Electronic Journal of Differential Equations, Vol. 2003(2003), No. 87, pp. 1–64.
ISSN: 1072-6691. URL: http://ejde.math.swt.edu or http://ejde.math.unt.edu
ftp ejde.math.swt.edu (login: ftp)

THE GENERALIZED AIRY DIFFUSION EQUATION

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ABSTRACT. Solutions of a generalized Airy diffusion equation and an associated nonlinear partial differential equation are obtained. Trigonometric type functions are derived for a third order generalized radial Euler type operator. An associated complex variable theory and generalized Cauchy-Euler equations are obtained. Further, it is shown that the Airy expansions can be mapped onto the Bessel Calculus of Bochner, Cholewinski and Haimo.

1. INTRODUCTION

The Airy diffusion equation arises from the radial part of a third order Laplace type operator on n-dimensional Euclidean space. In the one-dimensional case, the Airy Diffusion equation of Widder [33] is obtained. The difficult problem of representation of solutions encountered by Widder persists in the generalized Airy equation case.

In this paper we obtain a sequence of polynomial solutions of the Airy diffusion equation, which are analogous to the heat polynomials of Widder or the heat polynomials associated with the generalized heat polynomials of Cholewinski and Haimo [11] or of L. R. Bragg [8]. In the classical cases the heat polynomials are modified Hermite polynomials and therefore have an orthogonality relation with respect to a positive measure. The diffusion polynomial solutions obtained in this paper are 3-parity polynomials and therefore by a result of Daboul and Rathie [14] they can not be orthogonal in the usual sense.

We also relate the solutions of the generalized Airy equation to solutions of a nonlinear diffusion type partial differential equation. The diffusion polynomial solutions lead to dispersive waves which vanish at infinity.

Let $F(x_1, x_2, \dots, x_n) = F(r)$ be a radial function on n-dimensional Euclidean space, where $r = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$. Then a calculation shows that

$$\sum_{k=1}^{n} \frac{r}{x_k} \frac{\partial^3 F}{\partial x_k^3} = \frac{\partial^3 F}{\partial r^3} + \frac{3(n-1)}{r} \frac{\partial^2 F}{\partial r^2} - \frac{3(n-1)}{r^2} \frac{\partial F}{\partial r} \tag{1.1}$$

2000 *Mathematics Subject Classification.* 33C10, 35C05, 44A15.
Key words and phrases. Diffusion polynomial solutions, Bessel calculus, positive definite kernels.

Submitted February 3, 2003. Published August 22, 2003.

Let ν be a fixed nonnegative number, we define a linear third order differential operator ϑ_ν by

$$\vartheta_\nu = \frac{d^3}{dz^3} + \frac{3\,\nu}{z}\,\frac{d^2}{dz^2} - \frac{3\,\nu}{z}\,\frac{d}{dz} \tag{1.2}$$

The generalized Airy diffusion equation is defined by

$$\vartheta_\nu\, u(x,t) = \frac{\partial u(x,t)}{\partial t} \tag{1.3}$$

If $\nu = 0$, we have the Airy equation of Widder [33], and if $\nu = n-1$ we have a radial diffusion on n-dimensional Euclidean space. If ν is not an integer, we have an analogue of the situation encountered by Bochner [7], Weinstein [31, 32] and others in the Bessel function case. The operator ϑ_ν can be factored as

$$\vartheta_\nu = \frac{d}{dz}\,(z^{-3\nu}\,\frac{d}{dz}\,z^{3\nu}\,\frac{d}{dx}) = \frac{d}{dz}\,\Delta_z(\frac{2}{3}\,\nu) \tag{1.4}$$

where

$$\Delta_z(\mu) = \frac{d^2}{dz^2} + \frac{2\mu}{z}\,\frac{d}{dz} \tag{1.5}$$

is the radial part of the Laplace operator on n-dimensional Euclidean space R^n with $\mu = \frac{n-1}{2}$.

A number of solutions of the third order radial diffusion are shown to be related to solutions of the radial heat equation. In fact we show that the source solution of the generalized Airy diffusion equation is mapped onto the source solution of the radial heat equation. In the case that $\nu = 0$, that is the one dimensional radial diffusion, the source kernel is mapped onto the normal distribution function.

2. Preliminary Results

Let ν be a fixed nonnegative number. A simple calculation shows that

$$\vartheta_\nu x^{3n} = 3^3 n(n+\nu-1/3)(n-2/3)x^{3(n-1)} \tag{2.1}$$

and therefore ϑ_ν acts as a delta operator on the "basic" sequence $\{x^{3n}\}_{n=0}^{\infty}$. By iteration we find that

$$\vartheta_\nu^k x^{3n} = 3^{3k}\,\frac{\Gamma(n+1)\Gamma(n+\nu+2/3)\Gamma(n+1/3)}{\Gamma(n-k+1)\Gamma(n-k+\nu+2/3)\Gamma(n-k+1/3)}x^{3(n-k)} \tag{2.2}$$

For $k = n$, we get

$$\begin{aligned}\vartheta_\nu^n x^{3n} &= 3^3 n!\,\frac{\Gamma(n+1/3)}{\Gamma(1/3)}\,\frac{\Gamma(n+\nu 2/3)}{\Gamma(\nu+2/3)}\\ &= 3^{3n}(1)_n(1/3)_n(\nu+2/3)_n\\ &= \alpha(3n,\nu) := \alpha_{3n}(\nu),\end{aligned} \tag{2.3}$$

where

$$(a)_n = a(a+1)(a+2)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{2.4}$$

is the Pockhammer rising factorial function.

A Humbert type of Bessel function $G_\nu(z)$ is defined as the hypergeometric function

$$\begin{aligned}G_\nu(z) &=_0 F_2(1/3,\nu+2/3\,|\,(z/3)^3)\\ &= \sum_{n=0}^{\infty}\frac{\Gamma(1/3)\Gamma(\nu+2/3)z^{3n}}{3^{3n}\,n!\,\Gamma(n+1/3)\Gamma(n+\nu+2/3)}\end{aligned} \tag{2.5}$$

for $z \in \mathbb{C}$, the complex numbers. A calculation using Stirling's formula, shows that

$$\frac{\log \alpha_{3n}(\nu)}{3n \log 3n} \to 1 \quad \text{as } n \to \infty$$

and

$$\frac{3n}{(\alpha_{3n}(\nu))^{\frac{1}{3n}}} \to e \quad \text{as } n \to \infty$$

It follows that $G_\nu(z)$ is an entire function of order one and of exponential type 1. By Marichev [25], we also have the asymptotic estimate

$$G_\nu(z) \sim \frac{\Gamma(1/3)\Gamma(\nu+2/3)3^\nu}{\sqrt(3)\,2\pi}\,\frac{e^z}{z^\nu} \quad \text{as } z \to \infty,\ |\arg z| < \pi \tag{2.6}$$

Furthermore, for $\mathcal{G}_\nu(z)$ defined by $\mathcal{G}_\nu(z) = G_\nu(-z)$ we have

$$\begin{aligned}\mathcal{G}_\nu(z) &= G_\nu(-z) =_0 F_2[1/3, \nu + 2/3 \,|\, -(z/3)^3] \\ &\sim \frac{\Gamma(1/3)\Gamma(\nu+2/3)}{\sqrt{3}\,\pi}\,\frac{1}{(z/3)^\nu}\, e^{z/2} \cos(z\,\frac{\sqrt{3}}{2} - \frac{\nu\,\pi}{9})\end{aligned} \tag{2.7}$$

with $\arg z = 0$. Using the estimate in [3, p. 47],

$$\frac{\Gamma(\nu+2/3)}{\Gamma(\nu+2/3+n)} \sim (\nu+2/3)^{-n} \ \sim\ \nu^{-n} \quad \text{as } \nu \to \infty$$

and the Lebesgue Dominated Convergence Theorem, it follows that

$$G_\nu(3\,\nu^{1/3}\,(\frac{x}{2})^{2/3}) \to 2^{-2/3}\,\Gamma(1/3)x^{2/3}\,I_{-2/3}(x) \quad \text{as } \nu \to \infty \tag{2.8}$$

where $I_\nu(x)$ is a modified Bessel function of the first kind. Thus for large ν, behaves as an entire modified Airy Bessel function. Recall that

$$\Gamma(\nu+1)\,(z/2)^{-\nu}\,I_\nu(z) = \sum_{m=0}^{\infty} \frac{z^{2m}}{2^{2m}\,m!\,(\nu+1)_m} \tag{2.9}$$

is an entire function for $\nu > -1$.

In the Widder case of $\nu = 0$, we get upon application of the Gauss cubic factorial equation that

$$G_0(z) = \sum_{k=0}^{\infty} \frac{z^{3k}}{(3k)!} = \frac{1}{3}\,(e^z + 2\,e^{-z/2} \cos \frac{\sqrt{3}}{2}\,z) \tag{2.10}$$

Using (2.1) it readily follows that

$$\vartheta_\nu\, G_\nu(xy) = \vartheta_x^\nu\, G_\nu(xy) = y^3\, G_\nu(xy) \tag{2.11}$$

Thus G_ν should play the role of the exponential function in a calculus associated with the ϑ_ν operator.

Next we define a generalized addition formula associated with the ϑ_ν operator. This addition in terms of hypergeometric functions is an analogue of the addition for Bessel functions presented by Bochner [7]. If x and y are arbitrary complex numbers and n is a nonnegative integer, we define

$$\begin{aligned}
&(x \oplus_\nu y)^{3n} \\
&= x^{3n}\, {}_3F_2\left[\begin{matrix} -n, -n+2/3, -n+1/3-\nu \\ 1/3, \nu+2/3 \end{matrix}\;\middle|\; -(\frac{y}{x})^3\right] \\
&= \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(\nu+2/3)\Gamma(1/3)\Gamma(n+\nu+2/3)\, y^{3k} x^{3(n-k)}}{\Gamma(k+1/3)\Gamma(n-k+1/3)\Gamma(n-k+\nu+2/3)\Gamma(k+\nu+2/3)}
\end{aligned} \tag{2.12}$$

It readily follows that $(x \oplus_\nu y)^{3n}$ is a solution of the partial differential equation

$$\frac{\partial^3 u}{\partial x^3} + \frac{3\nu}{x}\frac{\partial^2 u}{\partial x^2} - \frac{3\nu}{x^2}\frac{\partial u}{\partial x} = \frac{\partial^3 u}{\partial y^3} + \frac{3\nu}{y}\frac{\partial^2 u}{\partial y^2} - \frac{3\nu}{y^2}\frac{\partial u}{\partial y} \tag{2.13}$$

which satisfies the boundary conditions $u(x,0) = x^{3n}$ and $u(0,y) = y^{3n}$. (2.13) can be considered as a third order Euler-Poisson-Darboux equation, see [13] or Weinstein [32]. The ordinary wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2} \tag{2.14}$$

has solutions $f(x+y)$ corresponding to a boundary value function $f(x)$, whereas (2.13) has in general solutions $f(x \oplus_\nu y)$ corresponding to boundary value functions with 3rd order symmetry. Using Whipple's equation, see Henrici [20], p. 43, we also have that

$$(x \oplus_\nu y)^{3n} = (x^3+y^3)^n\, {}_3F_2\left[\begin{matrix} -n/2, \frac{1-n}{2}, -n+\nu \\ 1/3, \nu+2/3 \end{matrix}\;\middle|\; \frac{4(xy)^3}{(x^3+y^3)^2}\right] \tag{2.15}$$

a result that relates the cubic addition associated with ϑ_ν to ordinary addition. Thus $(x \oplus_\nu y)^{3n} \sim (x^3+y^3)^n$ as $x \to \infty$ for y fixed, for the ${}_3F_2$ polynomial goes to its constant term as x goes to infinity.

Furthermore, a calculation employing Equations 2.2 and 2.3 yields the operational equation

$$G_\nu(y\,\vartheta_x(\nu)^{1/3})x^{3n} = (x \oplus_\nu y)^{3n} \tag{2.16}$$

This equation is the 3rd order analogue of the extremely important equation

$$e^{yD}x^n = (x+y)^n \tag{2.17}$$

where $D = \frac{d}{dx}$ is the derivative operator. In the case of $\nu = 0$, we get the binomial formula

$$(1 \oplus_0 x)^{3n} = \sum_{k=0}^{n} \binom{3n}{3k} x^{3k} \tag{2.18}$$

In the particular case of $x = 1$, we get the evaluation

$$(1 \oplus_0 1)^{3n} = \sum_{k=0}^{n} \binom{3n}{3k} = \frac{1}{3}\,(2^{3n} + (-1)^n 2) \tag{2.19}$$

See [18, p. 3]. Furthermore, a series multiplication yields

$$G_\nu(x)\, G_\nu(y) = G_\nu(x \oplus_\nu y) \tag{2.20}$$

which is the 3rd order analogue of the fundamental relation $e^x\, e^y = e^{x+y}$. Using Stirling's formula, we also obtain the third order binomial limit

$$(1 \oplus_\nu \frac{x}{n})^{3n} \to G_\nu(x) \quad \text{as } n \to \infty \tag{2.21}$$

Once again this is the analogue of the elementary calculus result

$$(1+\frac{x}{n})^n \to e^x \quad \text{as } n \to \infty \tag{2.22}$$

The function

$$\mathcal{G}_\nu(z) = G_\nu(-z) = G_\nu(\beta_3 z) ={}_0 F_2\big[\frac{1}{3}, \nu+2/3| -(\frac{z}{3})^3\big]$$

where $\beta_3 = e^{i\pi/3}$ is a primitive root of -1, which plays the role of e^{-x} in many calculations. However, it is not the multiplicative inverse of $G_\nu(z)$ as is seen in the following development. By Erdélyi [17, Volume 1, p. 186], we have

$${}_0F_2(a,b|z)\,{}_0F_2(a,b|-z) = {}_3F_8\left[\begin{array}{c} \frac{1}{3}(a+b-1), \frac{1}{3}(a+b+1) \\ a, b, \frac{a}{2}, \frac{a+1}{2}, \frac{b}{2}, \frac{b+1}{2}, \frac{a+b-1}{2}, \frac{a+b}{2} \end{array} \middle| -\frac{27}{64}z^2\right]$$

Taking $a = 1/3$ and $b = \nu + 2/3$, we get

$$\begin{aligned} &G_\nu(z)\,\mathcal{G}_\nu(z) \\ &={}_3 F_8\left[\begin{array}{c} \nu/3, \frac{\nu+1}{3}, \frac{\nu+2}{3} \\ 1/3, \nu+2/3, 1/6, 2/3, \frac{1}{2}(\nu+2/3), \frac{1}{2}(\nu+5/3), \frac{\nu}{2}, \frac{\nu+1}{2} \end{array} \middle| -\frac{z^6}{12^3}\right] \\ &= G_\nu((1 \oplus_\nu (-1))z) \end{aligned} \tag{2.23}$$

Since

$$\mathcal{G}_0(x) = \frac{1}{3}(e^{-x} + 2\,e^{x/2}\cos\frac{\sqrt{3}}{2}x)$$

we find that

$$G_0(x)\,\mathcal{G}_0(x) = \frac{1}{9}\{3 + 4\cosh(\frac{3}{2}x)\cos(\frac{\sqrt{3}}{2}x) + 2\cos(\sqrt{3}x)\}$$

The generalized translation of a function $f(x) \in C^\infty$ is defined by

$$G_\nu(y\,\vartheta_\nu^{1/3})\,f(x) = \sum_{n=0}^\infty \frac{y^{3n}}{\alpha_{3n}}\,\vartheta_\nu^n\,f(x) = f(x \oplus_\nu y)$$

provided that the infinite series converges locally uniformly in x and y. In Section 3 we show that if $f(x)$ is an entire function then $f(x \oplus_\nu y)$ is also an entire function in the variables x and y. The translation operator can also be defined for formal power series. Next we let $\mathcal{M}_\nu(R^+)$ be the collection of positive measures on R^+ such that the integral $\int_0^\infty e^{\tau\,y^2}\,d\gamma(y)$ is finite for $\tau \geq \tau_0 \geq 0$ and $d\gamma(y) \in \mathcal{M}_\nu$. If $\gamma(y)$ is an increasing function on R^+ with compact spectrum, that is the set of points of increase are contained in a compact set, then the Stieltjes measure $d\gamma(y)$ is in $\mathcal{M}_\nu(R^+)$. Let $d\gamma(y)$ be an element of $\mathcal{M}_\nu(R^+)$ and let $f(x) = \int_0^\infty \mathcal{G}_\nu(xy)\,d\gamma(y)$. Since $G_\nu(x)$ is an entire function of order one and type one, the integral converges locally uniformly in x.

By the uniform convergence, we also get

$$\vartheta_\nu^n\,f(x) = \int_0^\infty (-y)^{3n}\,\mathcal{G}_\nu(xy)\,d\gamma(y)$$

which is also uniformly convergent. Interchanging the summation and integration we get

$$G_\nu(z\,\vartheta_\nu^{1/3})f(x) = \int_0^\infty \mathcal{G}_\nu(zy)\,\mathcal{G}_\nu(xy)\,d\gamma(y) = f(x \oplus_\nu z)$$

The uniform convergence justifies the interchange of summation and integration. For x and z in a compact set; we have

$$\begin{aligned}\Big|\int_0^\infty G_\nu(zy)\,G_\nu(xy)\,d\gamma(y)\Big| &\le \int_0^\infty G_\nu(Ay)\,G_\nu(By)\,d\gamma(y)\\ &\le M\int_0^\infty e^{(A+B)y}\,d\gamma(y) < \infty\end{aligned}$$

The functions $G_\nu(z)$ and $\mathcal{G}_\nu(z)$ have Poisson type integral representations for $\nu \ge 1$ or $\operatorname{Re}\nu \ge 1$.

Theorem 2.1. *Let $\nu \ge 1$, then*

$$G_\nu(z) = \frac{\Gamma(\nu+2/3)}{3\,\Gamma(2/3)\,\Gamma(\nu)}\int_0^1 \tau\,(1+\tau^3)^{\nu-1}\,(e^{z\tau} + 2\,e^{-\frac12 z\tau}\cos\frac{\sqrt3}{2}\,z\tau)\,d\tau \tag{2.24}$$

and

$$\mathcal{G}_\nu(z) = \frac{\Gamma(\nu+2/3)}{3\,\Gamma(2/3)\,\Gamma(\nu)}\int_0^1 \tau\,(1+\tau^3)^{\nu-1}\,(e^{-z\tau} + 2\,e^{\frac12 z\tau}\cos\frac{\sqrt3}{2}\,z\tau)\,d\tau \tag{2.25}$$

Proof. Using Gauss' cubic equation $(3n)! = \frac{3^{3n+1/2}}{2\pi}\,n!\,\Gamma(n+1/3)\,\Gamma(n+2/3)$ and the integral representation of the Beta function, the general term of the $G_\nu(z)$ can be written as

$$\begin{aligned}&\frac{\Gamma(1/3)\,\Gamma(\nu+2/3)\,z^{3n}}{3^{3n}\,n!\,\Gamma(n+1/3)\,\Gamma(n+\nu+2/3)}\\ &= \frac{\sqrt3}{2\pi}\,\frac{\Gamma(1/3)\,\Gamma(\nu+2/3)}{\Gamma(\nu)}\,\frac{z^{3n}}{(3n)!}\int_0^1 t^{\nu-1}\,(1-t)^{n-1/3}\,dt\end{aligned}$$

Now the series $\sum_{n=1}^\infty \frac{z^{3n}}{(3n)!}t^{\nu-1}\,(1-t)^{n-1/3}$ converges uniformly with respect to t in the interval $[0,1]$ for $\nu \ge 1$. Since the term with $n = 0$ converges, we can interchange the summation. Hence

$$\begin{aligned}G_\nu(z) &= \frac{\sqrt3}{2\pi}\,\frac{\Gamma(1/3)\,\Gamma(\nu+2/3)}{\Gamma(\nu)}\int_0^1 t^{\nu-1}\,(1-t)^{-1/3}\sum_{n=1}^\infty \frac{z^{3n}\,(1-t)^n}{(3n)!}\,dt\\ &= \frac{\Gamma(\nu+2/3)}{\Gamma(2/3)\,\Gamma(\nu)}\int_0^1 t^{\nu-1}\,(1-t)^{-1/3}\,G_0((1-t)^{1/3}\,z)\,dt\end{aligned}$$

With the change of variable $\tau = (1-t)^{1/3}$, we get

$$G_\nu(z) = \frac{\Gamma(\nu+2/3)}{\Gamma(2/3)\,\Gamma(\nu)}\int_0^1 \tau\,(1-\tau^3)^{\nu-1}\,(e^{z\tau} + 2\,e^{-\frac12 z\tau}\cos\frac{\sqrt3}{2}\,z\tau)\,d\tau$$

(2.25) also follows from this result. □

The integral representations (2.24) and (2.25), yield the inequalities

$$|G_\nu(x)| = G_\nu(x) \le \frac{\Gamma(\nu+2/3)}{\Gamma(2/3)\,\Gamma(\nu)}\,3\,e^x$$

and

$$|\mathcal{G}_\nu(x)| \le \frac{\Gamma(\nu+2/3)}{\Gamma(2/3)\,\Gamma(\nu)}\,3\,e^{x/2}$$

for $x \ge 0$ and $\nu \ge 1$.

Next we introduce the entire Bessel functions

$$\mathbf{J}(z) = 2^{\nu-1/2}\,\Gamma(\nu+1/2)\,z^{1/2-\nu}\,J_{\nu-1/2}(z), \tag{2.26}$$

$$\mathbf{I}(z) = 2^{\nu-1/2}\,\Gamma(\nu+1/2)\,z^{1/2-\nu}\,I_{\nu-1/2}(z) \tag{2.27}$$

where $J_{\nu-1/2}(z)$ is the ordinary Bessel function of order $\nu - 1/2$ and $I_{\nu-1/2}(z)$ is the Bessel function of imaginary argument. We have the series expansion

$$\mathbf{J}_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n\,z^{2n}}{b_{2n}(\nu)} \tag{2.28}$$

where $b_{2n}(\nu) = 2^{2n}\,n!\,(\nu+1/2)_n = 2^{2n}\,n!\,\frac{\Gamma(\nu+1/2+n)}{\Gamma(\nu+1/2)}$. The Bessel function $J_\mu(z)$ has the asymptotic expansion

$$J_\mu(z) \simeq (\frac{2}{\pi\,z})^{1/2}\cos(z - \frac{1}{2}\,\mu\,\pi - \pi/4),\ -\pi < \arg z < \pi \tag{2.29}$$

see Erdelyi [16, Volume 2, p. 85]. In the next section it is shown that the source solution for the third order diffusion is an ordinary Bessel function.

3. The Source Function

The formal Dirac delta function associated with our third order calculus is

$$\mathcal{D}_\nu(x) = \int_0^\infty \mathcal{G}_\nu(xy)\,d\eta_\nu(y) \tag{3.1}$$

where $d\eta_\nu(y) = \frac{y^{3\nu+1}\,dy}{3^{\nu-1/3}\,\Gamma(\nu+2/3)}$. This is analogous to the classical representation

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixy}\,dy \tag{3.2}$$

or the Bessel representation

$$D_\nu(x) = \int_0^\infty \mathbf{J}_\nu(xy)\,d\mu_\nu(y) \tag{3.3}$$

see [11]. In general, solutions of the third order diffusion equation

$$\frac{\partial}{\partial t}\,u(x,t) = \frac{\partial^3 u}{\partial x^3} + \frac{3\nu}{x}\,\frac{\partial^2 u}{\partial x^2} - \frac{3\nu}{x^2}\,\frac{\partial \nu}{\partial x} = \vartheta_\nu\,u(x,t) \tag{3.4}$$

are formally given by the semigroup operation

$$u(x,t) = e^{t\,\vartheta_\nu}\,f(x) \tag{3.5}$$

with $u(x,0) = f(x)$.

We say that a function $u(x,t)$ in $C^3([0,a])$ in x and $C^1(0 \le t \le \sigma)$ in t is in $\mathcal{H}_\nu([0,a]\times[0,\sigma])$ if

$$\frac{\partial}{\partial t}\,u(x,t) = \vartheta_\nu\,u(x,t)$$

in the set $[0,a]\times[0,\sigma]$. We call a function in the class $\mathcal{H}_\nu$ an Airy or ν-diffusion. Taking $f(x) = \mathcal{D}_\nu(x)$, we get

$$\begin{aligned}\mathcal{K}_\nu(x,t) = e^{t\,\vartheta_\nu}\,\mathcal{D}_\nu(x) &= e^{t\,\vartheta_\nu}\int_0^\infty \mathcal{G}_\nu(xy)\,d\eta_\nu(y)\\ &= \int_0^\infty e^{-tx^3}\,\mathcal{G}_\nu(xy)\,d\eta_\nu(y)\end{aligned} \tag{3.6}$$

Using the series expansion (2.6), term by term integration, and a change of variables, we get

$$\mathcal{K}_\nu(x,t) = \frac{1}{(3t)^{\nu+2/3}} \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{3n} t^{-n}}{3^{3n}\, n!\,(1/3)_n} \tag{3.7}$$

Since $\mathcal{G}$ is an entire function of order 1, the term by term integration is valid based on the absolutely and uniform convergence of the integral. Further, we have the representations

$$\begin{aligned} \mathcal{K}_\nu(x,t) &= \frac{1}{(3t)^{\nu+2/3}}\, {}_0F_1(1/3\,|\, \frac{-x^3}{27t}) \\ &= \frac{\Gamma(4/3)}{3^{\nu+2/3}} t^{\nu+1} x\, J_{-2/3}(\frac{2x^{3/2}}{\sqrt{27t}}) \\ &= \frac{1}{(3t)^{\nu+3/2}}\, \mathbf{J}_{-1/6}(\frac{2x^{3/2}}{\sqrt{27t}}) \end{aligned} \tag{3.8}$$

Based on the asymptotic expansion (2.29), we see that

$$\mathcal{K}_\nu(x,t) \sim \frac{\Gamma(2/3)}{\pi}\, 3^{\nu+3/2} t^{\nu+19/12}\, (\frac{x}{3})^{1/4} \cos(\frac{2x^{3/2}}{\sqrt{27t}} + \frac{\pi}{12}) \tag{3.9}$$

as $x \to \infty$. In the classical case for the ordinary heat equation and for the radial heat equation the source solutions are Gaussian type functions and go to zero at infinity of order e^{-cx^2}. In contrast, the source solution $\mathcal{K}_\nu(x,t)$ oscillates between infinite values for large values of x.

Applying the translation operator to the third order delta function we formally get

$$\begin{aligned} G_\nu(z\vartheta_\nu)\, \mathcal{D}_\nu(x) &= \mathcal{D}_\nu(x \oplus_\nu z) \\ &= \int_0^\infty G_\nu(z\vartheta_x)\, \mathcal{G}_\nu(xy)\, d\eta_\nu(y) \\ &= \int_0^\infty \mathcal{G}_\nu(x \oplus_\nu z)\, d\eta_\nu(y) \\ &= \int_0^\infty \mathcal{G}_\nu(xy)\, \mathcal{G}_\nu(zy)\, d\eta_\nu(z) \end{aligned} \tag{3.10}$$

Hence the Poisson type integral gives solutions of the third order diffusion equation (3.4) with $u(x,0) = f(x)$ has the formal solution

$$u(x,t) = e^{t\vartheta_\nu}\, f(x) = \int_0^\infty \mathcal{K}_\nu(x \oplus_\nu z, t)\, f(z)\, d\eta_\nu(z) \tag{3.11}$$

Indeed the Poisson type integral gives solutions of the third order diffusion equation (3.4) provided the function has suitable behavior at infinity. It is easy to see that if f is a C^∞ function with compact support in $[0,\infty)$, then $u(x,t)$ given by (3.11) is a solution of the third order diffusion equation. The case $\nu = 0$ is associated with Widder's Airy transform [33].

To study the integral transform (3.11) we need to have order estimates on the translated kernel $\mathcal{K}_\nu(x \oplus_\nu z, t)$. Let

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^x\,(1-t)^{y-1}\,dt$$

denote the beta function. From the integral representation for the beta function, it readily follows that if $0 < x \le x_1$ and $0 < y \le y_1$, then

$$B(x_1, y_1) \le B(x, y)$$

or equivalently

$$\frac{\Gamma(x+y)}{\Gamma(x)\,\Gamma(y)} \le \frac{\Gamma(x_1+y_1)}{\Gamma(x_1)\,\Gamma(y_1)} \tag{3.12}$$

Theorem 3.1. *If $\nu \ge 0$ and $x, y \ge 0$, then*

$$(x \oplus_\nu y)^{3n} \le \Gamma(1/3)\,\Gamma(\nu+2/3)\,(n+2\{\nu\}+1)^{3(\{\nu\}+1)}\,(x+y)^{3(n+\{\nu\})} \tag{3.13}$$

where $\{\nu\} = \operatorname{ceil}(\nu)$.

Proof. . For $k \ne 0$ or n, using (3.11) we see that

$$\begin{aligned}
&\frac{(\nu+2/3)_n}{(\nu+2/3)_{n-k}\,(\nu+2/3)_k}\\
&= \frac{\Gamma(\nu+2/3)\,\Gamma(n+\nu+2/3)}{\Gamma(n+2\nu+4/3)}\,\frac{\Gamma(n+2\nu+4/3)}{\Gamma(n-k+\nu+2/3)\,\Gamma(k+\nu+2/3)}\\
&\le \frac{\Gamma(\nu+2/3)}{(n+\nu+2/3)_{[\nu+2/3]}}\,\frac{\Gamma(n+2\{\nu\}+2)}{\Gamma(n-k+\{\nu\}+1)\,\Gamma(k+\{\nu\}+1)}\\
&\le \frac{\Gamma(\nu+2/3)}{(n+\nu+2/3)_{[\nu+2/3]}}\,\frac{\Gamma(n+2\{\nu\}+2)}{\Gamma(n+\{\nu\}+1)}\binom{n+\{\nu\}}{k}\\
&\le \Gamma(\nu+2/3)\,\frac{(n+\{\nu\}+1)_{\{\nu\}+1}}{(n+\nu+2/3)_{[\nu+2/3]}}\binom{n+\{\nu\}}{k}
\end{aligned} \tag{3.14}$$

where $[x]$ is the greatest integer less than or equal to x. Employing the asymptotic expansion for the quotients of Gamma functions, it follows that

$$\frac{(n+\{\nu\}+1)_{\{\nu\}+1}}{(n+\nu+2/3)_{[\nu+2/3]}} \sim n^{\{\nu\}+1-[\nu+2/3]} = O(n^2)$$

as $n \to \infty$. In the same manner, we get

$$\frac{(1/3)_n}{(1/3)_{n-k}\,(1/3)_k} \le \frac{\Gamma(1/3)\,\Gamma(n+1/3)}{\Gamma(n+2/3)}\,(n+1)\binom{n}{k} \le \Gamma(1/3)\,(n+1)\binom{n}{k}$$

Note that $\Gamma(n+1/3)/\Gamma(n+2/3) \sim n^{-1/3}$ as $n \to \infty$. Next we obtain an estimate on the binomial. We have

$$\begin{aligned}
\binom{n}{k} &= \frac{\Gamma(n+1)}{\Gamma(n+2)}\,\frac{\Gamma(n+2)}{\Gamma(n-k+1)\,\Gamma(k+1)}\\
&\le \frac{1}{n+1}\,\frac{\Gamma(n+2\{\nu\}+2)}{\Gamma(n-k+\{\nu\}+1)\,\Gamma(k+\{\nu\}+1)}\\
&\le \frac{1}{n+1}\,\frac{\Gamma(n+2\{\nu\}+2)}{\Gamma(n+\{\nu\}+1)}\,\frac{\Gamma(n+\{\nu\}+1)}{\Gamma(n-k+\{\nu\}+1)\,k!}\\
&= \frac{1}{n+1}\,(n+\{\nu\}+1)_{\{\nu\}+1}\binom{n+\{\nu\}}{k}
\end{aligned} \tag{3.15}$$

It follows that

$$\begin{aligned}\begin{bmatrix}\alpha_{3n}\\ \alpha_{3k}\end{bmatrix} &= \binom{n}{k}\frac{(1/3)_n}{(1/3)_{n-k}\,(1/3)_k}\,\frac{(\nu+2/3)_k}{(\nu+2/3)_{n-k}\,(\nu+2/3)_k}\\ &\le \frac{\Gamma(\nu+2/3)\,\Gamma(1/3)}{(1+n)}\,\frac{\Gamma(n+1/3)}{\Gamma(n+2/3)}\,\frac{(n+\{\nu\}+1)^3_{\{\nu\}+1}}{(n+\nu+2/3)_{[\nu+2/3]}}\binom{n+\{\nu\}}{k}^3\\ &\le \Gamma(\nu+2/3)\,\Gamma(1/3)\,(n+2\{\nu\}+1)^{3(\{\nu\}+1)}\binom{n+\{\nu\}}{k}^3\end{aligned}\tag{3.16}$$

Thus we find that

$$\begin{aligned}&(x\oplus_\nu y)^{3n}\\ &= x^{3n}+y^{3n}+\sum_{k=1}^{n-1}\binom{\alpha_{3n}(\nu)}{\alpha_{3k}(\nu)}x^{3k}\,y^{3(n-k)}\\ &\le \Gamma(1/3)\,\Gamma(\nu+1/3)\,(n+2\{\nu\}+1)^{3(\{\nu\}+1)}\sum_{k=0}^{n}\binom{n+\{\nu\}}{k}^3 x^{3k}\,y^{3(n-k)}\\ &\le \Gamma(1/3)\,\Gamma(\nu+1/3)\,(n+2\{\nu\}+1)^{3(\{\nu\}+1)}\,(x+y)^{3(n+\{\nu\})}\end{aligned}\tag{3.17}$$

Let $f(z) = \sum_{n=0}^{\infty} a_n\, z^{3n}$ be an entire function, then by the Cauchy-Hadamard formula

$$\rho_r = [\limsup_{n\to\infty} |a_n|^{1/3n}]^{-1} = \infty$$

Consider $f(z\oplus w) = \sum_{n=0}^{\infty} a_n\,(z\oplus_\nu w)^{3n}$, we have

$$|f(z\oplus w)| \le \Gamma(1/3)\,\Gamma(\nu+2/3)\,(|z|+|w|)^{3\{\nu\}}\sum_{n=0}^{\infty}|a_n|\,(n+2\{\nu\}+1)^{3(\{\nu\}+1)}\,(|z|+|w|)^{3n}$$

Since $\limsup(n+2\{\nu\}+1)^{\frac{3(\{\nu\}+1)}{3n}} = 1$, it follows that

$$[\limsup |a_n|^{\frac{1}{3n}}\,(n+2\{\nu\}+1)^{\frac{3(\{\nu\}+1)}{3n}}]^{-1} = [\limsup |a_n|^{1/3n}]^{-1} = \infty$$

Thus $f(z\oplus w)$ is an entire function in the z and w variables.

Further, let $\nu_0 = \limsup_{n\to\infty} 3n\,|a_n|^{\frac{\rho}{3n}}$. Recall that if $0 < \nu_0 < \infty$ then $f(z)$ is of order ρ and type τ if and only if $\nu = e\tau\rho$. If $\nu_0 = 0$ then f is of growth $(\rho, 0)$ and if $\nu_0 = \infty$ then f is of growth not less than (ρ,∞).

If $f(z)$ is of growth (ρ,τ), we find that

$$\limsup_{n\to\infty} 3n\,|a_n(n+2\{\nu\}+1)^{3(\{\nu\}+1)}|^{\frac{\rho}{3n}} = \limsup_{n\to\infty} 3n\,|a_n|^{\frac{\rho}{3n}} = e\tau\rho$$

Thus for fixed complex w, $f(z\oplus w)$ is also of growth (ρ,τ). The translated functions occur as kernels in various integral transforms. Thus the convergence of the transforms can be related to the growth properties of non-translated functions $f(z)$.

Applying the above results to $\mathcal{K}_\nu(x,t)$ and using Stirling's formula, it follows that

$$\rho = \limsup_{n\to\infty}\frac{3n\log 3^n}{log(\frac{1}{3^{3n}\,t^n\,(1/3)_n\,n!})} = 3/2$$

and

$$\nu_0 = \limsup_{n\to\infty}\frac{3n}{|3^{3n}t^n\,(1/3)_n\,n!|^{\frac{\rho}{3n}}} = \frac{e}{(3t)^{1/2}} = e\tau\,\frac{3}{2}$$

It follows that $\mathcal{K}_\nu(x,t)$ is of growth $(3/2, \frac{2}{\sqrt{27t}})$ for $t>0$. Since the derivative of a function$f$ of growth (ρ,τ) is also of growth (ρ,τ), see [6], p. 13, it follows that all the derivatives of $\mathcal{K}_\nu(x,t)$ are also of growth $(3/2, \frac{2}{\sqrt{27t}})$ for $t>0$. □

Now

$$\frac{\partial}{\partial t}\mathcal{K}_\nu(x,t) = \frac{1}{3^{\nu+2/3}\,t^{\nu+5/3}}\sum_{n=0}^{\infty}\frac{(-1)^{n+1}\,(n+\nu+2/3)\,x^{3n}t^{-n}}{3^{3n}\,n!\,(1/3)_n}$$

and a simple calculation shows that $\frac{\partial}{\partial t}\mathcal{K}_\nu(x,t)$ is of order $3/2$. Since $\lim_{n\to\infty}(n+\nu+2/3)^{\frac{1}{2n}}=1$, it also follows that $\frac{\partial}{\partial t}\mathcal{K}_\nu$ is also of type $2/\sqrt{27t}$ for $t>0$. This result is used to confirm local uniform convergence in the following theorem.

Theorem 3.2. *Let $f(x)$ be a function on $R^+=[0,\infty)$ such that $f(x)=O(e^{-cx^\rho})$ with $\rho>3/2$ and $c>0$. Then*

$$u(x,t)=\int_0^\infty \mathcal{K}_\nu(x\oplus_\nu y,t)\,f(y)\,d\eta_\nu(y) \tag{3.18}$$

is a solution of the Airy diffusion equation, that is, $u(x,t)\in\mathcal{H}_\nu$ for $t>0$.

Proof. Based on the estimates for $\mathcal{K}_\nu(x\oplus_\nu y,t)$ and its derivatives it follows that the integral in (3.18) and the integrals with kernels $\frac{\partial^n}{\partial x^n}\mathcal{K}_\nu(x\oplus_\nu y,t)$ locally converge uniformly, thus the operations of differentiation and integration can be exchanged. Thus

$$\begin{aligned}\vartheta_\nu\,u(x,t) &= \int_0^\infty \vartheta_\nu\,\mathcal{K}_\nu(x\oplus_\nu y,t)\,f(y)\,d\eta_\nu(y)\\ &= \int_0^\infty \frac{\partial}{\partial t}\mathcal{K}_\nu(x\oplus_\nu y,t)\,f(y)\,d\eta_\nu(y)\\ &= \frac{\partial}{\partial t}\,u(x,t)\end{aligned} \tag{3.19}$$

Since the function $\mathcal{K}_\nu(x\oplus_\nu y,t)$ is not an approximate identity kernel, the boundary value $f(x)$ of $u(x,t)$ given by (3.18) is only recaptured in a formal way. □

Next we present an example of a ν-Airy heat function which is not an entire function of the space variable for a fixed t. We let

$$H_1(t)=\frac{2}{3^{(3\nu+5)/2}\,\Gamma(\nu+2/3)}t^{\frac{\nu}{2}+\frac{5}{6}}\,K_{\nu-\frac13}(2\sqrt{\frac{t}{27}})$$

where $K_{\nu-\frac13}$ is a modified Bessel function. By Erdélyi [17] we have the Mellin transform

$$\int_0^\infty t^{s-1}H_1(t)\,dt = 3^{3s}\,\frac{\Gamma(s+1)\Gamma(s+\nu+2/3)}{\Gamma(\nu+2/3)}$$

for $\operatorname{Re}s>-1$. The function $H_2(t)=x^{1/3}e^{-x}/\Gamma(1/3)$ has the Mellin transform

$$\int_0^\infty t^{s-1}H_2(t)\,dt=\frac{\Gamma(s+1/3)}{\Gamma(1/3)}$$

for $\operatorname{Re}s>-1/3$. Thus the mellin convolution of $H_1(x)$ and $H_2(x)$ is given by

$$\begin{aligned}\mathcal{O}_\nu(x) &= \int_0^\infty H_2(x/t)H_1(t)\,\frac{dt}{t}\\ &= \frac{2x^{1/3}}{3^{(3\nu+5)/2}\,\Gamma(1/3)\Gamma(\nu+2/3)}\int_0^\infty t^{\frac12\,(\nu-1)}\,e^{-\frac{x}{t}}\,K_{\nu-1/3}(2\sqrt{\frac{t}{27}})\,dt\end{aligned}$$

for $x \geq 0$. By the Mellin convolution theorem [30] it follows that

$$\int_0^\infty x^{s-1} \mathcal{O}_\nu(x)\,dx = 3^{3s}\,\frac{\Gamma(s+1)\Gamma(s+1/3)\Gamma(s+\nu+2/3)}{\Gamma(1/3)\Gamma(\nu+2/3)} = \alpha_{3s}(\nu)$$

for $\operatorname{Re} s > -1/3$. Letting

$$\phi(t) = \int_0^\infty e^{-tx} \mathcal{O}_\nu(x) x^{-1}\,dx$$

it follows that

$$\phi^{(n)}(t) = (-1)^n \int_0^\infty e^{-tx} x^{n-1} \mathcal{O}_\nu(x)\,dx$$

for $t \geq 0$ and $n = 0, 1, 2, \dots$. Thus $\phi(t) \in C^\infty(R^+)$ and $|\phi^{(n)}(t)| \leq \alpha_{3n}(\nu)$ for $t \geq 0$. Hence

$$u(x,t) = \sum_{n=0}^\infty \phi^{(n)}(t)\,\frac{x^{3n}}{\alpha_{3n}(\nu)}$$

is a ν-Airy diffusion for $-1 < x < 1$, $t > 0$. Further, we have

$$u(x,0) = \sum_{n=0}^\infty \phi^{(n)}(0)\,\frac{x^{3n}}{\alpha_{3n}(\nu)} = \sum_{n=0}^\infty (-1)^n x^{3n} = \frac{1}{1+x^3}$$

for $-1 < x < 1$. Hence the analytic extension $u(z,0)$ has singularities on the unit circle at the cube roots of -1. Thus $u(z,0)$ can not be extended to an entire function. It is holomorphic in the unit circle.

4. Time Series Solutions

Solutions for the partial differential equation $\vartheta_\nu\, u = u_t$ are obtained for the Cauchy data given on the t-axis, that is,

$$u(0,t) = g(t) \quad \text{and} \quad u_x(0,t) = 0$$

Let D_t stand for differentiation with respect to t. The function

$$u(x,t) = G_\nu(x\,D_t^{1/3})\,g(t) \tag{4.1}$$

Gives a formal solution of the ν-diffusion equation we have

$$\begin{aligned} \vartheta_\nu\, u(x,t) &= \vartheta_\nu\, G_\nu(x\,D_t^{1/3})\,g(t) \\ &= D_t\, G_\nu(x\,D_t^{1/3})\,g(t) \\ &= D_t\, u(x,t) \end{aligned}$$

Using the power series expansion of $G_\nu(x)$, we define

$$G_\nu(x\,D_t^{1/3})\,g(t) = \sum_{n=0}^\infty \frac{x^{3n}}{\alpha_{3n}}\,g^{(n)}(t) \tag{4.2}$$

provided the series converges locally uniformly for g a C^∞ function. Term by term x-differentiation by the ϑ_ν operator gives

$$\sum_{n=1}^\infty \frac{x^{3(n-1)}}{\alpha_{3(n-1)}}\,g^{(n)}(t) = \sum_{n=0}^\infty \frac{x^{3n}}{\alpha_{3n}}\,g^{(n+1)}(t) \tag{4.3}$$

Thus the series gives the formal solution.

Let $C_{00}^\infty(0,\infty)$ be the class of functions $f(x)$ infinitely differentiable and vanishing outside a compact subset of $0 < x < \infty$.

Theorem 4.1. *If $f \in C_{00}^{\infty}(0,\infty)$ then the function*

$$\hat{f}(t) = \int_0^\infty f(x)\,\mathcal{G}_\nu(xt)\,d\eta_\nu(x) \tag{4.4}$$

is a continuous function in $\mathcal{L}^1((0,\infty), d\eta_\nu(x))$.

Proof. Clearly the integral converges locally uniformly and defines a continuous function. Since $\vartheta_\nu\, G_\nu(xt) = -t^3\, G_\nu(xt)$, we find by integration by parts that

$$\begin{aligned}
t^3\,\hat{f}(t) &= -\int_0^\infty f(x)\,\vartheta_\nu\,\mathcal{G}_\nu(xt)\,d\eta_\nu(x)\\
&= \int_0^\infty \Big(\frac{\partial}{\partial x}x^{3\nu}\,\frac{\partial}{\partial x}x^{-3\nu}\,\frac{\partial}{\partial x}x^{3\nu+1}\,f(x)\Big)\,G_\nu(xt)\,\frac{dx}{c_\nu}\\
&= \frac{1}{c_\nu}\int_0^\infty \vartheta_{-\nu}x^{3\nu+1}\,f(x)\,\mathcal{G}_\nu(xt)\,dx
\end{aligned}$$

Clearly $\vartheta_{-\nu}x^{3\nu+1}\,f(x)$ is in $C_{00}^{\infty}(0,\infty)$. Repeated applications yield

$$t^{3n}\,\hat{f}(t) = \frac{1}{c_\nu}\int_0^\infty \vartheta_{-\nu}^{n}\,(x^{3\nu+1}\,f(x))\,\mathcal{G}_\nu(xt)\,dx$$

Hence for n sufficiently large, we have

$$\int_1^\infty |\hat{f}(t)|\,\frac{t^{3\nu+1}}{c_\nu}\,dt = \int_1^\infty t^{3n}\,|\hat{f}(t)|\,\frac{t^{-3n+3\nu+1}}{c_\nu}\,dt < M_2 < \infty$$

and it follows that

$$\begin{aligned}
\int_0^\infty |\hat{f}(t)|\,d\eta_\nu(t) &= \int_0^1 |\hat{f}(t)|t^{3\nu+1}\,\frac{dt}{c_\nu} + \int_1^\infty t^{3n}\,|\hat{f}(t)|\,\frac{x^{-3n+3\nu+1}}{c_\nu}\,dt\\
&< M_1 + M_2 < \infty\,.
\end{aligned}$$

Next we let $g(t) = e^{-ty^3}$, then

$$G_\nu(x\,D_t^{1/3})\,e^{-ty^3} = \sum_{n=0}^{\infty}\frac{x^{3n}}{\alpha_{3n}}\,(-y)^{3n}\,e^{-ty^3} = \mathcal{G}_\nu(xy)\,e^{-ty^3} \tag{4.5}$$

□

Theorem 4.2. *Let $\phi(x)$ be an entire function of growth $(2,\tau)$ and let*

$$g(t) = \int_0^\infty e^{-ty^3}\,\phi(y)\,d\eta_\nu(y) \tag{4.6}$$

then

$$G_\nu(x\,D_t^{1/3})\,g(t) = \int_0^\infty e^{-ty^3}\,\phi(y)\,\mathcal{G}_\nu(xy)\,d\eta_\nu(y) = u(x,t) \tag{4.7}$$

Proof. Clearly the integral (4.6) and all of its derivatives converge uniformly for $t > 0$. Thus

$$\begin{aligned}
G_\nu(x\,D_t^{1/3})\,g(t) &= \sum_{n=0}^{\infty}\frac{x^{3n}}{\alpha_{3n}}\int_0^\infty (-y)^{3n}\,e^{-ty^3}\,\phi(y)\,d\eta_\nu(y)\\
&= \int_0^\infty e^{-ty^3}\,\phi(y)\,\mathcal{G}_\nu(xy)\,d\eta_\nu(y)
\end{aligned}$$

Since $\mathcal{G}_\nu(z)$ is an entire function of order one, the uniform convergence allows the interchange of summation and integration. □

Corollary 4.3. *If $\phi(x) \in C^{\infty}_{00}(0,\infty)$ and if*
$$g(t) = \int_0^\infty e^{-ty^3}\,\phi(y)\,d\eta_\nu(y)$$
then
$$G_\nu(x\,D_t^{1/3})\,g(t) = \int_0^\infty e^{-ty^3}\,\phi(y)\,\mathcal{G}_\nu(xy)\,d\eta_\nu(y) = u(x,t) \tag{4.8}$$
is a ν-diffusion for $t > 0$. Moreover, $u(x,t)$ is in $\mathcal{L}^1(R^+, d\eta_\nu)$ and $u(x,0) = \hat{\phi}(x)$.

Proof. The integrability follows from Theorem 4.1. Next we define $g_a(t)$ for $a > 1/2$ by
$$g_a(t) = \begin{cases} \exp(-t^{-a}) & \text{for } t > 0 \\ 0 & \text{for } t \le 0 \end{cases} \tag{4.9}$$
Then by an application of Cauchy's integral formula, there is a $\theta = \theta(a)$ such that
$$|g_a^{(k)}(t)| = \frac{k!}{(\theta t)^k}\exp(\frac{1}{2}t^{-a}) \tag{4.10}$$
see for example Widder [34, p. 46]. Let $u_a(x,t) = G_\nu(x\,D_t^{1/3})\,g(t)$. Since $\frac{k!}{(3k)!} \le \frac{1}{(2k)!}$, we find using Gauss' cubic equation that for $t > 0$ and arbitrary complex x
$$\begin{aligned}
|u_a(x,t)| &\le \sum_{k=0}^\infty \frac{|x|^{3k}\,|g^{(k)}(t)|}{\alpha_{3k}} \\
&\le \sum_{k=0}^\infty \frac{k!}{\alpha_{3k}}\,\frac{|x|^{3k}}{(\theta t)^k}\exp(-\frac{1}{2}t^{-a}) \\
&\le \sum_{k=0}^\infty \frac{k!}{(3k)!}\,\frac{(2/3)_k}{(\nu+2/3)_k}\,\frac{|x|^{3k}}{(\theta t)^k}\exp(-\frac{1}{2}t^{-a}) \\
&\le \sum_{k=0}^\infty \frac{1}{(2k)!}\,\frac{|x^{3/2}|^{2k}}{(\theta t)^k}\exp(-\frac{1}{2}t^{-a}) \\
&= \cosh(\frac{|x^{3/2}|}{(\theta t)^{1/2}})\exp(-\frac{1}{2}t^{-a}) \\
&\le \exp(\frac{|x^{3/2}|}{(\theta t)^{1/2}} - \frac{1}{2}t^{1/2-a}) \\
&= \exp(\frac{1}{t^{1/2}}\{\frac{|x|^{3/2}}{\theta^{1/2}} - \frac{1}{2}t^{1/2-a}\})
\end{aligned} \tag{4.11}$$
or
$$|u_a(x,t)| \le \exp\Big(\frac{1}{t^{1/2}}\{\frac{|x|^{3/2}}{\theta^{1/2}} - \frac{1}{2}t^{1/2-a}\}\Big) \tag{4.12}$$
Hence the series defining $u_a(x,t)$ converges uniformly and absolutely for $t > 0$ and x in an arbitrary compact set. The previous inequality shows that
$$\lim_{t\to 0^+} u_a(x,t) = 0$$
locally uniformly. It follows that
$$u_a(x,t) = \sum_{n=0}^\infty \frac{x^{3n}}{\alpha_{3n}(\nu)}\,|g_a^{(n)}(t)| \tag{4.13}$$

is a nontrivial solution of the ν-diffusion equation with boundary values $u_a(x,0)=0$ for every $a>1$. Therefore the solutions of the ν-diffusion boundary value problem are not necessarily unique.

Inequality (4.12) shows that $u_a(x,t)$ is an entire function of growth $(3/2, 1/t)$ for $t>0$. However, u_a is not analytic in t since $u_a(0,t)$ vanishes for all $t\le 0$ and $u_a(0,t)=g_a(t)>0$ for $t>0$. The $u_a(x,t)$ are Tychonoff type solutions of the ν-diffusion equation. $\square$

5. ϑ_ν-Trigonometric Functions

The ordinary trigonometric functions play an important role in the theory of the heat and wave equations on Euclidean spaces. In order to study solutions of equations connected with the ϑ_ν operator, it is necessary to develop an associated trigonometric theory.

We define the ν-hyperbolic functions by the series

$$G_\nu(z)=\sum_{n=0}^{\infty}\frac{z^{3n}}{\alpha_{3n}}=\sum_{k=0}^{\infty}\frac{z^{6k}}{\alpha_{6k}}+\sum_{k=0}^{\infty}\frac{z^{6k+3}}{\alpha_{6k+3}}:=\cosh_\nu(z)+\sinh_\nu(z) \tag{5.1}$$

It follows that $\cosh_\nu(z)$ and $\sinh_\nu(z)$ are entire functions of order one. Since

$$\alpha_{6k}(\nu)=6^{6k}\,k!\,(1/2)_k\,(1/6)_k\,(2/3)_k\,(1/2+1/3)_k\,(\nu/2+5/6)_k \tag{5.2}$$

and

$$\alpha_{6k+3}(\nu)=9(\nu+2/3)\,6^{6k}\,k!\,(3/2)_k\,(2/3)_k\,(7/6)_k\,(\nu/2+5/6)_k\,(\nu/2+4/3)_k \tag{5.3}$$

we obtain the hypergeometric representations

$$\cosh_\nu(z):=\sum_{k=0}^{\infty}\frac{z^{6k}}{\alpha_{6k}}={}_0F_5(1/2,1/6,2/3,\nu/2+1/3,\nu/2+5/6\,|\,(\frac{x}{6})^6) \tag{5.4}$$

and

$$\begin{aligned}\sinh_\nu(z)&:=\sum_{k=0}^{\infty}\frac{z^{6k+3}}{\alpha_{6k+3}}\\&=\frac{x^3}{9(\nu+2/3)}\,{}_0F_5(3/2,2/3,7/6,\nu/2+5/6,\nu/2+4/3\,|\,(\frac{x}{6})^6)\end{aligned} \tag{5.5}$$

In particular for $\nu=0$, a calculation shows that

$$\cosh_0(z)=\frac{1}{3}\cosh z+\frac{2}{3}\cos(\frac{\sqrt{3}}{2}\,z)\cosh(z/2)\,, \tag{5.6}$$

$$\sinh_0(z)=\frac{1}{3}\sinh z+\frac{2}{3}\cos(\frac{\sqrt{3}}{2}\,z)\sinh(z/2) \tag{5.7}$$

Since

$$\cosh_\nu(ax)=\frac{1}{2}\,\{G_\nu(ax)+G_\nu(-ax)\} \tag{5.8}$$

we get

$$\vartheta_\nu\cos h_\nu(ax)=a^3\sinh_\nu(ax)\,, \tag{5.9}$$

$$\vartheta_\nu\sinh_\nu(ax)=a^3\cos h_\nu(ax) \tag{5.10}$$

Thus $\cosh_\nu(ax)$ and $\sinh_\nu(ax)$ are solutions of the harmonic type equation

$$(\vartheta_\nu^2-a^6)\,y(x)=0 \tag{5.11}$$

We note that $\cosh_\nu$ is an even function and $\sinh_\nu$ is an odd function. Since $\mathcal{G}_\nu(x) = G_\nu(-x) = \cosh_\nu(x) - \sinh_\nu(x)$, we get

$$G_\nu(x)\,G_\nu(-x) = \cosh^2_\nu(x) - \sinh^2_\nu(x) = G_\nu((1 \oplus_\nu (-1))x) \tag{5.12}$$

See Equation (2.3) for a hypergeometric representation.

In terms of the umbral ν-translation, we obtain a number of identities. We define an umbral multiplication in terms of repeated addition. We have

$$(2 \odot_\nu x)^{3n} = (x \oplus_\nu x)^{3n} = x^{3n} \sum_{k=0}^{n} \binom{\alpha_{3n}(\nu)}{\alpha_{3k}(\nu)} \tag{5.13}$$

and, for $m \geq 2$, we have the generalized ν-multinomial

$$\begin{aligned}(m \odot_\nu x)^{3n} &= (x \oplus_\nu \cdots \oplus_\nu x)^{3n} \\ &= x^{3n} \cdot \sum_{\substack{\ell_1 + \ell_2 + \cdots + \ell_m = n \\ \ell_j \geq 0}} \frac{\alpha_{3n}(\nu)}{\alpha_{3\ell_1}\,\alpha_{3\ell_2}\,\dots\,\alpha_{3\ell_m}}\end{aligned} \tag{5.14}$$

Now

$$\begin{aligned} G_\nu(x)\,G_\nu(\pm y) &= G_\nu(x \oplus_\nu (\pm y)) \\ &= \cosh_\nu(x \oplus_\nu (\pm y)) + \sinh_\nu(x \oplus_\nu (\pm y)) \\ &= \cosh_\nu(x)\cos h_\nu(y) \pm sinh_\nu(x)\sinh_\nu(y) \\ &\quad + \sinh_\nu(x)\cos h_\nu(y) \pm \cosh_\nu(x)\sinh_\nu(y)\,. \end{aligned} \tag{5.15}$$

Equating the even and odd parts, we obtain the angle-sum and angle-difference ν-relations,

$$\cosh_\nu(x \oplus_\nu (\pm y)) = cosh_\nu(x)\cos h_\nu(y) \pm \sinh_\nu(x)\sinh_\nu(y)\,, \tag{5.16}$$

$$\sinh_\nu(x \oplus_\nu (\pm y)) = \sinh_\nu(x)\cos h_\nu(y) \pm \cosh_\nu(x)\sinh_\nu(y)\,. \tag{5.17}$$

Thus we also get the multiple angle relations

$$\sinh_\nu(2 \odot_\nu x) = 2\sinh_\nu(x)\cos h_\nu(x) \tag{5.18}$$

$$\cosh_\nu(2 \odot_\nu x) = \cosh^2_\nu(x) + \sinh^2_\nu(x)\,. \tag{5.19}$$

Using the binomial theorem, we obtain the general multiple angle relations

$$\cosh_\nu(n \odot_\nu x) = \sum_{\ell=0}^{[n/2]} \binom{n}{2\ell} \sinh^{2\ell}_\nu(x)\cos h^{n-2\ell}_\nu(x) \tag{5.20}$$

$$\sinh_\nu(n \odot_\nu x) = \sum_{\ell=0}^{[n/2]-1} \binom{n}{2\ell+1} \sinh^{2\ell+1}_\nu(x)\cosh^{n-2\ell-1}_\nu(x) \tag{5.21}$$

In terms of hypergeometric functions, using umbral variables, Equation (5.20) yields the identity

$$\begin{aligned} &{}_0F_5\big(1/2, 1/6, 2/3, \nu/2 + 1/3, \nu/2 + 5/6 \,|\, \big(\frac{n \odot_\nu x}{6}\big)^6\big) \\ &= \sum_{\ell=0}^{[n/2]} \binom{n}{2\ell} \frac{x^{6\ell}}{9^{2\ell}\,(\nu + 2/3)^{2\ell}}\, {}_0F_5(3/2, 2/3, 7/6, \nu/2 + 5/6, \nu/2 + 4/3 \,|\, (\frac{x}{6})^6)^{2\ell} \\ &\quad \times {}_0F_5(1/2, 1/6, 2/3, \nu/2 + 1/3, \nu/2 + 5/6 \,|\, (\frac{x}{6})^6)^{n-2\ell}\,. \end{aligned} \tag{5.22}$$

Also from (5.1) we get the hypergeometric identity

$$\begin{aligned}
&{}_0F_2\big(1/3,\nu+2/3\,|\,\frac{(n\odot_\nu x)^3}{9}\big)\\
&=\sum_{\ell=0}^{n}\binom{n}{\ell}\frac{x^{3\ell}}{9^\ell\,(\nu+2/3)^\ell}\,{}_0F_5(3/2,2/3,7/6,\nu/2+5/6,\nu/2+4/3\,|\,(\frac{x}{6})^6)^\ell \qquad (5.23)\\
&\quad\times{}_0F_5(1/2,1/6,2/3,\nu/2+1/3,\nu/2+5/6\,|\,(\frac{x}{6})^6)^{n-\ell}\,.
\end{aligned}$$

Next we introduce an umbral subtraction for ν-translation. The generalized subtraction associated with the Bessel functions was developed by Cholewinski [9]. Since $G_\nu(0)=1$, we can define a "ν-Mobius" sequence $\{\gamma_{3k}(\nu)\}_{k=0}^{\infty}$ by the equation

$$G_\nu(\gamma_u x)=\frac{1}{G_\nu(x)}\quad\text{or}\quad G_\nu(\gamma_u x)G_\nu(x)=1\,. \tag{5.24}$$

By multiplication of series, we must have

$$\sum_{k=0}^{n}\binom{\alpha_{3n}}{\alpha_{3k}}\gamma_{3k}=\delta_{n,0}, \tag{5.25}$$

the Kronecker delta. For $n=0$, we get $\gamma_0=1$. Hence the sequence $\{\gamma_{3k}(\nu)\}$ is obtained inductively by the equation

$$\gamma_{3n}(\nu)=-\sum_{k=0}^{n-1}\binom{\alpha_{3n}}{\alpha_{3k}}\gamma_{3k}(\nu) \tag{5.26}$$

The first four values are $\gamma_0=1$, $\gamma_3(\nu)=-1$, $\gamma_6(\nu)=-1+8(\nu+5/3)/(\nu+2/3)$, and $\gamma_9(\nu)=-1-126(\nu+8/3)(\nu+2)/(\nu+2/3)^2$. Further, we get

$$G_\nu(x)\,G_\nu(\gamma_u y)=G_\nu(x\oplus_\nu\gamma_u y)=\cosh_\nu(x\oplus_\nu\gamma_u y)+\sinh_\nu(x\oplus_\nu\gamma_u y)\,. \tag{5.27}$$

Multiplying the series on the left and equating even and odd parts, it follows that

$$\cosh_\nu(x\oplus_\nu\gamma_u y)=\cosh_\nu(x)\cosh_\nu(\gamma_u y)+\sinh_\nu(x)\sinh_\nu(\gamma_u y)=1 \tag{5.28}$$

$$\sinh_\nu(x\oplus_\nu\gamma_u y)=\sinh_\nu(x)\cosh_\nu(\gamma_u y)+\cosh_\nu(x)\sinh_\nu(\gamma_u y)=0 \tag{5.29}$$

The $\cosh_\nu$ identity above is an analogue of the familiar result $\cosh^2 x-\sinh^2 x=1$. Of course Equation (5.28) can be expressed as a complicated hypergeometric identity. With the obvious notation we get the ν-hyperbolic tangent identity

$$1+tanh_\nu(x)tanh_\nu(\gamma_u x)=sech_\nu(x)\,sech_\nu(\gamma_u x)$$

Using Equations (5.18) and (5.19), we obtain the multiple angle relation

$$\tanh_\nu(2\odot_\nu x)=\frac{2\,\tanh_\nu(x)}{1+\tanh_\nu^2(x)}\,. \tag{5.30}$$

In the case $\nu=0$, the ν-hyperbolic tangent is

$$\tanh_0(x)=\frac{\sinh x-2/3\cos\frac{\sqrt{3}}{2}x\sinh\frac{x}{2}}{\cosh x+2/3\cos\frac{\sqrt{3}}{2}x\cosh\frac{x}{2}}=\tanh(x)\frac{\Big(1-\frac{\cos\frac{\sqrt{3}}{2}x}{3\cosh\frac{x}{2}}\Big)}{\Big(1+(2/3)\frac{\cos\frac{\sqrt{3}}{2}x\cosh\frac{x}{2}}{\cosh x}\Big)}$$

Next

$$\tanh_\nu(x)=\frac{1-G_\nu(-x)\,G_\nu(\gamma_u x)}{1+G_\nu(-x)\,G_\nu(\gamma_u x)} \tag{5.31}$$

which implies that

$$\tanh_\nu(x) = 1 + 2\sum_{n=1}^{\infty}(-1)^n \frac{G_\nu(-x)^n}{G_\nu(x)^n} \tag{5.32}$$

which converges for $x > 0$, since $|G_\nu(-x)| < G_\nu(x)$ for $x > 0$. In umbral notation Equation (5.32) can be written as

$$\tanh_\nu(x) = 1 + 2\sum_{n=1}^{\infty}(-1)^n G_\nu(n \odot_\nu (x(-1 \oplus_\nu \gamma_u)))\,. \tag{5.33}$$

This is the analogue of the elementary formula

$$\tanh(x) = 1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-2nx}, \qquad \text{for } \operatorname{Re} x > 0 \tag{5.34}$$

A complex ν-exponential function $E_\nu(x)$ is defined by the equation

$$E_\nu(x) = G_\nu(\omega_6 x) = \sum_{n=0}^{\infty} \frac{i^n x^{3n}}{\alpha_{3n}(\nu)} \tag{5.35}$$

where $\omega_6 = e^{\frac{\pi}{6}i} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$. The function $E_\nu(x)$ is the e^{ix} in the the ν-calculus associated with ϑ_ν, it is an entire function of order one. We define generalized sine and cosine functions by the equation

$$E_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{6k}}{\alpha_{6k}(\nu)} + i\sum_{k=0}^{\infty} \frac{(-1)^k x^{6k+3}}{\alpha_{6k+3}(\nu)} = \cos_\nu(x) + i\sin_\nu(x)\,. \tag{5.36}$$

From our previous results, we have

$$\cos_\nu(x) = \cosh_\nu(\omega_6 x)\,, \tag{5.37}$$

$$\sinh_\nu(\omega_6 x) = i\sin_\nu(x)\,. \tag{5.38}$$

The hypergeometric representations are

$$\cos_\nu(x) ={}_0 F_5\big(1/2, 1/6, 2/3, \nu/2 + 1/3, \nu/2 + 5/6 \,|\, -(\frac{x}{6})^6\big)\,, \tag{5.39}$$

$$\sin_\nu(x) = \frac{x^3}{9(\nu + 2/3)}\, {}_0F_5\big(3/2, 2/3, 7/6, \nu/2 + 5/6, \nu/2 + 4/3 \,|\, -(\frac{x}{6})^6\big)\,. \tag{5.40}$$

The trigonometric identities for the ν-hyperbolic functions are easily converted to identities for the ν-sine and cosine functions. We have

$$\cos_\nu(x \oplus_\nu y) = \cos_\nu(x)\cos_\nu(y) - \sin_\nu(x)\sin_\nu(y)\,, \tag{5.41}$$

$$\sin_\nu(x \oplus_\nu y) = \sin_\nu(x)\cos_\nu(y) + \sin_\nu(x)\cos_\nu(y)\,. \tag{5.42}$$

In particular,

$$\cos_\nu(2 \odot_\nu x) = \cos_\nu^2(x) - \sin_\nu^2(x)\,, \tag{5.43}$$

$$\sin_\nu(2 \odot_\nu x) = 2\sin_\nu(x)\cos_\nu(x)\,. \tag{5.44}$$

In terms of hypergeometric functions, (5.42) yields the identity

$$\begin{aligned}
&{}_0F_5\Big(1/2, 1/6, 2/3, \nu/2+1/3, \nu/2+5/6 \,|\, -(\frac{x\oplus_\nu y}{6})^6\Big)\\
&={}_0\,F_5\Big(1/2, 1/6, 2/3, \nu/2+1/3, \nu/2+5/6 \,|\, -(\frac{x}{6})^6\Big)\\
&\quad\times {}_0F_5\Big(1/2, 1/6, 2/3, \nu/2+1/3, \nu/2+5/6 \,|\, -(\frac{y}{6})^6\Big)\\
&\quad-\frac{(xy)^6}{9^2\,(\nu+2/3)}\,{}_0F_5\Big(3/2, 2/3, 7/6, \nu/2+5/6, \nu/2+4/3 \,|\, -(\frac{x}{6})^6\Big)\\
&\quad\times {}_0F_5\Big(3/2, 2/3, 7/6, \nu/2+5/6, \nu/2+4/3 \,|\, -(\frac{y}{6})^6\Big)\,.
\end{aligned} \tag{5.45}$$

In the case $\nu = 0$, the equations are

$$\sin_0(x) = -\frac{1}{3}\sin x + \frac{2}{3}\sin\frac{x}{2}\cosh\frac{\sqrt{3}}{2}x\,, \tag{5.46}$$

$$\cos_0(x) = -\frac{1}{3}\cos x + \frac{2}{3}\cos\frac{x}{2}\cosh\frac{\sqrt{3}}{2}x\,. \tag{5.47}$$

The ν-derivatives are

$$\vartheta_\nu \sin_\nu(ax) = a^3\cos_\nu(ax)\,, \tag{5.48}$$

$$\vartheta_\nu \cos_\nu(ax) = -a^3\sin_\nu(ax)\,. \tag{5.49}$$

Thus $\sin_\nu(ax)$ and $\cos_\nu(ax)$ are solutions of the differential equation

$$\vartheta_\nu\, y(x) + a^6\, y(x) = 0\,.$$

Complex variable type addition equations are

$$\cos_\nu(x\oplus_\nu \omega_6\, y) = \cos_\nu(x) cosh_\nu(y) - i\sin_\nu^2(x)\sinh_\nu(y)\,,$$
$$\sin_\nu(x\oplus_\nu \omega_6\, y) = \sin_\nu(x)\cosh_\nu(y) - i\cos_\nu(x)\sinh_\nu(y)$$

which are analogues of the complex variable formulas for $\sin(z)$ and $\cos(z)$ with $z = x + i\,y$.

Theorem 5.1 (A ν-DeMoivre's Formula)**.** *Let $\nu > 0$ and n be a positive integer. Then*

$$(\cos_\nu(x) + i\sin_\nu(x))^n = \cos_\nu(n\odot_\nu x) + i\sin_\nu(n\odot_\nu x)$$

Proof. We have $E_\nu(x)^n = E_\nu(n\odot_\nu x)$ and the equation follows. □

Since $E_\nu(\gamma_u z)\,E_\nu(z) = G_\nu(\gamma_u\,\omega_6\,z)\,G_\nu(\omega_6\,z) = 1$, it follows that

$$\cos_\nu(z)\cos_\nu(\gamma_u z) - \sin_\nu(z)\sin_\nu(\gamma_u z) = 1\,,$$
$$\sin_\nu(z)\cos_\nu(\gamma_u z) + \sin_\nu(\gamma_u z)\cos_\nu(z) = 0\,.$$

In the case that $\nu = 0$,

$$E_0(z) = \frac{1}{3}\,(e^{-iz} + 2\,e^{\frac{i\,z}{2}}\cosh(\frac{\sqrt{3}}{2}\,z)\,.$$

Obviously, the functions $G_0(x)$ and $E_0(x)$ are not periodic in the ordinary sense. We will obtain umbral periods associated with the ν-translation.

Theorem 5.2. *For $\nu \geq 0$, the function $\mathcal{G}_\nu$ takes on the values +1 and -1 for infinitely many real values of x.*

Proof. The theorem easily follows from the asymptotic expansion of $\mathcal{G}_\nu$. We have

$$\begin{aligned}\mathcal{G}_\nu(x) &= {}_0F_2\left(1/3, \nu+2/3\,|(-\frac{x}{3})^3\right)\\ &\sim \frac{\Gamma(1/3)\Gamma(\nu+2/3)}{\pi\sqrt{3}}\,(\frac{x}{3})^{-\nu}\,e^{x/2}\cos\big(\frac{\sqrt{3}}{2}x-\frac{\nu\pi}{2}\big)\end{aligned}\tag{5.50}$$

as $x\to\infty$, see Marichev [25], for example. Hence the theorem follows. □

Since the zeros of an entire function cannot have a finite limit point, there is a smallest $x_{2\pi}(\nu) > 0$ such that $\mathcal{G}_\nu(x_{2\pi}(\nu)) = 1$ and, likewise, a smallest $x_\pi(\nu) > 0$ such that $\mathcal{G}_\nu(x_\pi(\nu)) = -1$. Next we let $P_{2\pi}(\nu) = -x_{2\pi}(\nu)\,e^{-\frac{i\pi}{6}}\,e^{\frac{2\pi i}{3}}\cdots = -x_{2\pi}(\nu)\,i$, then

$$E_\nu(P_{2\pi}(\nu)) = G_\nu(-x_{2\pi}(\nu)\,e^{-\frac{i\pi}{6}}\,\omega_6) = \mathcal{G}_\nu(x_{2\pi}(\nu)) = 1\,.$$

Hence

$$E_\nu(z\oplus_\nu P_{2\pi}(\nu)) = E_\nu(z)\,E_\nu(P_{2\pi}(\nu)) = E_\nu(z)$$

and therefore $P_{2\pi}(\nu)$ is an umbral period of E_ν. In the case $\nu = 0$, $P_{2\pi}(0)$ is approximately -5.549831i.

By Theorem 2.24, we obtain Poisson type representations for the trigonometric functions of this section. For $\nu \ge 1$, we have

$$E_\nu(x) = \frac{\Gamma(\nu+2/3)}{\Gamma(2/3)\,\Gamma(\nu)}\int_0^1 \tau\,(1-\tau^3)^{\nu-1}\,(e^{-ix\tau}+2e^{\frac{ix\tau}{2}}cosh\,\frac{\sqrt{3}}{2}x\,\tau)\,d\tau\,,\tag{5.51}$$

$$\begin{aligned}\cosh_\nu(x) &= \frac{\Gamma(\nu+2/3)}{\Gamma(2/3)\,\Gamma(\nu)}\int_0^1 \tau\,(1-\tau^3)^{\nu-1}\,(\cosh(x\,\tau)+2\cos(\frac{\sqrt{3}}{2}\,x\,\tau)\cosh(\frac{x\,\tau}{2})\,d\tau\,,\\ \sinh_\nu(x) &= \frac{\Gamma(\nu+2/3)}{\Gamma(2/3)\,\Gamma(\nu)}\int_0^1 \tau\,(1-\tau^3)^{\nu-1}(\sinh(x\,\tau)-2\cos(\frac{\sqrt{3}}{2}\,x\,\tau)\sinh(\frac{x\,\tau}{2})\,d\tau\,,\\ \cos_\nu(x) &= \frac{\Gamma(\nu+2/3)}{\Gamma(2/3)\,\Gamma(\nu)}\int_0^1 \tau\,(1-\tau^3)^{\nu-1}\,(\cos(x\,\tau)+2\cos(\frac{\sqrt{3}}{2}\,x\,\tau)\cos h(\frac{x\,\tau}{2})\,d\tau\,,\\ \sin_\nu(x) &= \frac{\Gamma(\nu+2/3)}{\Gamma(2/3)\,\Gamma(\nu)}\int_0^1 \tau\,(1-\tau^3)^{\nu-1}(-sin(x\,\tau)+2\sin(\frac{\sqrt{3}}{2}x\tau)\cosh(\frac{x\,\tau}{2})\,d\tau\,.\end{aligned}$$

From these equations, one can obtain order estimates for the individual functions, for example

$$E_\nu(x) = O(e^{\sqrt{3}x/2})\quad\text{as } x\to\infty$$

by (5.51).

In the following example we need to know the smallest real zero of $\cos_\nu(2x)$. We have

$$\begin{aligned}\cos_\nu(2x) &= {}_0F_5\big(1/2,1/6,2/3,\nu/2+1/3,\nu/2+5/6\,|\,-(\frac{x}{3})^6\big)\\ &\sim \frac{\Gamma(1/2,1/6,2/3,\nu/2+1/3,\nu/2+5/6)}{\sqrt{6}\,(2\pi)^{5/2}}\,2\,(\frac{x}{2})^{-3}\,e^{\sqrt{3}x}\cos(x-\frac{\nu\pi}{6})\end{aligned}$$

as $x\to\infty$, where

$$\Gamma(1/2,1/6,2/3,\nu/2+1/3,\nu/2+5/6) = \Gamma(1/2)\,\Gamma(1/6)\,\Gamma(2/3)\,\Gamma(\frac{\nu}{2}+1/3)\,\Gamma(\frac{\nu}{2}+5/6)\,.$$

It follows from the asymptotic expansion that $\cos_\nu(2x)$ has infinitely many real zeros. Since $\cos_\nu(z)$ is entire and $\cos_\nu(0) = 1$, it follows that there is a smallest positive value x_0 such that $\cos_\nu(2x_0) = 0$.

Finally, we present a nontrivial application of the ν-cosine function. Consider the sixth order diffusion equation given by

$$\vartheta_\nu^2\, u(x,t) = \frac{\partial}{\partial t}\, u(x,t)\,. \tag{5.52}$$

Using (5.48) and (5.49), it follows that the function

$$u(x,t) = e^{-t}\cos_\nu(x) - e^{-64t}\cos_\nu(2x)$$

is a solution of the previous equations such that $u(x,0) = \cos_\nu(x) - \cos_\nu(2x)$ and $u_t(x,0) = -\cos_\nu(x) + 64\cos_\nu(2x)$, both being entire functions.

Let x_0 be the first positive root of $cos_\nu(2x)$, then for $-x_0 < x < x_0$, we have that $u(x,t) = 0$ on the curve

$$t = \frac{1}{63}\log\frac{cos_\nu(2x)}{cos_\nu(x)}$$

Thus the uniqueness of solutions for the diffusion equation (5.52) fails.

6. ν-Complex Functions

The trigonometric identities of Section 5 involving the $\cos_\nu(x \oplus_\nu \omega_0\, y)$ and $\sin_\nu(x \oplus_\nu \omega_0\, y)$ suggest the possibility of an analytic function theory associated with the ϑ_ν operator. The rudiments of such a theory is obtained in this section.

The operator limit

$$\frac{G_\nu(h\,\vartheta^{1/3}) - 1}{\frac{h^3}{\alpha_3(\nu)}} \;\to\; \vartheta_x \tag{6.1}$$

gives a second way to compute $\vartheta_x\, f(x)$ for suitable $f(x)$. We know that $\vartheta_x x^{3n} = 3^3\, n(n+\nu-1/3)(n-2/3)x^{3(n-1)} = d_{3n}(\nu)\, x^{3(n-1)}$. Consider the quotient

$$\begin{aligned}\frac{(x\oplus_\nu h)^{3n} - x^{3n}}{\frac{h^3}{\alpha_3}} &= \sum_{k=1}^{n}\alpha_3\binom{\alpha_{3n}}{\alpha_{3k}}x^{3(n-k)}\,h^{3(k-1)}\\ &\to \alpha_3\binom{\alpha_{3n}}{\alpha_3}x^{3(n-1)} = d_{3n}(\nu)x^{3(n-1)}\end{aligned}$$

as $h \to 0$. Thus

$$\lim_{h\to 0}\frac{(x\oplus_\nu h)^{3n} - x^{3n}}{\frac{h^3}{\alpha_3}} = d_{3n}(\nu)x^{3(n-1)}\,. \tag{6.2}$$

Hence we define

$$\vartheta_x\, f(x) = \lim_{h\to 0}\frac{f(x\oplus_\nu h) - f(x)}{\frac{h^3}{\alpha_3}} \tag{6.3}$$

provided the limit exists for suitable functions. It is clear that (6.3) is valid for polynomials and entire functions of the variable x^3.

Next we introduce an umbral complex variable $z_\nu \simeq (x\oplus_\nu\omega_6 y)$, where $\omega_6 = e^{i\pi/6}$. We define

$$z_\nu^{3n} = (x\oplus_\nu\omega_6 y)^{3n}\,.$$

Further, we let $\Delta z_\nu \simeq (\Delta x \oplus_\nu \omega_6 \Delta y)$, and we define the ν-derivative of a suitable function f at a z_ν by

$$\begin{aligned} \vartheta_x f(z_\nu) &= \lim_{\Delta z\to 0} \frac{f(x\oplus_\nu \omega_6 y)\oplus_\nu (\Delta x\oplus_\nu \omega_6\Delta y)) - f(x\oplus_\nu \omega_6 y)}{\frac{(\Delta x\oplus_\nu\omega_6\Delta y)^3}{\alpha_3(\nu)}} \\ &= \lim_{\Delta z\to 0 \text{ or } \Delta x,\Delta y\to 0} \frac{f(z_\nu\oplus_\nu \Delta z_\nu) - f(z_\nu)}{\frac{(\Delta z_\nu)^3}{\alpha_3(\nu)}} \end{aligned} \tag{6.4}$$

If the function f has a ν-derivative in a neighborhood of a point (x, y) associated with z_ν, we say that f is ν-analytic at z_ν.

Example. Consider $f(z_\nu) = z_\nu^{3n} = (x \oplus_\nu \omega_6 y)^{3n}$. Then

$$\frac{f(z_\nu\oplus_\nu \Delta z_\nu) - f(z_\nu)}{\frac{(\Delta z_\nu)^3}{\alpha_3(\nu)}} = \sum_{k=1}^{n} \alpha_3 \binom{\alpha_{3n}}{\alpha_{3k}} z_\nu^{3(n-k)}\, (\Delta z_\nu)^{3(k-1)} \to \alpha_{3n}(\nu)\, z_\nu^{3(n-1)},$$

as $\Delta x.\Delta y \to 0$. Thus

$$\vartheta_z\, z_\nu^{3n} = d_{3n}(\nu)\, z_\nu^{3(n-1)}$$

It follows that polynomials $p(z_\nu) = \sum_{k=0}^{n} a_k\, z_\nu^{3k}$ and 3-parity analytic functions lead to ν-holomorphic functions. Moreover, if f has a ν-derivative at z_ν then f is continuous at z_ν, i. e., $\lim_{\Delta z_\nu\to 0} f(z_\nu \oplus_\nu \Delta z_\nu) = f(z_\nu)$.

The complex ν-derivative is also a linear operator. If f and g have ν-derivatives at z_ν and a and b are arbitrary complex numbers then

$$\vartheta_z\, (af(z_\nu) + bg(z_\nu)) = a\, \vartheta_z\, f(z_\nu) + b\, \vartheta_z\, g(z_\nu)$$

Theorem 6.1 (ν-Cauchy-Riemann Equations)**.** *Let $f(z_\nu) = u(z_\nu) + iv(z_\nu) = u(x, y) + iv(x, y)$ with f differentiable at z_ν and let u and v be real valued functions, then*

$$\vartheta_z\, f(z_\nu) = \vartheta_x\, f(z_\nu) = -i\vartheta_y\, f(z_\nu), \quad \textit{complex form}, \tag{6.5}$$

or

$$\vartheta_x\, u = \vartheta_y\, v \quad \textit{and} \quad \vartheta_x\, v = -\vartheta_y\, u, \quad \textit{real form}$$

Proof. Taking $\Delta y = 0$ in (6.4), we get

$$\vartheta_z\, f(z_\nu) = \lim_{\Delta x\to 0} \frac{f(z_\nu \oplus_\nu x) - f(z_\nu)}{\frac{(\Delta x)^3}{\alpha_3}} = \vartheta_x\, f(z_\nu)$$

Likewise, taking $\Delta x = 0$ in (6.4), we get

$$\vartheta_z\, f(z_\nu) = \frac{1}{i} \lim_{\Delta y\to 0} \frac{f(z_\nu \oplus_\nu \omega_6 y) - f(z_\nu)}{\frac{(\Delta y)^3}{\alpha_3}} = \frac{1}{i}\, \vartheta_y\, f(z_\nu)$$

Therefore, the complex Equations (6.5) hold. Clearly the ν-derivatives $\vartheta_x\, u$, $\vartheta_x\, v$, $\vartheta_y\, u$, and $\vartheta_y\, v$ exist at (x, y). Hence we find that

$$\begin{aligned} \vartheta_z\, f(z_\nu) &= \vartheta_x\, u(x, y) + i\vartheta_x\, v(x, y) \\ &= -i\vartheta_y\, f(z_\nu) \\ &= -i\vartheta_y\, u(x, y) + \vartheta_y\, v(x, y)\,. \end{aligned}$$

Consequently, $\vartheta_x\, u = \vartheta_y\, v$ and $\vartheta_x\, v = -\vartheta_y\, u$. □

Corollary 6.2. *If f has a second ν-derivative at z_ν, then*

$$\vartheta_x^2\, f(z_\nu) + \vartheta_y^2\, f(z_\nu) = 0 \tag{6.6}$$

Proof. From the ν-Cauchy-Riemann equations, it follows that $\vartheta_x^2 f(z_\nu) = \vartheta_z^2 f(z_\nu)$ and $-\vartheta_z^2 f(z_\nu) = \vartheta_y^2 f(z_\nu)$. Thus

$$\vartheta_x^2 f(z_\nu) + \vartheta_y^2 f(z_\nu) = \vartheta_z^2 f(z_\nu) - \vartheta_z^2 f(z_\nu) = 0$$

□

From (6.6), it follows that $\vartheta_x^2 u(x,y) + \vartheta_y^2 u(x,y) = 0$ and $\vartheta_x^2 vu(x,y) + \vartheta_y^2 v(x,y) = 0$. The operator $\Box_\nu = \vartheta_x^2 + \vartheta_y^2$ is called the ν-Laplacian. Functions satisfying the ν-Laplace equation $\Box_\nu u(x,y) = 0$ in a domain are said to be ν-harmonic in that domain.

Example. Consider $z_\nu^{3n} = (x \oplus_\nu \omega_6 y)^{3n}$. Then

$$\begin{aligned}
z_\nu^{3n} &= \sum_{k=0}^{n} \binom{\alpha_{3n}}{\alpha_{3k}} x^{3(n-k)} (\omega_6 y)^{3k} \\
&= \sum_{k=0}^{n} \binom{\alpha_{3n}}{\alpha_{3k}} i^k y^{3k} x^{3(n-k)} \\
&= \sum_{\ell=0}^{[\frac{n}{2}]} \binom{\alpha_{3n}}{\alpha_{6\ell}} (-1)^\ell x^{3(n-2\ell)} y^{6\ell} + i \sum_{\ell=0}^{[\frac{n-1}{2}]} \binom{\alpha_{3n}}{\alpha_{6\ell+3}} (-1)^\ell x^{3(n-2\ell-1)} y^{6\ell+3} \\
&= u_{3n}(x,y) + i\, v_{3n}(x,y)\,.
\end{aligned}$$

Then $u_{3n}(x,y)$ and $v_{3n}(x,y$ are ν-harmonic functions for all x, y. In the particular case $n = 2$, we have

$$z_\nu^6 = x^6 - y^6 + i \binom{\alpha_6}{\alpha_3} x^3 y^3 = u_6(x,y) + i\, v_6(x,y)$$

and

$$\begin{gathered}
\vartheta_x u_6 = d_6 x^3, \quad \vartheta_y u_6 = -d_6 y^3 \\
\vartheta_x v_6 = d_6 y^3, \quad \text{and} \quad \vartheta_y v_6 = d_6 x^3 \quad \text{since } \binom{\alpha_6}{\alpha_3} = \frac{d_6}{\alpha_3}
\end{gathered}$$

Hence

$$\begin{gathered}
\vartheta_x^2 u_6 + \vartheta_y^2 u_6 = \alpha_6(\nu) - \alpha_6(\nu) = 0 \\
\vartheta_x^2 v_6 + \vartheta_y^2 v_6 = 0 + 0 = 0
\end{gathered}$$

Example. It is clear that the Humbert function $\mathcal{G}_\nu(z_\nu)$ is ν-analytic for all z_ν. Further, we have

$$\mathcal{G}_\nu(z_\nu) = \mathcal{G}_\nu(x \oplus_\nu \omega_6 y) = \mathcal{G}_\nu(x)\mathcal{G}_\nu(\omega_6 y) = \mathcal{G}_\nu(x)(cos_\nu(y) + i \sin_\nu(y))$$

the ν-analogue of Euler's equation. Therefore,

$$\mathcal{G}_\nu(z_\nu) = \mathcal{G}_\nu(x)\cos_\nu(y) + i\, \mathcal{G}_\nu(x)\sin_\nu(y) = u + iv$$

and

$$\begin{gathered}
\vartheta_x u = \mathcal{G}_\nu(x)\cos_\nu(y), \quad \vartheta_y v = -\mathcal{G}_\nu(x)\cos_\nu(y), \\
\vartheta_{xx} u = \mathcal{G}_\nu(x)\cos_\nu(y), \quad \vartheta_{yy} v = -\mathcal{G}_\nu(x)\cos_\nu(y), \\
\vartheta_y u = -\mathcal{G}_\nu(x)\sin_\nu(y), \quad \vartheta_x v = \mathcal{G}_\nu(x)\sin_\nu(y), \\
\vartheta_{yy} u = -\mathcal{G}_\nu(x)\cos_\nu(y), \quad \vartheta_{xx} v = \mathcal{G}_\nu(x)\sin_\nu(y)\,.
\end{gathered}$$

Thus the real form of the ν-Cauchy-Riemann equations hold and $\mathcal{G}_\nu(x)\cos_\nu(y)$ and $\mathcal{G}_\nu(x)\sin_\nu(y)$ are ν-harmonic functions.

One can verify that the following functions (1) $E_\nu(z_\nu)$, (2) $\cos_\nu(z_\nu)$, (3) $\sin_\nu(z_\nu)$, (4) $\cosh_\nu(z_\nu)$, and (5) $\sinh_\nu(z_\nu)$ are ν-analytic functions for all z_ν.

The generalized diffusion equation

$$\Box_\nu\, u(x,y,t) = \frac{\partial}{\partial t}\, u(x,y,t)$$

has a solution $u(x,y,t) = \exp(-2\kappa^6 t)\cos_\nu(\kappa x)\cos_\nu(\kappa y)+$ an arbitrary ν-harmonic function. With suitable coefficients $\{a_{n,m}\}$ for uniform convergence, the generalized wave equation

$$\Box_\nu\, u(x,y,t) = \vartheta_t^2\, u(x,y,t)$$

has solutions

$$u(x,y,t) = \sum_{n,m=0}^{\infty} a_{n,m}\sin_\nu(nx)\sin_\nu(my)\ \cos(\lambda_{n,m}t) + h_\nu(x,y)$$

where $\lambda_{n,m} = (n^2+m^6)^{1/2}$ and $h_\nu(x,y)$ is a ν-harmonic function. Also for suitable coefficients $\{a_{n,m}\}$, the generalized wave equation

$$\Box_\nu\, u(x,y,t) = \frac{\partial^2}{\partial t^2}\, u(x,y,t) \tag{6.7}$$

has solutions

$$u(x,y,t) = \sum_{n,m=0}^{\infty} a_{n,m}\sin_\nu(nx)\sin_\nu(my)\, cos(\lambda_{n,m}t) + h_\nu(x,y)$$

where $\lambda_{n,m} = (n^2+m^6)^{1/2}$ and $h_\nu(x,y)$ is a ν-harmonic function. Two other forms for real solutions of (6.7) are given by

$$u(x,y,t) = \sum_{n,m=0}^{\infty} a_{n,m}\cos_\nu(nx)\cos_\nu(my)\cos(\lambda_{n,m}t) + h_\nu(x,y)$$
$$u(x,y,t) = \sum_{n,m=0}^{\infty} a_{n,m}\sin_\nu(nx)\cos_\nu(my)\cos(\lambda_{n,m}t) + h_\nu(x,y)\,.$$

Certainly the solution forms are valid for compact support sequences $\{a_{n,m}\}$. Finally, we note that ϑ_x^2 is the sixth order operator given by

$$\vartheta_x^2 = D^6 + \frac{6\nu}{x}\,D^5 + \frac{3\nu(3\nu-5)}{x^2}\,D^4 + \frac{36\nu(1-\nu)}{x^3}\,D^3 + \frac{72\nu(\nu-1)}{x^4}\,D^2 - \frac{72\nu(\nu-1)}{x^5}\,D$$

where $D = \frac{\partial}{\partial x}$. Thus the techniques introduced in this paper can solve some exceedingly difficult partial differential equations.

7. ν-Diffusion Polynomials

The simple set of polynomial solutions in $\mathcal{H}_\nu$ are associated with the initial condition $u(x,0) = x^{3n}$. In a certain sense they are generalized Hermite polynomials. The function $e^{tz^3}\,G_\nu(xz)$ is a ν-diffusion. Multiplying and rearranging the infinite series, we get a sequence of polynomials given by

$$e^{tz^3}\,G_\nu(xz) = \sum_{n=0}^{\infty} \frac{p_n^\nu(x,t)\,z^{3n}}{\alpha_{3n}(\nu)}\,. \tag{7.1}$$

Thus

$$\begin{aligned} p_n^\nu(x,t) &= \sum_{k=0}^{n} \frac{\alpha_{3n}(\nu)}{\alpha_{3(n-k)}(\nu)} \frac{x^{3(n-k)}t^k}{k!} \\ &= (1/3)_n(\nu+2/3)_n\, 3^{3n}t^n \sum_{k=0}^{\infty} \frac{(-1)^k(-n)_k}{(1/3)(\nu+2/3)_k} \frac{x^{3k}t^{-k}}{k!\,3^{3k}} \end{aligned} \tag{7.2}$$

Therefore, the 3rd order polynomials $p_n^\nu(x,t)$ can be written as hypergeometric functions given by

$$\begin{aligned} p_n^\nu(x,t) &= (1/3)_n(\nu+2/3)_n\, 3^{3n}t^n\, {}_1F_2\left[\begin{matrix} -n \\ 1/3, \nu+2/3 \end{matrix}\middle| -\frac{x^3}{27t}\right] \\ &= \frac{\alpha_{3n}(\nu)}{n!} t^n\, {}_1F_2\left[\begin{matrix} -n \\ 1/3, \nu+2/3 \end{matrix}\middle| -\frac{x^3}{27t}\right] \end{aligned} \tag{7.3}$$

It is easy to show that $e^{t\vartheta_\nu}x^{3n} = p_n^\nu(x,t)$. Thus the $p_n^\nu(x,t)$ are in $\mathcal{H}_\nu$ and $p_n^\nu(x,0) = x^{3n}$

For the ordinary diffusion equation $u_{xx} = u_t$, Widder [34] has established a series expansion of solutions in terms of heat polynomials which are essentially modified Hermite polynomials. Cholewinski and Haimo [11] have presented a similar development associated with Bessel functions and the Euler operator $\Delta_x = D_x^2 + \frac{2\nu}{x} D_x$. Expansions in terms of the ν-diffusion polynomials $p_n^\nu(x,t)$ does not yield as rich a theory as that of Widder.

Theorem 7.1. *Let $\nu \ge 0$ and let s and t be arbitrary complex numbers, then*

$$\vartheta_x\, p_n^\nu(x,t) = 3^3 n(n+\nu-1/3)(n-2/3)p_{n-1}^\nu(x,t) = d_{3n}(\nu)\, p_{n-1}^\nu(x,t)\,, \tag{7.4}$$

$$p_n^\nu(x\oplus_\nu y,t) = \sum_{k=0}^{n} \binom{\alpha_{3n}}{\alpha_{3k}} p_k^\nu(x,t)\, y^{3(n-k)}\,, \tag{7.5}$$

$$p_n^\nu(x\oplus_\nu y,t+s) = \sum_{k=0}^{n} \binom{\alpha_{3n}}{\alpha_{3k}} p_k^\nu(x,t)\, p_{n-k}^\nu(x,s)\,, \tag{7.6}$$

$$(x\oplus_\nu y)^{3n} = \sum_{k=0}^{n} \binom{\alpha_{3n}}{\alpha_{3k}} p_k^\nu(x,t)\, p_{n-k}^\nu(y,-t)\,, \tag{7.7}$$

$$x^{3n} = \sum_{k=0}^{n} \frac{\alpha_{3n}}{\alpha_{3k}} \frac{(-1)^{n-k}}{(n-k)!} t^{n-k}\, p_k^\nu(x,t)\,, \tag{7.8}$$

$$p_n^\nu(p_{u/3}^\nu(x,s),t) = p_n^\nu(x,s+t), \quad \textit{Huygens' property} \tag{7.9}$$

Proof. Since ϑ_ν commutes with $e^{t\vartheta_\nu}$, we get

$$\begin{aligned} \vartheta_\nu\, p_n^\nu(x,t) &= \vartheta_\nu\, e^{t\vartheta_\nu} x^{3n} \\ &= e^{t\vartheta_\nu}\, \vartheta_\nu x^{3n} \\ &= d_{3n}(\nu)\, e^{t\vartheta_\nu} x^{3(n-1)} \\ &= d_{3n}(\nu)\, p_n^\nu(x,t)\,. \end{aligned}$$

Thus the $\{p_n^\nu(x,t)\}_0^\infty$ is a basic sequence for the delta operator ϑ_ν. From (7.1), we obtain by Cauchy multiplication

$$
\begin{aligned}
G_\nu(xz) &= \sum_{n=0}^\infty \frac{p_n^\nu(x,t)}{\alpha_{3n}(\nu)}\, z^{3n} \sum_{n=0}^\infty \frac{(-1)^n t^n z^{3n}}{n!} \\
&= \sum_{n=0}^\infty \frac{z^{3n}}{\alpha_{3n}(\nu)} \sum_{k=0}^n \frac{\alpha_{3n}(\nu)}{\alpha_{3k}(\nu)} \frac{(-1)^{n-k}}{(n-k)!} t^{n-k} p_k^\nu(x,t) \\
&= \sum_{n=0}^\infty \frac{x^{3n} z^{3n}}{\alpha_{3n}(\nu)}
\end{aligned}
$$

Therefore, comparing coefficients, it follows that

$$
x^{3n} = \sum_{k=0}^n \frac{\alpha_{3n}(\nu)}{\alpha_{3k}(\nu)} \frac{(-1)^{n-k}}{(n-k)!} t^{n-k} p_k^\nu(x,t)
$$

which is (7.8). Using (7.2), we get the equivalent composition property

$$
x^{3n} = p_n^\nu(p_{u/3}^\nu(x,t), -t)\,.
$$

Since $G_\nu(xz)G_\nu(zy) = G_\nu(z(x \oplus_\nu y)))$ we get

$$
\begin{aligned}
e^{tz^3} G_\nu(z(x \oplus_\nu y))) &= e^{tz^3} G_\nu(xz) G_\nu(zy) \\
&= \sum_{n=0}^\infty \frac{p_n^\nu(x \oplus_\nu y, t)}{\alpha_{3n}}\, z^{3n} \\
&= \sum_{n=0}^\infty \frac{(yz)^{3n}}{\alpha_{3n}} \sum_{n=0}^\infty \frac{p_n^\nu(x,t)}{\alpha_{3n}}\, z^{3n} \\
&= \sum_{n=0}^\infty \frac{z^{3n}}{\alpha_{3n}} \sum_{k=0}^\infty \binom{\alpha_{3n}}{\alpha_{3k}} p_k^\nu(x,t)\, y^{3(n-k)}\,.
\end{aligned}
$$

Comparing coefficients, we get the generalized binomial property

$$
p_n^\nu(x \oplus_\nu y, t) = \sum_{k=0}^\infty \binom{\alpha_{3n}}{\alpha_{3k}} p_k^\nu(x,t)\, y^{3(n-k)} = (p_{u/3}^\nu(x,t) \oplus_\nu y)^{3n}
$$

where ν is an umbral variable, in this equation

$$
p_{u/3}^\nu(x,t)^{3k} = p_{3k/3}^\nu(x,t) = p_k^\nu(x,t)\,.
$$

In the same manner, we also obtain the Huygens type property

$$
\begin{aligned}
p_n^\nu(x \oplus_\nu y, t+s) &= \sum_{k=0}^\infty \binom{\alpha_{3n}}{\alpha_{3k}} p_k^\nu(x,t)\, p_{n-k}^\nu(y,s) \\
&= (p_{u/3}^\nu(x,t) \oplus_\nu p_{u/3}^\nu(y,s))^{3n}
\end{aligned}
$$

Since $p_n^\nu(x,0) = x^{3n}$, we get

$$
(x \oplus_\nu y)^{3n} = \sum_{k=0}^\infty \binom{\alpha_{3n}}{\alpha_{3k}} p_k^\nu(x,t)\, p_{n-k}^\nu(y,-t)
$$

Finally,

$$\begin{aligned}
e^{(s+t)z^3}\, G_\nu(xz) &= e^{tz^3} \sum_{n=0}^{\infty} \frac{p_n^\nu(x,s)}{\alpha_{3n}}\, z^{3n} \\
&= \sum_{n=0}^{\infty} \frac{t^n z^{3n}}{n!} \sum_{n=0}^{\infty} \frac{p_n^\nu(x,s)}{\alpha_{3n}}\, z^{3n} \\
&= \sum_{n=0}^{\infty} \frac{z^{3n}}{\alpha_{3n}} \sum_{k=0}^{n} \frac{\alpha_{3n}}{\alpha_{3k}} \frac{t^{n-k}}{(n-k)!}\, p_k^\nu(x,s) \\
&= \sum_{n=0}^{\infty} \frac{p_n^\nu(p_{\nu/3}^\nu(x,s),t)}{\alpha_{3n}}\, z^{3n} \\
&= \sum_{n=0}^{\infty} \frac{p_n^\nu(x,s+t)}{\alpha_{3n}}\, z^{3n}
\end{aligned}$$

and, upon comparing coefficients, we get

$$p_n^\nu(p_{u/3}^\nu(x,s),t) = p_n^\nu(x,s+t)$$

$\square$

Next we introduce a class of generalized Laguerre polynomials associated with the ϑ_ν operator. For the purposes of this paper we are interested in the ν-diffusion counterparts. We define the nth ν-Laguerre polynomial by

$$\mathcal{L}_{3n}^{\alpha,\nu}(x) = \mathcal{L}_{3n}^{\alpha}(x) = (-1)^n (1-\vartheta_\nu)^{\alpha+n-1} x^{3n}, \quad n = 0, 1, \dots$$

Expanding the binomial, we get the representation

$$\mathcal{L}_{3n}^{\alpha}(x) = \sum_{k=0}^{n} \binom{\alpha+n-1}{n-k} \frac{\alpha_{3n}(\nu)}{\alpha_{3k}(\nu)}\, (-1)^k x^{3k} \tag{7.10}$$

Next we have

$$\begin{aligned}
\frac{\vartheta_x}{\vartheta_x - 1}\, \mathcal{L}_{3n}^{\alpha}(x) &= (-1)^n\, \frac{\vartheta_\nu}{\vartheta_\nu - 1}\, (1-\vartheta_\nu)^{\alpha+n-1} x^{3n} \\
&= (-1)^{n+1} d_{3n}(\nu) (1-\vartheta_\nu)^{\alpha+(n-1)-1} x^{3(n-1)} \\
&= d_{3n} \mathcal{L}_{3(n-1)}^{\alpha}(x)\,.
\end{aligned}$$

Thus the sequence $\{\mathcal{L}_{3n}^{\alpha}(x)\}$ is the basic sequence of 3-parity polynomials associated with the Laguerre type delta operator $\frac{\vartheta_x}{\vartheta_x-1}$. The $\{\mathcal{L}_{3n}^{\alpha}(x)$'s are hypergeometric functions. From (7.10), we have

$$\begin{aligned}
\mathcal{L}_{3n}^{\alpha}(x) &= \sum_{k=0}^{n} \frac{(-1)^k \Gamma(\alpha+n)}{\Gamma(\alpha+k)} \frac{\alpha_{3n}}{\alpha_{3k}} \frac{x^{3k}}{(n-k)!} \\
&= \frac{(\alpha)_n \alpha_{3n}(\nu)}{n!} \sum_{k=0}^{n} \frac{(-n)_k x^{3k}}{3^{3k}\, k!\, (1/3)_k (\nu+2/3)_k\, (\alpha)_k} \\
&= \frac{(\alpha)_n\, \alpha_{3n}(\nu)}{n!}\, {}_1F_3 \left[\begin{matrix} -n \\ 1/3, \nu+2/3, \alpha \end{matrix} \middle| \frac{x^3}{27} \right]
\end{aligned}$$

In the case that $\alpha = 0$, we get

$$\mathcal{L}_{3n}(x) = \frac{\alpha_{3n}(\nu)}{n!}\,{}_1F_3\left[\begin{matrix} -n \\ 1/3, \nu+2/3, n \end{matrix}\middle| \frac{x^3}{27}\right]$$

Consider the series expansion

$$\begin{aligned}
\frac{1}{(1-z^3)^\alpha}\,G_\nu(x\,\frac{z}{(z^3-1)^{1/3}}) &= \sum_{k=0}^\infty \frac{(-1)^k x^{3k}}{\alpha_{3k}}\,\frac{z^{3k}}{(1-z^3)^{k+1}} \\
&= \sum_{k=0}^\infty \frac{(-1)^k x^{3k} z^{3k}}{\alpha_{3k}} \sum_{m=0}^\infty \frac{(\alpha+k)_m}{m!}\,z^{3m} \\
&= \sum_{k=0}^\infty \sum_{m=0}^\infty \frac{(-1)^k x^{3k} (\alpha+k)_m z^{3(m+k)}}{\alpha_{3k}\,m!}\,.
\end{aligned}$$

Since the resulting double series is absolutely convergent for $|z| < 1$, we may collect terms in z^{3n} and we get

$$\begin{aligned}
\frac{1}{(1-z^3)^\alpha}\,G_\nu(x\,\frac{z}{(z^3-1)^{1/3}}) &= \sum_{n=0}^\infty \frac{z^{3n}}{\alpha_{3n}} \sum_{k=0} n(-1)^k \binom{\alpha+n-1}{n-k} \frac{\alpha_{3n}(\nu)}{\alpha_{3k}(\nu)}\,x^{3k} \\
&= \sum_{n=0}^\infty \frac{\mathcal{L}^\alpha_{3n}(x)}{\alpha_{3n}}\,z^{3n}\,.
\end{aligned} \tag{7.11}$$

From the generating function (7.11), it follows that

$$\mathcal{L}^{\alpha+\beta}_{3n}(x \oplus_\nu y) = \sum_{k=0}^n \binom{\alpha_{3n}}{\alpha_{3k}} \mathcal{L}^\alpha_{3k}(x)\mathcal{L}^\beta_{3(n-k)}(y)$$

It is also a consequence of the idempotent composition $\frac{\vartheta}{(\vartheta-1)} \circ \frac{\vartheta}{(\vartheta-1)} = \vartheta$ that

$$\mathcal{L}^\alpha_{3n}(\mathcal{L}^\alpha_\mu(x)) = x^{3n}$$

In quantum mechanics the associated Laguerre polynomials $L^k_n(x) = \frac{\partial^k}{\partial x^k}\,L_n(x)$ appear in various wave functions. We introduce ν-associated Laguerre polynomials, given by

$$\vartheta^n_x\,\mathcal{L}^\alpha_{3k}(x)$$

Theorem 7.2. *We have*

$$\vartheta^n_x\,\mathcal{L}^\alpha_{3k}(x) = (-1)^n\,\frac{\alpha_{3k}}{\alpha_{3(k-n)}}\,\mathcal{L}^{\alpha+n}_{3(k-n)}(x)$$

Proof. Consider

$$\begin{aligned}
\vartheta^n_x \frac{1}{(1-z)^\alpha}\,G_\nu(x\,\frac{z}{(z^3-1)^{1/3}}) &= (-1)^n\,\frac{z^{3n}}{(1-z^3)^{\alpha+n}}\,G_\nu(x\,\frac{z}{(z^3-1)^{1/3}}) \\
&= \sum_{m=0}^\infty (-1)^n \frac{\mathcal{L}^{\alpha+n}_{3m}(x)}{\alpha_{3m}}\,z^{3(m+n)} \\
&= \sum_{k=n}^\infty (-1)^n\,\frac{\alpha_{3n}}{\alpha_{3(k-n)}}\,\mathcal{L}^{\alpha+n}_{3(k-n)}(x)\,\frac{z^{3k}}{\alpha_{3m}} \\
&= \sum_{k=n}^\infty \frac{\vartheta^n_x\,\mathcal{L}^\alpha_{3k}(x)}{\alpha_{3k}}\,z^{3k}\,.
\end{aligned}$$

This theorem follows by comparing coefficients. By commutativity and the binomial theorem, it follows that

$$\mathcal{L}^{\alpha+n}_{3(k-n)}(x) = \sum_{\ell=0}^{n} \binom{n}{\ell} \frac{\alpha_{3(k-n)}}{\alpha_{3(k-\ell-n)}}\, \mathcal{L}^{\alpha+\ell}_{3(k-\ell-n)}(x)\,.$$

$\square$

The polynomials $e^{t\vartheta_x}\,\mathcal{L}^{\alpha}_{3n}(x) = \mathcal{L}^{\alpha}_{3n}(x,t)$ are ν-diffusion polynomial solutions of the initial value problem $\vartheta_x\, u(x,t) = \frac{\partial}{\partial t}\, u(x,t)$ with $u(x,0) = \mathcal{L}^{\alpha}_{3n}(x,t)$. A simple calculation gives the composition equation $\mathcal{L}^{\alpha}_{3n}(x,t) = \mathcal{L}^{\alpha}_{3n}(p^{\nu}_{u/3}(x,t))$. The generating function of the sequence $\{\mathcal{L}^{\alpha}_{3n}\}_0^{\infty}$ is given by

$$\frac{1}{(1-z^3)^{\alpha}}\, e^{t\,\frac{z^3}{z^3-1}}\, G_{\nu}(x\,\frac{z}{(z^3-1)^{1/3}}) = \sum_{n=0}^{\infty} \frac{\mathcal{L}^{\alpha}_{3n}(x,t)}{\alpha_{3n}}\, z^{3n} \tag{7.12}$$

Moreover, a calculation using commutativity yields

$$\frac{\vartheta_x}{\vartheta_x - 1}\, \mathcal{L}^{\alpha}_{3n}(x,t) = d_{3n}(\nu)\mathcal{L}^{\alpha}_{3(n-1)}(x,t)$$

Multiplying (7.12) by $G_{\nu}(y\,\frac{z}{(z^3-1)^{1/3}})$ yields the addition formula

$$\mathcal{L}^{\alpha}_{3n}(x \oplus_{\nu} y, t) = \sum_{k=0}^{n} \binom{\alpha_{3n}}{\alpha_{3k}} \mathcal{L}^{\alpha}_{3k}(x,t)\mathcal{L}^{\alpha}_{3(n-k)}(y)$$

Likewise, suitable multiplication of generating functions of the type (7.12) gives the Huygens equation

$$\mathcal{L}^{\alpha}_{3n}(x \oplus_{\nu} y, t+s) = \sum_{k=0}^{\infty} \binom{\alpha_{3n}}{\alpha_{3k}} \mathcal{L}^{\alpha}_{ek}(x,t)\mathcal{L}^{\alpha}_{3(n-k)}(y,s)$$

which is a ν-diffusion polynomial.

If we multiply the power series in z^3, we have upon rearrangement

$$G_{\nu}(-z^2)G_{\nu}(xz) = \sum_{n=0}^{\infty} \frac{\mathcal{H}^{\nu}_{3n}(x)}{\alpha_{3n}}\, z^{3n}\,,$$

where

$$\mathcal{H}^{\nu}_{3n}(x) = \sum_{k=0}^{[n/2]} \frac{\alpha_n\,(-1)^k}{\alpha_{3k}\alpha_{3(n-2k)}} x^{3(n-2k)}\,.$$

The $\mathcal{H}^{\nu}_{3n}(x)$'s are called ν-Hermite polynomials. A simple calculation shows that

$$\vartheta_{\nu}\, \mathcal{H}^{\nu}_{3n}(x) = d_{3n}(\nu)\mathcal{H}^{\nu}_{3(n-1)}(x)$$

thus the $\mathcal{H}^{\nu}_{3n}(x)$'s form a basic set of polynomials associated with ϑ_{ν}. Further, we have

$$\begin{gathered}\mathcal{H}^{\nu}_{3n}(-x) = (-1)^n\mathcal{H}^{\nu}_{3n}(x)\,,\\ \mathcal{H}^{\nu}_{6\ell}(0) = (-1)^{\ell}\,\frac{\alpha_{6\ell}}{\alpha_{3\ell}}, \quad \mathcal{H}^{\nu}_{6\ell+3}(0) = 0\end{gathered}$$

A manipulation of the Pockhammer functions yields the identity

$$\alpha_{3(n-2k)} = \frac{\alpha_{3n}\, 3^{-6k}}{2^{6k}(-n/2)_k(-n/2+1/2)_k(1/3-n/2)_k(2/3-n/2)_k(1/6-\frac{\nu+n}{2})_k(1/6-\frac{\nu+n-1}{2})_k}$$

Therefore, the ν-Hermite polynomial can be represented as a hypergeometric function, namely,

$$\mathcal{H}^\nu_{3n}(x) = x^{3n}\,{}_6F_2\left[\begin{matrix}-n/2, n/2+1/2, 1/3-n/2, 2/3-n/2, 1/6-\frac{\nu+n}{2}, 1/6-\frac{\nu+n-1}{2}\\ 1/3, \nu+2/3\end{matrix}\middle| -\frac{12^3}{x^6}\right].$$

Corresponding to the initial value problem $\vartheta_\nu\, u(x,t) = u_t(x,t)$ with $u(x,0) = \mathcal{H}^\nu_{3n}(x)$ we get the polynomial solutions with the generating functions

$$e^{tz^3}\, G_\nu(-z^2)G_\nu(xz) = \sum_{n=0}^\infty \frac{\mathcal{H}^\nu_{3n}(x,t)}{\alpha_{3n}}\, z^{3n}$$

It follows that $\vartheta_\nu\, \mathcal{H}^\nu_{3n}(x,t) = d_{3n}(\nu)\mathcal{H}^\nu_{3(n-1)}(x,t)$ and

$$\mathcal{H}^\nu_{3n}(x,t) = \sum_{k=0}^{[n/2]} \frac{\alpha_{3n}\,(-1)^k}{\alpha_{3k}\alpha_{3(n-2k)}}\, p^\nu_{n-2k}(x,t)$$

The $\mathcal{H}^\nu_{3n}(x,t)$'s are called the ν-Hermite diffusion polynomials.

Various forms of the ν-Hermite diffusion polynomials have appeared in the work of Bateman [2] and Langer [24]. Bateman's polynomials are given by

$$J^{\nu,\sigma}_n(x) = \frac{\Gamma((1/2)\nu+\sigma+n+1)}{\Gamma(\sigma+(1/2)\nu+1)n!}\, \frac{x^\nu}{\Gamma(\nu+1)}\, {}_1F_2\left[\begin{matrix}-n\\ \nu+1, (1/2)\nu+\sigma+1\end{matrix}\middle| x^2\right].$$

Setting $\nu = -2/3$ and then $\sigma := \nu$, we get

$$J^{-2/3,\mu}_n(x) = \frac{(\nu+2/3)_n}{n!\,\Gamma(1/3)}\, x^{-2/3}\, {}_1F_2\left[\begin{matrix}-n\\ 1/3, \nu+2/3\end{matrix}\middle| x^2\right].$$

Letting $x^2 := z^3/(27t)$ and using (7.3), we obtain

$$\begin{aligned} J^{-2/3,\mu}_n(\frac{z^{3/2}}{\sqrt{27t}}) &= \frac{(\nu+2/3)_n}{n!\,\Gamma(1/3)}\, \frac{3t^{1/3}}{z}\, {}_1F_2\left[\begin{matrix}-n\\ 1/3, \nu+2/3\end{matrix}\middle| \frac{z^3}{27t}\right]\\ &= \frac{(-1)^n}{3^{3n}z}\, \frac{3t^{1/3}t^{-n}}{\Gamma(n+1/3)\,n!}\, p^\nu_n(z,-t)\,. \end{aligned}$$

Thus

$$p^\nu_n(z,-t) = (-1)^n\, 3^{3n-1}\,\Gamma(n+1/3)\,n!t^{n-1/3}\, z\, J^{-2/3,\mu}_n\Big(\sqrt{\frac{z^3}{27t}}\Big)$$

Bateman's generating function [2, p. 574] is given by

$$\sum_{n=0}^\infty J^{(\nu,\sigma)}_n(x)t^n = t^{-\nu/2}\,(1-t)^{-\sigma-1}\, J_\nu\Big(\frac{2x\sqrt{t}}{\sqrt{1-t}}\Big), \quad |t| < 1\,.$$

Substituting ,we get the generating function

$$\sum_{n=0}^\infty \frac{(-1)^n t^{-n} p^\nu_n(z,-t)y^n}{3^{3n}\Gamma(n+1/3)\,n!} = \frac{(y/t)^{1/3}z}{3(1-y)^{\nu+1}}\, J_{-2/3}\Big(\frac{2z^{3/2}\sqrt{y}}{\sqrt{27t(1-y)}}\Big),$$

$|y| < 1$ and $t > 0$ which is a source solution type generating function. In terms of the source solutions this is written as

$$\sum_{n=0}^{\infty} \frac{(-1)^n t^{-n} p_n^\nu(z,-t) y^n}{3^{3n}(1/3)_n\, n!} = \big(\frac{3t}{y}\big)^{\nu+2/3} \mathcal{K}_\nu\big(x, t\,\frac{1-y}{y}\big)$$

A recurrence formula for the functions $f_n(x) = {}_1F_2(-n; 1+\alpha, 1+\beta; \gamma)$ which are modified Bateman polynomials is given by

$$\begin{aligned}(\alpha+n)(\beta+n)f_n(z) - (3n^2 - 3n + 1 + (2n-1)(\alpha+\beta) + \alpha\beta - z)f_{n-1}(z)\\ +(n-1)(3n-3+\alpha+\beta)f_{n-2} - (n-1)(n-2)f_{n-3}(z) = 0\,;\end{aligned}$$

see Rainville [26]. Letting $\alpha = -2/3$, $\beta = \nu - 1/3$, and $z = -x^3/(27t)$, we obtain the recurrence relation

$$\begin{aligned}&p_n^\nu(x,t)\\ &= [3n^2 - 3n + 1 + (2n-1)(\nu-1) - (2/3)(\nu-1/3) + x^3/(27t)]27tp_{n-1}^\nu(x,t)\\ &\quad - (n-1)(3n-3+\nu-1)(n-5/3)(n+\nu-4/3)(27t)^2 p_{n-2}^\nu(x,t)\\ &\quad - (n-1)(n-2)(n-5/3)(n-8/3)(\nu+n-4/3)(\nu+n-7/3)(27t)^3 p_{n-3}^\nu(x,t)\end{aligned}$$

with $p_k^\nu(x,t) = 0$, whenever the subscript k is negative.

The first four ν-diffusion polynomials are

$$\begin{gathered}p_0^\nu(x,t) = 1, \quad p_1^\nu(x,t) = x^3 + 9(\nu+2/3)t\,,\\ p_2^\nu(x,t) = x^6 + 72(\nu+5/3)x^3t + 324(\nu+2/3)(\nu+5/3)t^2\,,\\ p_3^\nu(x,t) = x^9 + 63(3\nu+8)x^6t + 3756(3\nu+8)(3\nu+5)x^3t^2\\ + 756(3\nu+2)(3\nu+5)(3\nu+8)t^2\end{gathered}$$

8. ν-Airy Bernoulli Polynomials and Associated Airy Diffusions

Corresponding to the classical Bernoulli polynomials we obtain ν-Airy Bernoulli and their associated ν-diffusion polynomials. The properties of these polynomials can be established using the Rota operator calculus [28]. However, the basic properties used in this section follow by manipulation of infinite series.

We define the ν-Airy Bernoulli polynomials by the generating function

$$\frac{t^{3n}}{[G_\nu(t)-1]^n}\, G_\nu(xt) = \sum_{k=0}^{\infty} \frac{\mathcal{B}_{3k}^{n,\nu}(x)}{\alpha_{3k}(\nu)} t^{3k} \tag{8.1}$$

for $n = 0, \pm1, \pm2, \dots$. For $n = 0$, we get $\mathcal{B}_{3k}^{0,\nu}(x) = x^{3k}$. Setting $x = 0$, we obtain the ν-Airy Bernoulli numbers given by

$$\frac{t^{3n}}{[G_\nu(t)-1]^n}\, G_\nu(xt) = \sum_{k=0}^{\infty} \frac{\mathcal{B}_{3k}^{n,\nu}(0)}{\alpha_{3k}(\nu)} t^{3k}\,. \tag{8.2}$$

Let $\delta_\nu = G_\nu(\vartheta_x^{1/3}) - 1$ be a ν-difference operator. In a suitable Rota calculus, we have

$$\mathcal{B}_{3k}^{n,\nu}(x) = \Big[\frac{\vartheta_x}{\delta_\nu}\Big]^n x^{3k}\,. \tag{8.3}$$

This is easy to verify directly. Furthermore, from Equations 8.1 and 8.2, we get the ν-binomial representation

$$\mathcal{B}^{n,\nu}_{3k}(x) = \sum_{\ell=0}^{k} \binom{\alpha_{3k}}{\alpha_{3\ell}} \mathcal{B}^{n,\nu}_{3(k-\ell)}(x^{3\ell}) = (\mathcal{B}^{n,\nu}_{\nu} \oplus_\nu x)^{3k} \tag{8.4}$$

using umbral notation. It also follows from (8.1) that

$$\begin{aligned} \vartheta_x \, \mathcal{B}^{n,\nu}_{3k}(x) &= 3^3 k(k+\nu-1/3)(k-2/3)\mathcal{B}^{n,\nu}_{3(k-1)}(x) \\ &= d_{3k}(\nu)\mathcal{B}^{n,\nu}_{3(k-1)}(x)\,. \end{aligned} \tag{8.5}$$

Using the binomial theorem, we get

$$\begin{aligned} \delta^n_\nu \, G_\nu(xt) &= [G_\nu(\vartheta_x^{1/3}) - 1]^n \, G(xt) \\ &= \sum_{k=0}^{n} \binom{n}{k} G_\nu(\vartheta_x^{1/3})^k (-1)^{n-k} G_{nu}(xt) \\ &= \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} G_\nu(t)^k G_{nu}(xt) \\ &= [G_\nu(t) - 1]^n \, G_\nu(xt)\,. \end{aligned} \tag{8.6}$$

Applying δ_ν to (8.1), we get

$$\begin{aligned} \frac{t^{3n}}{[G_\nu(t)-1]^n} \, G_\nu(xt) &= \sum_{k=0}^{\infty} \frac{\delta_\nu \mathcal{B}^{n,\nu}_{3k}(x)}{\alpha_{3k}(\nu)} t^{3k} \\ &= \sum_{k=0}^{\infty} \frac{\mathcal{B}^{n-1,\nu}_{3k}(x)}{\alpha_{3k}(\nu)} t^{3(k+1)} \\ &= \sum_{k=0}^{\infty} d_{3k}(\nu) \frac{\mathcal{B}^{n-1,\nu}_{3k}(x)}{\alpha_{3k}(\nu)} t^{3k}\,. \end{aligned} \tag{8.7}$$

Comparing coefficients, we get

$$\delta_\nu \mathcal{B}^{n,\nu}_{3k}(x) = \mathcal{B}^{n,\nu}_{3k}(x \oplus_\nu 1) - \mathcal{B}^{n,\nu}_{3k}(x) = d_{3k}(\nu)\mathcal{B}^{n-1,\nu}_{3(k-1)}(x)\,. \tag{8.8}$$

This is an analogue of the basic difference equation for the classical Bernoulli polynomials.

The equation

$$\frac{t^{3(n+m)}}{[G_\nu(t)-1]^{n+m}} \, G_\nu(t(x \oplus_\nu y)) = \sum_{k=0}^{\infty} \frac{\mathcal{B}^{n+m,\nu}_{3k}(x \oplus_\nu y)}{\alpha_{3k}(\nu)} t^{3k} \tag{8.9}$$

yields the identity

$$\begin{aligned} \mathcal{B}^{n+m,\nu}_{3k}(x \oplus_\nu y) &= \sum_{\ell=0}^{k} \binom{\alpha_{3k}}{\alpha_{3\ell}} \mathcal{B}^{n,\nu}_{3\ell}(x)\mathcal{B}^{m,\nu}_{3(k-\ell)}(y) \\ &= (\mathcal{B}^{n,\nu}_{\nu}(x) \oplus_\nu \mathcal{B}^{m,\nu}_{\nu}(y))^{3k} \end{aligned} \tag{8.10}$$

a binomial of Rota type. Taking $m = -n$, we get the hypergeometric result

$$(x \oplus_\nu y)^{3k} = \sum_{\ell=0}^{k} \binom{\alpha_{3k}}{\alpha_{3\ell}} \mathcal{B}^{n,\nu}_{3\ell}(x)\mathcal{B}^{m,\nu}_{3(k-\ell)}(y) \tag{8.11}$$

Next we obtain solutions of the ν-Airy heat equation corresponding to the initial value polynomials $\mathcal{B}_{3k}^{n;\nu}(x)$. They are given by

$$\begin{aligned} e^{t\vartheta_\nu}\mathcal{B}_{3k}^{n,\nu}(x) &= \mathcal{B}_{3k}^{n,\nu}(x,t) \\ &= \sum_{\ell=0}^{k}\frac{t^\ell}{\ell!}\frac{\alpha_{3k}(\nu)}{\alpha_{3(k-1)}(\nu)}\mathcal{B}_{3(k-\ell)}^{n,\nu}(x) \\ &= p_{3k}^{\nu}(\mathcal{B}_\nu^{n,\nu}(x),t) \end{aligned} \tag{8.12}$$

where the $p_{ek}^{\nu}(x,t)$ are the ν-diffusion polynomials of Section 7. The ν-Bernoulli heat polynomials are also given by the generating functions

$$\frac{t^{3n}}{[G_\nu(t)-1]^n}\,e^{ty^3}G_\nu(xy) = \sum_{k=0}^{\infty}\frac{\mathcal{B}_{3k}^{n,\nu}(x,t)}{\alpha_{3k}(\nu)}\,y^{3k}\,. \tag{8.13}$$

The generalized Huygens property for the ν-Bernoulli heat polynomials follows from (8.13), adn we get

$$\begin{aligned} \mathcal{B}_{3k}^{n+m,\nu}(x\oplus_\nu y, t+s) &= \sum_{\ell=0}^{k}\binom{\alpha_{3k}}{\alpha_{3\ell}}\mathcal{B}_{3\ell}^{n,\nu}(x,t)\mathcal{B}_{3(k-\ell)}^{m,\nu}(y,s) \\ &= (\mathcal{B}_\nu^{n,\nu}(x,t)\oplus_\nu \mathcal{B}_\nu^{m,\nu}(y,s))^{3k} \end{aligned} \tag{8.14}$$

Taking $m=-n$, we get the relation

$$p_{3k}^{\nu}(x\oplus_\nu y, t+s) = \sum_{\ell=0}^{k}\binom{\alpha_{3k}}{\alpha_{3\ell}}\mathcal{B}_{3\ell}^{n,\nu}(x,t)\mathcal{B}_{3(k-\ell)}^{-n,\nu}(y,s)\,. \tag{8.15}$$

Letting $s=-t$, it follows that

$$(x\oplus_\nu y)^{3k} = \sum_{\ell=0}^{k}\binom{\alpha_{3k}}{\alpha_{3\ell}}\mathcal{B}_{3\ell}^{n,\nu}(x,t)\mathcal{B}_{3(k-\ell)}^{-n,\nu}(y,-t)\,. \tag{8.16}$$

Since δ_ν and $e^{t\vartheta_x}$ commute, it follows from (8.8) that

$$\delta_\nu^n\mathcal{B}_{3k}^{n,\nu}(x,t) = d_{3k}(\nu)\mathcal{B}_{3(k-1)}^{n-1,\nu}(x,t)\,. \tag{8.17}$$

For $k\geq n\geq 0$, it follows that

$$\delta_\nu^n\mathcal{B}_{3k}^{n,\nu}(x,t) = \frac{\alpha_{3k}(\nu)}{\alpha_{3(k-n)(\nu)}}\,p_{3(k-n)}^{\nu}(x,t)\,. \tag{8.18}$$

The ν-Bernoulli heat polynomials can be expressed in terms of ν-diffusion polynomials. From (8.13) or 8.14, we get

$$\mathcal{B}_{3k}^{n,\nu}(x,t) = \sum_{\ell=0}^{k}\binom{\alpha_{3k}}{\alpha_{3\ell}}\mathcal{B}_{3(k-\ell)}^{n,\nu}(x,t)p_{3\ell}^{\nu}(x,t) \tag{8.19}$$

which expresses $\mathcal{B}_{3k}^{n;\nu}(x,t)$ as a sum of hypergeometric functions.

In order to calculate $\mathcal{B}_{3k}^{n;\nu}(x,t)$, we have to be able to calculate the ν-Airy Bernoulli numbers. The elementary relation

$$\frac{t^{3n}}{[G_\nu(t)-1]^n}\,\frac{t^{-3n}}{[G_\nu(t)-1]^{-n}} = 1 \tag{8.20}$$

implies that

$$(\mathcal{B}_\nu^{n,\nu}(x,t)\oplus_\nu\mathcal{B}_\nu^{m,\nu}(y,s))^{3k} = \delta_{0,k}, k=0,1,\dots \tag{8.21}$$

where $\delta_{0,k}$ is the Kronecker delta function. In multinomial form (8.21) can also be written as

$$\sum_{\underline{\ell}=(\ell_0,\ell_1,\dots,\ell_n),\underline{\ell}\in N^{n+1},|\underline{\ell}|=k} \binom{\alpha_{3k}}{\alpha_{3\underline{\ell}}} \mathcal{B}^{-1,\nu}_{3\ell_0} \mathcal{B}^{-1,\nu}_{3\ell_1} \cdots \mathcal{B}^{n,\nu}_{3\ell_{n-1}} = \delta_{0,k} \,. \tag{8.22}$$

Since $\mathcal{B}^{-1,\nu}_{3k}(0) = d_{3k}(\nu)$, (8.22) yields the recursion formula

$$\mathcal{B}^{n,\nu}_{3k} = -\frac{\alpha_3^n(\nu)\alpha_{3k}}{\alpha_{3(k+n)}} \sum \binom{\alpha_{3(k+n)}}{\alpha_{3\underline{\ell}}} \mathcal{B}^{n,\nu}_{3\ell_n} \,, \tag{8.23}$$

where the summation is taken over $|\underline{\ell}| = k+n$, $0 \le \ell_n < k$, $\ell_i > 0$, $i = 0, 1, \dots, n-1$, $\underline{\ell} = (\ell_0, \ell_1, \dots, \ell_n) \in N_+^{n+1}$.

9. A Primitive Integral for ϑ_ν

Associated with the differential operator ϑ_ν, we introduce a formal or primitive indefinite integral that commutes with ϑ_ν up to constants. Recall that $d_\alpha(\nu) = \alpha(\alpha + 3\nu - 1)(\alpha - 2)$ and $d_{\alpha+3}(\nu) = (\alpha + 3)(\alpha + 3\nu + 2)(\alpha + 1)$. We define

$$\begin{aligned} \int x^\alpha \,\partial_\nu(x) &= \frac{x^{\alpha+3}}{d_{\alpha+3}(\nu)} + c_1x^2 + c_2x^{1-3\nu} + c_3, \quad \text{if } \alpha \neq -1, -3, -(3\nu + 2) \\ &= \frac{x^{\alpha+3}}{(\alpha + 1)(\alpha + 3)^2} + c_1x^2 + c_2 \ln x + c_3, \quad \text{for } \nu = 1/3 \end{aligned} \tag{9.1}$$

$$\begin{aligned} \int x^{-3} \,\partial_\nu(x) &= \frac{lnx}{2(1 - 3\nu)} + c_1x^2 + c_2x^{1-3\nu} + c_3, \quad \nu \neq 1/3 \\ &= -\frac{1}{4} (lnx)^2 + c_2x^2 + c_2 \, lnx + c_3, \quad \text{for } \nu = 1/3 \end{aligned}$$

$$\begin{aligned} \int x^{-1} \,\partial_\nu(x) &= \frac{x^2 \, lnx}{2(3\nu + 1)} + c_1x^2 + c_2x^{1-3\nu} + c_3, \quad \nu \neq -1/3 \\ &= -\frac{1}{4}x^2 \, lnx + c_1x + c_2 \, lnx + c_3, \quad \nu = 1/3 \end{aligned}$$

and

$$\int x^{-(3(n+2)} \,\partial_\nu(x) = \frac{x^{1-3\nu}}{(1 + 3\nu)(3\nu - 1)} + c_1x^2 + c_2x^{1-3\nu} + c_3, \quad \nu \neq 1/3 \,.$$

In each of the above, we have $\vartheta_\nu \int x^\alpha \,\partial_\nu(x) = x^\alpha$. The indefinite integral is extended to 3-parity polynomials and formal power series in x^3. In which case we take c_1 and c_2 to be zero. We are primarily interested in the case of α a nonnegative integer divisible by three. If $p(x) = \sum_{k=0}^n a_k x^{3k}$, then a simple calculation shows that

$$p(x) + c = \int \vartheta_x \, p(x) \,\partial_\nu(x) = \vartheta_x \int p(x) \,\partial_\nu(x) + c$$

In general, as in elementary calculus, if $\vartheta \, f(x) = F(x)$, we take $\int F(x) \,\partial_\nu(x) = f(x) + c$.

The functions $G_\nu(x)$ and $\mathcal{G}_\nu(x)$ play the roles of e^x and e^{-x} for the primitive indefinite integral.

Theorem 9.1. *For $\nu \geq 0$, we have*

$$\int G_\nu(x)\,\partial_\nu(x) = G_\nu(x) + c\,,$$
$$\int \mathcal{G}_\nu(x)\,\partial_\nu(x) = -\mathcal{G}_\nu(x) + c\,.$$

Proof. We have $\vartheta_\nu\, G_\nu(x) = G_\nu(x)$, or working with the power,

$$\int \sum_{n=0}^{\infty} \frac{x^{3n}}{\alpha_{3n}}\,\partial_\nu(x) = \sum_{n=0}^{\infty} \frac{x^{3n+3}}{\alpha_{3n} d_{3(n+1)}} + c = \sum_{n=0}^{\infty} \frac{x^{3n}}{\alpha_{3n}} + c = G_\nu(x) + c$$

since $d_{3(n+1)}\alpha_{3n} = \alpha_{3(n+1)}(\nu)$. $\square$

ν-Primitive Integrals:

$$\int \cosh_\nu(x)\,\partial_\nu(x) = \sinh_\nu(x) + c\,,$$
$$\int \sinh_\nu(x)\,\partial_\nu(x) = \cosh_\nu(x) + c\,,$$
$$\int E_\nu(x)\,\partial_\nu(x) = -i\,E_\nu(x) + c\,,$$
$$\int \sin_\nu(x)\,\partial_\nu(x) = -\ cos_\nu(x) + c\,,$$
$$\int \cos_\nu(x)\,\partial_\nu(x) = \sin_\nu(x) + c\,,$$
$$\int p_n^\nu \nu^(x,t)\,\partial_\nu(x) = \frac{p_{n+1}^\nu(x,t)}{d \quad_{3(n+1)}} + c\,,$$
$$\int \mathcal{L}(x)_{3(k-n)}^{\alpha+n}\,\partial_\nu(x) = (-1)^n\,\frac{\alpha_{3(k-n)}}{\alpha_{3k}}\,\mathcal{L}(x)_{3k}^{\alpha} + c\,,$$
$$\int \mathcal{H}_{3n}^\nu(x)\,\partial_\nu(x) = \frac{\mathcal{H}_{3(n+1)}^\nu(x)}{d_{3(n+1)}} + c\,.$$

Actually, the constant c can be replaced by the general solution of the differential equation $\vartheta_\nu\, y(x) = 0$. In general this is given by $y(x) = c_1 x^2 + c_2 x^{1-3\nu} + c_3$, depending on the parameters as in (9.1).

A generalized ν-definite integral associated with ϑ_ν is defined by

$$\begin{aligned}\int_{y\oplus a}^{x\oplus b} z^{3n}\,\partial_\nu(x) &= \frac{z^{3(n+1)}}{d_{3(n+1)}}\Big|_{y\oplus a}^{x\oplus b}\\ &= \frac{(x\oplus b)^{3(n+1)} - (y\oplus a)^{3(n+1)}}{d_{3(n+1)}(\nu)}\,.\end{aligned}$$

The definite integral is extended by linearity to $\mathcal{P}_3$ the polynomials is x^3 and also to the formal power series $\mathcal{F}_3$ in x^3. The ν-definite integral $\int_{y\oplus a}^{x+\oplus b}$ commutes with

ϑ_ν on $\mathcal{P}_3$ (or $\mathcal{F}_3$). We have

$$\begin{aligned}\vartheta_x \int_{x\oplus a}^{x\oplus b} z^{3n}\,\partial_\nu(z) &= \vartheta_x\,\frac{(x\oplus a)^{3(n+1)}-(x\oplus b)^{3(n+1)}}{\alpha_{3(n+1)}}\\ &= (x\oplus a)^{3n}-(x\oplus b)^{3n}\\ &= \int_{x\oplus a}^{x\oplus b}\vartheta_z\, z^{3n}\,\partial_\nu(z)\,.\end{aligned}$$

The result extends to $\mathcal{P}_3$ by linearity. It is easy to show that

$$\int_a^b (z\oplus b)^{3n}\,\partial_\nu(z) = \int_{a\oplus b}^{x\oplus b} z^{3n}\,\partial_\nu(z) \tag{9.2}$$

and therefore by linearity $\int_a^b p(z\oplus b)\,\partial_\nu(z) = \int_{a\oplus b}^{x\oplus b} p(z)\,\partial_\nu(z)$. Note that (9.2) is the ν-analogue of the elementary change of variable formula

$$\int_a^x f(z+b)\,dz = \int_{a+b}^{x+b} f(z)\,dz\,.$$

10. An Orthogonality Relation

Since the ν-diffusion polynomials $p_n(x,t)$ are three-parity polynomials they cannot be orthogonal in the usual sense with respect to a measure, see [14]. In this section we obtain a multiple integral orthogonality relation.

By Erdélyi, [17, p. 218,], we have

$$\int_0^\infty e^{-pt}t^{\beta-1}\,{}_1F_2(-n;\alpha+1,\beta|\lambda t)\,dt = n!\,\frac{\Gamma(\beta)}{(\alpha+1)_n}\,p^{-\beta}\,L_n^\alpha(\lambda/p) \tag{10.1}$$

where $\operatorname{Re} p>0$ and L_n^α is a Laguerre polynomial. Since

$$p_n^\nu(x,t) = \frac{\alpha_{3n}(\nu)}{n!}t^n\,{}_1F_2(-n;1/3,\nu+2/3|-\frac{x^3}{27t})$$

a change of variables in (10.1) yields the following result.

Theorem 10.1. *Let $\nu>0$. Then*

$$\frac{(-1)^n}{3^{3n}t^{n+1/3}\Gamma(n+1/3)\,n!}\int_0^\infty e^{-\frac{z^3}{27t\tau}}\,p_n^\nu(z,-t)\,dz = \tau^{1/3}L_n^{\nu-1/3}(\tau),\quad n=0,1,\dots$$

By the usual orthogonality of the Laguerre polynomials, we obtain

$$\int_0^\infty L_m^{\nu+1/3}(\tau)L_n^{\nu-1/3}(\tau)e^{-\tau}\,\tau^{\nu-1/3}\,d\tau = \frac{\Gamma(\nu+n+2/3)}{n!}\,\delta_{n,m} \tag{10.2}$$

where $\delta_{n,m}$ is the Kronecker delta and $Re(\nu-1/3)>-1$. Thus (10.2) is valid for $\nu\ge 0$.

Theorem 10.2. *For $\nu\ge 0$, we have*

$$\begin{aligned}&\frac{1}{\sigma_n(t)}\int_0^\infty\int_0^\infty\int_0^\infty \exp(-\frac{z^3+y^3}{27t\tau})e^{-\tau}p_n^\nu(z,-t)p_m^\nu(y,-t)\tau^{\nu-1}\,d\tau\,dz\,dy\\ &=\alpha_{3n}(\nu)\delta_{n,m}\end{aligned} \tag{10.3}$$

$n=0,1,2,\dots,$ where $\sigma_n(t)=3^{3n}t^{2n+2/3}\Gamma(n+1/3)\Gamma(1/3)\Gamma(\nu+2/3)$.

Proof. Using Theorem 10.1 and Fubini's Theorem, we get

$$
\begin{aligned}
&\int_0^\infty \tau^{1/3} L_m^{\nu-1/3}(\tau) L_n^{\nu-1/3}(\tau) e^{-\tau}\tau^{\nu-1}\,d\tau \\
&= \frac{1}{(3^{3n}t^{n+1/3}\Gamma(n+1/3)\,n!)^2} \\
&\quad\times \int_0^\infty e^{\tau} t^{\nu-1}\,d\tau \int_0^\infty e^{-\frac{z^3}{27t\tau}} p_n^\nu(z,-t)\,dz \int_0^\infty e^{-\frac{y^3}{27t\tau}} p_m^\nu(y,-\tau)\,dy \\
&= \frac{1}{(3^{3n}t^{n+1/3}\Gamma(n+1/3)\,n!)^2} \\
&\quad\times \int_0^\infty \int_0^\infty \int_0^\infty e^{-\frac{z^3+y^3}{27t\tau}} e^{-\tau} p_m^\nu(z,-t) p_n^\nu(y,-t) \tau_{\nu-1}\,d\tau\,dz\,dy \\
&= \frac{\Gamma(\nu+n+2/3)}{n!}\,\delta_{n,m}\,.
\end{aligned}
$$

Adjusting the constants yields the theorem. ☐

By Erdelyi, [17, p. 149], we have the relation

$$\int_0^\infty e^{-t} t^{\nu-1} e^{-\frac{\alpha}{4t}}\,dt = z(\frac{\alpha}{4})^{\nu/2} K_\nu(\alpha^{1/2})$$

where K_ν is a modified Bessel function. With a suitable change of variables, we obtain

$$\int_0^\infty e^{-\frac{z^3+y^3}{27t\tau}} \tau^{\nu-1} e^{-\tau}\,d\tau = 2\big[\frac{z^3+y^3}{27t}\big]^{\nu/2} K_\nu\Big(2\big(\frac{z^3+y^3}{27t}\big)^{1/2}\Big)\,.$$

Substitution into (10.3), gives the equation

$$
\begin{aligned}
&\frac{2}{\sigma_n(t)} \int_0^\infty \int_0^\infty \Big[\frac{z^3+y^3}{27t}\Big]^{\nu/2} K_\nu\Big[2\big(\frac{z^3+y^3}{27t}\big)^{1/2}\Big]\, p_n^\nu(z,-t) p_m^\nu(y,-t)\,dz\,dy \\
&= \alpha_{3n}(\nu)\delta_{n,m}
\end{aligned}
$$

for $n = 0,1,2,\dots$. Under suitable conditions on the sequence $\{a_{3n}\}_0^\infty$, the function

$$u(x,t) = \sum_{m=0}^\infty a_{3m} p_m^\nu(x,t)$$

is in $\mathcal{H}_\nu(0,\infty)$ for $t > 0$. In the usual calculus manner the coefficients are formally determined by

$$a_{3n} = \frac{1}{\alpha_{3n}} \int_0^\infty \int_0^\infty \Big[\frac{x^3+y^3}{27t}\Big]^{\nu/2} K_\nu\Big[2\big(\frac{z^3+y^3}{27t}\big)^{1/2}\Big] u(z,-t) p_n^\nu(y,-t)\,dz\,dy$$

We will consider convergence criteria in the next section.

11. ν-Diffusion Polynomial Expansions

The diffusion polynomials $p_n^\nu(x,t)$ satisfy the ν-Airy diffusion equation $\vartheta_\nu u(x,t) = D_t u(x,t)$. Since the partial differential equation is linear, finite linear combinations of the diffusion polynomials are also solutions in $\mathcal{H}_\nu(0 \le x < \infty)$, for all t. Hence we expect to obtain infinite series expansions

$$u(x,t) = \sum_{m=0}^\infty a_{3m} p_m^\nu(x,t)$$

with possible convergence in a strip $|t| < \sigma$.

Theorem 11.1. *Let $\nu \geq 0$, then*

$$|p_n^\nu(x,t)| \leq M\,\alpha_{3n} e^{|x|+|t|} \tag{11.1}$$

Proof. Since

$$e^{-tz^3} G_\nu(xz) = \sum_{n=0}^\infty \frac{p_n^\nu(x,t)}{\alpha_{3n}(\nu)} z^{3n}$$

is an entire function, we find by Cauchy's Theorem that

$$p_n^\nu(x,t) = \frac{\alpha_{3n}}{2\pi i} \int_\Gamma \frac{e^{-tz^3} G_\nu(xz)}{z^{3n+1}}\,dz\,,$$

where Γ is the unit circle, $|z| = 1$. Since $G_\nu(z)$ is an entire function of order one, we have $|G_\nu(xz)| \leq Me^{|xz|}$. Therefore,

$$|p_n^\nu(x,t)| \leq M\,\frac{\alpha_{3n}}{2\pi} \int_\Gamma e^{|xz|-|t||z|^3}\,|dz| \leq M\alpha_{3n}(\nu) e^{|x|+|t|}$$

Since $|G_\nu(xz)| \leq G_\nu(|x||z|) = G_\nu(|x|)$ on $|z| = 1$, we also get the estimate

$$|p_n^\nu(x,t)| \leq \alpha_{3n} G_\nu(|x|) e^{|t|}$$

□

Theorem 11.2. *Suppose $\sum_{n=0}^\infty |a_n| < \infty$, then the series*

$$\sum_{n=0}^\infty a_n\,\frac{p_n^\nu(x,t)}{\alpha_{3n}}$$

converges absolutely and locally uniformly for $|x| < \infty$ and $|t| < \infty$.

Proof. By the estimate (11.1), it follows that

$$\Big|\sum_{n=0}^\infty a_n\,\frac{p_n^\nu(x,t)}{\alpha_{3n}}\Big| \leq Me^{|x|+|t|} \sum_{n=0}^\infty |a_n| < \infty$$

Thus the series converges absolutely and locally uniformly by the Weierstrass M-test. Since $\frac{\vartheta_\nu}{\alpha_{3n}}\,p_n^\nu(x,t) = \frac{p_{n-1}^\nu(x,t)}{\alpha_{3(n-1)}}$, it also follows that the differentiated series converges locally uniformly and therefore the series represents a function in $\mathcal{H}_\nu$ for $|x| < \infty$ and $|t| < \infty$. □

Let

$$\mathcal{R}_\nu(z) = \sum_{n=0}^\infty \frac{z^{3n}}{3^{3n}(1/3)_n(\nu+2/3)_n} = {}_1F_2\left(\begin{matrix}1\\ 1/3, \nu+2/3\end{matrix}\Big|\frac{z^3}{27t}\right).$$

Then using Stirling's formula we get

$$\limsup_{n\to\infty} \frac{3n\log 3n}{\log\left(3^{3n}(1/3)_n(\nu+2/3)_n\right)} = 3/2$$

Therefore, $\mathcal{R}_\nu$ is an entire function of order $3/2$. Furthermore,

$$\limsup_{n\to\infty} 3n\Big|\frac{1}{3^{3n}(1/3)_n(\nu+2/3)_n}\Big|^{\frac{3/2}{2n}} = \frac{e}{\sqrt{3}}.$$

Therefore, $\mathcal{R}_\nu(x)$ is of type $\frac{2}{3\sqrt{3}}$.

Lemma 11.3. *Let $\delta > 0$. Then*

$$\frac{p_n^\nu(|x|,|t|)}{\alpha_{3n}(\nu)} \le \frac{\delta^n}{n!}\,(1+\frac{|t|}{\delta})^n \mathcal{R}_\nu(\frac{|x|}{\delta^{1/3}})$$

Proof. We have

$$\begin{aligned}\frac{p_n^\nu(|x|,|t|)}{\alpha_{3n}} &\le \frac{\delta^n}{n!}\sum_{n=0}^{\infty}\Big(\frac{|t|}{\delta}\Big)^{n-k}\frac{\frac{|x|^{3k}}{\delta^k}}{3^{3k}(1/3)_k(\nu+2/3)_k}\\ &\le \mathcal{R}_\nu(\frac{|x|}{\delta^{1/3}})\,\frac{\delta^n}{n!}\sum_{n=0}^{\infty}\binom{n}{k}\Big(\frac{|t|}{\delta}\Big)^{n-k}\\ &= \frac{\delta^n}{n!}\,(1+\frac{|t|}{\delta})^n\mathcal{R}_\nu(\frac{|x|}{\delta^{1/3}})\end{aligned}$$

since

$$\frac{\frac{|x|^{3k}}{\delta^k}}{3^{3k}(1/3)_k(\nu+2/3)_k} < \mathcal{R}_\nu(\frac{|x|}{\delta^{1/3}})$$

□

Lemma 11.4.

$$p_n^\nu(x,t) \ge \frac{\alpha_{3n}}{n!}t^n, \quad \textit{for } t, x > 0$$

Proof. Since the coefficients of p_n^ν are positive, it follows that $p_n^\nu(x,t) \ge p_n^\nu(0,t) = \frac{\alpha_{3n}}{n!}t^n$. □

Theorem 11.5. *If the series $\sum_{n=0}^\infty a_n p_n^\nu(x_0,t_0)$ converges for $t_0 > 0$ and $x_0 > 0$, then the series*

$$\sum_{n=0}^{\infty} a_n p_n^\nu(x,t) \quad \textit{and} \quad \sum_{n=0}^{\infty} a_n d_{3n}(\nu) p_{n-1}^\nu(x,t)$$

converge absolutely and locally uniformly in the strip $|t| < t_0$ and $\sum_{n=0}^\infty a_n p_n^\nu(x,t)$ is in $\mathcal{H}_\nu(R_+)$ for $|t| < t_0$.

Proof. Since the general term of a convergent series must go to zero,

$$\lim_{n\to\infty} a_n p_n^\nu(x_0,t_0) = 0\,.$$

By Lemma 11.4, it therefore follows that

$$a_n = O\big(\frac{n!}{\alpha_{3n}t_0^n}\big)\,.$$

Using Lemma 11.3, we get for $\delta > 0$

$$\begin{aligned}&\sum_{n=0}^{\infty} a_n d_{3n} p_{n-1}^\nu(x,t)\ell M \sum_{n=1}^{infty}\frac{n!}{\alpha_{3n}t_0^n}\,\frac{\alpha_{3n}}{n!}\,(\delta+|t|)^n\mathcal{R}_\nu(\frac{|x|}{\delta^{1/3}})\\ &\le M\mathcal{R}_\nu(\frac{|x|}{\delta^{1/3}})\sum_{n=0}^{\infty}(\frac{\delta+|t|}{t_0})^n\end{aligned}$$

which converges for $\delta+|t| < t_0$. Since $\delta > 0$ is arbitrary it converges for $(\delta+|t|) < t_0$, and as before for $|t| < t_0$. The Weierstrass M-test provides the local uniform convergence.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of order ρ, $\rho > 0$, and of type $0 < \sigma < \infty$. The type is determined by

$$\limsup_{n\to\infty} \frac{3n}{e\rho} |a_n|^{\frac{\rho}{3n}} = \sigma$$

see for example Boas, [6, p. 11]. Therefore,

$$|a_n| \le M \left(\frac{e\sigma\rho}{3n}\right)^{3n/\rho} \tag{11.2}$$

□

Theorem 11.6. *If $f(z) = \sum_{n=0}^{\infty} a_n z^{3n}$ is an entire function of order ρ with $0 < \rho < 3/2$ and of type σ, $0 < \sigma < \infty$, then*

$$u(x,t) = \sum_{n=0}^{\infty} a_n p_n^{\nu}(x,t) \tag{11.3}$$

is in $\mathcal{H}_\nu(R)$ in the strip $|t| < 1/(\sigma\rho)^{3/\rho}$ and $u(x,0) = f(x)$.

Proof. Using (11.2) and Lemma 11.3, for $\delta > 0$ we obtain

$$\sum_{n=0}^{\infty} a_n p_n^{\nu}(x,t) \ell M \sum_{n=0}^{\infty} \left(\frac{e\sigma\rho}{3n}\right)^{3n/\rho} \frac{\alpha_{3n}}{n!} (\delta + |t|)^n \mathcal{R}_\nu(\frac{|x|}{\delta^{1/3}} \tag{11.4}$$

Using Stirling's formula, we get the estimate

$$\left(\frac{e\sigma\rho}{3n}\right)^{3n/\rho} 3^{3n}(1/3)_n(\nu+2/3)_n \sim \Big[\frac{e^{1-\frac{2}{3}\rho}3^{\rho-1}}{n^{1-\frac{2}{3}\rho-\frac{\nu\rho}{3n}}}\Big]^{3n/\rho} \frac{2\pi(\sigma\rho)^{3n/\rho}}{\Gamma(1/3)\Gamma(\nu+2/3)}$$

Now

$$\Big[\frac{e^{1-\frac{2}{3}\rho}3^{\rho-1}}{n^{1-\frac{2}{3}\rho-\frac{\nu\rho}{3n}}}\Big]^{3n/\rho} \frac{2\pi(\sigma\rho)^{3n/\rho}}{\Gamma(1/3)\Gamma(\nu+2/3)} = O(1)$$

for $0 < \rho < 3/2$. Thus the series in (11.4) is dominated by

$$M_t \mathcal{R}_\nu(\frac{|x|}{\delta^{1/3}}) \sum_{n=0}^{\infty} \{(\sigma\rho)^{3/\rho}(\delta + |t|)\}^n$$

which converges for $(\sigma\rho)^{3/\rho}(\delta + |t|) < 1$. Since $\delta > 0$ is arbitrary we get absolute and local uniform convergence for $|t| < \frac{1}{(\sigma\rho)^{3/\rho}}$, by the Weierstrass M-test. Since the order and type of an entire functions is not changed by taking derivatives, a similar type argument shows that the derived series

$$\sum_{n=1}^{\infty} a_n d_{2n} p_{n-1}^{\nu}(x,t)$$

also converges absolutely and locally uniformly for $|t| < \frac{1}{(\sigma\rho)^{3/\rho}}$. It follows that $u(x,t)$ given by (11.3) is in $\mathcal{H}_\nu$ in the stated strip. □

In the classical case developed by Widder [33] or in the Bessel function case treated by Bragg [8] and Cholewinski and Haimo [11], a series expansion of the type given by (11.3), leads to an integral representation of $u(x,t)$. In both of those cases the representation depends on the fact that the diffusion polynomials can be represented by Gaussian type integrals in terms of the source function. In the $\mathcal{H}_\nu$ case the corresponding integrals diverge.

The theorem does not imply that the given strip is the best possible. For example with $G_\nu(x) = \sum_{n=0}^{\infty} \frac{x^{3n}}{\alpha_{3n}}$, an entire function of order one and type one, we get

$$u(x,t) = e^t G_\nu(x) = \sum_{n=0}^{\infty} \frac{p_n^\nu(x,t)}{\alpha_{3n}}$$

which converges for all x and t. The theorem gives the strip $|t| < 1$.

Next we consider a sequence of complex numbers $\{a_n\}_0^\infty$ for which $|a_n|\alpha_{3n}(\nu) = O(1)$. Using Stirling's formula, we find that

$$\limsup_{n\to\infty} \frac{3n \log 3n}{\log \frac{1}{|a_n|}} \le 1$$

Thus the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function of order less than or equal to one. Likewise another calculation using Stirling's formula yields

$$\limsup_{n\to\infty} \frac{3n}{e} |a_n|^{1/3n} \le 1$$

Thus $f(z)$ is of growth $\le \{1,1\}$. We let $f_r(z) = \sum_{n=0}^{\infty} a_n r^{3n} z^{3n}$ with $0 < r < 1$ be the "Abel means" of f.

Theorem 11.7. *If $|a_n|\alpha_{3n} = O(1)$, then*

$$u_r(x,t) = \sum_{n=0}^{\infty} a_n r^{3n} p_n^\nu(x,t)$$

is an entire function in the variables x and t and it is in $\mathcal{H}_\nu$ for all x and t.

Proof. By Theorem 11.1, we get the domination

$$\begin{aligned}
\sum_{n=0}^{\infty} |a_n| r^{3n} |p_n^\nu(x,t)| &\le M e^{|x|+|t|} \sum_{n=0}^{\infty} |a_n|\alpha_{3n} r^{3n} \\
&\le M e^{|x|+|t|} \sum_{n=0}^{\infty} r^{3n} \\
&= M e^{|x|+|t|} \frac{1}{1-r^3} < \infty
\end{aligned}$$

for $0 < r < 1$. Once again the Weierstrass M-test gives absolute and locally uniform convergence for x and t in C, the complex numbers.

In the case $a_n = \frac{1}{\alpha_{3n}(\nu)}$, we get $G_\nu(rz) = \sum_{n=0}^{\infty} \frac{r^{3n} z^{3n}}{\alpha_{3n}}$ and we recover the generating function

$$u_r(x,t) = e^{tr^3} G_\nu(rx) = \sum_{n=0}^{\infty} \frac{p_n^\nu(x,t)}{\alpha_{3n}} r^{3n} .$$

□

12. Associated Functions

A sequence of ν-associated functions is defined as

$$Q_n^\nu(x,t) = (-\vartheta_\nu)^n \mathcal{K}_\nu(x,t) = (-1)^n \frac{\partial^n}{\partial t^n} \mathcal{K}_\nu(x,t) ,$$

where $\mathcal{K}_\nu(x,t)$ is the source solution of $\vartheta_\nu u = u_t$. Expansions in terms of the $Q_n^\nu(x,t)$ are related to Laurent expansions for analytic functions, see Widder [34] for the ordinary heat equation or Cholewinski and Haimo [11] for the Bessel function case. Since $-\vartheta_\nu\, Q_n^\nu(x,t) = (-\vartheta_\nu)^{n+1}\,\mathcal{K}_\nu(x,t) = Q_{n+1}^\nu(x,t)$, and $\frac{\partial}{\partial t}\, Q_n^\nu(x,t) = -Q_{n+1}^\nu(x,t)$, it follows that $\vartheta_\nu\, Q_n^\nu(x,t) = \frac{\partial}{\partial t}\, Q_n^\nu(x,t)$ for $t > 0$.

By our previous integral representation,

$$\mathcal{K}_\nu(x,t) = \int_0^\infty e^{-ty^3}\mathcal{G}_\nu(xy)\, d\eta_\nu(y), \quad t > 0$$

Since $\mathcal{G}_\nu(xy)$ and its derivatives are entire functions of growth $\{1,1\}$, we obtain and integral representation for $Q_n^\nu(x,t)$, namely,

$$Q_n^\nu(x,t) = \int_0^\infty y^{3n}e^{-ty^3}\mathcal{G}_\nu(xy)\, d\eta_\nu(y) \tag{12.1}$$

The growth condition yields the necessary domination integrals for the absolute and local uniform convergence. Further, Fubini's Theorem yields the series expansion

$$\begin{aligned} Q_n^\nu(x,t) &= \sum_{m=0}^\infty \frac{(-1)^m x^{3m}}{\alpha_{3m}} \int_0^\infty e^{-ty^3} y^{3n+3m+3\nu+1}\,\frac{dy}{c_\nu} \\ &= \sum_{m=0}^\infty \frac{(-1)^m\Gamma(n+m+\nu+2/3)}{\alpha_{3m}(\nu)3^{\nu+2/3}\Gamma(\nu+2/3)}\,\frac{x^m}{t^{n+m+\nu+2/3}} \\ &= \frac{(\nu+2/3)_n t^{-n}}{(3t)^{\nu+2/3}}\,{}_1F_2\left[\begin{matrix} n+\nu+2/3 \\ 1/3, \nu+2/3\end{matrix}\,\middle|\, -\frac{x^3}{27t}\right]. \end{aligned} \tag{12.2}$$

A direct calculation employing the coefficients of $Q_n^\nu(x,t)$ shows that Q_n^ν is an entire function of growth $\{3/2, \frac{2}{3\sqrt{3}|t|^{1/2}}\}$, for $t \neq 0$, which is the growth of $\mathcal{K}_\nu(x,t)$.

Next we obtain an upper bound of $|Q_n^\nu(x,t)|$ that applies to the variables ν, n, x, and t.

Lemma 12.1. *Let $\nu \geq 0$. Then*

$$|Q_n^\nu(x,t)| \leq M(\nu+2/3)_n\,\frac{e^{x^2/2}}{t^{n+\nu+2/3}}, \quad t > 0 \tag{12.3}$$

Proof. Since $\mathcal{G}_\nu(x)$ is of growth $\{1,1\}$, we have

$$|\mathcal{G}_\nu(xy)| \leq Me^{|x||y|} \leq Me^{\frac{|x|^2+|y|^2}{2}}.$$

Let $\epsilon > 0$. Since $y^2/2 \leq \epsilon y^3$, we get

$$\begin{aligned} |Q_n^\nu(x,t)| &\leq Me^{|x|^2/2}\int_0^\infty y^{3n}e^{-ty^3}e^{y^2/2}\, d\eta_\nu(y) \\ &\leq M\,\frac{e^{|x|^2/2}}{\Gamma(\nu+3/2)}\int_0^\infty e^{-(t-\epsilon)y^3}y^{3n+3\nu+1}\, dy \\ &= Me^{|x|^2/2}\,\frac{\Gamma(n+\nu+2/3)}{(t-\epsilon)^{n+\nu+2/3}}, \quad \text{for } t > \epsilon > 0\,. \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we obtain inequality 12.3. $\square$

The integral (12.1) leads to a number of generating functions involving the associated functions Q_n^ν. We have for $t > 0$

$$\sum_{n=0}^\infty \frac{Q_n^\nu(x,t)}{n!}\, z^n = \mathcal{K}_\nu(x, -z+t) \quad \text{with } z < t$$

for

$$\sum_{n=0}^{\infty} \frac{Q_n^{\nu}(x,t)}{n!}\, z^n = \int_0^{\infty} e^{zy^3-ty^3} \mathcal{G}_{\nu}(xy)\, d\eta_{\nu}(y)$$

The integral is dominated by

$$M \int_0^{\infty} e^{(z-t)y^2} e^{|xy|}\, d\eta_{\nu}(y) \quad \text{for } z-t<0\,.$$

Therefore, the interchange of summation and integration is valid for $z-t<0$. In a similar fashion, it follows that

$$\sum_{n=0}^{\infty} \frac{(-1)^n Q_n^{\nu}(x,t)}{\alpha_{3n}} z^{3n} = \int_0^{\infty} e^{-ty^3} \mathcal{G}_{\nu}(xy)\mathcal{G}_{\nu}(yz)\eta_{\nu}(y)\,. = \mathcal{K}(x \oplus_{\nu} z, t)$$

This series is also the symbolic time series solution given by $\mathcal{G}_{\nu}(zD_t^{1/3})\mathcal{K}_{\nu}(x,t)$.

Let $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of growth $\{3,\sigma\}$,that is for $\epsilon > 0$

$$|\phi(z)| \le M e^{(\sigma+\epsilon)|z|^3}\,.$$

Theorem 12.2. *If ϕ is of growth $\{3,\sigma\}$, then*

$$u(x,t) = \int_0^{\infty} e^{-ty^3} \mathcal{G}_{\nu}(xy)\phi(y)\, d\eta_{\nu}(y)$$

is in $\mathcal{H}_{\nu}$ for $t > \sigma \ge 0$.

Proof. Since $\mathcal{G}_{\nu}$ is of growth $\{1,1\}$, we find that

$$\begin{aligned} \Big| \int_0^{\infty} \mathcal{G}_{\nu}(xy) e^{-ty^2} \phi(y)\, d\eta_{\nu}(y) \Big| &\le M \int_0^{\infty} e^{xy} e^{-ty^3} e^{(\sigma+\epsilon)} y^3\, d\eta_{\nu}(y) \\ &< \infty, \quad \text{for} \quad t > \sigma + \epsilon\,. \end{aligned} \tag{12.4}$$

Since $\epsilon > 0$ is arbitrary, the integral converges absolutely and locally uniformly. Since derivatives of $\mathcal{G}_{\nu}$ are also of growth $\{1,1\}$, similar domination integrals allow the interchange of integration and differentiation. It readily follows that u is in $\mathcal{H}_{\nu}$ for $t > \sigma \ge 0$. $\square$

The integral representation also leads to an infinite series for $u(x,t)$. By (12.4), we also have

$$\Big| \int_0^{\infty} e^{-ty^3} \mathcal{G}_{\nu}(xy) \sum_{n=0}^{\infty} a_n y^3\, d\eta_{\nu}(y) \Big| \le M \int_0^{\infty} e^{xy} e^{-ty^3} e^{(\sigma+\epsilon)} y^3\, d\eta_{\nu}(y) < \infty\,.$$

Thus by Fubini's Theorem and domination for the differentiate integrals we get

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} an \int_0^{\infty} \mathcal{G}_{\nu}(xy) y^{3n} e^{-ty^3}\, d\eta_{\nu}(y) \\ &= \sum_{n=0}^{\infty} (-1)^n a_n \vartheta_x^n \int_0^{\infty} \mathcal{G}_{\nu}(xy) e^{-ty^3}\, d\eta_{\nu}(y) \\ &= \sum_{n=0}^{\infty} a_n (-\vartheta_{\nu})^n \mathcal{K}_{\nu}(x,t) \\ &= \sum_{n=0}^{\infty} a_n Q_n^{\nu}(x,t) \end{aligned}$$

which converges of all x and $t > \sigma$.

Theorem 12.3. *If the series*

$$\sum_{n=0}^{\infty} b_n Q_n^\nu(x,t)$$

converges absolutely for $t > \sigma$, then $\phi(y) = \sum_{n=0}^{\infty} b_n y^{3n}$ is of growth $\{3, \sigma\}$.

Proof. By the alternating series test

$$1 - \frac{(xy)^3}{\alpha_3} \le \mathcal{G}_\nu(xy), \quad x, y \ge 0\,.$$

Hence we get

$$\int_0^\infty (1 - \frac{(xy)^3}{3}) y^{3n} e^{-ty^3}\, d\eta_\nu(y) \le Q_n^\nu(x,t)\,.$$

Next

$$\int_0^\infty e^{-ty^3} y^{3n}\, d\eta_\nu(y) = \frac{(\nu+2/3)_n}{3^{\nu+2/3} t^{n+\nu+2/3}}\,.$$

Therefore,

$$\frac{1}{(3t)^{\nu+2/3}} \sum_{n=0}^{\infty} \frac{|b_n|}{t^n} ((\nu+2/3)_n - \frac{x^3}{\alpha_3}\frac{(\nu+2/3)_{n+1}}{t}) \le \sum_{n=0}^{\infty} |b_n||Q_n^\nu(x,t)|$$

and $\lim_{n\to\infty} \frac{|bn|}{t^n} (\nu+2/3)_{n+1}^{1/n} = 0$. By Stirling's formula $(\nu+2/3)_n^{1/n}$ and $(\nu + 2/3)_{n+1}^{1/n} \sim ne^{-1}$. It follows that $|b_n|^{1/n} \le \frac{(\sigma+\epsilon)}{\nu+2/3)_n^{1/n}} \le \frac{e}{n}(\sigma+\epsilon)$ since the series converges for $t = \sigma + \epsilon$. Consequently

$$\limsup_{n\to\infty} 3n|b_n|^{\frac{3}{3n}} \le \sigma + \epsilon$$

and therefore $\epsilon > 0$ arbitrary implies that $\phi(y)$ is of growth type $\{3, \sigma\}$. ☐

Example. Let $\phi(x) = e^{ax^3}$, which is of growth $\{3, |a|\}$. Applying the bound (12.3), Stirling's formula and the root test to the series

$$\sum_{n=0}^{\infty} \frac{a^n}{n!} Q_n^\nu(x,t)$$

we get

$$\limsup_{n\to\infty} \Big\{\frac{|a|^n(\nu+2/3)_n}{n!t^n}\Big\}^{1/n} = \frac{|a|}{t}\,.$$

Thus the convergence follows for $0 < |a| < t$.

13. Bessel Calculus Connections

The elements of the calculus associated with ϑ_ν can be associated with the elements of the calculus associated with the Bessel calculus. Formally the associations can be obtained through the use of a generalized Hadamard product. However, we present most of the results as integral representations between the elements of the respective calculi.

Let $\nu \ge 0$ and $\mu = \nu + 1/6$, then the Bessel coefficient is $b_{2n}(\mu) = b_{2n}(\nu+1/6) = 2^{2n}\, n!\,(\nu+2/3)_n$. Thus we have the coefficient relation $\alpha_{3n}(\nu) = (\frac{27}{4})^n (1/3)_n b_{2n}(\nu)$. We associated with a ν-exponential power series

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{\alpha_{3n}(\nu)} z^{3n}$$

the power series

$$f_{\#}(z) = \sum_{n=0}^{\infty} \frac{a_n}{b_{2n}(\mu)} z^{2n} .$$

Theorem 13.1. *Let $\nu \geq 0$ and $f(z) = \sum_{n=0}^{\infty} \frac{a_n}{\alpha_{3n}(\nu)} z^{3n}$ be an entire function of order ρ_f with $0 \leq \rho_f < 3$. then $f_{\#}(z) = \sum_{n=0}^{\infty} \frac{a_n}{b_{2n}(\mu)} z^{2n}$ is an entire function of order $\rho_{f_{\#}} = 2\rho_f/(3-\rho_f)$.*

Proof. By Stirling's formula it follows that

$$\frac{log \frac{\alpha_{3n}(\nu)}{b_{2n}(\mu)}}{3n \log 3n} \to 1/3 \quad \text{as } n \to \infty .$$

Since $\limsup_{n\to\infty} \frac{3n \log 3n}{\log |\frac{\alpha_{3n}}{a_n}|} = \rho_f$, we find that

$$\frac{3n \log 3n}{\log |\frac{\alpha_{3n}}{b_{2n}} \frac{b_{2n}}{a_n}|} = \frac{1}{\frac{\log\left|\frac{\alpha_{3n}}{b_{2n}}\right|}{3n \log 3n} + \frac{2}{3} \frac{log\left|\frac{b_{2n}}{a_n}\right|}{2n \log(\frac{e}{2} 2n)}} \simeq \frac{1}{1/3 + \frac{2}{3} \frac{1}{\rho_{f_{\#}}}} = \rho_f .$$

Solving for $\rho_{f_{\#}}$, we get

$$\rho_{f_{\#}} = \frac{2\rho_f}{3-\rho_f} .$$

$\square$

Example. consider the entire function $f(z) = \exp(z^3)$ of order 3. We have

$$f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^{3n} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{3n} (1/3)_n (\nu + 2/3)_n}{\alpha_{3n}(\nu)} z^{3n}$$

and

$$f_{\#}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{3n} (1/3)_n (\nu + 2/3)_n}{b_{2n}(\mu)} z^{2n} =_1 F_0(1/3| - \frac{27}{4} z^2) .$$

Since ${}_1F_0(1/3|z)$ converges for $|z| < 1$, $f_{\#}$ is holomorphic in the disk $|z| < 2/3$. In this case $f_{\#}$ is not an entire function.

Example. The function $G_\nu(z) = \sum_{n=0}^{\infty} \frac{z^{3n}}{\alpha_{3n}}$ is an entire function of order 1. We have

$$G_{\nu\,\#}(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{b_{2n}(\mu)} = \mathbf{I}_{\nu+1/6}(z)$$

is also a known entire function of order 1. The functions $G_\nu(z)$ and $\mathbf{I}_\mu(z)$ are zeta functions in their respective calculi. We also have $\mathcal{G}_{\nu\,\#}(z) = \mathbf{J}_\mu(z)$.

Proposition 13.2. *Let $\nu \geq 0$ and $f(z) = \sum_{n=0}^{\infty} \frac{a_n}{\alpha_{3n}(\nu)} z^{3n}$ be an entire function of order $\{3, \sigma\}$ with $\sigma > 0$, then the integral*

$$\int_0^{\infty} e^{-\frac{4zy^3}{27x^2}} f(zy)\, dy \tag{13.1}$$

converges absolutely and locally uniformly for $0 < (xz)^2 < \frac{4}{27\sigma}$ with $z > 0$.

Proof. For $\epsilon > 0$, we have the domination

$$\Big|\int_0^\infty e^{-\frac{4zy^3}{27x^2}}\, f(zy)\, dy\Big| \le M \int_0^\infty e^{-\frac{4zy^3}{27x^2}}\, e^{(\sigma+\epsilon)z^3y^3}\, dy$$

and the theorem follows. □

If $f(z)$ is an entire function of order $0 \le \rho_f < 3$, then it readily follows that the integral (13.1) converges absolutely and locally uniformly for $z, x > 0$. Thus in this case summations can be interchanged with the integration.

Proposition 13.3.

$$\int_0^\infty e^{-\frac{4zy^3}{27x^2}}\, y^{3n}\, dy = \frac{3^{3n}x^{2n+2/3}}{2^{2n+2/3}z^{n+1/3}}\,\Gamma(n+1/3)$$

Proof. With the change of variables $s = y^3$, the integral reverts to a gamma function integral representation. □

Theorem 13.4. *Let $f(z) = \sum_{n=0}^\infty \frac{a_n}{\alpha_{3n}(\nu)}\, z^{3n}$ be an entire function of order ≤ 3, then*

$$\frac{z^{1/3}2^{2/3}}{x^{2/3}\Gamma(1/3)} \int_0^\infty e^{-\frac{4zy^3}{27x^2}}\, f(zy)\, dy = \sum_{n=0}^\infty \frac{a_n}{b_{2n}(\mu)}\,(xz)^{2n} = f_\#(zx) \tag{13.2}$$

for $x, z > 0$.

Proof. Since f is of order less than three, we can invert the integration and summation in the integral 13.2. the result the follows by a term for term application of Proposition 13.3. □

The next result shows that the source solution in the ν-calculus is related to the Gaussian source solution of the radial heat equation

$$\Delta_x(u)\, u(x,t) = \frac{\partial^2 u}{\partial x^2} + \frac{2\mu}{x}\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t}, \quad \mu = \nu + 1/6 \tag{13.3}$$

see Cholewinski and Haimo [11].

Theorem 13.5. *Let $\nu \ge 0$. Then*

$$\frac{3^{\nu+2/3}x^{-2/3}}{2^\nu\Gamma(1/3)} \int_0^\infty 2^{-\frac{4y^3}{27x^2}}\, \mathcal{K}_\nu(y,t)\, dy = \frac{1}{(2t)^{(\nu+1/6)+1/2}}\, e^{-\frac{x^2}{4t}}$$

for $t > 0$ and $x \ne 0$.

Proof. Since $\mathcal{K}_\nu(y,t)$ is an entire function of order $3/2$, we can interchange the summations and integrations in the following equations. Using Proposition 13.3, we obtain

$$\begin{aligned}
\int_0^\infty e^{-\frac{4y^3}{27x^2}}\, \mathcal{K}_\nu(y,t)\, dy &= \frac{1}{(3t)^{\nu+2/3}} \sum_{n=0}^\infty \frac{(-1)^n t^{-n}}{3^{3n}\, n!\,(1/3)_n} \int_0^\infty e^{-\frac{4y^3}{27x^2}}\, y^{3n}\, dy \\
&= \frac{\Gamma(1/3)x^{2/3}}{(3t)^{\nu+2/3}2^{2/3}} \sum_{n=0}^\infty \frac{(-1)^n x^{2n} t^{-n}}{2^{2n}\, n!} \\
&= \frac{\Gamma(1/3)x^{2/3}2^\nu}{3^{\nu+2/3}}\, \frac{1}{(2t)^{\nu+2/3}}\, e^{-\frac{x^2}{4t}}
\end{aligned}$$

□

Next we let $P_{n,m}(x,t)$ denote the radial heat polynomials of Cholewinski and Haimo [11] and of Bragg [8]. We have

$$P_{n,\mu}(x,t)=\sum_{k=0}^{n}2^{2k}\binom{n}{k}\frac{\Gamma(\mu+1/2+n)}{\Gamma(\mu+1/2+n-k)}\,x^{2(n-k)}t^{k}\,.$$

Theorem 13.6. *Let $p_n^\nu(x,t)$ denote the ν- diffusion polynomials associated with ϑ_ν. Then*

$$\frac{2^{2n+2/3}x^{-2/3}}{3^{3n}\Gamma(n+1/3)}\int_0^\infty e^{-\frac{4y^3}{27x^2}}\,p_n^\nu(y,t)\,dy=P_{n,\nu+1/6}(x,t)$$

The proof follows by using (7.3) and Proposition 13.3, the result follows by interchanging the finite summation and integration.

Theorems 13.5 and 13.6 show that ν-diffusion polynomials and radial heat functions can in some cases be related by an integral equation.

Let $W_{n,\nu}(x,t)$ denote the Appell transform of $P_{n,\mu}(x,t)$ given by

$$\begin{aligned}W_{n,\nu}(x,t)&=\frac{1}{(2t)^{\nu+1/2}}e^{-\frac{x^2}{4t}}P_{n,\mu}(x/t,-1/t)\\&=t^{-2n}G_\nu(x,t)P_{n,\mu}(x,-t)\\&=\frac{(-1)^n2^{2n}(\nu+1/2)_n}{2^{\nu+1/2}t^{n+\nu+1/2}}\,{}_1F_1\left[\begin{matrix}n+\nu+1/2\\ \nu+1/2\end{matrix}\,\middle|\,-\frac{x^2}{4t}\right];\end{aligned}$$

see Cholewinski and Haimo [11]. $W_{n,\mu}(x,t)$ is a solution of the radial heat equation, (13.3), and plays the role of $z^{-(n+1)}$ in radial heat expansions, see for example Widder [34] for the classical heat theory.

Theorem 13.7. *Let $Q_n^\nu(x,t)$ be the ν-associated function given by* (12.2)*, then*

$$\begin{aligned}&\frac{(-1)^n3^{\nu+2/3}2^{2n-\nu}}{\Gamma(1/3)x^{2/3}}\int_0^\infty e^{-\frac{4y^3}{27x^2}}Q_n^\nu(y,t)\,dy\\&=W_{n,\nu+1/6}(x,t)\\&=t^{-2n}\,\frac{1}{(2t)^{\nu+1/6+1/2}}\,e^{-\frac{x^2}{4t}}P_{n,\nu+1/6}(x,-t)\end{aligned}$$

Since $Q_n^\nu(y,t)$ is an entire function of order 3/2, the proof follows by a term for term application of Proposition 13.3.

This result shows that in a ν-Appell transform theory the functions $Q_n^\nu(y,t)$ play the role of the $W_{n,\mu}(x,t)$ in the radial heat equation. The generalized Hankel translation defined by Bochner [7] or Delsarte [15] is given by

$$\begin{aligned}f(x\oslash_\mu y)&=\frac{\Gamma(\mu+1/2)}{\Gamma(\mu)\Gamma(1/2)}\int_0^\pi f(\{x^2+y^2-2xycos\,\theta\}^{1/2})\sin^{2\mu+1}\,\theta\,d\theta\\&=\int_0^\infty f(z)D_\mu(x,y,z)\,d\mu(z)\,,\end{aligned}\tag{13.4}$$

where

$$D_\mu(x,y,z)=\frac{2^{3\mu-5/2}\Gamma(\mu+1/2)^2}{\Gamma(\mu)\,\pi^{1/2}}\,(xyz)^{1-2\mu}\Delta(x,y,z)^{2\mu-2}$$

and $\Delta(x,y,z)$ is the area of a triangle with sides x,y,z is g there is such a triangle, and otherwise $D(x,y,z)=0$. The measure is given by

$$d\mu(x)=\frac{x^{2\mu}}{2^{\mu-1/2}\Gamma(\mu+1/2)}\,dx$$

In the case $f(z) = z^{2n}$, we get the Bessel binomial

$$\begin{aligned}(x \oslash_\mu y)^{2n} &= \int_0^\infty z^{2n} D_\mu(x,y,z)\, d\mu(z)\\ &= y^{2n}\, {}_2F_1\left[\begin{matrix}-n, -n-\mu+1/2\\ \mu+1/2\end{matrix}\middle|\frac{x^2}{y^2}\right].\end{aligned}$$

Hirschman developed the Banach algebra for $\mathcal{L}^1[0,\infty,d\mu]$ with translations given by (13.4), see [21]. At the present time a Banach algebra associated with ϑ_ν and the translation $(x \oplus_\nu y)^{3n}$ is unknown. In this paper the ν-translation $f(x \oplus_\nu y)$ is defined for a restricted class of functions.

Theorem 13.8. *Let $\nu \geq 0$ and let $\mu = \nu + 1/2$. Then*

$$\frac{2^{2n+4/3}(xy)^{-2/3}}{\Gamma(1/3)^2 3^{3n}(1/3)_n}\int_0^\infty\int_0^\infty e^{-\frac{4}{27}\left(\frac{w^3}{y^2}+\frac{z^3}{x^2}\right)}(z \oplus_\nu w)^{3n}\, dz\, dw = (x \oslash_\mu y)^{2n}$$

for $x, y > 0$.

The proof follows by interchanging the finite sums in the iterated integrals and by Proposition 13.3.

The integral relation given by Theorem 13.8 can be extended by a number pf classes of functions $f(z)$. For example if f is defined by the Stieltjes integral

$$f(z) = \int_0^\infty \mathcal{G}_\nu(zx)\, d\beta(x)\,, \tag{13.5}$$

where $\beta(x)$ is increasing and bounded with compact support for $d\beta(x)$ in $(0,\infty)$. Then

$$|f(z)| \leq \mathcal{G}_\nu(a|z|)\int_0^a d\beta(x) \leq M e^{a|z|}$$

where a is is the least upper bound of the support of $d\beta(x)$. In this case f is an entire function of growth $\{1, a\}$ and it follows that

$$f(z \oplus_\nu w) = \int_0^\infty \mathcal{G}_\nu(zx)\mathcal{G}_\nu(wx)\, d\beta(x)$$

with

$$|f(z \oplus_\nu w)| \leq \mathcal{G}_\nu(zx)\mathcal{G}_\nu(wx)\int_0^a d\beta(x) \leq M e^{a|z|+a|w|}\,.$$

We define

$$f_b(x) = \int_0^\infty \mathbf{J}_\mu(xt)\, d\beta(t^{2/3})\,.$$

Thus $f_b(x)$ is an entire function which is bounded on the real axis. We have

$$|f_b(x)| \leq \int_0^\infty |\mathbf{J}_\mu(xt)|\, d\beta(t^{2/3}) \leq \int_0^\infty d\beta(t^{2/3}) < \infty$$

since $|\mathbf{J}_\mu(xt)| \leq 1$. From the Hankel translation theory it follows that

$$f_b(x \oslash_\mu y) = \int_0^\infty \mathbf{J}_\mu(tx)\mathbf{J}_\mu(ty)\, d\beta(t^{2/3})\,.$$

Theorem 13.9. *Let $f(z)$ be defined by (13.5). Then for $x, y > 0$,*

$$\frac{2^{4/3}}{(xy)^{2/3}\Gamma(1/3)^2}\int_0^\infty\int_0^\infty e^{-\frac{4}{27}\left(\frac{w^3}{y^2}+\frac{z^3}{x^2}\right)} f(z \oplus_\nu w)\, dz\, dw = f_b(x \oslash_\mu y)\,. \tag{13.6}$$

Proof. The integral in (13.6) is dominated by

$$M\int_0^\infty\int_0^\infty e^{-\frac{4}{27}\left(\frac{w^3}{y^2}+\frac{z^3}{x^2}\right)}\,dw\,dz<\infty$$

and therefore converges absolutely and locally uniformly. Hence we can invert the following iterated integrals. We note that by Theorem 13.4, it follows that

$$\int_0^\infty e^{-\frac{4}{27}\frac{z^3}{x^2}}\mathcal{G}_\nu(tz)\,dz=\frac{x^{2/3}\Gamma(1/3)}{2^{2/3}}\,\mathbf{J}_\mu(xt^{3/2})$$

We have

$$\begin{aligned}
&\int_0^\infty\int_0^\infty e^{-\frac{4}{27}\left(\frac{w^3}{y^2}+\frac{z^3}{x^2}\right)}\int_0^\infty \mathcal{G}_\nu(zt)\mathcal{G}_\nu(wt)\,d\beta(t)\,dw\,dz\\
&=\int_0^\infty d\beta(t)\int_0^\infty \mathcal{G}_\nu(wt)e^{-\frac{4}{27}\frac{w^3}{y^2}}\,dw\int_0^\infty \mathcal{G}_\nu(zt)e^{-\frac{4}{27}\frac{z^3}{x^2}}\,dz\\
&=\frac{x^{2/3}\Gamma(1/3)}{2^{2/3}}\int_0^\infty \mathbf{J}_\mu(xt^{3/2})\,d\beta(t)\int_0^\infty \mathcal{G}_\nu(wt)e^{-\frac{4}{27}\frac{w^3}{y^2}}\,dw\\
&=\frac{(xy)^{2/3}\Gamma(1/3)^2}{2^{4/3}}\int_0^\infty \mathbf{J}_\mu(xt^{3/2})\mathbf{J}_\mu(yt^{3/2})\,d\beta(t)\\
&=\frac{(xy)^{2/3}\Gamma(1/3)^2}{2^{4/3}}\,f_b(x\oslash_\nu y),\quad\text{with the change of variables } t_*=t^{3/2}
\end{aligned}$$

Hence the theorem follows. □

The basic translation kernels $(x\oslash_\nu y)^{2n}$ and $(x\oplus_\nu y)^{3n}$ can also be related to each other by Beta function integrals.

Theorem 13.10. *Let $\nu\ge 0$ and let $\nu_*=\nu+1/6$. Then*

$$(x\oslash_{\nu_*}y)^{2n}=\int_0^1 t^{-2/3}(1-t)^{-2/3}(x^{2/3}t^{1/3}\oplus_\nu y^{2/3}(1-t)^{1/3})^{3n}\,dt/B(1/3,n+1/3)\,.$$

Proof. The generalized binomials are related by

$$\begin{aligned}
\binom{\alpha_{3n}(\nu)}{\alpha_{3k}(\nu)}&=\binom{b_{2n}(\nu_*)}{b_{2k}(\nu_*)}\frac{(1/3)_n}{(1/3)_k(1/3)_{n-k}}\\
&=\binom{b_{2n}(\nu_*)}{b_{2k}(\nu_*)}\frac{B(1/3,n+1/3)}{B(1/3+k,n-k+1/3)}\,,
\end{aligned}$$

where $B(p,q)$ is the Beta function. We have

$$\begin{aligned}
&\int_0^1 t^{-2/3}(1-t)^{-2/3}(x^{2/3}t^{1/3}\oplus_\nu y^{2/3}(1-t)^{1/3})^{3n}\,dt\\
&=\sum_{k=0}^n\binom{\alpha_{3n}(\nu)}{\alpha_{3k}(\nu)}x^{2k}y^{2(n-k)}\int_0^1 t^{k-2/3}(1-t)^{n-k-2/3}\,dt\\
&=\sum_{k=0}^n\binom{b_{2n}(\nu_*)}{b_{2k}(\nu_*)}\frac{B(1/3,n+1/3)}{B(1/3+k,n-k+1/3)}\,B(1/3+k,1/3+n-k)x^{2k}y^{2(n-k)}\\
&=B(1/3,n+1/3)\sum_{k=0}^n\binom{b_{2n}(\nu_*)}{b_{2k}(\nu_*)}x^{2k}y^{2(n-k)}\\
&=B(1/3,n+1/3)(x\oslash_{\nu_*}y)^{2n}
\end{aligned}$$

and the theorem follows. $\square$

The basis addition formula for Bessel functions is given by the integral

$$\mathbf{J}_{\nu_*}(x \oslash_{\nu_*} y) = \int_0^\infty \mathbf{J}_{\nu_*}(z) D_{\nu_*}(x,y,z)\, d\mu_{\nu_*}(z) = \mathbf{J}_{\nu_*}(x)\mathbf{J}_{\nu_*}(y)\,.$$

Using the previous theorem, a calculation shows that

$$\begin{aligned}\mathbf{J}_{\nu_*}(x \oslash_\nu y) &= \frac{1}{B(1/3,1/3)} \int_0^1 t^{-2/3}(1-t)^{-2/3} \\ &\quad \times {}_1F_2 \left[\begin{matrix} 2/3 \\ 1/3, \nu+2/3 \end{matrix} \middle| \frac{(x^{2/3}t^{1/3} \oplus_\nu y^{2/3}(1-t)^{1/3})^3}{4} \right] dt \\ &= \mathbf{J}_{\nu_*}(x)\mathbf{J}_{\nu_*}(y)\end{aligned}$$

14. Generalized Positive Definite Kernels

Bochner [7] obtained a positive definite theory associated with Bessel functions. Bochner's main result was extended by Cholewinski, Haimo and Nussbaum [12]. The positive definite results depend on the Banach algebra associated with kernel functions, see Hirschman [21]. In this section we obtain partial results concerning positive definite functions associated with the $(x \oplus_\nu y)$ translations. Banach algebra results are not available in this latter case.

Let $f(x)$ be a function on $0 \le x < \infty$ for which the ν-translation function $f(x \oplus_\nu y)$ is well defined. The function f is said to be ν-positive definite if

$$\sum_{i,j=1}^n a_i \bar{a}_j f(x_i \oplus_\nu y_j) \ge 0 \tag{14.1}$$

for all finite sets $0 < x_1, x_2, \dots, x_n$ and complex $a_1, a_2, \dots, a_n$. We write $f \in PD_\nu$. Since the discrete sum 14.1 implies its continuous counterpart, see for example Widder [34], p. 270, we have for suitable real valued continuous functions $\phi(x)$ that

$$I(\phi) = \int_0^\infty \int_0^\infty \phi(x)\phi(y) f(x_i \oplus_\nu y_j)\, d\eta_\nu(y)\, d\eta_\nu(x) \ge 0$$

Let $\phi(x)$ be a function on $0 \le x < \infty$ such that the integral

$$\int_0^\infty |\phi(x)|\, \mathcal{G}_\nu(xz)\, d\eta_\nu(x) \tag{14.2}$$

converges locally uniformly for z in R^+. We define the LT_ν transform of ϕ by

$$LT_\nu(\phi) = \hat{\phi}(z) = \int_0^\infty \phi(x)\, \mathcal{G}_\nu(xz)\, d\eta_\nu(x) \tag{14.3}$$

This is a generalization of the Laplace transform or of the Hankel transform in the Bessel function case. Clearly this transform is linear on the functions for which 14.2 converges. If $\beta(t)$ is any function on $0 \le t < \infty$ for which the integral

$$\int_0^\infty \mathcal{G}_\nu(xt)\, d\beta(t)$$

converges locally uniformly for x in R^+, the LT_ν-Stieltjes transform of $\beta(t)$ is given by

$$\hat{\beta}_S(x) = \int_0^\infty \mathcal{G}_\nu(xt)\, d\beta(t)$$

Examples. A. Let $\phi(x) \in C_{00}^{\infty}(R^+)$, with $\phi(x) = 0$ for $x \geq a$ then

$$|\hat{\phi}(z)| \leq \int_0^{\infty} |\phi(x)||\mathcal{G}_\nu(xz)|\, d\eta_\nu(x) = \int_0^a |\phi(x)||\mathcal{G}_\nu(xz)|\, d\eta_\nu(x) \leq M\, e^{a|z|}\,.$$

Thus ϕ is an entire function in z of order 1 and type a.
B. Let $\beta(x)$ be an increasing function on $0 \leq x < \infty$, which is constant for $x \geq a$, then

$$|\hat{\beta}_S(x)| \leq \int |\mathcal{G}_\nu(xt)|\, d\beta(t) \leq \int_0^a |\mathcal{G}_\nu(xt)|\, d\beta(t) \leq M\, e^{a|x|}\,.$$

Thus $\hat{\beta}_S$ has an extension to an entire function in x
C. Let ϕ be a continuous function on $0 \leq x < \infty$, such that $|\phi(x)| \leq M \exp(-x^\rho)$ with $\rho > 2$. Then $\hat{\phi}(z)$ is given by 14.3 is an entire function in z.

Let $\beta(z)$ be an increasing function as in Example B and ϕ be an element of $C_{00}^{\infty}(R^+)$. The function

$$f(x \oplus_\nu y) = \int_0^{\infty} \mathcal{G}_\nu(xz)\mathcal{G}_\nu(yz)\, d\beta(z)$$

is well-defined and, using Fubini's Theorem, we get

$$\begin{aligned} I(\phi) &= \int_0^{\infty}\int_0^{\infty} \phi(x)\bar{\phi}(y) f(x \oplus_\nu y)\, d\eta_\nu(x) d\eta_\nu(y) \\ &= \int_0^{\infty} d\beta(z) \int_0^{\infty} \phi(x)\mathcal{G}_\nu(xz)\, d\eta_\nu(x) \int_0^{\infty} \bar{\phi}(y)\mathcal{G}_\nu(yz)\, d\eta_\nu(y) \\ &= \int_0^{\infty} |\hat{\phi}(z)|^2\, d\beta(z) > 0 \end{aligned} \tag{14.4}$$

Theorem 14.1. *Let*

$$f(x) = \int_0^{\infty} \mathcal{G}(xz)\, d\beta(z)$$

with β given by Example B, the f is in PD_ν.

Proof. Let $0 < x_1, x_2, \dots, x_n$ and $a_1, a_2, \dots, a_n$ be arbitrary, then

$$\sum_{i,j=0}^{n} a_i \bar{a}_j f(x_i \oplus_\nu x_j) = \int_0^{\infty} |\sum_{i=1}^{n} a_i \mathcal{G}_\nu(x_i z)|^2\, d\beta(z) \geq 0\,.$$

□

Example. The source kernel $K_\nu(x;t)$ is ν-positive definite. We have

$$K_\nu(x;t) = \int_0^{\infty} \exp(-ty^3)\mathcal{G}_\nu(xy)\, d\eta_\nu(y)$$

and

$$\sum_{i,j=0}^{n} a_i \bar{a}_j K_\nu(x_i \oplus_\nu x_j;t) = \int_0^{\infty} \exp(-ty^3)|\sum_{i=1}^{n} a_i \mathcal{G}_\nu(x_i y)|^2\, d\eta_{nu}(y) \geq 0$$

Following S. Bernstein [5] we define a function $f(x)$ on $0 < a < x < b$ to be ν-absolutely monotonic if it has non-negative ϑ_ν-derivatives of all orders, i.e., $\vartheta_\nu^k f(x) \geq 0$, $0 < a < x < b$; $k = 0, 1, 2, \dots$.

Examples. The following functions are ν-absolutely monotonic:

(1) $f(x)=\sum_{k=0}^{n} a_k x^{3k}$ with $a_k \ge 0$ on $0\le x<\infty$.
(2) $f(x)=\sum_{k=0}^{\infty} a_k x^{3k}$ for $0\le x\le\rho$ with $a_k\ge 0$.
(3) $f(x)=\int_0^\infty \mathcal{G}_\nu(xy)\,d\beta(y)$ for $0\le x<\infty$ with $\beta(y)$ increasing and constant for $y\ge a>0$. Functions given this way are also in PD_ν on $0\le x<\infty$.

Theorem 14.2. *Let $\phi\in C_{00}^\infty(0,\infty)$ be a non-negative function and let*

$$u(x\oplus_\nu y;t)=\int_0^\infty \exp(-tz^3)\phi(z)\mathcal{G}_\nu(yz)\mathcal{G}_\nu(xz)\,d\eta_\nu(z)\,.$$

Then $u(x\oplus_\nu y;t)$ is a ν-positive definite Airy diffusion.

Proof. By Theorem 4.1, $u(x\oplus_\nu y;t)$ is in $\mathcal{L}^1((0,\infty),d\eta_\nu(x))$ for $t>0$ and $0\le y<\infty$, and clearly u is in $\mathcal{H}_\nu(R^+,t>0)$, $0\le y<\infty$. Let $0<x_1,x_2,\dots,x_n$ and let $\alpha_1,\alpha_2,\dots,\alpha_n$ be arbitrary complex numbers, then

$$\begin{aligned}\sum_{i,j=1}^{n}\alpha_i\bar\alpha_j u(x_i\oplus_\nu x_j;t)&=\int_0^\infty \exp(-ty^3)\phi(y)\sum_{i,j=1}^{n}\alpha_i\bar\alpha_j\mathcal{G}_\nu(x_iy)\mathcal{G}_\nu(x_jy)\,d\eta_\nu(y)\\&=\int_0^\infty \exp(-ty^3)\phi(y)|\sum_{i=1}^{n}\alpha_i\mathcal{G}_\nu(x_iy)|^2\,d\eta_\nu(y)\ge 0\end{aligned}\tag{14.5}$$

since $\phi(y)\ge 0$. Note that Inequality (14.5) is also valid for $t=0$. $\square$

Corollary 14.3. *The function $u(x\oplus_\nu \omega_6 y;t)=u(z_\nu;t)$ is a ν-analytic function of the umbral variable z_ν.*

Proof. We have

$$u(x\oplus_\nu \omega_6 y;t)=u(z_\nu;t)=\int_0^\infty \exp(-tw^3)\phi(w)\mathcal{G}_\nu(xw)G_\nu(yw)\,d\eta_\nu(w)\,.$$

Interchanging differentiation and integration, we have

$$\vartheta_x\,u(z_\nu;t)=\int_0^\infty \exp(-tw^3)\phi(w)(-w)^3\mathcal{G}_\nu(xw)G_\nu(yw)\,d\eta_\nu(w)=-\vartheta_y\,u(z_\nu;t)\,.$$

Thus the ν-Cauchy-Riemann equations hold for $x,y\ge 0$ and therefore u is ν-analytic in the "z_ν-umbral plane". $\square$

Corollary 14.4. *The function*

$$v(x\oplus_\nu y;t)=\int_0^\infty \exp(tw^3\phi(w)\mathcal{G}_\nu(xw)\mathcal{G}_\nu(yw)\,d\eta_\nu(w)$$

is a ν-positive definite Airy diffusion. Moreover, v is a ν-absolutely monotonic in the x (or y) variable. The function $v(z_\nu;t)$ is also ν-analytic.

The above results are also valid for $\beta(t)$ increasing and constant for $t\ge a>0$. In this case we have

$$u(x\oplus_\nu y;t)=\int_0^\infty \exp(-tw^3\mathcal{G}_\nu(xw)\mathcal{G}_\nu(yw)\,d\beta(t)\,,$$
$$v(x\oplus_\nu y;t)=\int_0^\infty \exp(tw^3\mathcal{G}_\nu(xw)\mathcal{G}_\nu(yw)\,d\beta(t)\,.$$

In general functions defined by integrals of the type

$$f(x) = \int_0^\infty \mathcal{G}_\nu(xy)\, d\beta(y)$$

are absolutely monotonic for suitable increasing functions $\beta(y)$. However, the representation is not necessarily unique as the following example demonstrates.

Example. Let

$$s_n = \int_0^\infty u^{3n}\, 3u^2 \exp(-\frac{1}{2}\, u) \sin \frac{\sqrt{3}}{2}\, u\, du\,.$$

A calculation shows that $s_n = 3(3n+2)! \sin \pi(n+1) = 0$, for $n = 0, 1, 2, \dots$. Next we have

$$\mu_n = 3 \cdot 2^{3n+3}! = \int_0^\infty u^{3n}\, d\beta(u)$$

where $\beta(u) = \int_0^u 3\exp(-\frac{1}{2}t)t^2\, dt$. Let

$$\alpha(u) = \int_0^u 3t^2 \exp(-\frac{1}{2}t)\,(1 - \sin \frac{\sqrt{3}}{2}t)\, dt\,.$$

Then

$$\int_0^\infty u^{3n}\, d\alpha(u) = 3 \cdot 2^{3(n+1)}\,(3n+2)! = \int_0^\infty u^{3n}\, d\beta(u)$$

with $\beta(u) \neq \alpha(u)$. Consider

$$f(x) = \int_0^\infty \mathcal{G}_\nu(xu)\, d\beta(u)\,.$$

We have

$$|f(x)| \leq M \int_0^\infty u^2 \exp((-\frac{1}{2} + x)u)\, du$$

for $0 \leq x < \frac{1}{2}$. It easily follows that f is absolutely monotonic for $0 \leq x < 1/2$. However

$$f(x) = \sum_{n=0}^\infty \frac{3 \cdot 2^{3(n+1)}\,(3n+2)!}{3^{3n}\, n!\,(\frac{1}{3})_n\,(\nu + \frac{2}{3})_n} x^{3n} = 6^3 \sum_{n=0}^\infty 2^{3n}\, \frac{(n+\frac{1}{3})(\frac{2}{3})_{n+1}}{(\nu+\frac{2}{3})_n} x^{3n}$$

which converges for $|x| < 1/2$, and

$$f(x) = \int_0^\infty \mathcal{G}_\nu(xu)\, d\beta(u) = \int_0^\infty \mathcal{G}_\nu(xu)\, d\alpha(u)\,.$$

Theorem 14.5. *Let $\alpha_c(t)$ be an increasing function on $0 \leq t < \infty$ with compact support in $[0, a]$, $a > 0$, and let*

$$f(z) = \int_0^\infty \mathcal{G}_\nu(zt)\, d\alpha_c(t)$$

If $f(x_0) = 0$ for some $x_0 > 0$, then $f(x) \equiv 0$.

Proof. We have

$$0 \leq \mu_n = \int_0^\infty t^{3n}\, d\alpha_c(t) \leq a^{3n} \int_0^a d\alpha_c(t) \leq a^{3n} M$$

and

$$f(z) = \int_0^\infty \mathcal{G}_\nu(zt)\, d\alpha_c = \sum_{n=0}^\infty M \frac{\mu_n\, z^{3n}}{\alpha_{3n}(\nu)}$$

with f an entire function. Since $\mu_n \geq 0$, $f(x_0) = 0$ implies $\mu_n = 0$ for $n = 0, 1, 2, \dots$. $\square$

Corollary 14.6. *Let*

$$u(x,t) = \int_0^\infty \exp(ty^3)\, \mathcal{G}_\nu(xy)\, d\alpha_c(y)\,. \tag{14.6}$$

If $u(x_0, t_0) = 0$ for some $x_0 > 0$ and $t_0 \geq 0$, then $u(x,t) = 0$.

Proof. Clearly $u(x,t)$ is an absolutely monotonic ν-Airy diffusion. The theorem implies that $u(x, t_0) = 0$ for $x \geq 0$. Hence we obtain

$$0 \leq \int_0^\infty d\alpha_c(y) \leq \int_0^\infty \exp(t_0 y^3)\, \mathcal{G}_\nu(xy)\, d\alpha_c(y) = 0$$

and therefore $d\alpha_c(y)$ is the zero measure.

Thus this ν-Airy diffusions given by the representation 14.6 are positive functions if $\alpha_c(y)$ is different from a constant. $\square$

Theorem 14.7. *Let $\alpha_c(y)$ be an increasing function with compact support and let*

$$f(x) = \int_0^\infty \mathcal{G}_\nu(xy)\, d\alpha_c(y)\,. \tag{14.7}$$

Then

$$f(x) \geq 0, \quad \begin{vmatrix} f(x) & \vartheta f(x) \\ \vartheta f(x) & \vartheta'' f(x) \end{vmatrix} \geq 0, \quad \begin{vmatrix} f(x) & \vartheta f(x) & \vartheta^2 f(x) \\ \vartheta f(x) & \vartheta^2 f(x) & \vartheta^3 f(x) \\ \vartheta^2 f(x) & \vartheta^3 f(x) & \vartheta^4 f(x) \end{vmatrix} \geq 0, \dots \tag{14.8}$$

and

$$f'(x) \geq 0, \quad \begin{vmatrix} \vartheta f(x) & \vartheta^2 f(x) \\ \vartheta^2 f(x) & \vartheta^3 f(x) \end{vmatrix} \geq 0, \quad \begin{vmatrix} \vartheta f(x) & \vartheta^2 f(x) & \vartheta^3 f(x) \\ \vartheta^2 f(x) & \vartheta^3 f(x) & \vartheta^4 f(x) \\ \vartheta^3 f(x) & \vartheta^4 f(x) & \vartheta^5 f(x) \end{vmatrix} \geq 0, \dots$$

for $0 \leq x < \infty$.

Proof. From the integral representation (14.7), it follows that

$$\vartheta^n f(x) = \int_0^\infty y^{3n}\, \mathcal{G}_\nu(xy)\, d\alpha_c(y)\,,$$

$$[\sum_{i,j=0}^n \vartheta_\nu^{i+j} f(x) a_i a_j = \int_0^\infty \mathcal{G}_\nu(xy) \left(\sum_{i=0}^n a_i t^{3i} \right)^2 d\alpha_c(y) \geq 0$$

$$\sum_{i,j=0}^n \vartheta_\nu^{i+j+m} f(x) a_i a_j = \int_0^\infty \mathcal{G}_\nu(xy) y^{3m} \left(\sum_{i=0}^n a_i t^{3i} \right)^2 d\alpha_c(y) \geq 0$$

for $m \geq 1$. Since the quadratic forms are positive definite the associated determinants are non-negative. Letting $\vartheta^n f(x) = \mu_n = \int_0^\infty y^{3n}\, d\alpha_c(y)$, we get that the quadratic forms

$$\sum_{i,j=0}^n \mu_{i+j} a_i a_j, \quad \sum_{i,j=0}^n \mu_{i+j+1} a_i a_j$$

are non-negative. The ν-generating function of the moments $\{\mu_n\}_0^\infty$ is

$$f(x) = \sum_{n=0}^\infty \frac{\mu_n}{\alpha_{3n}(\nu)} x^{3n}$$

an entire function.

Let

$$u(x,t) = \int_0^\infty \exp(ty^3)\, \mathcal{G}(xy)\, d\alpha_c(y) \tag{14.9}$$

then the determinants given by (14.8) with $f(x)$ being replaced by $u(x,t)$ for fixed $t \ge 0$ are non-negative. Since $u(x,t)$ is absolutely monotonic in the usual Bernstein sense, it follows that the determinants

$$u(x,t) \ge 0, \quad \begin{vmatrix} u & D_t u \\ D_t u & D_t^2 u \end{vmatrix} \ge 0, \quad \begin{vmatrix} u & D_t u & D_t^2 u \\ D_t u & D_t^2 u & D_t^3 u \\ D_t^2 u & D_t^3 & uD_t^4 u \end{vmatrix} \ge 0, \dots \tag{14.10}$$

and

$$D_t u(x,t) \ge 0, \quad \begin{vmatrix} D_t u & D_t^2 u \\ D_t^2 u & D_t^3 u \end{vmatrix} \ge 0, \quad \begin{vmatrix} D_t u & D_t^2 u & D_t^3 u \\ D_t^2 u & D_t^3 u & D_t^4 u \\ D_t^3 u & D_t^4 u & D_t^5 u \end{vmatrix} \ge 0, \dots$$

are also non-negative. We also obtain that the ν-diffusions given by equation (14.9) is logarithmically convex for

$$\frac{\partial^2}{\partial t^2} \log u(x,t) = \frac{u(x,t)\, \frac{\partial^2}{\partial t^2} u(x,t) - \left(\frac{\partial}{\partial t} u(x,t)\right)^2}{u(x,t)^2} \ge 0$$

by (14.10). From the extension of (14.10) to $D_t^m u(x,t)$, etc., we also get

$$\frac{u(x,t)}{D_t u(x,t)} \ge \frac{D_t u(x,t)}{D_t^2 u(x,t)} \ge \frac{D_t^2 u(x,t)}{D_t^3 u(x,t)} \ge \dots .$$

Thus $D_t^m u(x,t)$ is also logarithmically convex. □

15. ν- Associated Nonlinear Equations

Associated with the ν-Airy diffusion equation we obtain a non-linear partial differential equation which is linearized by the ν-Airy equation.

We let $\Delta(\nu) = D_x^2 + \frac{2\nu}{x} D_x = x^{-2\nu}\, Dx^{2\nu}\, D$ denote the Euler-radial operator with $\nu \ge 0$. Further we define a non-linear differential operator by

$$\begin{aligned} K_\nu(\phi) &= x^{-3\nu}\, D_x x^{3\nu}\, D_x \phi - \frac{3\nu}{x^2}\,\phi - \frac{\nu}{x}\,\phi^2 + \frac{1}{9}\,\phi^3 - \phi\phi_x \\ &= \frac{\partial^2 \phi}{\partial x^2} + \frac{3\nu}{x}\,\frac{\partial \phi}{\partial x} - \frac{3\nu}{x^2}\,\phi - \frac{\nu}{x}\,\phi^2 + \frac{1}{9}\,\phi^3 - \phi\,\frac{\partial \phi}{\partial x} \end{aligned} \tag{15.1}$$

Theorem 15.1. *Let $\nu \ge 0$, if $u(x,t)$ is a solution of the ν-Airy equation*

$$\vartheta_\nu u(x,t) = \frac{\partial u(x,t)}{\partial t}$$

then $\phi(x,t) = -3\,\frac{\partial}{\partial x}\, u(x,t)/u(x,t)$ is a solution of the non-linear partial differential equation

$$\frac{\partial \phi(x,t)}{\partial t} = \frac{\partial}{\partial x}\, K_\nu(\phi) = \vartheta_\nu \phi - \phi\Delta(\nu)\phi + (\frac{1}{3}\,\phi^2 - \phi_x)(\phi_x - \frac{3\nu}{x^2}) + \frac{6\nu}{x^3}\,\phi \tag{15.2}$$

Proof. Letting $u_x = -\frac{1}{3}\,\phi u$, we get $\frac{u_{xx}}{u} = \frac{1}{9}\,\phi^2 - \frac{1}{3}\,\phi_x$. Hence

$$\frac{\partial u}{\partial t} = D_x x^{-3\nu} D_x x^{3\nu} u_x$$

implies that

$$\begin{aligned}-3\frac{\partial u}{\partial t} &= D_x(\frac{3\nu}{x}\,\phi u + u\phi_x + \phi u_x) \\ &= D_x(-\frac{9\nu}{x}\,u_x + u\phi_x + \phi u_x) \\ &= u(\frac{9\nu}{x^2}\,\frac{u_x}{u} - \frac{\phi u}{x}\,\frac{u_{xx}}{u} + \phi\,\frac{u_{xx}}{u} + 2\phi_x\,\frac{u_x}{u} + \phi_{xx})\end{aligned}$$

It follows that

$$-3\frac{\partial \ln u}{\partial t} = \phi_{xx} + \frac{3\nu}{x}\,\phi_x - \frac{3\nu}{x^2}\,\phi + \frac{1}{9}\,\phi^3 - \frac{\nu}{x}\,\phi^2 - \phi\phi_x = K_\nu(\phi)$$

and therefore,

$$\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial x}\,K_\nu(\phi)$$

□

In the case that $\nu = 0$, we have the equation

$$\frac{\partial \phi}{\partial t} = \frac{\partial^3 \phi}{\partial x^3} - \phi\,\frac{\partial^2 \phi}{\partial x^2} + \frac{1}{3}\,\phi^3\phi_x - \Big(\frac{\partial \phi}{\partial x}\Big)^2 \tag{15.3}$$

which linearizes to the Airy equation $\frac{\partial \phi}{\partial t} = \frac{\partial^3 \phi}{\partial x^2}$. The KdV equation

$$\frac{\partial \phi}{\partial t} + 6\phi\,\frac{\partial \phi}{\partial x} + \frac{\partial^3 \phi}{\partial x^3} = 0$$

linearizes to the Airy equation $\frac{\partial \phi}{\partial t} + \frac{\partial^3 \phi}{\partial x^3} = 0$. Replacing t by $-t$ gives the form of the Airy equation of this paper.

We note that if $\phi(x,t)$ is a solution of the non-linear equation (15.2) then formally

$$u(x,t) = B(t)\exp(-\frac{1}{3}\int_0^x \phi(y,t)\,dy)$$

is a solution of the ν-Airy equation.

The differential form of (15.2) suggests a conservation of "mass" for suitable solutions. If $\phi(x,t)$ is a solution of (15.2) such that ϕ and all of its derivatives vanish at $\pm\infty$, i.e., $\phi \in C_0^\infty(R)$, then

$$\int_{-\infty}^{\infty} \frac{\partial \phi(x,t)}{\partial t}\,dx = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} \phi(x,t)\,dx = K_\nu(\phi)|_{-\infty}^{\infty} = 0$$

which implies the conservation form

$$\int_{-\infty}^{\infty} \phi(x,t)\,dx = \text{constant},$$

a conservation of mass interpretation.

Associated with the ν-diffusion polynomials $p_n^\nu(x,t)$ we obtain an infinite sequence of rational function solutions of (15.2). We have

$$p_n^\nu(x,t) = x^{3n}\,{}_3F_0\big[-n, \frac{2}{3}-n, \frac{1}{3}-\nu-n \,|\, \big(-\frac{3}{x}\big)^3 t\big]\,.$$

Therefore,

$$\begin{aligned}\omega_n^\nu(x,t) &= -3\,\frac{\partial}{\partial x}\,p_n^\nu(x,t)/p_n^\nu(x,t)\\ &= -\frac{9n}{x}\;\frac{{}_3F_0\big[1-n,2/3-n,1/3-\nu-n\,|\,\big(-\frac{3}{x}\big)^3t\big]}{{}_3F_0\big[-n,2/3-n,1/3-\nu-n\,|\,\big(-\frac{3}{x}\big)^3t\big]}\end{aligned} \tag{15.4}$$

is a solution of the non-linear partial differential equation (15.2). From this equation it follows that

$$\omega_n^\nu(x,t)\sim-\frac{9n}{x}$$

as $x\to\infty$. The first four ω_n^ν's are given in the table:

$$\omega_0^\nu(x,t)=0\,,$$

$$\omega_1^\nu(x,t)=\frac{-9x^2}{x^3+9(\nu+2/3)t}$$

$$\omega_2^\nu(x,t)=-\frac{18x^5+648(\nu+5/3)x^2t}{x^6+72(\nu+5/3)x^2t+324(\nu+2/3)t^2}\,,$$

$$\omega_3^\nu(x,t)=-3\frac{9x^8+1215(\nu+8/3)x^5t+4536(\nu+5/3)_2x^2t^2}{x^9+\frac{405}{2}(\nu+9/3)x^6t+1512(\nu+5/3)_2x^3t^2+120\cdot3^6(\nu+2/3)_3t^3}\,.$$

The source solution of the ν-Airy equation also yields a solution of (15.2). We have

$$\mathcal{K}_\nu(x,t)=\frac{2^{2/3}\,\Gamma(\frac{4}{3})}{3^{\nu+2/3}t^{\nu+1}}\cdot J_{-2/3}\left(\frac{2x^{3/2}}{\sqrt{27t}}\right)$$

and therefore

$$\omega_\nu(x,t)=-3\,\frac{\partial}{\partial x}\,\mathcal{K}_\nu(x,t)/\mathcal{K}_\nu(x,t)=-\left(\frac{3x}{t}\right)^{1/2}\frac{J_{1/3}\left(\frac{2x^{3/2}}{\sqrt{27t}}\right)}{J_{-2/3}\left(\frac{2x^{3/2}}{\sqrt{27t}}\right)}$$

Let $z=\frac{2x^{3/2}}{\sqrt{27t}}$. The recursion formula for Bessel functions $J_{\nu-1}(z)+J_{\nu+1}(z)=\frac{2\nu}{z}\,J_\nu(z)$ gives a continued fraction representation of the solution $\omega_\nu(x,t)$. We have

$$\omega_\nu(x,t)=-\left(\frac{3x}{t}\right)^{1/2}\left\{-\frac{4}{3z}-\cfrac{1}{-\frac{10}{3z}-\cfrac{1}{-\frac{16}{3z}-\cfrac{1}{-\frac{22}{z}-\cfrac{J_{-\frac{11}{3}}(z)}{J_{-\frac{14}{3}}(z)}}}}\right\}$$

Thus the non-linear equation (15.2) has continued fraction solutions.

The associated functions $Q_n^\nu(x,t)$ also yield solutions of (15.2). We have

$$\begin{aligned}W_n^\nu(x,t) &= -3\,\frac{\partial}{\partial x}\,Q_n^\nu(x,t)/Q_n^\nu(x,t)\\ &= \frac{(n+\nu+2/3)}{(\nu+2/3)}\cdot\frac{x^2}{9t}\cdot\frac{{}_1F_2\left[\begin{matrix}n+\nu+5/3\\4/3,\nu+5/3\end{matrix}\middle|-\frac{x^3}{27t}\right]}{{}_1F_2\left[\begin{matrix}n+\nu+2/3\\1/3,\nu+2/3\end{matrix}\middle|-\frac{x^3}{27t}\right]}\,.\end{aligned} \tag{15.5}$$

Using the asymptotic estimates of Marichev [25, p. 71], we get

$$W_n^\nu(x,t) \sim \frac{x^2}{27t}$$

as $x \to \infty$ for $t > 0$.

Next we will show that the non-linear partial differential equation (15.2) has time periodic solutions. Let

$$C_\nu(x,t) = \exp(ity^3)\mathcal{G}_\nu(\omega_6 xy) = \exp(ity^3)E_\nu(xy)$$

which is a complex solution of the ν-Airy equation, where $\omega_6 = \exp(i\pi/6)$. Since

$$\frac{\partial}{\partial x}\,\mathcal{G}_\nu(x,y) = \frac{x^2y^3}{3(\nu+2/3)}\,{}_0F_2[4/3,\nu+5/3\,|\,\Big(\frac{xy}{3}\Big)^3]$$

we find that

$$\begin{aligned} C_\nu(x,t) &= -3\,\frac{\frac{\partial}{\partial x}\,C_\nu(x,t)}{C_\nu(x,t)} \\ &= -i\,\frac{x^2y^3}{(\nu+2/2)}\cdot\frac{{}_0F_2[4/3,\nu+5/3\,|\,i\,\big(\frac{xy}{3}\big)^3]}{{}_0F_2[1/3,\nu+1/3\,|\,i\,\big(\frac{xy}{3}\big)^3]} \end{aligned}$$

a solution that is independent of t.

The function $C_\nu(x,t)$ is periodic in the t variable, we have

$$C_\nu(x,t+2\pi k/y^3) = C_\nu(x,t)$$

for $y \neq 0$. The real and imaginary parts of $C_\nu(x,t)$ are also ν-diffusions. We have

$$\exp(ity^3)E_\nu(xy) = (\cos ty^3 + i\sin ty^3)(cos_\nu(xy) + i\sin_\nu(xy))\,.$$

Hence

$$R_\nu(x,t) = \mathcal{R}e\exp(ity^3)E_\nu(xy) = \cos(ty^3)\cos_\nu(xy) - \sin(ty^3)\sin_\nu(xy)$$

and

$$I_\nu(x,t) = dm\exp(ity^3)\,E_\nu(xy) = \sin(ty^3)\cos_\nu(xy) + \sin_\nu(xy)\cos(ty^3) \tag{15.6}$$

are t-periodic ν-diffusions. It follows that

$$\mathcal{R}_\nu(x,i) = -3\,\frac{\cos(ty^3)\,\frac{\partial}{\partial x}\cos_\nu(xy) - \sin(ty^3)\,\frac{\partial}{\partial x}\sin_\nu(xy)}{\cos(ty^3)\cos_\nu(xy) - \sin(ty^3)\sin_\nu(xy)}$$

and

$$d_\nu(x,t) = -3\,\frac{\sin(ty^3)\,\frac{\partial}{\partial x}\cos_\nu(xy) + \cos(ty^3)\,\frac{\partial}{\partial x}\sin_\nu(xy)}{\sin(ty^3)\cos_\nu(xy) + \cos(ty^3)\sin_\nu(xy)}$$

are t-periodic solutions of the non-linear partial differential equation (15.2). Both of these functions can be expressed in terms of hypergeometric functions.

Actually, there exist a hierarchy of generalized Airy equations and the associated non-linear partial differential equations. We present a few of the equations and results without proofs. Let ν_1 and $\nu_2 \geq 0$, then the next simplest higher order Airy equation is given by

$$\begin{aligned} \frac{\partial u}{\partial t} &= (x^3\,\frac{\partial^6}{\partial x^6} + 3(\nu_1+\nu_2+3)x^2\,\frac{\partial^5}{\partial y^5} + 9(2\nu_1+2\nu_2+\nu_1\nu 2+2)x\,\frac{\partial^4}{\partial x^4} \\ &\quad + 3(6\nu_1\nu_2+3\nu_1+3\nu_2+2)\,\frac{\partial^3}{\partial x^3} - 9\,\frac{\nu_1\nu_2}{x}\,\frac{\partial^2}{\partial x^2} + 9\,\frac{\nu_1\nu_2}{x^2}\,\frac{\partial}{\partial x})\,u \end{aligned} \tag{15.7}$$

Let $\underline{\nu} = (\nu_1, \nu_2)$ and let

$$p_n^{\underline{\nu}}(x,t) = x^{3n}\, {}_6F_0(-n, n, 2/3-n, 2/3-n, 1/3-\nu_1-n\, 1/3-\nu_2-n \,|\, \frac{3^6}{x^3}t)$$

then $p_n^{\underline{\nu}}(x,t)$ is a polynomial solution of 15.7 for $n = 0, 1, 2, \dots$. Moreover, we have the generating function

$$\begin{aligned}&\exp(tz^3)\, {}_0F_5(1, 1/3, 1/3, \nu_1+2/3, \nu_2+2/3 \,|\, \big(\frac{xz}{9}\big)^3)\\ &= \sum_{n=0}^{\infty} \frac{p_n^{\underline{\nu}}(x,t)\, z^{3n}}{3^{6n}\,(n!)^2 (1/3)_n^2 (\nu_1+2/3)_n(\nu_2+2/3)_n}\,.\end{aligned}$$

Theorem 15.2. *Let ν_1 and $\nu_2 \geq 0$ and let $u(x,t)$ be a solution of the higher order Airy diffusion* (15.7)*. Then $\phi(x, `, t) = -3\,\frac{\partial}{\partial x}\, u(x,t)/u(x,t)$ is a solution of the non-linear equation*

$$\begin{aligned}\frac{\partial\phi}{\partial t} &= \frac{\partial}{\partial x}\,\vartheta_{\nu(2)} x^3 K_{\nu(1)} - \frac{1}{3}\,\frac{\partial}{\partial x}(x^3\, K_{\nu(1)}(\phi) K_{\nu(2)}(\phi)\\ &\quad - \frac{2}{3}\,\phi\,\frac{\partial^2}{\partial x^2} x^3 K_{\nu(1)}(\phi) + \frac{2}{3}\,\frac{\partial}{\partial x}(\frac{1}{3}\,\phi^2 - \phi_x)\,\frac{\partial}{\partial x}\,(x^3 K_{\nu(1)}(\phi))\end{aligned} \tag{15.8}$$

The functions

$$\begin{aligned}w_n^{\underline{\nu}}(x,t) &= -3\,\frac{\partial}{\partial x}\, p_n^{\underline{\nu}}(x,t)/p_n^{\underline{\nu}}(x,t)\\ &= -\frac{9n}{x}\cdot \frac{{}_6F_0(-n, 1-n, 2/3-n, 2/3-n, 1/3-\nu_1-n,\ 1/3-nu_2-n \,|\, \frac{3^6}{x^3}t)}{{}_6F_0(-n, -n, 2/3-n, 1/3-\nu_1-n,\ 1/3-nu_2-n \,|\, \frac{3^6}{x^3}t)}\end{aligned}$$

are rational function solutions of (15.8), $n = 0, 1, 2, \dots$. It can be shown that (15.8) also has t-periodic solutions.

16. Constant Coefficient ν-Differential Equations

In this section, solutions of polynomial operators with argument ϑ_ν are obtained in a number of cases. First of all we consider solutions of the differential equations

$$(\vartheta_\nu - a^3) y(x) = 0 \tag{16.1}$$

where a is some complex number. For a series of the form $\sum_{n=0}^{\infty} c_n x^{n+\lambda}$, we obtain

$$\sum_{n=0}^{\infty} c_n\{(3n+\lambda)(3n+\lambda+3\nu-1)(3n+\lambda-2)x^{3n+\lambda} - a^3x^3x^{3n+\lambda}\}$$

with indicial equations $f(\lambda) = \lambda(\lambda+3\nu-1)(\lambda-2)$ and skip number three. For $\nu \geq 0$ and ν different from $\frac{1}{3}$, we obtain three linearly independent solutions of

(16.1). They are, corresponding to the indicial roots 0, 2, and $1-3\nu$ respectively

$$G_\nu(ax) = \sum_{n=0}^{\infty} \frac{a^{3n}x^{3n}}{3^{3n}n!(1/3)_n(\nu+2/3)_n} =_0 F_2[1/3, \nu+2/3 \,|\, \Big(\frac{ax}{3}\Big)^3],$$

$$G_\nu^{(2)}(ax) = \sum_{n=0}^{\infty} \frac{a^{3n}x^{3n+2}}{3^{3n}n!(5/3)_n(\nu+4/3)_n} = x^2 \, {}_0F_2[5/3, \nu+4/3 \,|\, \Big(\frac{ax}{3}\Big)^3],$$

$$\begin{aligned} G_\nu^{(3)}(ax) &= \sum_{n=0}^{\infty} \frac{a^{3n}x^{3n+1-3\nu}}{3^{3n}n!(4/3-\nu)_n(2/3-\nu)_n n!} \\ &= x^{1-3\nu} \, {}_0F_2[4/3-\nu, 2/3-\nu \,|\, \Big(\frac{ax}{3}\Big)^3] \end{aligned}$$

In the latter case we must have ν such that $4/3-\nu$, $2/3-\nu \neq 0, -1, -2, \dots$. The function $G_\nu(ax)$ is the basic zeta function of this paper associated with ϑ_ν.

If $p(\vartheta_\nu = \sum_{k=0}^{n} a_k \vartheta_\nu^k$ is a monic polynomial operator, we find that $p(\vartheta_\nu G_\nu(rx) = p(r^3)G_\nu(rx)$. We let $p(r^3) = \prod_{k=1}^{n}(r^3 - r_k)$. If the r_k's are distinct, we get the general solution of the equation $p(\vartheta_\nu)y = 0$ is given by

$$y(x) = \sum_{k=1}^{n} \{c_{1k}G_\nu(r_k^{1/3}x) + c_{2k}G_\nu^{(2)}(r_k^{1/3}x) + c_{3k}G_\nu^{(3)}(r_k^{1/3}x)\}$$

We are interested in determining solutions in the repeated root case. Thus we need to obtain solutions of the equation $(\vartheta_\nu - a^3)^k y = 0$. Due to the failure of a nice Leibnitz type product formula for $\vartheta^k(uv)$, an exponential shift type equation is not available for our repeated root equation. In the classical case the exponential shift formula $(D-a)^k(e^{ax}v) = e^{ax}D^k v$, leads to solutions in the case of repeated roots for constant coefficient equations in D. A new method that works in a number of cases follows.

For m a non-negative integer, we define

$$\begin{aligned} G_{\nu-m}(ax) &= \sum_{n=0}^{\infty} \frac{a^{3n}x^{3n}}{3^{3n}n!(1/3)_n(\nu-m+2/3)_n}, \\ G_{\nu-m}^{(2)}(ax) &= \sum_{n=0}^{\infty} \frac{a^{3n}x^{3n+2}}{3^{3n}n!(5/3)_n(\nu-m+4/3)_n} \end{aligned}$$

with ν such that $(\nu-m+2/3)$ and $(\nu-m+4/3)$ are not zero or a negative integer. The ν-Airy operator can be written as

$$\vartheta_\nu = \vartheta_{\nu-m} + \frac{3m}{x^3}\mathcal{N}(\mathcal{N}-2)$$

where $\mathcal{N} = xD$ is the numbers operator.

Theorem 16.1. *For restricted* ν *and* $0 \le k \le m$

$$(\vartheta_\nu - a^3)^k G_{\nu-m}^{(2)}(ax) = \frac{m!}{(m-k)!} \, \frac{a^{3k}}{(\nu+2/3-m)_k} \, G_{\nu-(m-k)}^{(2)}(ax) \tag{16.2}$$

Proof. We consider the case $k=1$, the general case follows by iteration. By (16.2), it follows that

$$(\vartheta_\nu - a^3)^k G_{\nu-m}^{(2)}(ax) = \frac{3m}{x^3}\,\mathcal{N}(\mathcal{N}-2)G_{\nu-m}(ax)$$

where $\mathcal{N} = xD$ is the number operator. Since

$$\frac{3m}{x^3}\,\mathcal{N}(\mathcal{N}-2)x^{3n} = 3m(3n)(3n-2)x^{3(n-1)}$$

we get

$$\begin{aligned}\frac{3m}{x^3}\,\mathcal{N}(\mathcal{N}-2)G_{\nu-m}(ax) &= \sum_{n=1}^{\infty}\frac{m(n-2/3)x^{3(n-1)}a^{3n}}{3^{3(n-1)}(n-1)!(1/3)_n(\nu-m+2/3)_n}\\ &= \frac{ma^3}{(\nu-m+2/3)}\,G_{\nu-(m-1)}(ax)\end{aligned}$$

Since $(\vartheta_\nu - a^3)^m G_{\nu-(m-1)}(ax) = \frac{m!a^{3m}}{(\nu+2/3-m)_m}G_\nu(ax)$, $(\vartheta_\nu - a^3)^{m+1}G_{\nu-m}(ax) = 0$. Hence the equation

$$(\vartheta_\nu - a^3)^{m+1}y(x) = 0$$

has solutions $G_\nu(ax), G_{\nu-1}(ax), \dots, G_{\nu-m}(ax)$. □

In the case of repeated roots corresponding to $G_\nu^{(2)}(ax)$, we have the equations

$$(\vartheta_\nu - a^3)^m G_{\nu-m}^{(2)}(ax) = \frac{m!}{(m-k)!}\,\frac{a^{3k}}{(\nu+4/3-m)_k}\,G_{\nu-(m-k)}^{(2)}(ax)\,.$$

Therefore, $(\vartheta_\nu - a^3)^{m+1}y(x) = 0$ also has $G_\nu^{(2)}(ax), G_{\nu-1}^{(2)}(ax), \dots, G_{\nu-m}^{(2)}(ax)$ as solutions.

Based on the exponential shift equation $(D-a)^k x^n e^{ax} = \frac{n!}{(n-k)!}x^{n-k}e^{ax}$, we can find solutions of the eigenvalue problem for the wave operator $(\frac{\partial}{\partial x} - \frac{\partial}{\partial t})$. The eigenvalue problem

$$(\frac{\partial}{\partial x} - \frac{\partial}{\partial t})u(x,t) = \lambda u(x,t) \tag{16.3}$$

has polynomial type solutions $u_{n,\lambda}(x,t) = e^{\lambda x}(x+t)^n$. We will an analogue of this result for the corresponding ϑ_ν problem.

The corresponding eigenvalue problem in the ν-Airy case is given by the equation

$$(\vartheta_\nu - \frac{\partial}{\partial t})u(x,t) = \lambda^3 u(x,t) \tag{16.4}$$

Using (15.6), we find that solutions corresponding to $u(x,0) = G_{\nu-n}(\lambda x)$ for suitable ν are given by

$$\begin{aligned}C_{n,\lambda}^{\nu}(x,t) &= \exp(t(\vartheta_\nu - \lambda^3))G_{\nu-n}(\lambda x)\\ &= \sum_{k=0}^{\infty}\frac{t^k}{k!}(\vartheta_\nu - \lambda^3)^k G_{\nu-n}(\lambda x)\\ &= \sum_{k=0}^{\infty}\frac{t^k}{k!}\,\frac{n!\lambda^{3k}G_{\nu-(n-k)}(\lambda x)}{(n-k)!(\nu+2/3-n)_k}\\ &= \sum_{k=0}^{\infty}\binom{n}{k}\frac{t^k\lambda^{3k}G_{\nu-(n-k)}(\lambda x)}{(\nu+2/3-n)_k}\end{aligned} \tag{16.5}$$

Solutions of the eigenvalue equation (16.4) also lead to solutions of the previously encountered non-linear partial differential equation

$$\frac{\partial}{\partial t}\,\phi(x,t) = \frac{\partial}{\partial x}\,K_\nu(\phi)\,.$$

Let u_λ be a solution of the eigenvalue equation (16.4) and let

$$\phi_\lambda(x,t) = -3\,\frac{\frac{\partial}{\partial x}\,u_\lambda(x,t)}{u_\lambda(x,t)}\,.$$

Then

$$\frac{\partial u}{\partial t} + \lambda^3 u = \vartheta u = D(x^{-3\nu} Dx^{3\nu} u_x = -\frac{1}{3}\,Dx^{-3\nu}Dx^{3\nu}u\phi\,.$$

Therefore, as before we find that

$$-\frac{3}{u}\,(\frac{\partial u}{\partial t} + \lambda^3 u) = K_\nu(\phi)$$

and

$$\frac{\partial}{\partial t}\phi_\lambda(x,t) = DK_\nu(\phi) \tag{16.6}$$

Thus solutions of 16.6 for suitable ν and n a non-negative integer are given by

$$\begin{aligned}\phi^\nu_{n,\lambda}(x,t) &= -3\,\frac{\frac{\partial}{\partial x}\,C^\nu_{n,\lambda}(x,t)}{C^\nu_{n,\lambda}(x,t)}\\ &= -3\,\frac{\sum_{k=0}^n \binom{n}{k}\frac{t^k\lambda^{3k}}{(\nu+2/3-n)_k}\,\frac{\partial}{\partial x}\,G_{\nu-(n-k)}(\lambda x)}{\sum_{k=0}^n \binom{n}{k}\frac{t^k\lambda^{3k}}{(\nu+2/3-n)_k}\,G_{\nu-(n-k)}(\lambda x)}\end{aligned}$$

which is a quotient of sums of hypergeometric functions. That is

$$\phi^\nu_{n,\lambda}(x,t) = -x^2\,\frac{\sum_{k=0}^n \binom{n}{k}\frac{t^k\lambda^{3k}}{(\nu+2/3-n)_k}\,{}_0F_2[4/3,\nu+5/3-n+k\mid\,\big(\frac{\lambda x}{3}\big)^3]}{\sum_{k=0}^n \binom{n}{k} left\frac{t^k\lambda^{3k}}{(\nu+2/3-n)_k}\,{}_0F_2[4/3,\nu+5/3-n+k\mid\,\big(\frac{\lambda x}{3}\big)^3]}$$

for $n = 0,1,2,\dots$. In the same manner, we find that

$$C^{(2),\nu}_{n,\lambda}(x,t) = \sum_{k=0}^\infty \binom{n}{k}\frac{t^k\lambda^{3k}G^{(2)}_{\nu-(n-k)}(\lambda x)}{(\nu+4/3-n)_k}$$

is a solution of the eigenvalue problem (16.3) for $n = 0,1,2,\dots$. Hence for suitable ν we obtain solutions of the non-linear partial differential equation (16.6) given by

$$\begin{aligned}\phi^\nu_{n,\lambda}(x,t) &= -3\,\frac{\frac{\partial}{\partial x}\,C^{(2),\nu}_{n,\lambda}(x,t)}{C^{(2),\nu}_{n,\lambda}(x,t)}\\ &= -3\,\frac{\sum_{k=0}^n \binom{n}{k}\frac{t^k\lambda^{3k}}{(\nu+4/3-n)_k}\,\frac{\partial}{\partial x}\,G^{(2)}_{\nu-(n-k)}(\lambda x)}{\sum_{k=0}^n \binom{n}{k}\frac{t^k\lambda^{3k}}{(\nu+2/3-n)_k}\,G^{(2)}_{\nu-(n-k)}(\lambda x)}\\ &= -3\,\frac{\sum_{k=0}^n \binom{n}{k}\frac{t^k\lambda^{3k}}{(\nu+4/3-n)_k}\,{}_0F_2[2/3,\nu+4/3-n+k\,|\big(\frac{\lambda x}{3}\big)^3]}{\sum_{k=0}^n \binom{n}{k}\frac{t^k\lambda^{3k}}{(\nu+2/3-n)_k}\,{}_0F_2[5/3,\nu+4/3-n+k\,|\big(\frac{\lambda x}{3}\big)^3]}\end{aligned}$$

for $n = 0,1,2,\dots$.

References

[1] M. Abramowitz and I. A. Stegan, *Handbook of Mathematical Functions with Formulas and Graphs*, National Bureau pf Standards, 1964.

[2] H. Bateman, Two systems of polynomials for the solution of Laplace's integral equation, *Duke Math. J.*, 2(1936), 569-577.

[3] Bateman Manuscript Project, *Tables of Integral Transforms*, Volume 1, McGraw-Hill, New York, 1953.

[4] Bateman Manuscript Project, *Higher Transcendental Functions*, Volume 2, McGraw-Hill, New York, 1953.
[5] S. Bernstein, Sur les functions absolument monotones, *Acta Mathematica*, 51(???), 1-66.
[6] R.P. Boas, Jr. *Entire Functions*, Academic Press Inc., New York, 1954.
[7] Sturm-Liouville and heat equations whose eigenfunctions are ultraspherical polynomials or associated Bessel functions, *Proc. of the Conf. on Diff. Equations*, University of Maryland, 1955, 23-48.
[8] L. R. Bragg, The radial heat polynomials and related functions, *Trans. Amer. Math. Soc.*, 119(1965), 270-290.
[9] F. M. Cholewinski, *The Finite Calculus Associated with Bessel Functions*, Contemporary Math., Vol. 75, AMS, 1988.
[10] F. M. Cholewinski, A class of cubic Fock Spaces, *Technical Report # 639*, Clemson University, 1994, 1-146.
[11] F. M. Cholewinski and D. T. Haimo, Classical analysis and the generalized heat equation, *SIAM Review*, 10(1968), 67-80.
[12] F. M. Cholewinski, D. T. Haimo, and A. E. Nussbaum, A necessary and sufficient condition for the representation of a function as a Hankel-Stieltjes transform, *Studia Math.*, 36(1970), 269-274.
[13] E. T. Copson, On the Riemann-Green functions, *J. Ratl. Mech. Anal.*, 1(1953), 324-348.
[14] J. Daboul and P. N. Rathie, Generalized parity and γ- polynomials, *Int. J. Math. and Stat. Sci*, 5(1996), 179-196.
[15] J. Delsarte, Sur une extension de la formula de Taylor, *Jour. de Math. Pur. Appl.*, 17(1936), 213-231.
[16] A. Edrei, On the generating functions of totally positive sequences, *Jour. d'Analyse Mathematique*, 2(49), 104-109.
[17] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. Tricomi, *Higher Transcendental Functions*, McGraw-Hill, New York, 1953.
[18] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, Academic Press, New York, 1980.
[19] D. T. Haimo, Series representations of generalized temperature functions, *SIAM J. Appl. Math.*, 15(1967), 359-367.
[20] P. Henrici, *Applied and Computational Complex Analysis*, Volume 1, John Wiley & Sons, New York, 1974.
[21] I. I. Hirschman, Jr, Variation diminishing Hankel transforms, *J. Analyse Math.*, 8(1960-61), 307-336.
[22] I. I. Hirschman and D. V. Widder, *The Convolution Transform*, Princeton University Press, Princeton, NJ, 1955.
[23] S. Karlin, *Total Positivity*, Volume 1, Stanford University Press, Stanford, CA, 1968.
[24] R. E. Langer, The solutions of the differential equation $v''' + \lambda^2 z\, v' + 3\mu\, \lambda^2\, v = 0$, *Duke Math. J.*, 22(1955), 525-541.
[25] O. I. Marichev, *Handbook of Integral Transforms of Higher Transcendental Functions*, Ellis Harwood Ltd., London, 1983.
[26] E. D. Rainville, The contiguous function relations for ${}_pF_q$ with applications to Bateman's $J_n^{x,\nu}$ and Rice's $H_n(\zeta, \rho, \nu)$, *Bull. Amer. Math. Soc.*, 51(1945), 714-223.
[27] E. D. Rainville, *Special Functions*, MacMillian Company, New York, 1960.
[28] Gian-Carlo Rota, *Finite Operator Calculus*, Academic Press, New York, 1975.
[29] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Ellis Horwood Ltd., England, 1984.
[30] E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Oxford University Press, Oxford, 1937.
[31] A. Weinstein Generalized axially symmetric potential theory, *Bull. Amer. Math. Soc.*, 59(1953), 20-38.
[32] A. Weinstein, The generalized radiation problem and the Euler-Poisson-Darboux equation, *Summa Brasiliensis*, 3(1955), 125-145.
[33] D. V. Widder, The Airy transform, *Amer. Math. Monthly*, 86(1979), 271-277.
[34] D. V. Widder, *The Heat Equation*, Academic Press, New York, 1975.

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