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OSCILLATION FOR EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS AND DISTRIBUTED DELAY II: APPLICATIONS

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ABSTRACT. We apply the results of our previous paper "Oscillation of equations with positive and negative coefficients and distributed delay I: General results" to the study of oscillation properties of equations with several delays and positive and negative coefficients

$$\dot{x}(t)+\sum_{k=1}^{n} a_k(t)x(h_k(t))-\sum_{l=1}^{m} b_l(t)x(g_l(t))=0, \quad a_k(t)\geq 0, b_l(t)\geq 0,$$

integrodifferential equations with oscillating kernels and mixed equations combining two above equations. Comparison theorems, explicit non-oscillation and oscillation results are presented.

1. INTRODUCTION

The study of oscillation properties of delay differential equations with positive and negative coefficients began in the eighties [12, 14] and was inspired by the study of equations with oscillating coefficients. This research was later developed in [7, 13], neutral equations with positive and negative coefficients were studied in [10, 15, 17], see also recent publications [1, 11, 20]. In [14, 6, 7] the first order equation with two constant concentrated delays and a positive and a negative coefficient was studied, while paper [19] considered oscillation of integrodifferential equations with oscillatory kernels.

In [6] for the equation

$$\dot{x}(t)+a(t)x(t-\tau)-b(t)x(t-\sigma)=0, \quad t\geq t_0, \tag{1.1}$$

where $a(t)\geq 0,\ b(t)\geq 0$ are continuous functions, $\tau>\sigma>0$, the following result was obtained: Suppose

$$\int_{t-\tau+\sigma}^{t} b(s)ds\leq 1, \quad a(t)\geq b(t-\tau+\sigma), \tag{1.2}$$

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$$\liminf_{t\to\infty} \int_{t-\tau}^{t} [a(s) - b(s-\tau+\sigma)]ds > \frac{1}{e}. \tag{1.3}$$

Then all solutions of (1.1) are oscillatory.

In [17] the inequality (1.3) was improved:

$$\liminf_{t\to\infty} \Big(\int_{t-\tau}^{t} [a(s) - b(s-\tau+\sigma)]ds + \frac{1}{e} \int_{t-\tau+\sigma}^{t} b(s-\tau)ds \Big) > \frac{1}{e}. \tag{1.4}$$

Recently numerous publications on the oscillation of delay equations with positive and negative coefficients have appeared (in addition to [1, 4, 5, 11, 16, 20] see [9] and references therein). However all the publications except [8, 4] consider equations with constant delays only. Paper [4] deals with a more general case when the delays are not constant.

Our previous paper [3] gave a general insight into the problem. In [3] we considered the equation with a distributed delay

$$\dot{x}(t) + \int_0^t x(s)\, d_s R(t,s) - \int_0^t x(s)\, d_s T(t,s) = 0,\ t \geq 0, \tag{1.5}$$

where both $R(t,s)$ and $T(t,s)$ are nondecreasing in s for each t.

As special cases, (1.5) includes delay differential equations with variable concentrated delays, integrodifferential equations and mixed differential equations. The basic result of the paper [3] was the relation between the following properties for (1.5): the existence of a nonoscillatory solution of (1.5), the existence of an eventually positive solution of the corresponding differential inequality and the existence of a nonnegative solution of some nonlinear integral inequality which is explicitly constructed by (1.5). Theorems of this kind are well known and widely applied for delay differential equations with positive coefficients.

In the present paper we apply general results obtained in [3] to specific classes of equations. Section 2 contains preliminaries and relevant results from paper [3]. In Section 3 equations with positive and negative coefficients and several concentrated delays are considered. In Section 4 oscillation and nonoscillation results are deduced for integrodifferential equations with oscillatory kernels. Section 5 deals with mixed equations containing both several concentrated delays and an integral term.

2. Preliminaries and General Results

We consider a scalar delay differential equation (1.5) under the following conditions:

(a1) $R(t,\cdot)$, $T(t,\cdot)$ are left continuous functions of bounded variation and for each s their variations on the segment $[0,s]$

$$P_R(t,s) = var_{\tau\in[0,s]} R(t,\tau),\ P_T(t,s) = var_{\tau\in[0,s]} T(t,\tau) \tag{2.1}$$

are locally integrable functions in t, $R(t,s) = R(t,t+)$, $T(t,s) = T(t,t+)$, $t < s$;

(a2) $R(t,\cdot)$, $T(t,\cdot)$ are nondecreasing functions for each t, $R(t,s) \geq T(t,s)$ for each t,s;

(a3) For each t_1 there exist $s_1 = s(t_1) \leq t_1$, $r_1 = r(t_1) \leq t_1$, such that $R(t,s) = 0$ for $s < s_1$, $t > t_1$, $T(t,s) = 0$ for $s < r_1$, $t > t_1$; in addition, functions $s(t)$, $r(t)$ satisfy

$$\lim_{t\to\infty} s(t) = \infty, \lim_{t\to\infty} r(t) = \infty.$$

If (a3) holds then we can introduce the following functions

$$h(t)=\inf_s \{s : R(t,s)\neq 0\}, \quad g(t)=\inf_s \{s : T(t,s)\neq 0\}, \tag{2.2}$$

such that $\lim_{t\to\infty} h(t)=\infty$, $\lim_{t\to\infty} g(t)=\infty$, and (1.5) can be rewritten as

$$\dot{x}(t)+\int_{h(t)}^t x(s)\, d_sR(t,s)-\int_{g(t)}^t x(s)\, d_sT(t,s)=0,\ t\geq 0. \tag{2.3}$$

If (a2) and (a3) hold, then obviously $h(t)\leq g(t)$.

Together with (2.3) we consider for each $t_0\geq 0$ an initial-value problem

$$\dot{x}(t)+\int_{h(t)}^t x(s)\, d_sR(t,s)-\int_{g(t)}^t x(s)\, d_sT(t,s)=0,\ t\geq t_0. \tag{2.4}$$

$$x(t)=\varphi(t),\ t<t_0,\ x(t_0)=x_0, \tag{2.5}$$

where $\varphi(t)$ is a Borel measurable bounded function.

Definition. An absolutely continuous on each interval $[t_0,c]$ function $x:R\to R$ is called *a solution of problem* (2.4), (2.5), if it satisfies (2.4) for almost all $t\in[t_0,\infty)$ and equalities (2.5) for $t\leq t_0$.

Definition. For each $s\geq 0$ solution $X(t,s)$ of the problem

$$\dot{x}(t)+\int_{h(t)}^t x(s)\, d_sR(t,s)-\int_{g(t)}^t x(s)\, d_sT(t,s)=0,\ x(t)=0,\ t<s,\ x(s)=1, \tag{2.6}$$

is called *a fundamental function* of (2.3).

We assume $X(t,s)=0,\ 0\leq t<s$.

Definition. We will say that equation (2.3) *has a nonoscillatory solution* if for some $t_0,\varphi(t)$ and x_0 the solution of (2.4)-(2.5) is eventually positive or eventually negative. Otherwise all solutions of this equation are *oscillatory.*

Below we present the results obtained in [3] on oscillation of equation (2.3) with a distributed delay.

Consider together with (2.3) the following delay differential inequality

$$\dot{y}(t)+\int_{h(t)}^t y(s)\, d_sR(t,s)-\int_{g(t)}^t y(s)\, d_sT(t,s)\leq 0,\ t\geq 0. \tag{2.7}$$

The following theorem establishes sufficient nonoscillation conditions.

Lemma 2.1. [3] *Suppose (a1)-(a3) hold. Consider the following hypotheses:*

(1) *There exists $t_1\geq 0$ such that for $t\geq t_1$ the following inequality*

$$u(t)\geq\int_{h(t)}^t \exp\Big\{\int_s^t u(\tau)d\tau\Big\}\, d_sR(t,s)-\int_{g(t)}^t \exp\Big\{\int_s^t u(\tau)d\tau\Big\}\, d_sT(t,s) \tag{2.8}$$

has a nonnegative locally integrable solution (we assume $u(t)=0$ for $t<t_1$);

(2) *There exists $t_2\geq 0$ such that $X(t,s)>0,\ t\geq s\geq t_2$;*

(3) *Equation (2.3) has a nonoscillatory solution;*

(4) *Inequality (2.7) has an eventually positive solution.*

Then the implication (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) *is valid.*

Necessary nonoscillation (sufficient oscillation) conditions require some more constraints on R,T. Let

(a4) for any t $R(t,s) - T(t, s-h(t)+g(t))$ is nondecreasing in s and
$$\limsup_{t\to\infty} T(t,t+)[g(t)-h(t)] \le l < 1.$$

Lemma 2.2 ([3])**.** *Suppose (a1)–(a4) hold. Then hypotheses (1)–(4) of Theorem 1 are equivalent.*

To obtain other necessary oscillation conditions we consider the following form of equation (2.3):
$$\dot{x}(t) + \sum_{k=1}^{n} \int_{h_k(t)}^{t} x(s)\, d_s R_k(t,s) - \sum_{l=1}^{m} \int_{g_l(t)}^{t} x(s)\, d_s T_l(t,s) = 0,\ t \ge 0, \tag{2.9}$$
where R_k, T_l, h_k, g_l satisfy the following conditions:

(a1*) $R_k(t,\cdot)$, $T_l(t,\cdot)$ are left continuous functions of bounded variation and for each s their variations on the segment $[0,s]$ $P_{R_k}(t,s)$, $P_{T_l}(t,s)$ are locally integrable functions in t, $R_k(t,s) = R_k(t,t+)$, $T_l(t,s) = T_l(t,t+)$, $t < s$;

(a2*) $R_k(t,\cdot)$, $T_l(t,\cdot)$ are nondecreasing functions for each t, $\sum_k R_k(t,s) \ge \sum_l T_l(t,s)$ for each t, s;

(a3*) For each k, l $\lim_{t\to\infty} h_k(t) = \infty$, $\lim_{t\to\infty} g_l(t) = \infty$.

Denote
$$R(t,s) = \sum_{k=1}^{n} R_k(t,s), T(t,s) = \sum_{l=1}^{m} T_l(t,s), \tag{2.10}$$
$$h(t) = \max_k h_k(t), g(t) = \min_l g_l(t). \tag{2.11}$$

Let us also introduce the following additional constraints:

(add1) $m = n$, $R_k(t,s) \ge T_k(t,s)$ for each t, s, $k = 1, \dots, n$; for any t, k, function $R_k(t,s) - T_k(t, s-h_k(t)+g_k(t))$ is nondecreasing in s and
$$\limsup_{t\to\infty} \sum_{k=1}^{n} T_k(t,t+)[g_k(t) - h_k(t)] < 1.$$

(add2) $h(t) \le g(t), R(t,s) > T(t,s), R(t,s) - T(t, s-h(t)+g(t))$ is nondecreasing in s for any t and
$$\begin{aligned}\limsup_{t\to\infty} \Big[T(t,t+)[g(t)-h(t)] &+ \sum_{k=1}^{n} R_k(t,t+)\big(h(t)-h_k(t)\big) \\ &+ \sum_{l=1}^{m} T_l(t,t+)\big(g_l(t)-g(t)\big)\Big] < 1.\end{aligned} \tag{2.12}$$

Consider together with (2.9) the delay differential inequality
$$\dot{y}(t) + \sum_{k=1}^{n} \int_{h_k(t)}^{t} y(s)\, d_s R_k(t,s) - \sum_{l=1}^{m} \int_{g_l(t)}^{t} y(s)\, d_s T_l(t,s) \le 0,\ t \ge 0. \tag{2.13}$$

The next lemma establishes non-oscillation criteria for (2.9).

Lemma 2.3. [3] *Suppose R_k, T_l, h_k, g_k satisfy (a1*)-(a3*) and at least one of conditions (add1), (add2) hold. Then the following hypotheses are equivalent:*

(1) *There exists* $t_1 \geq 0$ *such that the inequality*

$$\begin{aligned} u(t) \geq \sum_{k=1}^{n} \int_{h_k(t)}^{t} \exp\big\{\int_{s}^{t} u(\tau)d\tau\big\}\, d_s R_k(t,s) \\ - \sum_{l=1}^{m} \int_{g_l(t)}^{t} \exp\big\{\int_{s}^{t} u(\tau)d\tau\big\}\, d_s T_l(t,s), \quad t \geq t_1, \end{aligned} \tag{2.14}$$

has a nonnegative locally integrable solution (we assume $u(t) = 0$ *for* $t < t_1$*);*

(2) *There exists* $t_2 \geq 0$ *such that* $X(t,s) > 0,\ t \geq s \geq t_2$*;*

(3) *Equation (2.9) has a nonoscillatory solution;*

(4) *Inequality (2.13) has an eventually positive solution.*

Lemmas 2.1–2.3 yield the following comparison result. Let us compare the oscillation properties of the equation

$$\dot{x}(t) + \sum_{k=1}^{n} \int_{\tilde{h}_k(t)}^{t} x(s)\, d_s L_k(t,s) - \sum_{l=1}^{m} \int_{\tilde{g}_l(t)}^{t} x(s)\, d_s D_l(t,s) = 0, \tag{2.15}$$

to the oscillation properties of (2.9).

Lemma 2.4 ([3])**.** (1) *If* $(a1^\star) - (a3^\star)$ *and anyone of the conditions (add1), (add2) hold for (2.15) (where* R_k, T_l *are changed by* L_k, D_l*),* $L_k(t,s) \geq R_k(t,s)$*,* $D_l(t,s) \leq T_l(t,s)$ *and (2.15) has a nonoscillatory solution, then (2.9) has a nonoscillatory solution.*

(2) *If* $(a1^\star) - (a3^\star)$ *and any one of the conditions (add1),(add2) hold for (2.9),* $L_k(t,s) \leq R_k(t,s)$*,* $D_l(t,s) \geq T_l(t,s)$ *and all solutions of (2.15) are oscillatory, then all solutions of (2.9) are oscillatory.*

Lemma 2.5 describes the asymptotic behavior of nonoscillatory solutions of (2.9).

Lemma 2.5 ([3])**.** *Suppose* $(a1^\star)$*-*$(a3^\star)$ *and anyone of the following conditions holds:*

(1) *(add1) is satisfied and for some* k

$$\int_0^\infty \big[R_k(t,t+) - T_k(t,t+)\big]dt = \infty; \tag{2.16}$$

(2) *(add2) is satisfied and*

$$\int_0^\infty \big[R(t,t+) - T(t,t+)\big]dt = \infty. \tag{2.17}$$

Then any nonoscillatory solution x *of (2.9) satisfies* $\lim_{t\to\infty} x(t) = 0$.

Consider the following two equations

$$\begin{aligned} \dot{x}(t) + \int_{h(t)}^{t} x(s)\, d_s\big[R(t,s) - T(t,s-h(t)+g(t))\big] \\ + \int_{g(t)}^{t} x(s)\big(\exp\big\{\int_{s+h(t)-g(t)}^{s} [R(\tau,\tau+) - T(\tau,\tau+)]d\tau\big\} - 1\big)\, d_s T(t,s) = 0 \end{aligned} \tag{2.18}$$

and

$$\begin{aligned}&\dot{x}(t)+\int_{g(t)}^{t} x(s)\Big(\exp\Big\{\int_{s-g(t)+h(t)}^{s}[R(\tau,\tau+)-T(\tau,\tau+)]d\tau\Big\}-1\Big)\\ &\times d_sR(t,s-g(t)+h(t))+\int_{g(t)}^{t} x(s)\, d_s\big[R(t,s-g(t)+h(t))-T(t,s)\big]=0.\end{aligned} \tag{2.19}$$

If (a1)-(a4) hold, then equations (2.18),(2.19) contains terms with positive coefficients only. The oscillation properties of these equations will be compared to the properties of (2.3).

Lemma 2.6 ([3]). *Suppose (a1)-(a4) hold for (2.3). If all solutions of either (2.18) or (2.19) are oscillatory, then all solutions of (2.3) are also oscillatory.*

Corollary 2.7. *Suppose (a1)-(a4) hold for (2.3) and at least one of the following four inequalities is satisfied:*

(1) $$\begin{aligned}&\liminf_{t\to\infty}\Big\{\int_{h(t)}^{t}\big[R(t,\tau)-T(t,\tau-h(t)+g(t))\big]d\tau\\ &+\int_{g(t)}^{t} d\tau\int_{h(\tau)}^{\tau}\Big(\exp\Big\{\int_{s+h(t)-g(t)}^{s}[R(u,u+)-T(u,u+)]du\Big\}-1\Big)\, d_sT(t,s)\Big\}>\frac{1}{e}\end{aligned}$$

(2) $$\begin{aligned}&\liminf_{t\to\infty}\Big\{\int_{h(t)}^{t}\big[R(t,\tau+)-T(t,\tau-h(t)+g(t))\big]\, d\tau\\ &\qquad+\int_{g(t)}^{t} d\tau\int_{h(\tau)}^{\tau}\Big(\int_{s+h(t)-g(t)}^{s}[R(u,u+)-T(u,u+)]du\Big)\, dsT(t,s)\Big\}>\frac{1}{e}\end{aligned}$$

(3) $$\begin{aligned}&\liminf_{t\to\infty}\Big\{\int_{g(t)}^{t} d\tau\int_{g(\tau)}^{\tau}\Big(\exp\Big\{\int_{s-g(t)+h(t)}^{s}[R(u,u+)-T(u,u+)]du\Big\}-1\Big)\\ &\qquad\times d_sR(t,s-g(t)+h(t))+\int_{g(t)}^{t}\big[R(t,\tau-g(t)+h(t))-T(t,\tau)\big]\, d\tau\Big\}>\frac{1}{e}\end{aligned}$$

(4) $$\begin{aligned}&\liminf_{t\to\infty}\Big\{\int_{g(t)}^{t} d\tau\int_{g(\tau)}^{\tau}\Big(\int_{s-g(t)+h(t)}^{s}[R(u,u+)-T(u,u+)]du\Big)\\ &\qquad\times d_sR(t,s-g(t)+h(t))+\int_{g(t)}^{t}\big[R(t,\tau-g(t)+h(t))-T(t,\tau)\big]\Big\}>\frac{1}{e}\end{aligned}$$

Then all solutions of (2.3) are oscillatory.

Similar results can be obtained for (2.9).

Lemma 2.8 ([3]). *Suppose R_k, T_l, h_k, g_l satisfy (a1***)-(a3***) and (add1) holds. If all solutions of either*

$$\begin{aligned}&\dot{x}(t)+\sum_{k=1}^{n}\int_{h_k(t)}^{t} x(s)\, d_s\big[R_k(t,s)-T_k(t,s-h_k(t)+g_k(t))\big]\\ &+\sum_{k=1}^{n}\int_{g_k(t)}^{t} x(s)\Big(\exp\Big\{\int_{s+h_k(t)-g_k(t)}^{s}[R_k(\tau,\tau+)-T_k(\tau,\tau+)]d\tau\Big\}-1\Big)\\ &\times d_sT_k(t,s)=0\end{aligned} \tag{2.20}$$

or

$$\begin{aligned}\dot{x}(t) + \sum_{k=1}^{n} \int_{g_k(t)}^{t} x(s)\Big(\exp\Big\{ \int_{s-g_k(t)+h_k(t)}^{s} [R_k(\tau,\tau+) - T_k(\tau,\tau+)] \times d\tau \Big\} - 1\Big) \\ \times d_s R_k(t, s - g_k(t) + h_k(t)) \\ + \sum_{k=1}^{n} \int_{g_k(t)}^{t} x(s)\, d_s \Big[R_k(t, s - g_k(t) + h_k(t)) - T_k(t,s) \Big] = 0\end{aligned} \tag{2.21}$$

are oscillatory, then all solutions of (2.9) are also oscillatory.

Corollary 2.9. *Suppose (a1*)-(a3*) and (add1) hold for (2.9) and at least one of the following inequalities is satisfied:*

(1)
$$\begin{aligned}\liminf_{t\to\infty} \sum_{k=1}^{n} \Big\{ \int_{h_k(t)}^{t} \big[R_k(t,\tau) - T(t, \tau - h_k(t) + g_k(t))\big] d\tau + \int_{g_k(t)}^{t} d\tau \\ \times \int_{h_k(\tau)}^{\tau} \big(\exp\big\{ \int_{s+h_k(t)-g_k(t)}^{s} [R_k(u,u+) - T_k(u,u+)] du \big\} - 1 \big)\, d_s T_k(t,s) \Big\} > \frac{1}{e}\end{aligned}$$

(2)
$$\begin{aligned}\liminf_{t\to\infty} \sum_{k=1}^{n} \Big\{ \int_{h_k(t)}^{t} \big[R_k(t,\tau+) - T_k(t, \tau - h_k(t) + g_k(t))\big]\, d\tau \\ + \int_{g_k(t)}^{t} d\tau \int_{h_k(\tau)}^{\tau} \big(\int_{s+h_k(t)-g_k(t)}^{s} [R_k(u,u+) - T_k(u,u+)] du \big)\, ds T_k(t,s) \Big\} > \frac{1}{e}\end{aligned}$$

(3)
$$\begin{aligned}\liminf_{t\to\infty} \sum_{k=1}^{n} \Big\{ \int_{g_k(t)}^{t} d\tau \int_{g_k(\tau)}^{\tau} \Big(\exp\big\{ \int_{s-g_k(t)+h_k(t)}^{s} [R_k(u,u+) - T_k(u,u+)] \\ \times du \big\} - 1 \Big) d_s R_k(t, s - g_k(t) + h_k(t)) \\ + \int_{g_k(t)}^{t} \big[R_k(t, \tau - g_k(t) + h_k(t)) - T_k(t,\tau) \big]\, d\tau \Big\} > \frac{1}{e}\end{aligned}$$

(4)
$$\begin{aligned}\liminf_{t\to\infty} \sum_{k=1}^{n} \Big\{ \int_{g_k(t)}^{t} d\tau \int_{g_k(\tau)}^{\tau} \big(\int_{s-g_k(t)+h_k(t)}^{s} [R_k(u,u+) - T_k(u,u+)] du \big) \\ \times d_s R_k(t, s - g_k(t) + h_k(t)) + \int_{g_k(t)}^{t} \big[R_k(t, \tau - g_k(t) + h_k(t)) - T_k(t,\tau) \big] \Big\} > \frac{1}{e}\end{aligned}$$

Then all solutions of (2.3) are oscillatory.

Let us proceed with nonoscillation conditions.

Lemma 2.10 ([3]). *Suppose (a1)-(a4) hold for (2.3) and there exists λ, $0 < \lambda < 1$, such that*

$$\limsup_{t\to\infty} \int_{h(t)}^{g(t)} [R(s,s+) - \lambda T(s,s+)]\, ds < \frac{1}{e} \ln \frac{1}{\lambda}, \tag{2.22}$$

$$\limsup_{t\to\infty} \int_{h(t)}^{t} [R(s,s+) - \lambda T(s,s+)] ds < \frac{1}{e}. \tag{2.23}$$

Then (2.3) has a nonoscillatory solution.

Lemma 2.11 ([3]). *Suppose $n = m$, conditions (a1*)-(a3*), (add1) and the following inequality*

$$\limsup_{t\to\infty} \sum_{k=1}^{n} \int_{h_k(t)}^{t} \big[R_k(s, s+) - \frac{1}{e} T_k(s, s+)\big] ds < \frac{1}{e} \tag{2.24}$$

hold. Then (2.9) has a nonoscillatory solution.

3. Equations with Concentrated Delays

Let us study oscillation properties of a delay differential equation with several variable concentrated delays

$$\dot{x}(t) + \sum_{k=1}^{n} a_k(t)x(h_k(t)) - \sum_{l=1}^{m} b_l(t)x(g_l(t)) = 0, t \ge 0. \tag{3.1}$$

This equation is a special case of (1.5) when we assume

$$R(t, s) = \sum_{k=1}^{n} a_k(t)\chi_{[h_k(t),\infty)}(s), \quad T(t, s) = \sum_{l=1}^{m} b_l(t)\chi_{[g_l(t),\infty)}(s), \tag{3.2}$$

where $\chi_{[c,d]}$ is a characteristic function of segment $[c, d]$.

An initial value problem, definitions of a solution, the fundamental solution, oscillatory and nonoscillatory solutions for equation (3.1) are the same as for (2.9).

The hypotheses of Lemma 2.1 are satisfied for the delay equation (3.1) if the following conditions hold:

(C1) $a_k(t) \ge 0$, $b_l(t) \ge 0$ are Lebesgue measurable essentially locally bounded functions;

(C2) For any $t \ge s \ge 0$
$$\sum_{k=1}^{n} a_k(t)\chi_{[h_k(t),\infty)}(s) \ge \sum_{l=1}^{m} b_l(t)\chi_{[g_l(t),\infty)}(s).$$

(C3) $h_k(t), g_l(t) : [0,\infty) \to \mathbf{R}$ are Lebesgue measurable functions, $h_k(t) \le t, g_l(t) \le t$, $\lim_{t\to\infty} h_k(t) = \infty$, $\lim_{t\to\infty} g_l(t) = \infty$.

Consider the inequality

$$\dot{y}(t) + \sum_{k=1}^{n} a_k(t)y(h_k(t)) - \sum_{l=1}^{m} b_l(t)y(g_l(t)) \le 0, t \ge 0. \tag{3.3}$$

The following proposition is an immediate consequence of Lemma 2.1.

Proposition 3.1. *Suppose (C1)–(C3) hold. Consider the following hypotheses:*

(1) *There exists $t_1 \ge 0$ such that the inequality*
$$u(t) \ge \sum_{k=1}^{n} a_k(t) \exp\Big\{\int_{h_k(t)}^{t} u(s)ds\Big\} - \sum_{l=1}^{m} b_l(t) \exp\Big\{\int_{g_l(t)}^{t} u(s)ds\Big\}, \ t \ge t_1 \tag{3.4}$$
has a nonnegative locally integrable solution (we assume $u(t) = 0$ for $t < t_1$);

(2) *There exists $t_2 \ge 0$ such that the fundamental function $X(t, s) > 0$, $t \ge s \ge t_2$;*

(3) *Equation (3.1) has a nonoscillatory solution;*

(4) *Inequality (3.3) has an eventually positive solution.*

Then the implications 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4) *are valid.*

Oscillation criteria for equations with several concentrated delays can be obtained as corollaries of Lemma 2.3. To this end denote

$$h(t) = \max_k h_k(t), \quad g(t) = \min_l g_l(t), \quad a(t) = \sum_{k=1}^{n} a_k(t), \quad b(t)) = \sum_{l=1}^{m} b_l(t).$$

Proposition 3.2 provides oscillation criteria.

Proposition 3.2. *Suppose (C1), (C3) and anyone of the following conditions*

(C4) $m = n$, $a_k \geq b_k$, $h_k(t) \leq g_k(t)$ *for* $k = 1, \dots, n$, $t \geq 0$,

$$\limsup_{t\to\infty} \Big\{ \sum_{k=1}^{n} b_k(t)[g_k(t) - h_k(t)] \Big\} < 1$$

(C5) $h(t) \leq g(t)$, $a(t) \geq b(t)$,

$$\limsup_{t\to\infty} \big\{ b(t)[g(t) - h(t)] + \sum_{k=1}^{n} a_k(t)[h(t) - h_k(t)] + \sum_{l=1}^{m} b_l(t)[g_l(t) - g(t)] \big\} < 1$$

hold. Then all four hypotheses of Proposition 3.1 are equivalent.

Remark. It is to be noted that (C5) is a special case of (C4), if we rewrite the left hand side of (3.1) as a sum of $(n+m+1)$ positive and $(n+m+1)$ negative terms:

$$\begin{aligned}
&\dot{x}(t) + \sum_{k=1}^{n} a_k(t)x(h_k(t)) - \sum_{l=1}^{m} b_l(t)x(g_l(t)) \\
&= \dot{x}(t) + \sum_{k=1}^{n} a_k(t)x(h_k(t)) - \sum_{k=1}^{n} a_k(t)x(h(t)) + \sum_{k=1}^{n} a_k(t)x(h(t)) \\
&\quad - \sum_{l=1}^{m} b_l(t)x(g(t)) + \sum_{l=1}^{m} b_l(t)x(g(t)) - \sum_{l=1}^{m} b_l(t)x(g_l(t)) \\
&= \dot{x}(t) + a(t)x(h(t)) - b(t)x(g(t)) + \sum_{k=1}^{n} a_k(t)\big[x(h_k(t)) - x(h(t))\big] \\
&\quad + \sum_{l=1}^{m} b_l(t)\big[x(g(t)) - x(g_l(t))\big]
\end{aligned}$$

We proceed with comparison results which are deduced from Lemma 2.4. To this end consider the equation

$$\dot{x}(t) + \sum_{k=1}^{n} \tilde{a}_k(t)x(\tilde{h}_k(t)) - \sum_{l=1}^{m} \tilde{b}_l(t)x(\tilde{g}_l(t)) = 0, t \geq 0. \tag{3.5}$$

Proposition 3.3. (1) *Suppose (C1), (C3) and either (C4) or (C5) hold for (3.5), where* a_k, b_l, h_k, g_l *are changed by* $\tilde{a}_k, \tilde{b}_l, \tilde{h}_k, \tilde{g}_l$*, respectively. If* $\tilde{a}_k(t) \geq a_k(t)$, $\tilde{b}_l(t) \leq b_l(t)$, $\tilde{h}_k(t) \leq h_k(t)$, $\tilde{g}_l(t) \geq g_l(t)$ *and (3.5) has a nonoscillatory solution, then (3.1) also has a nonoscillatory solution.*

(2) *Suppose (C1), (C3) and either (C4) or (C5) hold. If* $\tilde{a}_k(t) \leq a_k(t)$, $\tilde{b}_l(t) \geq b_l(t)$, $\tilde{h}_k(t) \geq h_k(t)$, $\tilde{g}_l(t) \leq g_l(t)$ *and all solutions of (3.5) are oscillatory, then all solutions of (3.1) are also oscillatory.*

To apply Proposition 3.3 consider the autonomous delay equation

$$\dot{y} + \sum_{i=1}^{n} c_i y(t-\delta_i) - \sum_{l=1}^{m} d_l y(t-\sigma_l) = 0, \ t \geq 0, \tag{3.6}$$

where the following conditions hold:

(A1) $c_i \geq 0, d_l \geq 0, \delta_i \geq 0, \sigma_l \geq 0$

and one of the two following conditions satisfied:

(A2) $n = m, c_i \geq d_i, \delta_i \geq \sigma_i, \sum_{i=1}^{n} d_i(\delta_i - \sigma_i) < 1$;

(A3) $c = \sum_{i=1}^{n} c_i \geq d = \sum_{l=1}^{m} d_l, \delta = \min \delta_i \geq \sigma = \max \sigma_l$,

$$d(\delta - \sigma) + \sum_{i=1}^{n} c_i(\delta_i - \delta) + \sum_{l=1}^{m} d_l(\sigma - \sigma_l) < 1;$$

Corollary 3.1. (1) *Suppose (C1)-(C3), (A1) and at least one of (A2), (A3) hold. If* $a_k(t) \leq c_k, h_k(t) \geq t - \delta_k, b_l(t) \geq d_l, g_l(t) \leq t - \sigma_l$ *and (3.6) has a nonoscillatory solution, then (3.1) also has a nonoscillatory solution.*

(2) *Suppose (C1), (C3), (A1) and at least one of (C4), (C5) hold. If* $a_k(t) \geq c_k, h_k(t) \leq t - \delta_k, b_l(t) \leq d_l, g_l(t) \geq t - \sigma_l$ *and all solutions of (3.6) are oscillatory, then all solutions of (3.1) are also oscillatory.*

Lemma 2.5 immediately implies the following result on the asymptotic behaviour of solutions.

Proposition 3.4. *Suppose(C1),(C3) and one of the following two conditions is satisfied*

(1) *(C4) holds and there exists such k that* $\int_0^\infty [a_k(t) - b_k(t)]dt = \infty$.

(2) *(C5) holds and* $\int_0^\infty [a(t) - b(t)]dt = \int_0^\infty \left[\sum_{k=1}^{n} a_k(t) - \sum_{l=1}^{m} b_l(t)\right]dt = \infty$.

Then any nonoscillatory solution of (3.1) satisfies $\lim_{t\to\infty} y(t) = 0$.

Proposition 3.5. *Suppose the hypotheses (C1), (C3), (C4) hold. If all solutions of anyone of the following equations are oscillatory*

$$\begin{aligned} &\dot{x}(t) + \sum_{k=1}^{n} [a_k(t) - b_k(t)] x(h_k(t)) \\ &+ \sum_{k=1}^{n} b_k(t) \Big(\exp\Big\{ \int_{h_k(t)}^{g_k(t)} [a_k(s) - b_k(s)] ds \Big\} - 1 \Big) x(g_k(t)) = 0, \end{aligned} \tag{3.7}$$

$$\dot{x}(t) + \sum_{k=1}^{n} \Big[a_k(t) - b_k(t) + a_k(t) \Big(\exp\Big\{ \int_{h_k(t)}^{g_k(t)} [a_k(s) - b_k(s)] ds \Big\} - 1 \Big) \Big] x(g_k(t)) = 0, \tag{3.8}$$

$$\dot{x}(t) + \sum_{k=1}^{n} [a_k(t) - b_k(t)] x(h_k(t)) + \sum_{k=1}^{n} b_k(t) x(g_k(t)) \int_{h_k(t)}^{g_k(t)} [a_k(s) - b_k(s)] ds = 0, \tag{3.9}$$

$$\dot{x}(t) + \sum_{k=1}^{n} \Big(a_k(t) - b_k(t) + a_k(t) \int_{h_k(t)}^{g_k(t)} [a_k(s) - b_k(s)] ds \Big) x(g_k(t)) = 0, \tag{3.10}$$

then all solutions of (3.1) are also oscillatory.

Remark. Since (C5) is a special case of (C4), similar equations can be presented if (C1),(C3),(C5) are satisfied.

Corollary 3.2. *Suppose (C1), (C3), (C4) are satisfied and at least one of the following conditions holds:*

$$\begin{aligned}(1)\quad \liminf_{t\to\infty}\Big\{ &\sum_{k=1}^{n}[a_k(t)-b_k(t)](t-h_k(t))\\ &+\sum_{k=1}^{n} b_k(t)\Big(\exp\Big\{\int_{h_k(t)}^{g_k(t)}[a_k(s)-b_k(s)]ds\Big\}-1\Big)(t-g_k(t))\Big\}>\frac{1}{e}\end{aligned}$$

$$\begin{aligned}(2)\quad \liminf_{t\to\infty}\Big\{ &\sum_{k=1}^{n}[a_k(t)-b_k(t)](t-h_k(t))\\ &+\sum_{k=1}^{n} b_k(t)(t-h_k(t))\int_{h_k(t)}^{g_k(t)}[a_k(s)-b_k(s)]ds\Big\}>\frac{1}{e}\end{aligned}$$

$$\begin{aligned}(3)\quad \liminf_{t\to\infty}\Big\{ &\sum_{k=1}^{n}\int_{g_k(t)}^{t}\big[a_k(s)-b_k(s)\\ &+a_k(s)\big(\exp\Big\{\int_{h_k(s)}^{g_k(s)}[a_k(\tau)-b_k(\tau)]d\tau\Big\}-1\big)\big]ds\Big\}>\frac{1}{e}\end{aligned}$$

$$(4)\quad \liminf_{t\to\infty}\Big\{\sum_{k=1}^{n}\int_{g_k(t)}^{t}\big[a_k(s)-b_k(s)+a_k(s)\int_{h_k(s)}^{g_k(s)}[a_k(\tau)-b_k(\tau)]d\tau\big]ds\Big\}>\frac{1}{e},$$

then all solutions of (3.1) are oscillatory.

Corollary 3.3. *Suppose (A1), (A2) and at least one of the following conditions hold:*

(1) $\sum_{k=1}^{n}\big[(a_k-b_k)\delta_k+b_k\big(e^{(a_k-b_k)(\delta_k-\sigma_k)}-1\big)\sigma_k\big]>1/e$;
(2) $\sum_{k=1}^{n}(a_k-b_k)\big[\delta_k+b_k\sigma_k(\delta_k-\sigma_k)\big]>1/e$;
(3) $\sum_{k=1}^{n}\big[a_k-b_k+a_k\big(e^{(a_k-b_k)(\delta_k-\sigma_k)}-1\big)\big]\sigma_k>1/e$;
(4) $\sum_{k=1}^{n}(a_k-b_k)\big[1+a_k(\delta_k-\sigma_k)\big]\sigma_k>1/e$.

Then all solutions of (3.6) are oscillatory.

Now we proceed with explicit nonoscillation conditions.

Proposition 3.6. *Suppose either (C1), (C3), (C4) hold and the following inequality is satisfied*

$$\limsup_{t\to\infty}\sum_{k=1}^{n}\int_{h_k(t)}^{t}\big[a_k(s)-\frac{1}{e}b_k(s)\big]ds<\frac{1}{e}.$$

or (C1), (C3), (C5) hold and

$$\begin{aligned}&\limsup_{t\to\infty}\Big\{\big(1-\frac{1}{e}\big)\sum_{k=1}^{n}\int_{h_k(t)}^{t}a_k(s)\,ds\\ &+\int_{h(t)}^{t}\big[\sum_{k=1}^{n}a_k(s)-\frac{1}{e}\sum_{l=1}^{m}b_k(s)\big]ds+\big(1-\frac{1}{e}\big)\sum_{l=1}^{m}\int_{g(t)}^{t}b_l(s)\,ds\Big\}<\frac{1}{e}.\end{aligned}$$

Then (3.1) has a nonoscillatory solution.

Denote

$$H(t) = \min_k h_k(t), \quad G(t) = \max_l g_l(t).$$

Proposition 3.7. *Suppose there exist* $\tilde{a}(t)$, $\tilde{b}(t)$, *such that*

$$\tilde{b}(t) \le b(t) \le a(t) \le \tilde{a}(t), \quad \text{where} \quad a(t) = \sum_{k=1}^{n} a_k(t),\ b(t) = \sum_{l=1}^{m} b_l(t),$$

there exist finite limits

$$B_{11} = \lim_{t\to\infty} \int_{H(t)}^{t} \tilde{a}(s)\,ds, \quad B_{12} = \lim_{t\to\infty} \int_{H(t)}^{t} \tilde{b}(s)\,ds, \tag{3.11}$$

$$B_{21} = \lim_{t\to\infty} \int_{G(t)}^{t} \tilde{a}(s)\,ds, \quad B_{22} = \lim_{t\to\infty} \int_{G(t)}^{t} \tilde{b}(s)\,ds, \tag{3.12}$$

and (C1)–(C3) hold. Suppose, in addition, that the system

$$\ln x_1 > x_1 B_{11} - x_2 B_{12} \tag{3.13}$$

$$\ln x_2 < x_1 B_{21} - x_2 B_{22} \tag{3.14}$$

has a positive solution (x_1, x_2) *such that eventually* $x_1\tilde{a}(t) \ge x_2\tilde{b}(t)$. *Then (3.1) has a nonoscillatory solution.*

Proof. Consider the function $u(t) = x_1\tilde{a}(t) - x_2\tilde{b}(t)$. Then $u(t)$ is nonnegative and the system (3.13)-(3.14) yields

$$x_1 > \exp\{x_1 B_{11} - x_2 B_{12}\}, \quad x_2 < \exp\{x_1 B_{21} - x_2 B_{22}\}.$$

Thus by definitions (3.11)-(3.12) there exists $t_1 > 0$, such that for $t \ge t_1$

$$x_1 \ge \exp\Big\{x_1 \int_{H(t)}^{t} \tilde{a}(s)\,ds - x_2 \int_{H(t)}^{t} \tilde{b}(s)\,ds\Big\} = \exp\Big\{\int_{H(t)}^{t} u(s)\,ds\Big\}$$

$$-x_2 \ge -\exp\Big\{x_1 \int_{G(t)}^{t} \tilde{a}(s)\,ds - x_2 \int_{G(t)}^{t} \tilde{b}(s)\,ds\Big\} = -\exp\Big\{\int_{G(t)}^{t} u(s)\,ds\Big\}$$

After multiplying the first inequality by $\tilde{a}(t)$, the second one by $\tilde{b}(t)$ and summation we have

$$\begin{aligned}
u(t) &= x_1\tilde{a}(t) - x_2\tilde{b}(t) \\
&\ge \tilde{a}(t)\exp\Big\{\int_{H(t)}^{t} u(s)\,ds\Big\} - \tilde{b}(t)\exp\Big\{\int_{G(t)}^{t} u(s)\,ds\Big\} \\
&\ge a(t)\exp\Big\{\int_{H(t)}^{t} u(s)\,ds\Big\} - b(t)\exp\Big\{\int_{G(t)}^{t} u(s)\,ds\Big\} \\
&= \sum_{k=1}^{n} a_k(t)\exp\Big\{\int_{H(t)}^{t} u(s)\,ds\Big\} - \sum_{l=1}^{m} b_l(t)\exp\Big\{\int_{G(t)}^{t} u(s)\,ds\Big\} \\
&\ge \sum_{k=1}^{n} a_k(t)\exp\Big\{\int_{h_k(t)}^{t} u(s)\,ds\Big\} - \sum_{l=1}^{m} b_l(t)\exp\Big\{\int_{g_l(t)}^{t} u(s)\,ds\Big\}.
\end{aligned}$$

By Proposition 3.1, (3.1) has a nonoscillatory solution. □

Example 1. Consider the equation

$$\dot{x}(t)+\sum_{k=1}^{n}\frac{a_k}{t}x\big(\frac{t}{\mu_k}\big)-\sum_{k=1}^{n}\frac{b_k}{t}x\big(\frac{t}{\nu_k}\big)=0,\ t\geq t_0>0, \tag{3.15}$$

where $a_k\geq b_k\geq 0$, $\mu_k\geq\nu_k>1$. We apply Corollary 3.2 (Ineq. 4):

$$\begin{aligned}
&\liminf_{t\to\infty}\big\{\sum_{k=1}^{n}\int_{t/\nu_k}^{t}\big(\frac{a_k-b_k}{s}+\frac{a_k}{s}\int_{s/\mu_k}^{s/\nu_k}\frac{a_k-b_k}{\tau}\,d\tau\big)ds\big\}\\
&=\liminf_{t\to\infty}\big\{\sum_{k=1}^{n}\int_{t/\nu_k}^{t}\big(\frac{a_k-b_k}{s}+\frac{a_k}{s}\big[\ln\frac{s}{\nu_k}-\frac{s}{\mu_k}\big](a_k-b_k)\big)ds\big\}\\
&=\liminf_{t\to\infty}\big\{\sum_{k=1}^{n}(a_k-b_k)\int_{t/\nu_k}^{t}\big(\frac{1}{s}+\frac{a_k}{s}\ln\frac{\mu_k}{\nu_k}\big)ds\big\}\\
&=\liminf_{t\to\infty}\big\{\sum_{k=1}^{n}(a_k-b_k)\big(1+a_k\ln\frac{\mu_k}{\nu_k}\big)\big[\ln t-\ln\frac{t}{\nu_k}\big]\big\}\\
&=\sum_{k=1}^{n}(a_k-b_k)\big(1+a_k\ln\frac{\mu_k}{\nu_k}\big)\ln\nu_k
\end{aligned}$$

Thus if

$$\sum_{k=1}^{n}(a_k-b_k)\big(1+a_k\ln\frac{\mu_k}{\nu_k}\big)\ln\nu_k>\frac{1}{e},$$

then all solutions of (3.15) are oscillatory. For nonoscillation results, we apply Proposition 3.6.

$$\begin{aligned}
\limsup_{t\to\infty}\big\{\sum_{k=1}^{n}\int_{t/\mu_k}^{t}\big[\frac{a_k}{s}-\frac{1}{e}\frac{b_k}{s}\big]ds\big\}&=\limsup_{t\to\infty}\big\{\sum_{k=1}^{n}\big(a_k-\frac{1}{e}b_k\big)\big(\ln t-\ln\frac{t}{\mu_k}\big)\big\}\\
&=\sum_{k=1}^{n}\big(a_k-\frac{1}{e}b_k\big)\ln\mu_k
\end{aligned}$$

Thus if $\sum_{k=1}^{n}\big(a_k-\frac{1}{e}b_k\big)\ln\mu_k<1/e$, then (3.15) has a nonoscillatory solution.

Example 2. Consider the equation

$$\dot{x}(t)+\sum_{k=1}^{n}\frac{a_k}{t}x\big(\frac{t}{\mu_k}\big)-\frac{b}{t}x\big(\frac{t}{\nu}\big)=0,\quad t\geq t_0>0, \tag{3.16}$$

where $\sum_{k=1}^{n}a_k\geq b\geq 0$, $\mu_k\geq\nu>1$. Unlike in Example 1, here the number of positive terms is not equal to the number of negative terms. This equation can be rewritten in one of the following forms:

(1) $\dot{x}(t)+\sum_{k=1}^{n}\frac{a_k}{t}x\big(\frac{t}{\mu_k}\big)-\sum_{k=1}^{n}\frac{a_k}{t}x\big(\frac{t}{\nu}\big)+\big(\sum_{k=1}^{n}\frac{a_k}{t}\big)x\big(\frac{t}{\nu}\big)-\frac{b}{t}x\big(\frac{t}{\nu}\big)=0$

(2) $\dot{x}(t)+\sum_{k=1}^{n}\frac{a_k}{t}x\big(\frac{t}{\mu_k}\big)-\sum_{k=1}^{n}\frac{b_k}{t}x\big(\frac{t}{\nu}\big)=0$, where $b_1=b$, $b_k=0, k>1$ and $a_1\geq b$.

(3) $\dot{x}(t)+\sum_{k=1}^{n}\frac{a_k}{t}x\big(\frac{t}{\mu_k}\big)-\sum_{k=1}^{n}\frac{\lambda_k b}{t}x\big(\frac{t}{\nu}\big)=0$, where $\lambda_k>0$, $\sum_{k=1}^{n}\lambda_k=1$ and $a_k\geq\lambda_k b$.

A computation similar to Example 1 yields that if anyone of the following three conditions holds

(1) $\big(\sum_{k=1}^{n}a_k-b\big)\ln\nu>\frac{1}{e}$

(2) $a_1 \geq b$ and $(a_1 - b)\big(1 + a_1 \ln \frac{\mu_1}{\nu}\big) \ln \nu + \big(\sum_{k=2}^n a_k\big)\big(1 + a_k \ln \frac{\mu_k}{\nu}\big) \ln \nu > 1/e$
(3) For each k $a_k \geq \lambda_k b$ and $\sum_{k=1}^n \big(a_k - \lambda_k b\big)\big(1 + a_k \ln \frac{\mu_k}{\nu}\big) \ln \nu > 1/e$.

then all solutions of (3.16) are oscillatory.

If anyone of the following three conditions holds

(1) $\sum_{k=1}^n a_k\big(1 - \frac{1}{e}\big) \ln \mu_k + \big(\sum_{k=1}^n a_k - b\big) \ln \nu < 1/e$,
(2) $a_1 \geq b$ and $\big(a_1 - \frac{1}{e} b\big) \ln \mu_1 + \sum_{k=2}^n a_k \ln \mu_k < \frac{1}{e}$,
(3) For each k $a_k \geq \lambda_k b$ and $\sum_{k=1}^n \big(a_k - \frac{1}{e}\lambda_k b\big) \ln \mu_k < 1/e$,

then (3.16) has a nonoscillatory solution.

4. Integrodifferential Equations

In this section we will study the following integrodifferential equation

$$\dot{x}(t) + \int_0^t K(t,s)x(s)\; ds - \int_0^t M(t,s)x(s)\; ds = 0, \tag{4.1}$$

(4.1) is a special case of (1.5) if we assume

$$R(t,s) = \int_0^s K(t,\zeta)\, d\zeta, \quad T(t,s) = \int_0^s M(t,\zeta)\, d\zeta. \tag{4.2}$$

The hypotheses of Lemma 2.1 are satisfied for the delay equation (4.1) if the following conditions hold:

(I1) $K(t,s)$, $M(t,s)$ are Lebesgue integrable over each finite square $[0,b] \times [0,b]$ functions;
(I2) There exist finite functions $h(t), g(t)$ such that $h(t) = \inf\{s | K(t,s) \geq 0\}$, $g(t) = \inf\{s | M(t,s) \geq 0\}$ and $\lim_{t\to\infty} h(t) = \infty$, $\lim_{t\to\infty} g(t) = \infty$;
(I3) For each t, s $K(t,s) \geq 0$, $M(t,s) \geq 0$ and
$$\int_{h(s)}^s K(t,\tau)\, d\tau \geq \int_{g(s)}^s M(t,\tau)\, d\tau.$$

For $t_0 \geq 0$ consider an initial-value problem

$$\dot{x}(t) + \int_{h(t)}^t K(t,s)x(s)\, ds - \int_{g(t)}^t M(t,s)x(s)\, ds = 0, \quad t > t_0. \tag{4.3}$$

$$x(t) = \varphi(t), \quad t < t_0,\ x(t_0) = x_0. \tag{4.4}$$

As in the general case we will say that (4.1) has a nonoscillatory solution if for some $t_0 \geq 0$, $\varphi(t)$ and x_0 the solution of (4.3)-(4.4) is eventually positive or eventually negative. Otherwise all solutions of (4.1) are oscillatory.

Consider in addition the inequality

$$u(t) \geq \int_{h(t)}^t K(t,s) \exp\big\{ \int_s^t u(\tau) d\tau \big\}\, ds - \int_{g(t)}^t M(t,s) \exp\big\{ \int_s^t u(\tau) d\tau \big\}\, ds. \tag{4.5}$$

Lemma 2.1 immediately implies the following proposition.

Proposition 4.1. *Suppose (I1)-(I3) hold. Consider the following hypotheses*

(1) *There exists $t_1 \geq 0$ such that inequality (4.5) has a nonnegative locally integrable solution;*
(2) *There exists $t_2 \geq 0$ such that $X(t,s) > 0$, $t \geq s \geq t_2$, where $X(t,s)$ is a fundamental function of (4.1);*
(3) *(4.1) has a nonoscillatory solution;*

(4) *Inequality*

$$\dot{y}(t) + \int_{t_0}^{t} K(t,s)y(s)\ ds - \int_{t_0}^{t} M(t,s)y(s)\ ds \le 0 \tag{4.6}$$

has an eventually positive solution.

Then the implications 1) ⇒ 2) ⇒ 3) ⇒ 4) *are valid.*

Now let us apply Lemma 2.2 to (4.1). Let us introduce the following additional assumption

(I4) $K(t,s) \ge M(t, s - h(t) + g(t))$ for each t,s and

$$\limsup_{t\to\infty} [g(t) - h(t)] \int_{g(t)}^{t} M(t,s)\, ds < 1.$$

Proposition 4.2. *Suppose (I1)–(I4) hold. Then all four hypotheses of Proposition 4.1 are equivalent.*

In addition to (4.1) consider the integrodifferential equation

$$\dot{x}(t) + \sum_{i=1}^{n} \int_{h_i(t)}^{t} K_i(t,s)x(s)\, ds - \sum_{l=1}^{m} \int_{g_l(t)}^{t} M_l(t,s)x(s)\, ds = 0, \tag{4.7}$$

where the following conditions hold:

(I1*) $K_i(t,s)$, $M_l(t,s)$ are Lebesgue integrable over each finite square $[0,b]\times[0,b]$ functions;

(I2*) There exist finite functions $h_i(t) = \inf\{s | K_i(t,s) \ge 0\}$, $g_l(t) = \inf\{s | M_l(t,s) \ge 0\}$ and $\lim_{t\to\infty} h_i(t) = \infty$, $\lim_{t\to\infty} g_l(t) = \infty$;

(I3*) For each t,s,i,l $K_i(t,s) \ge 0$, $M_l(t,s) \ge 0$,

$$\sum_{i=1}^{n} \int_{h_i(t)}^{t} K_i(t,\tau)\, d\tau \ge \sum_{l=1}^{m} \int_{g_l(t)}^{t} M_l(t,\tau)\, d\tau$$

and one of two following conditions is satisfied:

(I4*) $m = n$, $K_i(t,s) \ge M_i(t,s)$ for each t,s, $i = 1,\dots,n$, for any t,i $K_i(t,s) \ge M_i(t, s - h_i(t) + g_i(t))$ and

$$\limsup_{t\to\infty} \sum_{i=1}^{n} [g_i(t) - h_i(t)] \int_{g_i(t)}^{t} M_i(t,s)\, ds < 1.$$

(I5*) For each i,l,t,s $h(t) \le g(t)$, where h,g are defined in (2.11) and

$$K(t,s) = \sum_{i=1}^{m} K_i(t,s) \ge \sum_{l=1}^{l} M_l(t,s) = M(t,s),$$

we have $K(t,s) \ge M(t, s - h(t) + g(t))$ and

$$\begin{aligned} \limsup_{t\to\infty} \Big[(g(t) - h(t)) \int_{g(t)}^{t} M(t,s)\, ds + \sum_{i=1}^{n} (h(t) - h_i(t)) \int_{h_i(t)}^{t} K_i(t,s)\, ds \\ + \sum_{l=1}^{m} (g_l(t) - g(t)) \int_{g_l(t)}^{t} M_l(t,s)\, ds \Big] < 1. \end{aligned} \tag{4.8}$$

Proposition 4.3. *Suppose (I1*)–(I3*) and one of (I4*),(I5*) hold. Then the following hypotheses are equivalent:*

(1) *Inequality*
$$\dot y + \sum_{i=1}^{n} \int_{h_i(t)}^{t} K_i(t,s)y(s)\,ds - \sum_{l=1}^{m} \int_{g_l(t)}^{t} M_l(t,s)y(s)\,ds \le 0, \quad t \ge 0, \tag{4.9}$$
has an eventually positive solution;

(2) *There exists $t_2 \ge 0$ such that $X(t,s) > 0$, $t \ge s \ge t_2$, where $X(t,s)$ is a fundamental function of (4.7);*

(3) *(4.7) has a nonoscillatory solution.*

(4) *There exists $t_3 \ge 0$ such that for $t \ge t_3$ the inequality*
$$\begin{aligned} u(t) \ge & \sum_{i=1}^{n} \int_{h_i(t)}^{t} K_i(t,s) \exp\Big\{ \int_s^t u(\tau)d\tau \Big\}\,ds \\ & - \sum_{l=1}^{m} \int_{g_l(t)}^{t} M_l(t,s) \exp\Big\{ \int_s^t u(\tau)d\tau \Big\}\,ds \end{aligned} \tag{4.10}$$
has a nonnegative locally integrable solution.

We proceed with comparison results which are deduced from Lemma 2.4. To this end consider the equation
$$\dot x(t) + \sum_{i=1}^{n} \int_{\tilde h_k(t)}^{t} \tilde K_i(t,s)x(s)\ ds - \sum_{l=1}^{m} \int_{\tilde g_l(t)}^{t} \tilde M_l(t,s)x(s)\ ds = 0. \tag{4.11}$$
For (4.11) h_i,g_l are changed by $\tilde h_i, \tilde g_l$, respectively.

Proposition 4.4. (1) *Suppose (I1***)–(I3***) and at least one of (I4***),(I5***) hold, where K_i,M_l,h,g are changed by $\tilde K_i, \tilde M_l, \tilde h, \tilde g$, respectively. If $\tilde K_i(t,s) \ge K_i(t,s)$, $\tilde M_l(t,s) \le M_l(t,s)$ and (4.11) has a nonoscillatory solution, then (4.7) also has a nonoscillatory solution.*

(2) *Suppose (I1***)-(I3***) and at least one of (I4***),(I5***) hold. If $\tilde K_i(t,s) \le K_i(t,s)$, $\tilde M_l(t,s) \ge M_l(t,s)$ and all solutions of (4.11) are oscillatory, then all solutions of (4.7) are also oscillatory.*

Corollary 4.1. (1) *Suppose (I1***)–(I3***), (A1) and at least one of (A2), (A3) hold. If for each t, i, l*
$$\int_{h_i(t)}^{t} K_i(t,s)\,ds \le c_i, \ \int_{g_l(t)}^{t} M_l(t,s)\,ds \ge d_l$$
and (3.6) has a nonoscillatory solution, then (4.7) also has a nonoscillatory solution.

(2) *Suppose (I1***), (I3***), (A1) and at least on of (I4***), (I5***) hold. If for each t, i, l*
$$\int_{h_i(t)}^{t} K_i(t,s)\,ds \ge c_i, \ \int_{g_l(t)}^{t} M_l(t,s)\,ds \le d_l$$
and all solutions of (3.6) are oscillatory, then all solutions of (4.7) are also oscillatory.

Lemma 2.5 immediately implies the following result on the asymptotic behaviour of solutions.

Proposition 4.5. *Suppose (I1***)–(I3***) is satisfied and anyone of the following conditions holds:*

(1) *(I4⋆) is satisfied and for some* l, $\int_0^\infty \big[\int_{t_0}^t (K_l(t,s) - M_l(t,s))\,ds\big]\,dt = \infty$

(2) *(I5⋆) is satisfied and* $\int_0^\infty \big[\int_{t_0}^t (K(t,s) - M(t,s))\,ds\big]\,dt = \infty$.

Then any nonoscillatory solution y *of (4.7) satisfies* $\lim_{t\to\infty} y(t) = 0$.

Lemma 2.8 yields oscillation conditions for (4.7).

Proposition 4.6. *Suppose hypotheses (I1⋆)–(I4⋆) hold. If all solutions of anyone of the following two equations are oscillatory*

$$\begin{aligned}\dot{x}(t) + \sum_{l=1}^n \int_{h_l(t)}^t \big[K_l(t,s) - M_l(t, s - h_l(t) + g_l(t))\big]x(s)\,ds + \sum_{l=1}^n \int_{g_l(t)}^t M_l(t,s)x(s)\\ \times \Big(\exp\Big\{\int_{s+h_l(t)-g_l(t)}^s d\tau \int_{h_l(\tau)}^\tau \big[K_l(\tau,\zeta) - M_l(\tau,\zeta)\big]d\zeta\Big\} - 1\Big)ds = 0\,,\end{aligned} \tag{4.12}$$

$$\begin{aligned}&\dot{x}(t) + \sum_{l=1}^n \int_{g_l(t)}^t K_l(t, s + h_l(t) - g_l(t))x(s)\\ &\times \Big(\exp\Big\{\int_{s+h_l(t)-g_l(t)}^s d\tau \int_{h_l(\tau)}^\tau \big[K_l(\tau,\zeta) - M_l(\tau,\zeta)\big]d\zeta\Big\} - 1\Big)\\ &+ \sum_{l=1}^n \int_{g_l(t)}^t \big[K_l(t, s + h_l(t) - g_l(t)) - M_l(t,s)\big]x(s)\,ds = 0\,,\end{aligned} \tag{4.13}$$

then all solutions of (4.7) are also oscillatory.

Remark. Since (I5⋆) is a special case of (I4⋆), similar equations can be presented if (I1⋆)-(I3⋆),(I5⋆) are satisfied.

The oscillation conditions for integrodifferential equations with nonnegative kernels [2] lead to the following result.

Corollary 4.2. *Suppose the hypotheses (I1)-(I4) hold for (4.1) and at least one of the following inequalities is valid:*

$$\begin{aligned}1)\quad \liminf_{t\to\infty}\Big\{\int_{h(t)}^t \Big[\int_{h(s)}^s K(t,\tau)\,d\tau - \int_{g(s)}^s M(t, \tau - h(t) + g(t))\,d\tau\Big]ds\\ + \int_{g(t)}^t d\tau \int_{h(\tau)}^\tau \Big(\exp\Big\{\int_{s+h(t)-g(t)}^s \Big[\int_{h(u)}^u K(u,\tau)d\tau\\ - \int_{g(u)}^u M(u,\tau)d\tau\Big]du\Big\} - 1\Big)M(t,s)ds\Big\} > \frac{1}{e}\end{aligned}$$

$$\begin{aligned}2)\quad \liminf_{t\to\infty}\Big\{\int_{h(t)}^t \Big[\int_{h(s)}^s K(t,\tau)\,d\tau - \int_{g(s)}^s M(t, \tau - h(t) + g(t))\,d\tau\Big]\,ds\\ + \int_{g(t)}^t d\tau \int_{h(\tau)}^\tau \Big(\int_{s+h(t)-g(t)}^s \Big[\int_{h(u)}^u K(u,\tau)\,d\tau\\ - \int_{g(u)}^u M(u,\tau)\,d\tau\Big]du\Big)M(t,s)\,ds\Big\} > \frac{1}{e}\end{aligned}$$

$$3)\quad \liminf_{t\to\infty}\Big\{\int_{g(t)}^t d\tau \int_{g(\tau)}^\tau \Big(\exp\Big\{\int_{s-g(t)+h(t)}^s \Big[\int_{h(u)}^u K(u,\tau)d\tau$$

$$- \int_{g(u)}^{u} M(u,\tau) d\tau \big] du \} - 1 \Big) K(t, s - g(t) + h(t)) ds$$
$$+ \int_{g(t)}^{t} \big[\int_{h(s)}^{s} K(t, \tau - g(t) + h(t))\, d\tau - \int_{g(s)}^{s} M(t, \tau - h(t) + g(t))\, d\tau \big] ds \Big\} > \frac{1}{e}$$

4) $$\liminf_{t\to\infty} \Big\{ \int_{g(t)}^{t} d\tau \int_{g(\tau)}^{\tau} \Big(\int_{s-g(t)+h(t)}^{s} \Big[\int_{h(u)}^{u} K(u,\tau) d\tau - \int_{g(u)}^{u} M(u,\tau) d\tau \Big] du \Big) K(t, s - g(t) + h(t)) ds$$
$$+ \int_{g(t)}^{t} \big[\int_{h(s)}^{s} K(t, \tau - g(t) + h(t))\, d\tau - \int_{g(s)}^{s} M(t, \tau - h(t) + g(t))\, d\tau \big] ds \Big\} > \frac{1}{e}$$

Then all solutions of (4.1) are oscillatory.

Lemma 2.11 implies the following nonoscillation results for (4.7).

Proposition 4.7. *Suppose (I1⋆)–(I4⋆) and the following inequality*

$$\limsup_{t\to\infty} \sum_{l=1}^{n} \int_{h_l(t)}^{t} \big[\int_{h_l(s)}^{s} K_l(s,\tau)\, d\tau - \frac{1}{e} \int_{g_l(s)}^{s} M_l(s,\tau)\, d\tau \big] ds < \frac{1}{e} \tag{4.14}$$

hold. Then (4.7) has a nonoscillatory solution.

Proposition 4.8. *Suppose (I1⋆)–(I3⋆),(I5⋆) and the following inequality*

$$\limsup_{t\to\infty} \Big\{ \sum_{l=1}^{n} \big(1 - \frac{1}{e}\big) \int_{h_l(t)}^{t} ds \int_{h_l(s)}^{s} K_l(s,\tau)\, d\tau \int_{0}^{t} ds \big[\int_{0}^{s} K(s,\tau)\, d\tau - \frac{1}{e} \int_{0}^{s} M(s,\tau)\, d\tau \big] + \big(1 - \frac{1}{e}\big) \int_{g_l(t)}^{t} ds \int_{g_l(s)}^{s} M_l(s,\tau)\, d\tau \Big\} < \frac{1}{e}$$

hold. Then (4.7) has a nonoscillatory solution.

Similar to Proposition 3.7 the following result can be obtained. Let $H(t) = \min_i h_i(t), G(t) = \max_l g_l(t)$.

Proposition 4.9. *Suppose there exist $\tilde{K}(t,s)$, $\tilde{M}(t,s)$, such that*

$$\tilde{M}(t,s) \le M(t,s) \le K(t,s) \le \tilde{K}(t,s),$$

where $K(t,s) = \sum_{i=1}^{n} K_i(t,s)$, $M(t,s) = \sum_{l=1}^{m} M_l(t,s)$, the following limits exist and are finite:

$$B_{11} = \lim_{t\to\infty} \int_{H(t)}^{t} ds \int_{H(s)}^{s} \tilde{K}(s,\tau)\, d\tau, B_{12} = \lim_{t\to\infty} \int_{H(t)}^{t} ds \int_{H(s)}^{s} \tilde{M}(s,\tau)\, d\tau, \tag{4.15}$$

$$B_{21} = \lim_{t\to\infty} \int_{G(t)}^{t} ds \int_{G(s)}^{s} \tilde{K}(s,\tau)\, d\tau, B_{22} = \lim_{t\to\infty} \int_{G(t)}^{t} ds \int_{G(s)}^{s} \tilde{M}(s,\tau)\, d\tau, \tag{4.16}$$

and (I1)–(I4) hold for $\tilde{K}(t,s)$, $\tilde{M}(t,s)$. Suppose, in addition, that the system

$$\ln x_1 > x_1 B_{11} - x_2 B_{12} \tag{4.17}$$

$$\ln x_2 < x_1 B_{21} - x_2 B_{22} \tag{4.18}$$

has a positive solution (x_1, x_2) *such that eventually*

$$x_1 \int_{h(t)}^{t} \tilde{K}(t,s)\,ds \geq x_2 \int_{g(t)}^{t} \tilde{M}(t,s)\,ds.$$

Then (4.7) has a nonoscillatory solution.

Example 3. Consider the integrodifferential equation

$$\dot{x}(t) + \int_0^t L(t,s)x(s)\,ds = 0. \tag{4.19}$$

Let $\alpha > 0$,

$$L(t,s) = \begin{cases} \alpha \sin(s-t), & 0 \leq t-s \leq 2\pi, \\ 0, & \text{otherwise}, \end{cases}$$

$$K(t,s) = L^+(t,s) = \frac{1}{2}(|L(t,s)| + L(t,s)) = \alpha \sin(s-t)\chi_{[t-2\pi,t-\pi]}(s),$$

$$M(t,s) = L^-(t,s) = \frac{1}{2}(|L(t,s)| - L(t,s)) = -\alpha \sin(s-t)\chi_{[t-\pi,t]}(s).$$

Then $h(t) = t - 2\pi,\ g(t) = t - \pi, M(t, s+\pi) = K(t,s)$ and

$$\begin{aligned} &\limsup_{t\to\infty} \big\{[g(t)-h(t)] \int_{g(t)}^{t} M(t,s)\,ds\big\} \\ &= \limsup_{t\to\infty} \big\{[g(t)-h(t)] \int_{t-\pi}^{t} \alpha(-\sin(s-t))\,ds\big\} \\ &= \limsup_{t\to\infty} \big\{\pi\alpha\big(\cos 0 - \cos(-\pi)\big)\big\} = 2\pi\alpha. \end{aligned}$$

Thus the hypothesis (I4) holds for (4.19) if $\alpha < 1/(2\pi)$. Let us proceed to nonoscillation conditions for this equation. To this end we will apply Proposition 4.7. We have

$$\begin{aligned} \int_{h(t)}^{t} K(t,s)\,ds &= \int_{t-2\pi}^{t-\pi} \alpha \sin(s-t)\,ds = 2\alpha, \\ \int_{g(t)}^{t} M(t,s)\,ds &= -\int_{t-\pi}^{t} \alpha \sin(s-t)\,ds = 2\alpha, \end{aligned}$$

which after the substitution in (4.14) yields

$$\begin{aligned} \limsup_{t\to\infty} \int_{h(t)}^{t} \Big[\int_{h(s)}^{s} K(s,\tau)\,d\tau - \frac{1}{e}\int_{g(s)}^{s} M(s,\tau)\,d\tau\Big] ds &= [t-h(t)]2\alpha\big(1-\frac{1}{e}\big) \\ &= 2\pi 2\alpha \frac{e-1}{e} < \frac{1}{e}, \end{aligned}$$

which is satisfied when $4\pi\alpha(e-1) < 1$. Consequently, if $\alpha < \frac{1}{4\pi(e-1)}$, then (4.19) has a nonoscillatory solution.

Example 4. Let $0 < \alpha < \beta$. Consider equation (4.19) with

$$L(t,s) = \begin{cases} \alpha \sin(s-t), & 0 \leq t-s \leq \pi, \\ \beta \sin(s-t), & \pi \leq t-s \leq 2\pi, \\ 0, & \text{otherwise}, \end{cases}$$

Then $K(t,s)=\beta\sin(s-t)\chi_{[t-2\pi,t-\pi]}(s)$, $M(t,s)=\alpha\sin(s-t)\chi_{[t-\pi,t]}(s)$ and

$$\limsup_{t\to\infty}\big\{[g(t)-h(t)]\int_{g(t)}^t M(t,s)\,ds\big\}=2\pi\alpha.$$

The hypothesis (I4) holds for (4.19) if in addition $\alpha<\frac{1}{2\pi}$. Similarly to Example 1, if $4\pi(\beta e-\alpha)<1$, we have

$$\begin{aligned}\int_{h(t)}^t\big[\int_{h(s)}^s K(s,\tau)\,d\tau-\frac1e\int_{g(s)}^s M(s,\tau)\,d\tau\big]\,ds&=2\pi 2\big(\beta-\frac{\alpha}{e}\big)\\&=\frac{4\pi}{e}(\beta e-\alpha)<\frac1e\,.\end{aligned}$$

Consequently, if $\beta e-\alpha<\frac{1}{4\pi}$, then (4.19) has a nonoscillatory solution.

For this kernel we can also obtain oscillation conditions. Since

$$\int_{h(s)}^s K(t,\tau)\,d\tau-\int_{g(s)}^s M(t,\tau-h(t)+g(t))\,d\tau=2(\beta-\alpha)$$

for $t-\pi\le\tau\le t$, we have

$$\int_{h(u)}^u K(u,\tau)\,d\tau-\int_{g(u)}^u M(u,\tau)\,d\tau=2(\beta-\alpha).$$

Then after substituting these results into the first formula in Corollary 4.2 we have

$$\begin{aligned}&\liminf_{t\to\infty}\Big\{\int_{h(t)}^t\Big[\int_{h(s)}^s K(t,\tau)\,d\tau-\int_{g(s)}^s M(t,\tau-h(t)+g(t))\,d\tau\Big]ds\\&+\int_{g(t)}^t d\tau\int_{h(\tau)}^{\tau}\Big(\exp\Big\{\int_{s+h(t)-g(t)}^s\big[\int_{h(u)}^u K(u,\tau)\,d\tau\\&-\int_{g(u)}^u M(u,\tau)\,d\tau\big]du\Big\}-1\Big)M(t,s)\,ds\Big\}\\&\ge\int_{t-\pi}^t\Big[\int_{h(s)}^s K(t,\tau)\,d\tau-\int_{g(s)}^s M(t,\tau-h(t)+g(t))\,d\tau\Big]ds\\&\quad+2(\beta-\alpha)\int_{g(t)}^t d\tau\int_{h(\tau)}^{\tau}\big(e^{2\pi(\beta-\alpha)}-1\big)\,d_sT(t,s)\\&=2\pi(\beta-\alpha)+\big(e^{2\pi(\beta-\alpha)}-1\big)2\pi.\end{aligned}$$

Thus if $2\pi(\beta-\alpha+\exp\{2\pi(\beta-\alpha)\}-1)>1/e$, then all solutions of (4.19) are oscillatory.

5. Mixed Equations

In this section we will consider mixed equations

$$\begin{aligned}&\dot x(t)+\sum_{k=1}^n a_k(t)x(h_k(t))-\sum_{l=1}^m b_l(t)x(g_l(t))+\sum_{i=1}^r\int_0^t K_i(t,s)x(s)ds\\&-\sum_{j=1}^p\int_0^t M_j(t,s)x(s)ds=0,\quad t\ge t_0\ge0,\end{aligned}\tag{5.1}$$

with the initial conditions

$$x(t)=\varphi(t),\quad t<t_0,\ x(t_0)=x_0.\tag{5.2}$$

We assume that (C1)–(C3) and (I1*)–(I3*) hold. To avoid confusion, in (I2*) instead of h_k, g_l the following functions are introduced:

$$\tilde{h}_i = \inf\{s | K_i(t,s) \geq 0\}, \quad \tilde{g}_j = \inf\{s | M_j(t,s) \geq 0\}.$$

Consider in addition the following hypotheses:

(M1) $m = n$, $a_k \geq b_k$, $h_k(t) \leq g_k(t)$ for k, t, $t \geq 0$, r=p, $K_i(t,s) \geq M_i(t, s - \tilde{h}_i(t) + \tilde{g}_i(t))$ and

$$\limsup_{t\to\infty} \Big\{ \sum_{k=1}^{n} b_k(t)[g_k(t) - h_k(t)] + \sum_{i=1}^{r} [\tilde{g}_i(t) - \tilde{h}_i(t)] \int_{\tilde{g}_i(t)}^{t} M_i(t,s) ds \Big\} < 1.$$

(M2) $h(t) \leq g(t)$, $a(t) = \sum_{k=1}^{n} a_k(t) \geq b(t) = \sum_{l=1}^{m} b_l(t)$, $\tilde{h}_i(t) \leq \tilde{h}(t) \leq \tilde{g}(t) \leq \tilde{g}_j(t)$, for each i, j, t, $K(t,s) = \sum_{i=1}^{r} K_i(t,s) \geq M(t,s) = \sum_{j=1}^{p} M_j(t,s)$ for each t, s and

$$\begin{aligned}
&\limsup_{t\to\infty} \Big\{ b(t)[g(t) - h(t)] + \sum_{k=1}^{n} a_k(t)[h(t) - h_k(t)] + \sum_{l=1}^{m} b_l(t)[g_l(t) - g(t)] \\
&+ (\tilde{g}(t) - \tilde{h}(t)) \int_{\tilde{g}(t)}^{t} M(t,s)\, ds + \sum_{i=1}^{n} (\tilde{h}(t) - \tilde{h}_i(t)) \int_{\tilde{h}_i(t)}^{t} K_i(t,s)\, ds \\
&+ \sum_{l=1}^{m} (\tilde{g}_l(t) - \tilde{g}(t)) \int_{\tilde{g}_l(t)}^{t} M_l(t,s)\, ds \Big\} < 1.
\end{aligned}$$

Obviously two other combinations of (C4), (C5) with (I4*), (I5*) can be considered.

Proposition 5.1. *Suppose (C1)–(C3), (I1*)–(I3*) and at least one of hypotheses (M1), (M2) hold. Then the following hypotheses are equivalent.*

(1) *There exists $t_1 \geq 0$ such that for $t \geq t_1$ the inequality*

$$\begin{aligned}
u(t) \geq& \sum_{k=1}^{n} a_k(t) \exp\Big\{ \int_{h_k(t)}^{t} u(s) ds \Big\} - \sum_{l=1}^{m} b_l(t) \exp\Big\{ \int_{g_k(t)}^{t} u(s) ds \Big\} \\
&+ \sum_{i=1}^{r} \int_{\tilde{h}_i(t)}^{t} K_i(t,s) \exp\Big\{ \int_{s}^{t} u(\tau) d\tau \Big\}\, ds - \sum_{j=1}^{p} \int_{\tilde{g}_l(t)}^{t} M_j(t,s) \exp\Big\{ \int_{s}^{t} u(\tau) d\tau \Big\}\, ds
\end{aligned}$$

has a nonnegative locally integrable solution (we assume $u(t) = 0$ for $t < t_1$);

(2) *There exists $t_2 \geq 0$ such that the fundamental function of (5.1) $X(t,s) > 0$, $t \geq s \geq t_2$;*

(3) *Equation (5.1) has a nonoscillatory solution;*

(4) *The inequality*

$$\begin{aligned}
&\dot{y}(t) + \sum_{k=1}^{n} a_k(t) y(h_k(t)) - \sum_{l=1}^{m} b_l(t) y(g_l(t)) \\
&+ \sum_{i=1}^{r} \int_{0}^{t} K_i(t,s) y(s)\, ds - \sum_{j=1}^{p} \int_{0}^{t} M_j(t,s) x(s)\, ds \leq 0
\end{aligned} \tag{5.3}$$

has an eventually positive solution.

The comparison result for equation (5.1) combines Propositions 3.3 and 4.4. Consider the comparison equation

$$\begin{aligned} &\dot{x}(t) + \sum_{k=1}^{n} \tilde{a}_k(t) x(\bar{h}_k(t)) - \sum_{l=1}^{m} \tilde{b}_l(t) x(\bar{g}_l(t)) \\ &+ \sum_{i=1}^{r} \int_0^t \tilde{K}_i(t,s) x(s)\, ds - \sum_{j=1}^{p} \int_0^t \tilde{M}_j(t,s) x(s)\, ds = 0. \end{aligned} \tag{5.4}$$

Proposition 5.2. *1) Suppose (C1)-(C3),(I1*)-(I3*) and either (M1) or (M2) hold for (5.4), where $a_k, b_l, h_k, g_l, K_i, M_l$ are changed by $\tilde{a}_k, \tilde{b}_l, \bar{h}_k, \bar{g}_l, \tilde{K}_i, \tilde{M}_l$, respectively. If $\tilde{a}_k(t) \geq a_k(t)$, $\tilde{b}_l(t) \leq b_l(t)$, $\bar{h}_k(t) \leq h_k(t)$, $\bar{g}_l(t) \geq g_l(t)$, $\tilde{K}_i(t,s) \geq K_i(t,s)$, $\tilde{M}_l(t,s) \leq M_l(t,s)$ and (5.4) has a nonoscillatory solution, then (5.1) also has a nonoscillatory solution.*
2) Suppose (C1)-(C3),I1*)-(I3*) and either (M1) or (M2) hold for (5.1). If $\tilde{a}_k(t) \leq a_k(t)$, $\tilde{b}_l(t) \geq b_l(t)$, $\bar{h}_k(t) \geq h_k(t)$, $\bar{g}_l(t) \leq g_l(t)$, $\tilde{K}_i(t,s) \leq K_i(t,s)$, $\tilde{M}_l(t,s) \geq M_l(t,s)$ and all solutions of (5.4) are oscillatory, then all solutions of (5.1) are also oscillatory.

Proposition 5.3. *Suppose (C1)–(C3), (I1*)–(I3*), (M1) hold and either there exists such k that $\int_0^\infty \big[a_k(t) - b_k(t)\big] dt = \infty$ or there exists such i that*

$$\int_0^\infty \Big[\int_{\tilde{h}_i(t)}^t K_i(t,s)\, ds - \int_{\tilde{g}_i(t)}^t M_i(t,s)\, ds \Big] dt = \infty.$$

Then any nonoscillatory solution of (5.1) tends to zero at infinity.

Remark. Similar result can be obtained if (C1)–(C3), I1*)–(I3*), (M2) are satisfied.

Proposition 5.4 presents oscillation conditions for (5.1).

Proposition 5.4. *Suppose (C1)–(C3), (I1*)–(I3*), (M1) and the following inequality hold*

$$\begin{aligned} &\liminf_{t\to\infty} \Big\{ \sum_{k=1}^{n} [a_k(t) - b_k(t)](t - h_k(t)) \\ &+ \sum_{k=1}^{n} b_k(t) \Big(\exp\Big\{ \int_{h_k(t)}^{g_k(t)} [a_k(s) - b_k(s)] ds \Big\} - 1 \Big) (t - g_k(t)) \\ &+ \sum_{i=1}^{r} \int_{\tilde{h}_i(t)}^t \Big[\int_{\tilde{h}_i(s)}^s K_i(t,\tau)\, d\tau - \int_{\tilde{g}_i(s)}^s M_i(t, \tau - \tilde{h}_i(t) + \tilde{g}_i(t))\, d\tau \Big] ds \\ &+ \sum_{i=1}^{r} \int_{\tilde{g}_i(t)}^t d\tau \int_{\tilde{h}_i(\tau)}^\tau \Big(\exp\Big\{ \int_{s+\tilde{h}_i(t)-\tilde{g}_i(t)}^s \Big[\int_{\tilde{h}_i(u)}^u K_i(u,\tau) d\tau \\ &- \int_{\tilde{g}_i(u)}^u M_i(u,\tau) d\tau \Big] du \Big\} - 1 \Big) M_i(t,s) ds \Big\} > \frac{1}{e} \end{aligned}$$

Then all solutions of (5.1) are oscillatory.

Remark. Similarly to inequality 1) in Corollary 2.9 other oscillation conditions for (5.1) can be deduced using inequalities 2)-4) of this corollary.

Proposition 5.5 present nonoscillation conditions.

Proposition 5.5. *Suppose (C1)–(C3), (I1*)–(I3*), (M1) and the following inequality holds*

$$\begin{aligned}&\limsup_{t\to\infty}\Big\{\sum_{k=1}^n\int_{h_k(t)}^t\big[a_k(s)-\frac{1}{e}b_k(s)\big]\,ds\\&+\sum_{i=1}^r\int_{\tilde h_i(t)}^t ds\Big[\int_{\tilde h_i(s)}^s K_i(s,\tau)\,d\tau-\frac{1}{e}\int_{\tilde g_i(s)}^s M_i(s,\tau)\,d\tau\Big]\Big\}<\frac{1}{e}.\end{aligned}$$

Then (5.1) has a nonoscillatory solution.

Remark. Similar results are obtained (see Propositions 3.6, 4.7 and 4.8) if (M2) is satisfied instead of (M1).

As a final example, consider the following equation of the mixed type

$$\dot x(t)+\sum_{k=1}^n a_k(t)x(h_k(t))-\int_0^t K(t,s)x(s)\,ds=0, \tag{5.5}$$

under the following conditions:

(m1) $a_k\ge 0$ are Lebesgue measurable bounded functions, K is Lebesgue integrable over each finite square $[0,b]\times[0,b]$;

(m2) There exists finite function $g(t)=\inf\{s|K(t,s)>0\}$ and $\lim_{t\to\infty}g(t)=\infty$; $\lim_{t\to\infty}h_k(t)=\infty$ for each k;

(m3) For any $t\ge s\ge 0$, $\sum_{k=1}^n a_k(t)\chi_{[h_k(t),\infty)(s)\ge\int_{g(s)}^s K(t,\tau)\,d\tau}$.

Consider also the following hypothesis

(m4) There exist constants $c_1,\dots,c_n$, $\sum_{k=1}^n c_k=1$ such that

$$a_k(t)\chi_{[h_k(t),\infty)}(s)\ge c_k\int_{s-h_k(t)}^{s-h_k(t)+g(t)}K(t,\tau)\,d\tau$$

and

$$\limsup_{t\to\infty}\Big\{\Big[\sum_{k=1}^n c_k(g(t)-h_k(t))\Big]\int_{g(t)}^t K(t,s)\,ds\Big\}<1.$$

Proposition 5.6. *Suppose (m1)–(m3) hold. Consider the following hypotheses*

(1) *There exists $t_1\ge 0$ such that the inequality*

$$u(t)\ge\sum_{k=1}^n a_k(t)\exp\Big\{\int_{h_k(t)}^t u(s)ds\Big\}-\int_{g(t)}^t K(t,s)\exp\Big\{\int_s^t u(\tau)d\tau\Big\},\ t\ge t_1$$

has a nonnegative locally integrable solution (we assume $u(t)=0$ for $t<t_1$);

(2) *There exists $t_2\ge 0$ such that the fundamental function of (5.5) $X(t,s)>0$, $t\ge s\ge t_2$;*

(3) *Equation (5.5) has a nonoscillatory solution;*

(4) *The inequality*

$$\dot y(t)+\sum_{k=1}^n a_k(t)y(h_k(t))-\int_0^t K(t,s)y(s)\,ds\le 0 \tag{5.6}$$

has an eventually positive solution.

Then the implications $1)\Rightarrow 2)\Rightarrow 3)\Rightarrow 4)$ *are valid. If in addition (m4) holds then hypotheses 1)–4) are equivalent.*

To deduce a comparison result we introduce the equation

$$\dot{x}(t)+\sum_{k=1}^{n} b_k(t)x(\tilde{h}_k(t))-\int_0^t M(t,s)x(s)\,ds=0. \tag{5.7}$$

Proposition 5.7. *1) Suppose (m1)-(m4) hold, where a_k, h_k, K are changed by $b_k, \tilde{h}_k, M$, respectively. If $b_k(t)\ge a_k(t), \tilde{h}_k(t)\le h_k(t), M(t,s)\ge K(t,s)$ for each t,s,k and (5.7) has a nonoscillatory solution, then (5.5) also has a nonoscillatory solution.*
2) Suppose (m1)-(m4) hold. If $b_k(t)\le a_k(t), \tilde{h}_k(t)\ge h_k(t), M(t,s)\le K(t,s)$ for each t,s,k and all solutions of (5.7) are oscillatory, then all solutions of (5.5) are also oscillatory.

Proposition 5.8. *Suppose (m1)–(m4) hold and*

$$\int_0^{\infty}\Big[\sum_{k=1}^{n} a_k(t)-\int_{g(t)}^{t} K(t,s)\,ds\Big]dt=\infty.$$

Then any nonoscillatory solution x of (5.5) satisfies $\lim_{t\to\infty} x(t)=0$.

Note that Corollary 2.9, 1) implies the following result.

Proposition 5.9. *Suppose (m1)–(m4) and the following inequality hold*

$$\begin{aligned}\liminf_{t\to\infty}\Big\{&\sum_{k=1}^{n}\int_{h_k(t)}^{t}\big[a_k(s)-c_k\int_{s-h_k(t)}^{s-h_k(t)+g(t)} K(t,\tau)\,d\tau\big]ds+\sum_{k=1}^{n} c_k\int_{g(t)}^{t} d\tau\\ &\times\int_{h_k(\tau)}^{\tau}\Big(\exp\Big\{\int_{s+h_k(t)-g(t)}^{s}\big[a_k(u)-\int_{g(u)}^{u} K(u,\zeta)\,d\zeta\big]du\Big\}-1\Big)K(t,s)ds\Big\}>\frac{1}{e}.\end{aligned}$$

Then all solutions of (5.5) are oscillatory.

Remark. Similarly inequalities 2)-4) in Corollary 2.9 can be rewritten for (5.5).

Proposition 5.10. *Suppose (m1)–(m4) and the inequality*

$$\limsup_{t\to\infty}\Big\{\sum_{k=1}^{n}\int_{h_k(t)}^{t}\big[a_k(s)-\frac{c_k}{e}\int_{g(s)}^{s} K(s,\tau)\,d\tau\big]ds\Big\}<\frac{1}{e}$$

holds. Then (5.5) has a nonoscillatory solution.

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