HOPF-TYPE ESTIMATES FOR SOLUTIONS TO HAMILTON-JACOBI EQUATIONS WITH CONCAVE-CONVEX INITIAL DATA

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Abstract. We consider the Cauchy problem for the Hamilton-Jacobi equations with concave-convex initial data. A Hopf-type formula for global Lipschitz solutions and estimates for viscosity solutions of this problem are obtained using techniques of multifunctions and convex analysis.

1. Introduction

This paper is a continuation of the works [10] and [8], where the explicit solutions via Hopf-type formulas of the Cauchy problem to the Hamilton-Jacobi equations with concave-convex Hamiltonians were considered. Namely, we consider the Cauchy problem for the Hamilton-Jacobi equation

$$\frac{\partial u}{\partial t} + H(t, \frac{\partial u}{\partial x}) = 0 \quad \text{in} \quad U := \{t > 0, \ x \in \mathbb{R}^n\}$$

$$u(0, x) = \phi(x) \quad \text{on} \quad \{t = 0, \ x \in \mathbb{R}^n\}. \quad (1.2)$$

Here $\partial/\partial x = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$, the Hamiltonian $H = H(t, p)$ and $\phi = \phi(x)$ are given functions, and $u = u(t, x)$ is unknown.

In this paper we shall assume that $n = n_1 + n_2$ and that the variable $x \in \mathbb{R}^n$ is separated as $x = (x', x'')$ with $x' \in \mathbb{R}^{n_1}$, $x'' \in \mathbb{R}^{n_2}$, similarly for $p, q, \cdots \in \mathbb{R}^n$. In particular, the zero-vector in $\mathbb{R}^n$ will be $0 = (0', 0'')$, where $0'$ and $0''$ stand for the zero-vectors in $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively.

Definition. A function $g = g(x', x'')$ is called concave-convex if it is concave in $x' \in \mathbb{R}^{n_1}$ for each $x'' \in \mathbb{R}^{n_2}$ and convex in $x'' \in \mathbb{R}^{n_2}$ for each $x' \in \mathbb{R}^{n_1}$.

For results on the concave-convex functions the reader is referred to [7], [8], [10].

In [10, Chapter 10], Van, Tsuji and Thai Son proposed to examine a class of concave-convex functions in a more general framework where the discussion of the global Legendre transformation still make sense.

Bardi and Faggian [2] found explicit pointwise upper and lower bounds of Hopf-type for the viscosity solutions under the following hypotheses: $H$ depends only on
$p$ and is a concave-convex function given by the difference of convex functions,

$$H(p', p'') := H_1(p') - H_2(p''),$$

and $\phi$ is uniformly continuous. Also if $H \in C(\mathbb{R}^n)$ and $\phi = \phi(x)$ is concave-convex function given by special representation $\phi(x) = \phi_1(x) - \phi_2(x)$, where $\phi_1, \phi_2$ are convex and Lipschitz continuous.


The aim of this paper is to look for explicit global Lipschitz solution of the Cauchy problem (1.1)–(1.2) and to establish pointwise upper and lower bounds of viscosity solutions to the Hamilton-Jacobi equations with concave-convex hamiltonians via Hopf-type formulas.

The aim of this paper is to look for explicit global Lipschitz solution of the Cauchy problem (1.1)–(1.2) and to establish pointwise upper and lower bounds of viscosity solutions when the initial function $\phi(x) = \phi(x')$ is concave-convex on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.

**Definition.** A function $u = u(t, x)$ in $\text{Lip}(\bar{U})$ will be called a global Lipschitz solution of the Cauchy problem (1.1)–(1.2) if it satisfies (1.1) almost everywhere (a.e.) in $U$, with $u(0, x) = \phi(x)$ for all $x \in \mathbb{R}^n$.

2. **Hopf-type formula for global Lipschitz solutions**

We consider the Cauchy problem for the Hamilton-Jacobi equation

$$u_t + H(t, Du) = 0 \quad \text{in } U := \{ t > 0, \ x \in \mathbb{R}^n \} \tag{2.1}$$

$$u(0, x) = \phi(x) \quad \text{on } \{ t = 0, \ x \in \mathbb{R}^n \}, \tag{2.2}$$

where the Hamiltonian $H$ depends on the variable $t$ and the spatial derivatives $Du$.

We note that Van, Tsuji, Hoang and Thai Son [9], [10] have obtained a Hopf-type formula with the initial function $\phi = \phi(x)$ nonconvex and $H$ merely continuous. Moreover, a global Lipschitz solution of (2.1)–(2.2) is given by an explicit Hopf-type formula in the following case (see Chap. 9, [10]): The Hamiltonian (depends explicitly on $t$) $H = H(t, p)$ is continuous in $U_G := \{(t, p) : t \in (0, +\infty) \setminus G, \ p \in \mathbb{R}^n \}$ where $G$ is closed subset of $\mathbb{R}$ with Lebesgue measure zero; and, for each $N \in (0, +\infty)$ corresponds a function $g_N := g_N(t) \in L^\infty_{\text{loc}}(\mathbb{R})$ so that

$$\sup_{|p| \leq N} |H(t, p)| \leq g_N(t) \quad \text{for almost } t \in (0, +\infty);$$

while the initial function $\phi = \phi(x)$ satisfies one of the following two conditions:

1. $\phi = \phi_1 - \phi_2$, where $\phi_1, \phi_2$ are convex functions;
2. $\phi$ is minimum of a family of convex functions.

In this section, we look for explicit global Lipschitz solutions of problem (2.1)–(2.2), where $x \in \mathbb{R}^n$, $n = n_1 + n_2$, $x = (x', x'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and the initial-valued function $\phi = \phi(x) := \phi(x', x'')$ is a strictly concave-convex function on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.
Lemma 2.2. Let \( \text{lemmas 10.5 and 10.6 in [10].} \)

\[
\begin{align*}
  \lim_{|x''| \to +\infty} \phi(x', x'') &= +\infty \text{ for each } x' \in \mathbb{R}^{n_1}; \\
  \lim_{|x'| \to +\infty} \phi(x', x'') &= -\infty \text{ for each } x'' \in \mathbb{R}^{n_2}.
\end{align*}
\] (2.3) (2.4)

We now consider the Cauchy problem (2.1)–(2.2) with the following hypotheses:

(M1) The Hamiltonian \( H = H(t, p) \) is continuous in

\[ U_G := \{(t, p) : t \in (0, +\infty) \setminus G, p \in \mathbb{R}^n \} \]

with \( G \) be a closed subset of \( \mathbb{R} \) with Lebesgue measure 0. Moreover, for each \( N \in (0, +\infty) \) there corresponds a function \( g_N := g_N(t) \in L^\infty_{\text{loc}}(\mathbb{R}) \) so that

\[ \sup_{|p| \leq N} |H(t, p)| \leq g_N(t) \text{ for almost } t \in (0, +\infty); \]

(M2) The equality

\[ \sup_{p'' \in \mathbb{R}^{n_2}} \inf_{p' \in \mathbb{R}^{n_1}} \varphi(t, x, p) = \inf_{p'' \in \mathbb{R}^{n_2}} \sup_{p' \in \mathbb{R}^{n_1}} \varphi(t, x, p) \]

is satisfied in \( U \), where

\[ \varphi(t, x, p) := \langle p, x \rangle - \phi^*(p) - \int_0^t H(\tau, p) d\tau \] (2.5)

for \((t, x) = (t, x', x'') \in U, p = (p', p'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \). Here, \( \phi^* \) denotes the conjugate of \( \phi \) which is defined as in Section 3 later.

(M3) To each bounded subset \( V \) of \( U \) there corresponds a positive number \( N(V) \) so that

\[ \max_{|q''| \leq N(V)} \inf_{q' \in \mathbb{R}^{n_1}} \varphi(t, x, q', q'') > \inf_{q'' \in \mathbb{R}^{n_2}} \varphi(t, x, q', p''), \]

\[ \min_{|q'| \leq N(V)} \sup_{q'' \in \mathbb{R}^{n_2}} \varphi(t, x, q', q'') < \sup_{q'' \in \mathbb{R}^{n_2}} \varphi(t, x, p', q''), \]

whenever \((t, x) \in V, p \in (p', p'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \) and \( \min\{|p'|, |p''|\} > N(V) \).

The main result of this Section is as follows.

**Theorem 2.1.** Let \( \phi \) be a strictly concave-convex function on \( \mathbb{R}^n \) with (2.3)–(2.4) and assume M1–M3. Then the formula

\[ u(t, x) := \sup_{p'' \in \mathbb{R}^{n_2}} \inf_{p' \in \mathbb{R}^{n_1}} \varphi(t, x, p) = \inf_{p'' \in \mathbb{R}^{n_2}} \sup_{p' \in \mathbb{R}^{n_1}} \varphi(t, x, p), \] (2.6)

for \((t, x) \in U\), determines a global Lipschitz solution of the Cauchy problem (2.1)–(2.2).

To prove this theorem, we need the following lemmas, which are similar to the lemmas 10.5 and 10.6 in [10].

**Lemma 2.2.** Let \( \mathcal{O} \) be an open subset of \( \mathbb{R}^m \), and \( \eta = \eta(q, p) = \eta(q, p', p'') \) be a continuous function on \( \mathcal{O} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \) with the following properties:
Here we put satisfies all the assumptions of Lemma 2.2, where \( \eta \). We can verify that the function

\[
L_o \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}
\]

Proof of Theorem 2.1. Suppose that the conditions 1–2 in Lemma 2.2 are satisfied for a nonempty subset \( E \subseteq \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \) such that \( \eta(\xi, p) \) is finite on \( \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \) and \( \eta(\xi, p) \equiv -\infty \) on \( \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \). Moreover, for each bounded subset \( V \) of \( \mathcal{O} \), corresponds a positive number \( N(V) \) such that

\[
\max_{|q''| \leq N(V)} \inf_{q' \in \mathbb{R}^{n_1}} \eta(\xi, q', q'') > \inf_{q' \in \mathbb{R}^{n_1}} \eta(\xi, q', q'').
\]

and

\[
\min_{|q''| \leq N(V)} \sup_{q' \in \mathbb{R}^{n_1}} \eta(\xi, q', q'') < \sup_{q' \in \mathbb{R}^{n_1}} \eta(\xi, q', q''),
\]

whenever \( \xi \in V \), \( p = (p', p'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \) and \( \min\{|p'|, |p''|\} > N(V) \); (3) For each fixed \( p \) of \( E \), \( \eta \) is differentiable in \( \xi \in \mathcal{O} \) with continuous gradient

\[
\partial \eta/\partial \xi = \partial \eta(\xi, p)/\partial \xi
\]

on \( \mathcal{O} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \).

Then we have:

i. The function

\[
\psi = \psi(\xi) := \sup_{p' \in \mathbb{R}^{n_1}} \inf_{p'' \in \mathbb{R}^{n_2}} \eta(\xi, p) = \sup_{p' \in \mathbb{R}^{n_1}} \inf_{p'' \in \mathbb{R}^{n_2}} \eta(\xi, p)
\]

is a locally Lipschitz continuous on \( \mathcal{O} \).

ii. \( \psi = \psi(\xi) \) is directionally differentiable in \( \mathcal{O} \) with

\[
\partial_e \psi(\xi) = \max_{p'' \in L''(\xi)} \min_{p' \in L'(\xi)} \langle \partial \eta(\xi, p', p'')/\partial \xi, e \rangle
\]

where

\[
L'(\xi) := \{ p' \in \mathbb{R}^{n_1} : \sup_{p'' \in \mathbb{R}^{n_2}} \eta(\xi, p', p'') = \psi(\xi) \} \quad (2.7)
\]

\[
L''(\xi) := \{ p'' \in \mathbb{R}^{n_2} : \inf_{p' \in \mathbb{R}^{n_1}} \eta(\xi, p', p'') = \psi(\xi) \}. \quad (2.8)
\]

Lemma 2.3. Suppose that the conditions 1–2 in Lemma 2.2 are satisfied for a continuous function \( \eta = \eta(\xi, p', p'') \) on \( \mathcal{O} \times \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \). Then \( 2.7 \)–\( 2.8 \) determines the non-empty valued, closed, locally bounded multifunction \( L = L(\xi) := L'(\xi) \times L''(\xi), \xi \in \mathcal{O} \).

Proof of Theorem 2.1. We can verify that the function

\[
\eta = \eta(\xi, p) := \varphi(t, x, p)
\]

satisfies all the assumptions of Lemma 2.2, where

\[
E := \text{dom } \phi^* \neq 0, \quad m := 1 + n = 1 + n_1 + n_2, \quad \xi := (t, x).
\]

Here we put \( \mathcal{O} := \bar{U} \) and conclude that

\[
L(t, x) = L'(t, x) \times L''(t, x) = \{ p \in E : \varphi(t, x, p) = u(t, x) \}.
\]
determines a nonempty-valued, locally bounded, closed multifunction \( L = L(t,x) \) of \((t,x) \in \bar{U}\). Take arbitrary an \( r \in (0, +\infty) \) and denote
\[
V_r = \{(t,x) \in \bar{U} : t + |x| < r\}, \quad N_r = N(V_r).
\]
Let \( g_N = g_N(t) \) as be in the condition M1. Then for any two points \((t^1, x^1)\) and \((t^2, x^2)\) in \( V_r \), we may choose an element \( p = (p^1, p^2) \in L'(t^1, x^1) \times L''(t^2, x^2) \) of the nonempty set
\[
L'(t^1, x^1) \times L''(t^2, x^2) \subset \bar{B}^{n_1}(0', N_r) \times \bar{B}^{n_2}(0', N_r)
\]
and get
\[
u(t^2, x^2) - \nu(t^1, x^1) = \inf \varphi(t^2, x^2, p', p'') - \sup \varphi(t^1, x^1, p', p'') \leq \varphi(t^2, x^2, p', p'') - \varphi(t^1, x^1, p', p'') = \varphi(t^2, x^2, p) - \varphi(t^1, x^1, p) = (p, x^2 - x^1) + \int_{t_2}^{t_1} H(\tau, p) d\tau \leq N_r \cdot |x^2 - x^1| + s_r \cdot |t^2 - t^1|
\]
where \( s_r = \text{esssup}_{t \in (0, r)} g_N(t) \). Dually,
\[
u(t^1, x^1) - \nu(t^2, x^2) \leq N_r \cdot |x^2 - x^1| + s_r \cdot |t^2 - t^1|.
\]
Hence, \( \nu = \nu(t, x) \) is a locally Lipschitz continuous in \( \bar{U} \) and thus it be long to Lip(\( \bar{U} \)). Next, let \( e^o := (1,0,0,\ldots,0,0) \), \( e^1 := (0,1,0,\ldots,0,0) \), \ldots, \( e^n := (0,0,0,\ldots,0,1) \) \( \in \mathbb{R}^{n+1} \). We now replace in Lemma 2.2 the set \( \mathcal{O} := U_G \). From this lemma we see that \( u = u(t, x) \) is directionally differentiable in \( U_G \) with
\[
\frac{\partial e^o}{\partial x} u(t, x) = \max_{p' \in L'(t,x)} \min_{p'' \in L''(t,x)} \{-H(t,p), p \in L(t,x)\} = \min_{p' \in L'(t,x)} \max_{p'' \in L''(t,x)} \{-H(t,p), p \in L(t,x)\},
\]
\[
\frac{\partial e^o}{\partial t} u(t, x) = \max_{p' \in L'(t,x)} \min_{p'' \in L''(t,x)} \{H(t,p), p \in L(t,x)\} = \max_{p' \in L'(t,x)} \min_{p'' \in L''(t,x)} \{H(t,p), p \in L(t,x)\};
\]
and for \( 1 \leq i \leq n: \)
\[
\frac{\partial e_i}{\partial x} u(t, x) = \max_{p' \in L'(t,x)} \min_{p'' \in L''(t,x)} \{p_i, p \in L(t,x)\} = \min_{p' \in L'(t,x)} \max_{p'' \in L''(t,x)} \{p_i, p \in L(t,x)\},
\]
\[
\frac{\partial e_i}{\partial t} u(t, x) = \max_{p' \in L'(t,x)} \min_{p'' \in L''(t,x)} \{-p_i, p \in L(t,x)\} = \min_{p' \in L'(t,x)} \max_{p'' \in L''(t,x)} \{-p_i, p \in L(t,x)\}. \tag{2.9}
\]
Since \( u = u(t, x) \) is locally Lipschitz continuous in \( \bar{U} \), according to Rademacher’s Theorem, there exists a set \( \mathcal{Q} \subset U \) of \((n + 1)\) dimensional Lebesgue measure 0 such that \( u = u(t, x) \) is differentiable with
\[
\frac{\partial u(t, x)}{\partial \tau} = \frac{\partial e^o}{\partial \tau} u(t, x) = -\frac{\partial e^o}{\partial \tau} u(t, x),
\]
\[
\frac{\partial u(t, x)}{\partial x_i} = \frac{\partial e_i}{\partial x} u(t, x) = -\frac{\partial e_i}{\partial x} u(t, x) \tag{2.10}
\]
at any point \((t, x) \in U \setminus Q\). Hence, (2.9)–(2.10) show that the equalities for \(1 \leq i \leq n\),

\[
\frac{\partial u(t, x)}{\partial x_i} = \max_{p' \in L''(t, x)} \min_{p'' \in L'(t, x)} \{p_n, p \in L(t, x)\}
\]

\[
= \min_{p' \in L'(t, x)} \max_{p'' \in L''(t, x)} \{p_n, p \in L(t, x)\}
\]

\[
\frac{\partial u(t, x)}{\partial x} = \max_{p' \in L''(t, x)} \min_{p'' \in L'(t, x)} \{p_n, p \in L(t, x)\}
\]

\[
\frac{\partial u(t, x)}{\partial x} = \max_{p' \in L'(t, x)} \min_{p'' \in L''(t, x)} \{p_n, p \in L(t, x)\}
\]

hold for all \((t, x) \in U \setminus \{P := (G \times \mathbb{R}^n) \cup Q\} =: U_P\), this implies

\[
L(t, x) = \left\{ \frac{\partial u(t, x)}{\partial x} \right\}, \quad (t, x) \in U_P;
\]

and we obtain

\[
\frac{\partial u(t, x)}{\partial t} = \{-H(t, p), p \in L(t, x)\}.
\]

Thus,

\[
\frac{\partial u(t, x)}{\partial t} + H(t, \frac{\partial u(t, x)}{\partial x}) = -H(t, \frac{\partial u(t, x)}{\partial x}) + H(t, \frac{\partial u(t, x)}{\partial x}) = 0
\]

hold almost everywhere in \(U\). Furthermore

\[
u(0, x) = u(0, x', x'')
\]

\[
= \sup_{p' \in \mathbb{R}^{n_2}} \inf_{p'' \in \mathbb{R}^{n_1}} \{\langle p', x' \rangle + \langle p'', x'' \rangle - \psi^*(p', p'')\}
\]

\[
= \inf_{p' \in \mathbb{R}^{n_1}} \sup_{p'' \in \mathbb{R}^{n_2}} \{\langle p', x' \rangle + \langle p'', x'' \rangle - \psi^*(p', p'')\}
\]

\[
= (\psi^*(p', p'')) = \psi(x', x'') = \psi(x)
\]

for all \(x = (x', x'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\). From what has already been proved, we conclude that \(u = u(t, x)\) is a global Lipschitz solution of the Cauchy problem (2.1)–(2.2).

\[
\square
\]

**Remark 2.4.** If \(n_2 = 0\), we obtain the Hopf-type formulas of the Cauchy problem for the convex initial data as in Chapter 8 [10].

**Remark 2.5.** Assume (M1), (M2). Then (M3) is satisfied if

\[
\inf_{p' \in \mathbb{R}^{n_1}} \varphi(t, x, p', p'') \rightarrow -\infty \quad \text{locally uniformly in } (t, x) \in U \text{ as } |p''| \rightarrow +\infty
\]

and

\[
\sup_{p'' \in \mathbb{R}^{n_2}} \varphi(t, x, p', p'') \rightarrow +\infty \quad \text{locally uniformly in } (t, x) \in U \text{ as } |p'| \rightarrow +\infty
\]

i.e., if the following statement holds:

For any \(\lambda \) and \(\mu \) in \(\mathbb{R}\) and any bounded subset \(V \) of \(U\), there exists positive numbers \(N(\lambda, V)\) and \(N(\mu, V)\), respectively, so that

\[
\inf_{q' \in \mathbb{R}^{n_1}} \varphi(t, x, q', p'') < \lambda \quad \text{whenever } (t, x) \in V, |p''| > N(\lambda, V)
\]

and

\[
\sup_{q'' \in \mathbb{R}^{n_2}} \varphi(t, x, p', q'') > \mu \quad \text{whenever } (t, x) \in V, |p'| > N(\mu, V).
\]
Indeed, fix an arbitrary \( q^0 = (q^0, q''^0) \) in the domain of \( \phi \), which is not empty. Since the finite function \( \bar{U} \ni (t, x) \mapsto \varphi(t, x, q^0) \) is continuous, it follows that: for any bounded subset \( V \) of \( \bar{U} \),

\[
\lambda_V := \inf_{(t,x) \in V} \varphi(t,x,q^0) > -\infty, \\
\mu_V := \sup_{(t,x) \in V} \varphi(t,x,q^0) < +\infty.
\]

Under the hypothesis above, we certainly find a number \( N(\lambda, V) \geq |q''^0| \) (for each such \( V \)) so that

\[
\inf_{q'' \in \mathbb{R}^{n_2}} \varphi(t,x,q',q'') < \lambda_V = \inf_{(t,x) \in V} \varphi(t,x,q''^0, q''^0)
\]

when \( (t, x) \in V \) and \( |p''| > N(\lambda, V) \),

\[
\inf_{q'' \in \mathbb{R}^{n_2}} \varphi(t,x,q',p'') < \varphi(t,x,q''^0, q''^0)
\]

when \( (t, x) \in V \), \( |p''| > N(\lambda, V) \),

\[
\inf_{q'' \in \mathbb{R}^{n_2}} \varphi(t,x,q',q'') < \inf_{q'' \in \mathbb{R}^{n_1}} \varphi(t,x,q'', q'')
\]

when \( (t, x) \in V \), \( |p''| > N(\lambda, V) \),

\[
\inf_{q'' \in \mathbb{R}^{n_2}} \varphi(t,x,q',q'') < \max_{|q''| \leq N(\lambda, V)} \inf_{q'' \in \mathbb{R}^{n_1}} \varphi(t,x,q', q'')
\]

when \( (t, x) \in V \), \( |p''| > N(\lambda, V) \).

Analogously, we also obtain

\[
\sup_{q'' \in \mathbb{R}^{n_2}} \varphi(t,x,p',q'') > \min_{|q''| \leq N(\mu, V)} \sup_{q'' \in \mathbb{R}^{n_1}} \varphi(t,x,q'', q'')
\]

when \( (t, x) \in V \), \( |p'| > N(\mu, V) \), where \( N(\mu, V) \geq |q''^0| \). Hence (M3) is satisfied.

### 3. Hopf-type estimates for viscosity solutions

Consider the Cauchy problem for the Hamilton-Jacobi equation

\[
\begin{align*}
\frac{\partial u}{\partial t} + H\left(\frac{\partial u}{\partial x}\right) &= 0 \quad \text{in } U := \{t > 0, x \in \mathbb{R}^n\} \\
&\text{on } \{t = 0, x \in \mathbb{R}^n\}.
\end{align*}
\]

When \( H = H(p) \) is continuous and \( \phi = \phi(x) \) is uniformly continuous, the Cauchy problem \((3.1)-(3.2)\) has a unique viscosity solution \( u = u(t, x) \) which is in the space of continuous functions that are uniformly continuous in \( x \) uniformly in \( t \), \( UC_2([0, +\infty) \times \mathbb{R}^n) \) (see [5]). We also refer the readers to [4,5] for the definition and properties of viscosity solutions.

In the case of Lipschitz continuous and convex (or concave) initial data \( \phi \) and merely continuous Hamiltonian \( H \), or for convex \( \phi \) and Lipschitz continuous \( H \), the formula

\[
u(t,x) = \sup_{p \in \mathbb{R}^n} \{ (p,x) - \phi^*(p) - tH(p) \}
\]

determines a (unique) viscosity solution \( u = u(t,x) \in UC_2([0, +\infty) \times \mathbb{R}^n) \) of the problem \((3.1)-(3.2)\). Here \( \phi^* \) denotes the Legendre transform of \( \phi \) (see, [1,2]).
In this section we are interested in giving explicit pointwise upper and lower bounds for viscosity solutions where the initial function $\phi = \phi(x', x'')$ is concave-convex. First, we rewrite some main results on the conjugate of the concave-convex functions (for the details, see [10, Chapter 10]). Let $\phi = \phi(x', x'')$ be a concave-convex function on $\mathbb{R}^n \times \mathbb{R}^m$. Then
\[
\phi^*(p', x'') = \inf_{x' \in \mathbb{R}^n_1} \{ \langle x', p' \rangle - \phi(x', x'') \}
\]
and
\[
\phi^*(p', x'') = \sup_{x'' \in \mathbb{R}^m_2} \{ \langle x'', p'' \rangle - \phi(x', x'') \}
\]
is the Fenchel conjugate of $x'$-concave (resp. $x''$-convex) function $\phi(x', x'')$.

If $\phi = \phi(x', x'')$ is concave-convex function with conditions (2.3)–(2.4), then $\phi^*(p', x'')$ (resp. $\phi^*(x', p'')$) is concave (resp. convex) not only in $p' \in \mathbb{R}^n_1$ (resp. $p'' \in \mathbb{R}^m_2$) but also in the whole variable $(p', x'') \in \mathbb{R}^n_1 \times \mathbb{R}^m_2$ (resp. $(x', p'') \in \mathbb{R}^n_1 \times \mathbb{R}^m_2$) and
\[
\lim_{|p'| \to +\infty} \frac{\phi^*(p', x'')}{|p'|} = -\infty \quad \text{(resp. } \lim_{|p''| \to +\infty} \frac{\phi^*(x', p'')}{|p''|} = +\infty
\]
locally uniformly in $x'' \in \mathbb{R}^m_2$ (resp. $x' \in \mathbb{R}^n_1$). Besides the Fenchel “partial conjugate” $\phi^*$ and $\phi^*$, we consider two “total conjugate” of $\phi$:
\[
\varphi^*(p', p'') = \inf_{x' \in \mathbb{R}^n_1} \{ \langle x', p' \rangle + \phi^2(x', p'') \}
\]
and
\[
\varphi^*(p', p'') = \sup_{x'' \in \mathbb{R}^m_2} \{ \langle x', p' \rangle + \phi^2(x', p'') \}
\]
Therefore, the functions $\varphi^*$ and $\varphi^*$ are usually called the upper and lower conjugate, respectively, of $\phi$. Note that
\[
\varphi^* \leq \varphi^*.
\]
These functions are also concave-convex, and with (2.3)–(2.4) they coincide. In this situation, the Fenchel conjugate
\[
\phi^* := \varphi^* = \varphi^*
\]
of $\phi$ will simultaneously have the properties
\[
\lim_{|p'| \to +\infty} \frac{\phi^*(p', p'')}{|p''|} = +\infty \quad \text{for each } p' \in \mathbb{R}^n_1
\]
and
\[
\lim_{|p'| \to +\infty} \frac{\phi^*(p', p'')}{|p'|} = -\infty \quad \text{for each } p'' \in \mathbb{R}^m_2.
\]
If (2.3)–(2.4) are not assumed, the partial conjugates $\phi^*$ and $\phi^*$ are still concave and convex, respectively, but might be infinite somewhere, then the lower and upper conjugates $\varphi^*$ and $\varphi^*$ might not coincide. One can claim only that
\[
\phi^*(p', x'') < +\infty, \quad \forall (p', x'') \in \mathbb{R}^n_1 \times \mathbb{R}^m_2;
\]
\[
\phi^*(x', p'') > -\infty, \quad \forall (x', p'') \in \mathbb{R}^n_1 \times \mathbb{R}^m_2.
\]
Now let
\[ D_1 := \{ p' \in \mathbb{R}^{n_1} : \phi^*(p', x'') > -\infty \forall x'' \in \mathbb{R}^{n_2} \}, \]
\[ D_2 := \{ p'' \in \mathbb{R}^{n_2} : \phi^2(x', p'') < +\infty \forall x' \in \mathbb{R}^{n_1} \}, \]
hence for all \( x'' \in \mathbb{R}^{n_2} \), \( \phi^*(p', x'') \) is finite on \( D_1 \), and for all \( x' \in \mathbb{R}^{n_1} \), \( \phi^2(x', p'') \) is finite on \( D_2 \).

We now consider the Cauchy problem (3.1)-(3.2) with the hypothesis:

(M4) The Hamiltonian \( H = H(p) \) is continuous and the initial function \( \phi = \phi(x', x'') \) is concave-convex and Lipschitz continuous (without (2.3)-(2.4)).

For \((t, x) \in U\), we set
\[
\begin{align*}
    u_-(t, x) &:= \sup_{p' \in D_1} \inf_{p'' \in \mathbb{R}^{n_2}} \{ \langle p, x \rangle - \tilde{\phi}^*(p) - tH(p) \} \tag{3.3} \\
u_+(t, x) &:= \inf_{p' \in D_1} \sup_{p'' \in \mathbb{R}^{n_2}} \{ \langle p, x \rangle - \phi^*(p) - tH(p) \}. \tag{3.4}
\end{align*}
\]

**Remark 3.1.** The concave-convex function \( \phi = \phi(x', x'') \) is Lipschitz continuous in the sense: \( \phi(x', x'') \) is Lipschitz continuous in \( x' \in \mathbb{R}^{n_1} \) for each \( x'' \in \mathbb{R}^{n_2} \) and in \( x'' \in \mathbb{R}^{n_2} \) for each \( x' \in \mathbb{R}^{n_1} \).

Our estimates for viscosity solutions in this section read as follows:

**Theorem 3.2.** Assume (M4). Then the unique viscosity solution \( u = u(t, x) \in UC_2([0, +\infty) \times \mathbb{R}^n) \) of the Cauchy problem (3.1)-(3.2) satisfies on \( U \) the inequalities
\[
    u_-(t, x) \leq u(t, x) \leq u_+(t, x),
\]
where \( u_- \) and \( u_+ \) are defined by (3.3) and (3.4) respectively.

**Proof.** For each \( p' \in D_1 \), let
\[
\Phi(x; p') = \Phi(x', x''; p') := \langle x', p' \rangle - \phi^*(p', x'') - \phi^1(p', x'')
\]
\[
= \langle x', p' \rangle - \inf_{x'' \in \mathbb{R}^{n_2}} \{ \langle x', p' \rangle - \phi(x', x'') \}
\]
\[
\geq \phi(x', x'') \quad \text{for all } (x', x'') \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}.
\]
Since \( \phi^1(p', .) \) is a concave and finite, so \( -\phi^1(p', .) \) is convex and finite, it is convex and Lipschitz continuous function; therefore, \( \Phi(x; p') \) is convex and Lipschitz continuous with its Fenchel conjugate given by
\[
\Phi^*(p; p') = \Phi^*(p', p''; p') = \sup_{x \in \mathbb{R}^n} \{ \langle x, p \rangle - \Phi(x, p') \}
\]
\[
= \sup_{x \in \mathbb{R}^n} \{ \langle x', p' \rangle + \langle x'', p'' \rangle - \langle x', p' \rangle + \phi^*(p', x'') \}
\]
\[
= \begin{cases} 
+\infty & \text{if } (p', p'') \neq (p', p'') \\
\phi^*(p', p'') & \text{if } (p', p'') = (p', p'').
\end{cases}
\]

Next, consider the Cauchy problem
\[
\frac{\partial v}{\partial t} + H(\frac{\partial v}{\partial x}) = 0 \quad \text{in } U = \{ t > 0, x \in \mathbb{R}^n \},
\]
\[
v(0, x) = \Phi(x; p') \quad \text{on } \{ t = 0, x \in \mathbb{R}^n \}.
\]
This is the Cauchy problem with the continuous Hamiltonian $H = H(p)$ and the convex and Lipschitz continuous initial function $\Phi = \Phi(x; p')$ for each $p' \in D_1$, its unique viscosity solution $v = v(t, x) \in UC_x([0, +\infty) \times \mathbb{R}^n)$ is given by

$$v(t, x) = \sup_{p \in \mathbb{R}^n} \{\langle p, x \rangle - \Phi^*(p; p') - tH(p)\}$$

$$= \sup_{p'' \in \mathbb{R}^n} \{\langle p', x' \rangle + \langle p'', x'' \rangle - \Phi^*(p', p'') - tH(p', p'')\}$$

with the initial condition

$$v(0, x) = \Phi(x; p') \geq \phi(x) = u(0, x)$$

for each $p' \in D_1$ (see [1]). Hence, for each $p' \in D_1$, $v = v(t, x)$ is a (continuous) supersolution of the problem (3.1)–(3.2) (according to a standard comparison theorem for unbounded viscosity solutions (see [5])), that means

$$u(t, x) \leq v(t, x) \quad \text{for each } p' \in D_1,$$

and then

$$u(t, x) \leq \sup_{p \in D_1} \inf_{p'' \in \mathbb{R}^n} \{\langle p, x \rangle - \Phi^*(p) - tH(p)\}$$

$$u(t, x) \leq u_+(t, x) \quad \text{on } \bar{U}.$$

Dually, we also obtain $u(t, x) \geq u_-(t, x)$ on $\bar{U}$. Therefore, Theorem 3.2 has been proved. $\square$

**Corollary 3.3.** Assume (M1), (M2) for the case when $H(t, p)$ is not depending on $t$. Moreover, assume that $\phi = \phi(x', x'')$ is concave-convex and Lipschitz continuous function on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and satisfies the conditions (2.3)–(2.4). Then (2.6) determines the unique viscosity solution $u(t, x) \in UC_x([0, +\infty) \times \mathbb{R}^n)$ of the Cauchy problem (3.1)–(3.2).

**Proof.** Since $\phi = \phi(x', x'')$ is a concave-convex and Lipschitz continuous function so $\text{dom} \phi^*$ is a bounded and nonempty set. Independently of $(t, x) \in \bar{U}$, it follows that

$$\varphi(t, x, p', p'') \to -\infty \quad \text{whenever } |p''| \text{ is large enough}$$

$$\varphi(t, x, p', p'') \to +\infty \quad \text{whenever } |p'| \text{ is large enough}.$$ 

From Remark 2.5 implies that hypothesis (M3) hold. Then the conclusion follows from Theorem 3.2. $\square$

**References**


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