POSITIVE SOLUTIONS OF BOUNDARY-VALUE PROBLEMS FOR 2M-ORDER DIFFERENTIAL EQUATIONS

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Abstract. This article concerns the existence of positive solutions to the differential equation

\((-1)^m x^{(2m)}(t) = f(t, x(t), x'(t), \ldots, x^{(m)}(t)), \quad 0 < t < \pi,\)

subject to boundary condition

\(x^{(2i)}(0) = x^{(2i)}(\pi) = 0,\)

or to the boundary condition

\(x^{(2i)}(0) = x^{(2i+1)}(\pi) = 0,\)

for \(i = 0, 1, \ldots, m - 1\). Sufficient conditions for the existence of at least one positive solution of each boundary-value problem are established. Motivated by references [7, 17, 21], the emphasis in this paper is that \(f\) depends on all higher-order derivatives.

1. Introduction

The study of the existence of positive solutions of boundary-value problems for second-order and higher-order ordinary differential equations has gained prominence recently and is a rapidly growing field. This happens because of the applications of this problem, especially fourth-order differential equations; see for example the articles [5, 7, 9, 12, 13, 16, 17, 19, 20, 21] and the monographs [1, 2, 3].

For the second-order case, the existence of positive solutions of boundary-value problems for nonlinear differential equations has been studied by many authors. The differential equation

\[x''(t) + f(t, x(t)) = 0, \quad 0 < t < 1,\]

subjected to different boundary conditions has received much attention. Specially in seeking conditions on the nonlinearity \(f\) for which there are at least one, at least two, or at least three positive solutions. See for example [4, 8, 10, 11, 24].
However, there are not many publications about the existence of positive solutions of the differential equation

\[ x^n(t) + f(t, x(t), x'(t)) = 0, \quad 0 < t < 1, \tag{1.2} \]

under various boundary conditions. This because the presence of \( x' \) in the nonlinearity \( f \) causes considerable difficulties \([6, 17, 18, 22]\).

Recently, Chyan and Henderson \([7]\) studied the \(2m\)-order differential equation

\[ x^{(2m)}(t) = f(t, x(t), x''(t), \ldots, x^{(m-1)}(t)), \quad 0 < t < 1, \tag{1.3} \]

with either the Lidstone boundary condition

\[ x^{(2i)}(0) = x^{(2i)}(1) = 0 \quad \text{for} \quad i = 0, 1, \ldots, m - 1, \tag{1.4} \]

or with the focal boundary condition

\[ x^{(2i+1)}(0) = x^{(2i)}(1) = 0 \quad \text{for} \quad i = 0, 1, \ldots, m - 1. \tag{1.5} \]

They proved the existence of at least one positive solution when \( f \) is either super-linear or \( f \) is sub-linear.

Similar problems were also investigated by Palamides \([21]\) using an analysis of the corresponding field on the face-plane and the Sperner's Lemma. The method there is different from that in \([7, 17]\). In the papers mentioned above, the nonlinearity \( f \) depends on \( x, x'', \ldots, x^{(2(m-1))}\).

In this paper, we consider the \(2m\)-order differential equation

\[ (-1)^m x^{(2m)}(t) = f(t, x(t), x'(t), \ldots, x^{(m)}(t)), \quad 0 < t < \pi, \tag{1.6} \]

with either the Lidstone boundary conditions

\[ x^{(2i)}(0) = x^{(2i)}(\pi) = 0 \quad \text{for} \quad i = 0, 1, \ldots, m - 1, \tag{1.7} \]

or the focal boundary conditions

\[ x^{(2i)}(0) = x^{(2i+1)}(\pi) = 0 \quad \text{for} \quad i = 0, 1, \ldots, m - 1. \tag{1.8} \]

We assume \( f : [0, \pi] \times I_0 \times I_1 \times \cdots \times I_m \to [0, +\infty) \) is continuous, where \( I_0 = [0, +\infty) \), \( I_1 = R \), \( I_2 = (-\infty, 0] \), \ldots for BVP (1.6)–(1.7), and \( I_0 = I_1 = [0, +\infty) \), \( I_2 = I_3 = (-\infty, 0] \), \ldots for BVP (1.6) and (1.8). It is easy to check that if \( x(t) \) is a positive solution of BVP (1.6)–(1.7), then

\[ (-1)^m x^{(2m)}(t) \geq 0, \quad (-1)^{m-1} x^{(2(m-1))}(t) \geq 0, \quad \text{for} \quad t \in [0, \pi] \]

for \( t \in [0, \pi] \) if \( x(t) \) is a positive solution of BVP (1.6) and (1.8).

The emphasis of this paper is that \( f \) depends on each of the \( m \) higher-order derivatives; i.e., \( f \) depends on \( x, x', \ldots, x^{(m)}\). To obtain the main results, we need the following notation and an abstract existence theorem, whose proof can be found in the text books \([14, 23]\).

**Definition:** Let \( X \) be a real Banach space. A non-empty closed convex set \( P \subset X \) is called a cone of \( X \) if it satisfies the following conditions:

(i) \( x \in P \) and \( \lambda \geq 0 \) implies \( \lambda x \in P \).

(ii) \( x \in P \) and \( -x \in P \) implies \( x = 0 \).

Every cone \( P \subset X \) induces an ordering in \( X \), which is given by \( x \leq y \) if and only if \( y - x \in P \).
Let $X$ and $Y$ be Banach spaces, $L : \text{dom } L \subset X \to Y$ be a Fredholm operator of index zero, $P : X \to X$ and $Q : Y \to Y$ be projectors such that

$$\Im P = \Ker L, \Ker Q = \Im L, \quad X = \Ker L \oplus \Ker P, \quad Y = \Im L \oplus \Im Q.$$ 

It follows that

$$L|_{\text{dom } L \cap \Ker P} : \text{dom } L \cap \Ker P \to \Im L$$

is invertible, we denote the inverse of that map by $K_p$.

If $\Omega$ is an open bounded subset of $X$, $\text{dom } L \cap \bar{\Omega} \neq \Phi$, the map $N : X \to Y$ will be called $L$–compact on $\Omega$ if $QN(\Omega)$ is bounded and $K_p(I - Q)N : \Omega \to X$ is compact. Now, we present the fixed point theorem.

**Theorem 1.1** ([14, 23]). Let $X$ and $Y$ be Banach spaces, $K_1 \subset X$ and $K \subset Y$ be cones in $X$ and $Y$, respectively, and the operators $L$ and $N$ be defined above such that $NX \subset K$, $L^{-1}(K) \subset K_1$ and $\Ker L = \{0\}$. Let $\Omega_1$ and $\Omega_2$ be open bounded subsets in $X$ such that $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$. If $N : \bar{\Omega}_2 \to Y$ is $L$-compact on $\Omega_2$ and there is $h \in L^{-1}(K)$ with $h \neq 0$ such that

(i) $Lx \neq \lambda N x$ for $\lambda \in (0, 1)$ and $x \in \text{dom } L \cap \partial \Omega_1 \cap K_1$; $Lx - Nx \neq \lambda Lh$ for $\lambda > 0$ and $x \in \text{dom } L \cap \partial \Omega_2 \cap K_1$, or

(ii) $Lx - Nx \neq \lambda Lh$ for $\lambda > 0$ and $x \in \text{dom } L \cap \partial \Omega_1 \cap K_1$; $Lx \neq Nx$ for $\lambda \in (0, 1)$ and $x \in \text{dom } L \cap \partial \Omega_2 \cap K_1$,

then $Lx = Nx$ has at least one solution $x \in \text{dom } L \cap (\bar{\Omega}_2/\Omega_1) \cap K_1$.

2. Positive solutions of boundary-value problems

In this section, we present the main results and then give some examples to illustrate the main results.

**Theorem 2.1.** Suppose

(A) The following inequality holds uniformly in $t$:

$$\limsup_{\sum_{i=0}^m |x_i| \to +\infty} \frac{f(t, x_0, x_1, \ldots, x_m)}{\sum_{i=0}^m |x_i|} < \frac{1}{m + 1}.$$

(B) The following inequality holds uniformly in $t$:

$$\liminf_{\sum_{i=0}^m |x_i| \to 0} \frac{f(t, x_0, x_1, \ldots, x_m)}{\sum_{i=0}^m |x_i|} > 1.$$ 

Then BVP (1.6)–(1.7) has at least one positive solution.

**Proof.** Let $X = C^m[0, \pi]$ and $Y = C^0[0, \pi]$ be endowed with the norms $\|x\| = \max\{\|x\|_\infty, \|x\|_\infty, \cdots, \|x^{(m)}\|_\infty\}$ and $\|x\|_\infty = \max_{t \in [0, \pi]} |x(t)|$, respectively. For $x \in Y$, denote

$$\|x\|_1 = \int_0^\pi |x(t)| dt, \quad \|x\|_2 = \left( \int_0^\pi |x(t)|^2 dt \right)^{1/2}.$$ 

Define

$$\text{dom } L = \{x \in C^m[0, \pi] : x^{(2i)}(0) = x^{(2i)}(\pi) = 0, \ i = 0, 1, \ldots, m - 1\}.$$ 

Define the linear operator $L : \text{dom } L \cap X \to Y$ and the nonlinear operator $N : X \to Y$ by

$$Lx(t) = (-1)^m x^{(2m)}(t) \quad \text{for } x \in \text{dom } L \cap X,$$

$$N x(t) = f(t, x(t), x'(t), \ldots, x^{(m)}(t)) \quad \text{for } x \in X.$$
Then the differential equation (1.6) can be written as $Lx = Nx$. It is easy to see that $\text{Ker} L = \{ 0 \}$ and $\text{Im} L = Y$. Define the projectors $P : X \to X$ by $P x(t) = 0$ for all $t \in [0, \pi]$ and $Q : Y \to Y$ by $Q y(t) = 0$ for all $t \in [0, \pi]$, respectively. So $L$ is a Fredholm operator of index zero, and $L^{-1} : Y \to X \cap \text{dom} L$ can be written by

$$L^{-1} y(t) = \int_0^\pi G_m(s, t) y(s) ds,$$

where

$$G_0(s, t) = \begin{cases} \frac{s(\pi-t)}{\pi}, & 0 \leq s \leq t \\ \frac{t(\pi-s)}{\pi}, & 0 \leq t \leq s, \end{cases}$$

$$G_k(s, t) = \int_0^\pi G_0(s, u) G_{k-1}(u, t) du \quad \text{for} \quad k = 1, \ldots, m.$$

It is easy to check that $L^{-1}$ is completely continuous, together with that $N : X \to Y$ is continuous and bounded, it follows that $N$ is $L$-compact. We divide the proof into two steps.

**Step 1.** Prove the first part of (ii) in Theorem 1.1. By (B), there is $r > 0$ such that if $\sum_{i=0}^m |x_i| \leq r$, then

$$f(t, x_0, x_1, \ldots, x_m) > \sum_{i=0}^m |x_i| \geq x_0.$$

Choose

$$\Omega_1 = \{ x \in X : \|x\| \leq r/(m + 1) \},$$

$$K_1 = \{ x \in \text{dom} L \cap X : x(t) \geq 0 \quad \text{and} \quad (-1)^m x^{(2m)}(t) \geq 0 \quad \text{for} \quad t \in [0, \pi] \},$$

$$K = \{ x \in Y : x(t) \geq 0 \quad \text{for} \quad t \in [0, \pi] \}.$$

Then $\text{Ker} L = \{ 0 \}$, $NX \subset K$, $L^{-1}(K) \subset K_1$ and $K_1 \subset X$ and $K \subset Y$ are cones.

If $x \in \text{dom} L \cap \partial \Omega_1 \cap K_1$, then $\|x\| \leq r/(m + 1)$, so

$$\sum_{i=0}^m |x^{(i)}(t)| \leq \sum_{i=0}^m \|x^{(i)}\|_\infty \leq (m + 1)\|x\| \leq r.$$

It follows that

$$f(t, x(t), x'(t), \ldots, x^{(m)}(t)) \geq x(t) \quad \text{for} \quad t \in [0, \pi]. \quad (2.1)$$

Thus

$$\sin tf(t, x(t), x'(t), \ldots, x^{(m)}(t)) \geq x(t) \sin t \quad \text{for} \quad t \in [0, \pi].$$

Integrating the above inequality from 0 to $\pi$, we obtain

$$\int_0^\pi \sin tf(t, x(t), x'(t), \ldots, x^{(m)}(t)) dt \geq \int_0^\pi \sin tx(t) dt$$

$$= - \cos tx(t) \bigg|_0^\pi + \int_0^\pi x'(t) \cos t dt$$

$$= \sin tx'(t) \bigg|_0^\pi - \int_0^\pi \sin tx''(t) dt$$

$$= \ldots$$

$$= \int_0^\pi \sin t(-1)^m x^{(2m)}(t) dt.$$
i.e.,
\[ \int_0^\pi \sin tN x(t) \, dt \geq \int_0^\pi \sin tL x(t) \, dt. \] (2.2)

On the other hand, let \( h(t) \) be the unique solution of the following problem (it is easy to know, from \([7]\), that it has unique solution)
\[ (-1)^m x^{(2m)}(t) = 1, \quad 0 < t < \pi, \]
\[ x^{(2i)}(0) = x^{(2i)}(\pi) = 0 \quad i = 0, 1, \ldots, m - 1. \]

Then \( h \in \text{dom } L \) and \( L h(t) = 1 \). We will prove that
\[ L x - N x \neq \lambda L h \]
for \( \lambda > 0 \) and \( x \in \text{dom } L \cap \partial \Omega_1 \cap K_1 \). In fact, if there is \( \lambda_1 > 0 \) and \( x_1 \in \text{dom } L \cap \partial \Omega_1 \cap K_1 \) such that
\[ L x_1 - N x_1 = \lambda_1 L h, \]
then
\[ \int_0^\pi \sin tL x_1(t) \, dt = \int_0^\pi \sin tN x_1(t) \, dt + \lambda_1 \int_0^\pi \sin t \, dt \]
\[ > \int_0^\pi \sin tN x_1(t) \, dt, \]
which contradicts (2.2). So the first part of (ii) in Theorem 1.1 is satisfied.

**Step 2.** Prove the second part of (ii) in Theorem 1.1. Choose \( 1/(m + 1) > \epsilon > 0 \) and \( M > 0 \) such that
\[ f(t, x_0, x_1, x_2, \ldots, x_m) \leq \left( \frac{1}{m + 1} - \epsilon \right) \sum_{i=0}^m |x_i| + M \] (2.3)
for all \( t \in [0, \pi] \) and \( x_i \in I_i \) for \( i = 0, \ldots, m \). In fact, from (A), there is \( H > 0 \) such that
\[ f(t, x_0, x_1, x_2, \ldots, x_m) \leq \left( \frac{1}{m + 1} - \epsilon \right) \sum_{i=0}^m |x_i| \]
for \( t \in [0, \pi] \) and \( \sum_{i=0}^m |x_i| \geq H \), where \( x_i \in I_i \) for \( i = 0, 1, \ldots, m \). Let
\[ M = \max_{t \in [0, \pi], \sum_{i=0}^m |x_i| \leq H} f(t, x_0, x_1, x_2, \ldots, x_m), \]
then we have (2.3). So for \( x \in \text{dom } L \cap K_1 \), we have
\[ f(t, x(t), x'(t), \ldots, x^{(m)}(t)) \leq \left( \frac{1}{m + 1} - \epsilon \right) \left( \sum_{i=1}^m |x_i| + x(t) \right) + M. \]

In order to get \( \Omega_2 \), we now prove that the set
\[ S = \{ x \in \text{dom } L \cap K_1, \ L x = \lambda N x, \ 0 < \lambda < 1 \} \]
is bounded. In fact, if \( S \) is unbounded, then there is \( \lambda \in (0, 1) \), and \( x \in S \) such that \( x \) satisfies
\[ (-1)^m x^{(2m)}(t) = \lambda f(t, x(t), x'(t), \ldots, x^{(m)}(t)), \ t \in [0, \pi]. \] (2.4)
Thus
\((-1)^m x^{(2m)}(t)x(t) = \lambda x(t)f(t, x(t), x'(t), \ldots, x^{(m)}(t))\)
\leq \lambda \left( \frac{1}{m+1} - \epsilon \right) \left( x^2(t) + \sum_{i=1}^{m} x(t)|x^{(i)}(t)| \right) + x(t) M.

Integrating above inequality from 0 to \(\pi\), we get
\[\int_0^{\pi} (-1)^m x^{(2m)}(t) dt \leq \lambda \left( \frac{1}{m+1} - \epsilon \right) \int_0^{\pi} \left( x^2(t) + \sum_{i=1}^{m} x(t)|x^{(i)}(t)| \right) dt + M \int_0^{\pi} x(t) dt.\]

Since
\[(-1)^m \int_0^{\pi} x(t)x^{(2m)}(t) dt = (-1)^m \int_0^{\pi} x(t) dx^{(2m-1)}(t)\]
\[= (-1)^m x(t)x^{(2m-1)} \bigg|_0^{\pi} + (-1)^{m-1} \int_0^{\pi} x^{(2m-1)}(t)x'(t) dt\]
\[= (-1)^{m-1} \int_0^{\pi} x'(t) dx^{(2m-2)}(t)\]
\[= \ldots\]
\[= \int_0^{\pi} \left( x^{(m)}(t) \right)^2 dt,\]
we obtain
\[\|x^{(m)}\|_2^2 \leq \lambda \left( \frac{1}{m+1} - \epsilon \right) \left[ \int_0^{\pi} x^2(t) dt + \sum_{i=1}^{m} \int_0^{\pi} x(t)|x^{(i)}(t)| dt \right] + M \int_0^{\pi} x(t) dt\]
\[\leq \lambda \left( \frac{1}{m+1} - \epsilon \right) \left( \|x\|_2^2 + \sum_{i=1}^{m} \|x\|_2\|x^{(i)}\|_2 \right) + \pi M \|x\|_\infty.\]

Since \(x(t) \sim \sum_{n=1}^{\infty} a_n \sin nt\), where \(a_n\) is the Fourier coefficient of \(x\) and
\[x'(t) \sim \sum_{n=1}^{\infty} na_n \cos nt,\]
by Parseval equality, \(\|x\|_2 \leq \|x'\|_2\). Similarly, we have
\[\|x\|_2 \leq \|x'\|_2 \leq \cdots \leq \|x^{(m)}\|_2.\]

Again,
\[|x(t)| = |x(t) - x(0)| = |\int_0^{t} x'(s) ds|\]
\[\leq \int_0^{t} |x'(s)| ds \leq \int_0^{\pi} |x'(s)| ds\]
\[\leq \left( \int_0^{\pi} |x'(t)|^2 dt \int_0^{\pi} dt \right)^{1/2} = \pi^{1/2} \|x'\|_2.\]

Then we obtain \(\|x\|_\infty \leq \pi^{1/2} \|x'\|_2\). Thus
\[\|x^{(m)}\|_2^2 \leq \lambda \left( \frac{1}{m+1} - \epsilon \right) (m+1) \|x^{(m)}\|_2^2 + M \pi^{3/2} \|x^{(m)}\|_2.\]
Hence
\[ \|x^{(m)}\|_2 \leq \frac{M\pi^{3/2}}{\epsilon(m + 1)} =: c_1. \]

Thus, we obtain
\[ \|x\|_{\infty} \leq \pi^{1/2}\|x^{(m)}\|_2 \leq \frac{M\pi^2}{\epsilon(m + 1)} =: c_2, \]
\[ \|x^{(i)}\|_2 \leq \|x^{(m)}\|_2 \leq \frac{M\pi^{3/2}}{\epsilon(m + 1)} =: c_1 \quad \text{for } i = 0, 1, \ldots, m. \]

Similarly, we have
\[ \|x^{(i)}\|_{\infty} \leq \pi^{1/2}\|x^{(i+1)}\|_2 \leq \frac{M\pi^2}{\epsilon(m + 1)} =: c_2 \quad \text{for } i = 1, \ldots, m - 1. \]

From (2.3),
\[ |x^{(2m)}(t)| \leq \left( \frac{1}{m + 1} - \epsilon \right) \left( x(t) + \sum_{i=1}^{m} |x^{(i)}(t)| \right) + M \]
\[ \leq \left( \frac{1}{m + 1} - \epsilon \right) \left( \|x\|_{\infty} + \frac{m}{2} + \frac{1}{2} \sum_{i=1}^{m} |x^{(i)}(t)|^2 \right) + M \]
\[ \leq \left( \frac{1}{m + 1} - \epsilon \right) \left( c_2 + \frac{m}{2} + \frac{1}{2} \sum_{i=1}^{m} |x^{(i)}(t)|^2 \right) + M. \]

Integrating above inequality from 0 to \( \pi \), we get
\[ \|x^{(2m)}\|_1 \leq \pi \left( \frac{1}{m + 1} - \epsilon \right) (c_2 + \frac{m}{2}) + \frac{1}{2} \left( \frac{1}{m + 1} - \epsilon \right) c_2^2 + M\pi =: c_3. \]

Since \( x^{(2m-2)}(0) = x^{(2m-2)}(\pi) = 0 \), there is \( \xi \in [0, \pi] \) such that \( x^{(2m-1)}(\xi) = 0 \), thus
\[ |x^{(2m-1)}(t)| \leq \|x^{(2m)}\|_1. \]

So \( \|x^{(2m-1)}\|_{\infty} \leq c_3. \) Similarly, one gets
\[ \|x^{(2i-1)}\|_{\infty} \leq c_3, \quad i = 1, \ldots, m - 1. \]

This implies \( \|x\| \leq \max\{c_3, c_2, c_1\} + 1 \) for all \( x \in S. \)

Choose \( R > \max\{\max\{c_1, c_2, c_3\} + 1, r/(2m + 1)\} \). Let
\[ \Omega_2 = \{x \in X : \|x\| < R\}. \]

Then \( S \subset \Omega_2 \). So \( Lx \neq \lambda Nx \) for \( \lambda \in (0, 1) \) and \( x \in \text{dom } L \cap \partial \Omega_2 \cap K_1 \). Thus by Theorem 1.1, \( Lx = Nx \) has at least one solution \( x \in \text{dom } L \cap (\Omega_2/\Omega_1) \cap K_1 \). \( x \) is a solution of BVP (1.6)–(1.7).

Next, we prove that \( x(t) > 0 \) for \( t \in [0, \pi] \). Since \( (-1)^m x^{(2m)}(t) \geq 0 \) for all \( t \in [0, \pi] \), together with the boundary value conditions (1.7), we get \( x(t) \geq 0 \) and \( x''(t) \leq 0 \) for all \( t \in [0, \pi] \). If there is \( t_0 \in (0, \pi) \) such that \( x(t_0) = 0 \), then the
Theorem 2.2. Assume the following two conditions are satisfied:

(C) The inequality \( f(t, x_0, x_1, \ldots, x_m) \geq x_0 \) holds for all \((x_0, x_1, \ldots, x_m) \) in \(R^{m+1}\) and all \( t \) in \([0, \pi]\).

(D) The following inequality holds uniformly for \( t \) in \([0, \pi]\):
\[
\limsup_{\sum_{i=0}^{m} |x_i| \to 0} \frac{f(t, x_0, x_1, \ldots, x_m)}{\sum_{i=0}^{m} |x_i|} < \frac{1}{(m+1)\Delta_m}.
\]

Then BVP (1.6)–(1.7) has at least one positive solution.

Proof. We divide the proof of the theorem into two steps.

Step 1. To prove the first part of (i), choose \( r > 0 \) and \( \delta \in (0, 1/[(m+1)\Delta_m]) \) such that
\[
f(t, x_0, x_1, \ldots, x_m) \leq \delta \sum_{i=0}^{m} |x_i|
\]
for \( t \in [0, \pi] \) and \((x_0, x_1, \ldots, x_m) \) in \(R^{m+1}\) with \( \sum_{i=0}^{m} |x_i| \leq r \). Let
\[
\Omega_1 = \{ x \in \text{dom } L \cap K_1, \|x\| < \frac{r}{m+1} \}.
\]
For \( x \in \partial \Omega_1 \), we have \( \|x\| = \frac{r}{m+1} \), then
\[
\sum_{i=0}^{m} |x^{(i)}(t)| \leq \sum_{i=0}^{m} \|x^{(i)}\|_\infty \leq (m+1)\|x\| = r.
\]
So, we get
\[ f(t, x(t), x'(t), \ldots, x^{(m)}(t)) \leq \delta \sum_{i=1}^{m} |x^{(i)}(t)|, \quad \text{for } t \in [0, \pi]. \]

If \( Lx = \lambda Nx \) with \( \lambda \in (0, 1) \) has a solution \( x \in \operatorname{dom} L \cap K \cap \partial \Omega_{1} \), then
\[ x(t) = \lambda L^{-1} N x(t) = \lambda \int_{0}^{\pi} G_{m}(t, s)f(s, x(s), x'(s), \ldots, x^{(m)}(s))ds. \]

Hence, we get
\[
\|x\|_{\infty} = \lambda \max_{t \in [0, \pi]} \left[ \int_{0}^{\pi} G_{m}(t, s)f(s, x(s), x'(s), \ldots, x^{(m)}(s))ds \right]
\leq \delta \max_{t \in [0, \pi]} \left[ \int_{0}^{\pi} G_{m}(t, s) \sum_{i=0}^{m} |x^{(i)}(s)|ds \right]
\leq \delta \Delta_{1}(m + 1)\|x\|. \]

It is easy to check that
\[
\|x'\|_{\infty} = \lambda \max_{t \in [0, \pi]} \left[ -\int_{0}^{t} \frac{s}{\pi} G_{m-1}(t, s)f(s, x(s), x'(s), \ldots, x^{(m)}(s))ds \right.
+ \int_{t}^{\pi} \left( 1 - \frac{s}{\pi} \right) G_{m-1}(t, s)f(s, x(s), x'(s), \ldots, x^{(m)}(s))ds \right]
\leq \max_{t \in [0, \pi]} \left( \int_{0}^{t} \frac{s}{\pi} G_{m-1}(t, s)f(s, x(s), x'(s), \ldots, x^{(m)}(s))ds \right.
+ \int_{t}^{\pi} \left( 1 - \frac{s}{\pi} \right) G_{m-1}(t, s)f(s, x(s), x'(s), \ldots, x^{(m)}(s))ds \right)
\leq \max_{t \in [0, \pi]} \left( \int_{0}^{t} \frac{s}{\pi} G_{m-1}(t, s)ds + \int_{t}^{\pi} \left( 1 - \frac{s}{\pi} \right) G_{m-1}(t, s)ds \right) \delta(m + 1)\|x\|
\leq \Delta_{2}\delta(m + 1)\|x\|. \]

Finally, we can get \( \|x^{(m)}\|_{\infty} \leq \delta \Delta_{m}(m + 1)\|x\|. \) Hence, we have
\[ \|x\| \leq \delta \Delta_{m}(m + 1)\|x\|. \]

Thus \((m + 1)\delta \Delta_{m} \geq 1\), which contradicts \( \delta \in (0, 1/(\Delta_{m}(m + 1))] \). The first step is complete.

**Step 2.** Choose \( \Omega_{2} \) sufficiently large such that \( \Omega_{1} \subset \overline{\Omega_{1}} \subset \Omega_{2} \), by condition (C), we have that
\[ f(t, x_{0}, x_{1}, \ldots, x_{m}) \geq x_{0} \]
holds for all \( t \in [0, \pi] \) all \((x_{0}, x_{1}, \ldots, x_{m}) \in R^{m+1} \). Hence,
\[ f(t, x(t), x'(t), \ldots, x^{(m)}(t)) \geq x(t) \]
holds for all \( t \in [0, \pi] \), i.e. (2.1) holds. Similar to Step 1 in Theorem 1.1, we can get a contradiction, hence the second part of (i) in Theorem 1.1 is satisfied. It follows from (i) of Theorem 1.1 that BVP (1.6) and (1.8) has at least one positive solution \( x(t) \). The proof is complete. \( \square \)
Remark. Consider the boundary-value problem
\[ (-1)^m x^{(2m)}(t) = f(t, x(t), x'(t), \ldots, x^{(m)}(t)), \quad 0 < t < T, \]
\[ x^{(2i)}(0) = x^{(2i)}(T) = 0 \quad \text{for} \ i = 0, 1, \ldots, m - 1, \]  
(2.6)
where \( T > 0 \) is a constant, \( f \) and \( m \) are defined in (1.6)–(1.7). Let \( s = \pi t/T \), we transform BVP (2.6) into a BVP similar to BVP (1.6)–(1.7). Then a similar existence result can be obtained.

Theorem 2.3. Suppose (A) and (B) of Theorem 2.1 hold. Then BVP (1.6) and (1.8) has at least one positive solution.

Proof. Consider the boundary-value problem
\[ (-1)^m x^{(2m)}(t) = \begin{cases} f(t, x(t), x'(t), \ldots, x^{(m)}(t)), & \text{for } 0 \leq t \leq \pi, \\ f(2\pi - t, x(2\pi - t), -x'(2\pi - t), \ldots, \\ (-1)^m x^{(m)}(2\pi - t)), & \text{for } \pi \leq t \leq 2\pi, \\ \end{cases} \]
\[ x^{(2i)}(0) = x^{(2i)}(2\pi) = 0 \quad \text{for} \ i = 0, 1, \ldots, m - 1. \]

This problem is exactly similar to that of Theorem 2.1, we can obtain at least one positive solution \( x(t) \), which is defined on \([0, 2\pi]\), of above BVP and so \( x(t)(t \in [0, \pi]) \) is a positive solution of BVP (1.6) and (1.8). The proof completed. □

Theorem 2.4. Suppose Conditions (C) and (D) of Theorem 2.2 hold. Then BVP (1.6) and (1.8) has at least one positive solution.

The proof is similar to that of Theorem 2.3 and is omitted. Next, we present two examples to illustrate the main results.

Example 2.5. Consider the boundary-value problem
\[ x^{(4)}(t) = f(t, x(t), x'(t), x''(t)), \quad 0 < t < \pi, \]
\[ x(0) = x''(0) = x''(\pi) = x''(\pi) = 0, \]  
(2.7)
where \( f \) is a nonnegative continuous function. From Theorem 2.1, if
\[ \limsup_{|x|+|y|+|z| \to \infty} \frac{f(t, x, y, z)}{|x| + |y| + |z|} < \frac{1}{3}, \]
and
\[ \liminf_{|x|+|y|+|z| \to 0} \frac{f(t, x, y, z)}{|x| + |y| + |z|} > 1 \]
hold uniformly, then (2.7) has at least one positive solution.

Example 2.6. Consider the boundary-value problem
\[ x^{(6)}(t) = 2 \frac{2}{1 + |x(t)| + |x'(t)| + |x''(t)| + |x'''(t)| + |x''''(t)|}, \quad 0 < t < \pi, \]
\[ x(0) = x''(0) = x'''(0) = x''(\pi) = x'''(\pi) = x''(\pi) = x'''(\pi) = 0. \]  
(2.8)
It is easy to check that all conditions of Theorem 2.1 are satisfied. So (2.8) has at least one positive solution.

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