, \theta_{-t}(p), B\right) \tag{2.3}$$

for each $p \in P$.

3. Dynamics of synchronized systems

Consider two dissipative nonautonomous dynamical systems in $\mathbb{R}^d$, given by

$$\frac{dx}{dt} = f(p,x), \qquad p \in P, \tag{3.1}$$

with driving system $\theta_t : P \to P$, and

$$\frac{dy}{dt} = g(q,x), \quad q \in Q, \tag{3.2}$$

with driving system $\psi_t : Q \to Q$.

Suppose that both systems are sufficiently regular to ensure the forwards existence and uniqueness of solutions, so they generate nonautonomous dynamical systems on $P \times \mathbb{R}^d$ and $Q \times \mathbb{R}^d$, respectively. In particular, suppose that both satisfy a uniform dissipativity condition

$$\big\langle x, f(p,x)\big\rangle \leq K - L|x|^2, \quad p \in P, \quad \big\langle x, g(q,x)\big\rangle \leq K - L|x|^2, \quad q \in Q, \tag{3.3}$$

From these conditions we obtain the differential inequalities

$$\frac{1}{2}\frac{d}{dt}|x(t)|^2 \leq K - L|x(t)|^2, \quad \frac{1}{2}\frac{d}{dt}|y(t)|^2 \leq K - L|y(t)|^2$$

uniformly in $p \in P$ and $q \in Q$, respectively. Thus in both cases the closed ball

$$B_d[0, \sqrt{(K+1)/L}] := \{x \in \mathbb{R}^d \ ; \ |x|^2 \leq (K+1)/L\}$$

is uniformly absorbing, and positively invariant, so both systems have uniform pullback attractors in $\mathbb{R}^d$, respectively

$$\widehat{A}^{(f)} = \{A_p^{(f)} : p \in P\}, \quad \widehat{A}^{(g)} = \{A_q^{(g)} : q \in Q\}.$$

Consider now the dissipatively coupled system

$$\frac{dx}{dt} = f(p,x) + \nu(y-x), \quad \frac{dy}{dt} = g(q,x) + \nu(x-y) \tag{3.4}$$

with the product driving system $(\theta_t, \psi_t) : P \times Q \to P \times Q$. Here $\nu > 0$. By the uniform dissipativity condition (3.3) we have

$$\begin{aligned}
\frac{d}{dt}\left(|x|^2(t) + |y|^2(t)\right) &= 2\langle x(t), \frac{d}{dt}x(t))\rangle + 2\langle y(t), \frac{d}{dt}y(t))\rangle \\
&= 2\langle x(t), f(\theta_t p, x(t))\rangle + 2\langle x(t), \nu(y(t) - x(t)\rangle \\
&\quad +2\langle y(t), g(\psi_t, y(t))\rangle + 2\langle x(t), \nu(x(t) - y(t)\rangle \\
&\le 4K - 2L\left(|x(t)|^2 + |y(t)|^2\right)
\end{aligned}$$

from which it follows that the closed ball

$$B_{2d}[0, \sqrt{(2K+1)/L}] := \{x \in \mathbb{R}^{2d} \ ; \ |x|^2 \le (2K+1)/L\}$$

in $\mathbb{R}^{2d}$ is a uniform absorbing and positively invariant for the coupled system (3.4), so the coupled system (3.4) has a uniform pullback attractor in $\mathbb{R}^{2d}$ for each $\nu > 0$ which will be denoted by

$$\widehat{A}^{(\nu)} = \{A^{(\nu)}_{(p,q)} : (p,q) \in P \times Q\}.$$

In addition, writing

$$\left(x^{(\nu)}(t), y^{(\nu)}(t)\right) = \left(x^{(\nu)}(t,p,q,x_0,y_0), y^{(\nu)}(t,p,q,x_0,y_0)\right)$$

for the solution of the coupled system (3.4) with initial parameter value (p,q) and initial state (x_0, y_0) we obtain

Theorem 3.1. *For all finite $T_2 \ge T_1 > 0$, all $(x_0,y_0) \in B_{2d}[0, \sqrt{(2K+1)/L}]$ and all (p,q) we have*

$$\lim_{\nu\to\infty} \left|x^{(\nu)}(t) - y^{(\nu)}(t)\right| = 0 \quad \textit{uniformly in} \quad t \in [T_1, T_2] \tag{3.5}$$

The proof will be given in Section 5.

From this and the fact that a pullback attractor consists of the entire trajectories of a system it follows the statement of the next theorem.

Theorem 3.2. *Let* $\operatorname{Diag}\left(\mathbb{R}^d \times \mathbb{R}^d\right) = \left\{(x,x) \ \ x \in \mathbb{R}^d\right\}$. *Then*

$$\lim_{\nu\to\infty} H^*_{2d}\left(A^{(\nu)}_{(p,q)}, \operatorname{Diag}\left(\mathbb{R}^d \times \mathbb{R}^d\right) \bigcap B_{2d}[0, \sqrt{(2K+1)/L}]\right) = 0. \tag{3.6}$$

The proof will be given in Section 6.

In fact, we can say much more about the dynamics inside the pullback attractor $\widehat{A}^{(\nu)}$ and what happens as $\nu \to \infty$. Let

$$\left(x^{(\nu)}(t), y^{(\nu)}(t)\right) = \left(x^{(\nu)}(t,p,q,x_0,y_0), y^{(\nu)}(t,p,q,x_0,y_0)\right)$$

be an entire trajectory of the coupled system inside the pullback attractor $\widehat{A}^{(\nu)} = \{A^{(\nu)}_{(p,q)} : (p,q) \in P \times Q\}$, so

$$\left(x^{(\nu)}(t), y^{(\nu)}(t)\right) \in A^{(\nu)}_{(\theta_t p, \psi_t q)} \quad \text{for all} \quad t \in \mathbb{R}.$$

Theorem 3.3. *For any entire trajectory $\left(x^{(\nu)}(t), y^{(\nu)}(t)\right)$ of the coupled system inside the pullback attractor $\widehat{A}^{(\nu)}$ there exists convergent subsequences*

$$\lim_{\nu'\to\infty} x^{(\nu')}(t) = z(t), \quad \lim_{\nu'\to\infty} y^{(\nu')}(t) = z(t)$$

uniformly on compact time subintervals in $\mathbb{R}$, where $z(t)$ is a solution of the nonautonomous differential equation

$$\frac{dz}{dt} = \frac{1}{2}\left(f(p,z) + g(q,z)\right) \tag{3.7}$$

with the product driving system $(\theta_t, \psi_t) : P \times Q \to P \times Q$.

The proof will be given in Section 7.

It follows from the uniform dissipativity condition (3.3) that the closed ball

$$B_d[0, \sqrt{(K+1)/L}] := \{x \in \mathbb{R}^d \ ; \ |x|^2 \leq (K+1)/L\}$$

is uniformly absorbing, and positively invariant for the limiting system (3.7), so the limiting system (3.7) has a uniform pullback attractor

$$\widehat{A}^{(\infty)} = \{A^{(\infty)}_{(p,q)} : (p,q) \in P \times Q\}$$

in $\mathbb{R}^d$. From Theorems 3.2 and 3.3 we thus have the following statement.

Corollary 3.4. *Let* $\mathrm{Diag}\left(A^{(\infty)}_{(p,q)} \times A^{(\infty)}_{(p,q)}\right) = \left\{(x,x) \ ; \ x \in A^{(\infty)}_{(p,q)}\right\}$. *Then*

$$\lim_{\nu\to\infty} H^*_{2d}\left(A^{(\nu)}_{(p,q)}, \mathrm{Diag}\left(A^{(\infty)}_{(p,q)} \times A^{(\infty)}_{(p,q)}\right)\right) = 0. \tag{3.8}$$

The proof will be given in Section 7.

Remark 3.5. Similar results hold for parabolic partial differential equations. This was shown for the autonomous case in [2, 6]. The main difference in the nonautonomous case is the use of nonautonomous pullback attractors as above. For example, consider two nonautonomous reaction-diffusion equations

$$\frac{\partial u}{\partial t} = \Delta u + f(p,u), \quad \frac{\partial v}{\partial t} = \Delta v + g(q,v)$$

on a bounded domain Ω in $\mathbb{R}^n$ and, say, Dirichlet boundary conditions for which the driving systems are, respectively, $\theta_t : P \to P$ and $\psi_t : Q \to Q$. Assuming the same properties of f and g as above, the solutions generate asymptotic compact cocycle mappings in an appropriate Banach space X^α. Similarly, the coupled system

$$\frac{\partial u}{\partial t} = \Delta u + f(p,u) + \nu(v-u), \quad \frac{\partial v}{\partial t} = \Delta v + g(q,v) + \nu(u-v)$$

on the bounded domain Ω in $\mathbb{R}^n$ with the same boundary condition and coupled driving system $(\theta_t, \psi_t) : P \times Q \to P \times Q$ generates an asymptotic compact cocycle mapping in the product Banach space $X^\alpha \times X^\alpha$ with a pullback attractor $\widehat{A}^{(\nu)} = \{A^{(\nu)}_{(p,q)} : (p,q) \in P \times Q\}$, where the components sets $A^{(\nu)}_{(p,q)}$ are nonempty compact subsets of $X^\alpha \times X^\alpha$. These component sets are close to the diagonal product of the corresponding component sets of the pullback attractor $\widehat{A}^{(\nu)} = \{A^{(\infty)}_{(p,q)} : (p,q) \in P \times Q\}$ in X^α of the limiting system

$$\frac{\partial z}{\partial t} = \Delta z + \frac{1}{2}\left(f(p,z) + g(q,z)\right)$$

and appropriate counterparts of the above results in $\mathbb{R}^d$ can be established using similar technical details to those in [1, 2, 6].

4. An Example

Consider the scalar nonautonomous differential equations

$$\frac{dx}{dt} = -x + \alpha(t), \quad \frac{dy}{dt} = -y + \beta(t), \tag{4.1}$$

where α and β are bounded functions. Here the driving systems are defined by the shift operators defined by $\theta_t\alpha(\cdot) = \alpha(\cdot + t)$ and $\psi_t\beta(\cdot) = \beta(\cdot + t)$ for all $t \in \mathbb{R}$ and the base spaces P and Q are, respectively, the closed hulls of the functions α and β as in [4, 8, 9].

We note that both systems are strongly dissipative with

$$|x_1(t) - x_2(t)| \le e^{-t}|x_{0,1} - x_{0,2}|, \quad |y_1(t) - y_2(t)| \le e^{-t}|y_{0,1} - y_{0,2}|$$

for any pair of initial values. Thus both systems have singleton trajectory pullback attractors defined via

$$\bar{x}(t) = e^{-t}\int_{-\infty}^{t} e^s\alpha(s)\,ds, \quad \bar{y}(t) = e^{-t}\int_{-\infty}^{t} e^s\beta(s)\,ds,$$

i.e. with $A_p^{(f)} = \{\bar{x}(0)\}$ and $A_q^{(g)} = \{\bar{y}(0)\}$.

The limiting system

$$\frac{dz}{dt} = -z + \frac{1}{2}\left(\alpha(t) + \beta(t)\right)$$

is also strongly dissipative with a singleton trajectory pullback attractor given by

$$\bar{z}(t) = \frac{1}{2}e^{-t}\int_{-\infty}^{t} e^s\left(\alpha(s) + \beta(s)\right)\,ds = \frac{1}{2}\left(\bar{x}(t) + \bar{y}(t)\right)$$

i.e., the average of $\bar{x}$ and $\bar{y}$ (which is due to the linearity of the equations).

The synchronized system here is

$$\frac{dx}{dt} = -x + \alpha(t) + \nu(y - x), \quad \frac{dy}{dt} = -y + \beta(t) + \nu(x - y)$$

has general solution

$$\begin{pmatrix} x^{(\nu)}(t) \\ y^{(\nu)}(t) \end{pmatrix} = e^{A_\nu(t-t_0)}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \int_{t_0}^{t} e^{A_\nu(t-s)}\begin{pmatrix} \alpha(s) \\ \beta(s) \end{pmatrix}\,ds,$$

where

$$A_\nu = \begin{bmatrix} -1-\nu & -\nu \\ -\nu & -1-\nu \end{bmatrix}, \quad e^{A_\nu t} = \begin{bmatrix} e^{-t} + e^{-(1+\nu)t} & e^{-t} - e^{-(1+\nu)t} \\ e^{-t} - e^{-(1+\nu)t} & e^{-t} + e^{-(1+\nu)t} \end{bmatrix},$$

so

$$\begin{aligned} \begin{pmatrix} x^{(\nu)}(t) \\ y^{(\nu)}(t) \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} e^{-(t-t_0)}(x_0 + y_0) + e^{-(1+\nu)(t-t_0)}(x_0 - y_0) \\ e^{-(t-t_0)}(x_0 + y_0) - e^{-(1+\nu)(t-t_0)}(x_0 - y_0) \end{pmatrix} \\ &\quad + \frac{1}{2}\int_{t_0}^{t}\begin{pmatrix} e^{-(t-s)}(\alpha(s) + \beta(s)) + e^{-(1+\nu)(t-s)}(\alpha(s) - \beta(s)) \\ e^{-(t-s)}(\alpha(s) + \beta(s)) - e^{-(1+\nu)(t-s)}(\alpha(s) - \beta(s)) \end{pmatrix}\,ds, \end{aligned}$$

Taking the pullback convergence limit $t_0 \to -\infty$ we obtain the singleton trajectory in the pullback attractor, namely

$$\begin{pmatrix} x^{(\nu)}(t) \\ y^{(\nu)}(t) \end{pmatrix} = \frac{1}{2}\int_{-\infty}^{t}\begin{pmatrix} e^{-(t-s)}(\alpha(s) + \beta(s)) + e^{-(1+\nu)(t-s)}(\alpha(s) - \beta(s)) \\ e^{-(t-s)}(\alpha(s) + \beta(s)) - e^{-(1+\nu)(t-s)}(\alpha(s) - \beta(s)) \end{pmatrix}\,ds.$$

Thus as $\nu \to \infty$ we obtain

$$\begin{pmatrix} \bar{x}^{(\nu)}(t) \\ \bar{y}^{(\nu)}(t) \end{pmatrix} \quad \to \quad \frac{1}{2}\int_{t_0}^{t} e^{-(t-s)}(\alpha(s)+\beta(s))\,ds \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \bar{z}(t) \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Since the attractors here each consist of a single trajectory we in fact have continuous convergence of the attractors.

5. Proof of Theorem 3.1

As above we consider the solution

$$\left(x^{(\nu)}(t), y^{(\nu)}(t)\right) = \left(x^{(\nu)}(t,p,q,x_0,y_0), y^{(\nu)}(t,p,q,x_0,y_0)\right)$$

of the coupled system (3.4) with initial parameter value $(p,q) \in P \times Q$ and initial state $(x_0, y_0) \in B_{2d}[0, \sqrt{(2K+1)/L}]$. For the remainder of this section the index ν will be omitted. Then, from (3.4), it follows that $U(t) := x(t) - y(t)$ satisfies the equation

$$\frac{d}{dt}U(t) = -2\nu U(t) + f(\theta_t p, x(t)) - g(\psi_t q, y(t)),$$

so

$$U(t) = U(0)e^{-2\nu t} + e^{-2\nu t}\int_0^t e^{2\nu s}\left(f(\theta_s p, x(s)) - g(\psi_s q, y(s))\right) ds$$

and hence

$$|U(t)| \le |U(0)|e^{-2\nu t} + e^{-2\nu t}\int_0^t e^{2\nu s}\left(|f(\theta_s p, x(s))| + |g(\psi_s q, y(s))|\right) ds.$$

Now $B_{2d}[0, \sqrt{(2K+1)/L}]$ is positively invariant for the synchronized system, so if $(x_0, y_0) \in B_{2d}[0, \sqrt{(2K+1)/L}]$, then $x(t) - y(t) \in B_{2d}[0, \sqrt{(2K+1)/L}]$ for all $t \ge 0$. Moreover $B_{2d}[0, \sqrt{(2K+1)/L}]$, P and Q are compact, so by the continuity of f and g there exists a finite constant M such that

$$|f(p,x)| + |g(q,y)| \le M \quad \text{for all} \quad (x,y) \in B_{2d}[0, \sqrt{(2K+1)/L}], p \in P, q \in Q.$$

Thus

$$|U(t)| \le |U(0)|e^{-2\nu t} + e^{-2\nu t}\int_0^t e^{2\nu s} M\,ds.$$

from which it follows that

$$|U(t)| \le |U(0)|e^{-2\nu t} + \frac{M}{2\nu}\left(1 - e^{-2\nu t}\right).$$

Thus, reinserting the index ν,

$$\left|x^{(\nu)}(t) - y^{(\nu)}(t)\right| = |U(t)| \to 0 \quad \text{as} \quad \nu \to \infty$$

for all $t \in (0, T]$ with an arbitrary finite $T > 0$, and hence for any $t \in [T_1, T_2]$ for arbitrary finite $T_2 \ge T_1 > 0$.

6. Proof of Theorem 3.2

Let $\left(x_0^{(\nu)}, y_0^{(\nu)}\right) \in A^{(\nu)}_{(p,q)}$. Then there exists an entire trajectory $\left(x^{(\nu)}(t), y^{(\nu)}(t)\right) \in A^{(\nu)}_{(\theta_t p, \psi_t q)}$ for all $t \in \mathbb{R}$ with $\left(x^{(\nu)}(0), y^{(\nu)}(0)\right) = \left(x_0^{(\nu)}, y_0^{(\nu)}\right)$.

We apply Theorem 3.1 to this trajectory on a time interval $[-1, 1]$, say, i.e. considering it as starting at time -1 with parameters $(\theta_{-1}p, \psi_{-1}q)$ instead of time 0 with parameters (p, q). Thus we have convergence $\left|x^{(\nu)}(t) - y^{(\nu)}(t)\right| \to 0$ as $\nu \to \infty$ on the time interval $t \in [0, 1]$, and in particular at time $t = 0$, that is

$$\left|x_0^{(\nu)} - y_0^{(\nu)}\right| \to 0 \quad \text{as} \quad \nu \to \infty.$$

Since $\left(x_0^{(\nu)}, y_0^{(\nu)}\right) \in B_{2d}[0, \sqrt{(2K+1)/L}]$, which is compact, we have

$$\lim_{\nu\to\infty} H^*_{2d}\left(A^{(\nu)}_{(p,q)}, \operatorname{Diag}\left(\mathbb{R}^d \times \mathbb{R}^d\right) \bigcap B_{2d}[0, \sqrt{(2K+1)/L}]\right) = 0,$$

where $\operatorname{Diag}\left(\mathbb{R}^d \times \mathbb{R}^d\right) = \left\{(x, x)\ ;\ x \in \mathbb{R}^d\right\}$.

7. Proof of Theorem 3.3 and Corollary 3.4

Let $\left(x^{(\nu)}(t), y^{(\nu)}(t)\right)$ be an entire trajectory of the coupled system inside the pullback attractor $\widehat{A}^{(\nu)}$ for each $\nu \geq 0$ with $\left(x^{(\nu)}(0), y^{(\nu)}(0)\right) = \left(x_0^{(\nu)}, y_0^{(\nu)}\right) \in A^{(\nu)}_{(p,q)}$. Define

$$z^{(\nu)}(t) = \frac{1}{2}\left(x^{(\nu)}(t) + y^{(\nu)}(t)\right) \quad \text{for all} \quad t \in \mathbb{R}.$$

Then

$$\begin{aligned}
\frac{d}{dt}z^{(\nu)}(t) &= \frac{1}{2}\left(f\left(\theta_t p, x^{(\nu)}(t)\right) + g\left(\psi_t q, x^{(\nu)}(t)\right)\right) \\
&= \frac{1}{2}\left(f\left(\theta_t p, 2z^{(\nu)}(t) - y^{(\nu)}(t)\right) + g\left(\psi_t q, 2z^{(\nu)}(t) - x^{(\nu)}(t)\right)\right) \qquad (7.1)
\end{aligned}$$

Thus

$$\left|\frac{d}{dt}z^{(\nu)}(t)\right| \leq \frac{1}{2}\left(\left|f\left(\theta_t p, x^{(\nu)}(t)\right)\right| + \left|g\left(\psi_t q, y^{(\nu)}(t)\right)\right|\right) \leq M$$

where M is a finite bound on $|f(p, x)| + |g(q, y)|$ on the compact set $P \times Q \times B_{2d}[0, \sqrt{(2K+1)/L}]$. Thus the sequence of functions $z^{(\nu)}$ is equicontinuous on any compact time interval and has a uniformly convergent subsequence on this interval. By a diagonal sequence argument this can be extended to uniform convergence on all time intervals of the form $[-N, N]$. Thus

$$z^{(\nu')}(t) \to z(t) \quad \text{as} \quad \nu' \to \infty \quad \text{for all} \quad t \in \mathbb{R},$$

where $z(t)$ is continuous. Now, by Theorem 3.1,

$$z^{(\nu)}(t) - y^{(\nu)}(t) = \frac{1}{2}\left(x^{(\nu)}(t) - y^{(\nu)}(t)\right) \to 0,$$
$$z^{(\nu)}(t) - x^{(\nu)}(t) = \frac{1}{2}\left(y^{(\nu)}(t) - x^{(\nu)}(t)\right) \to 0$$

as $\nu \to \infty$ for all $t \in \mathbb{R}$. Hence

$$2z^{(\nu')}(t) - y^{(\nu')}(t) \to z(t), \quad 2z^{(\nu')}(t) - x^{(\nu')}(t) \to z(t)$$

as well as
$$x^{(\nu')}(t) \to z(t), \quad y^{(\nu')}(t) \to z(t)$$
as $\nu' \to \infty$ for all $t \in \mathbb{R}$. (There convergences here are in fact uniform on any compact interval in $\mathbb{R}$). Writing the differential equation (7.1) with ν' in integral form, i.e., as
$$\begin{aligned} &z^{(\nu')}(t) \\ &= z^{(\nu')}(t_0) + \frac{1}{2}\int_{t_0}^{t} \Big(f\big(\theta_s p, 2z^{(\nu')}(s) - y^{(\nu')}(s)\big) + g\big(\psi_s q, 2z^{(\nu')}(s) - x^{(\nu')}(s)\big)\Big)\, ds, \end{aligned}$$
by continuity it follows that
$$z(t) = z(t_0) + \frac{1}{2}\int_{t_0}^{t} \left(f\left(\theta_s p, z(s)\right) + g\left(\psi_s q, z(s)\right)\right) ds,$$
i.e., z is a solution of the nonautonomous differential equation (3.7), namely
$$\frac{dz}{dt} = \frac{1}{2}\left(f(p,z) + g(q,z)\right)$$
with the product driving system $(\theta_t, \psi_t) : P \times Q \to P \times Q$.

Finally, the assertion of Corollary 3.4 holds because $z(t)$ constructed above is an entire trajectory of the limiting system (3.7), so belongs to its pullback attractor. Specifically
$$z(t) \in A^{(\infty)}_{(\theta_t p, \psi_t q)} \quad \text{for all} \quad t \in \mathbb{R}.$$
In particular, this means that
$$\lim_{\nu\to\infty} H^*_{2d}\left(A^{(\nu)}_{(p,q)}, \mathrm{Diag}\left(A^{(\infty)}_{(p,q)} \times A^{(\infty)}_{(p,q)}\right)\right) = 0,$$
where $\mathrm{Diag}\left(A^{(\infty)}_{(p,q)} \times A^{(\infty)}_{(p,q)}\right) = \left\{(x,x)\ ;\ x \in A^{(\infty)}_{(p,q)}\right\}$.

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