ON NONNEGATIVE ENTIRE SOLUTIONS OF SECOND-ORDER SEMILINEAR ELLIPTIC SYSTEMS

TOMOMITSU TERAMOTO

Abstract. We consider the second-order semilinear elliptic system
\[ \Delta u_i = P_i(x)u_1^{\alpha_i} \quad \text{in} \ R^N, \quad i = 1, 2, \ldots, m \]
with nonnegative continuous functions \( P_i \). We establish nonexistence criteria of nonnegative nontrivial entire solutions for this system. We also proved a Liouville type theorem for nonnegative entire solutions.

1. Introduction

This paper concerns the second-order semilinear elliptic system
\[ \begin{align*}
\Delta u_1 &= P_1(x)u_2^{\alpha_1}, \\
\Delta u_2 &= P_2(x)u_3^{\alpha_2}, \\
& \quad \vdots \\
\Delta u_m &= P_m(x)u_{m+1}^{\alpha_m}, \quad u_{m+1} = u_1,
\end{align*} \]
(1.1)
where \( x \in R^N \), \( N \geq 1, m \geq 2, \) and \( \alpha_i > 0, \ i = 1, 2, \ldots, m \) are constants satisfying \( \alpha_1 \alpha_2 \cdots \alpha_m > 1 \), and the functions \( P_i(x) \) are nonnegative continuous functions on \( R^N \).

We are concerned with the problem of existence and nonexistence of nonnegative nontrivial entire solutions of (1.1). By an entire solution of (1.1) we mean a vector function \((u_1, u_2, \ldots, u_m) \in (C^2(R^N))^m\) which satisfies (1.1) at every point of \( R^N \).

The problem of existence and nonexistence of nonnegative entire solutions for the scalar equation
\[ \Delta u = f(x, u), \quad x \in R^N \]
has been investigated by many authors, and numerous results have been obtained (see e.g. [2, 5, 7, 9] and references therein). In particular, when \( f \) has the form \( f(x, u) = P(x)u^\alpha \) with \( \alpha > 0 \) and nonnegative function \( P \), critical decay rate of \( P \) to admit nonnegative entire solutions has been characterized. On the other hand, very little is known about this problem for elliptic system (1.1) except for the case \( m = 2 \). For \( m = 2 \) we refer to [3, 5, 6, 8, 12, 13, 14].

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In [3, 12, 14], the system (1.1) with \( m = 2 \) has been considered under the conditions \( \alpha_i \geq 1, \ i = 1, 2 \), and nonexistence criteria of nonnegative nontrivial entire solutions have been obtained. The result is described roughly as follows:

**Theorem 1.1.** Let \( N \geq 3 \), \( m = 2 \) and \( \alpha_i \geq 1, i = 1, 2 \). Suppose that \( P_i, \ i = 1, 2 \), satisfy

\[
P_i(x) \geq \frac{C_i}{|x|^{\lambda_i}}, \quad |x| \geq r_0 > 0, \quad i = 1, 2,
\]

where \( C_i > 0 \) and \( \lambda_i, \ i = 1, 2 \), are constants. If \( (\lambda_1, \lambda_2) \) satisfies

\[
\lambda_1 - 2 + \alpha_1(\lambda_2 - 2) \leq 0 \quad \text{or} \quad \lambda_2 - 2 + \alpha_2(\lambda_1 - 2) \leq 0,
\]

then the system (1.1) does not possess any nonnegative nontrivial entire solutions.

However, if \( \alpha_1 \) or \( \alpha_2 \) is less than 1, Theorem 1.1 cannot derive any information about the nonnegative nontrivial entire solutions. Recently, Teramoto and Usami [13] have proved a Liouville type theorem for nonnegative entire solutions of (1.1) with \( m = 2 \) under the condition \( \alpha_1 \alpha_2 > 1 \). The result is described as follows:

**Theorem 1.2.** Let \( N \geq 3 \), \( m = 2, \alpha_1 \alpha_2 > 1, 0 < \alpha_1 < 1 \). Suppose that \( P_i, \ i = 1, 2 \), satisfy (1.2) for some constants \( \lambda_i, \ i = 1, 2 \). If \( (\lambda_1, \lambda_2) \) satisfies

\[
\lambda_1 - 2 + \alpha_1(\lambda_2 - 2) \leq 0,
\]

then the system (1.1) does not possess nonnegative nontrivial entire solutions satisfying

\[
u_1(x) = O(\exp |x|^\rho) \quad \text{as} \quad |x| \to \infty \quad \text{for some} \ \rho > 0.
\]

The aim of this paper is to extend Theorems 1.1 and 1.2 to the system (1.1) with \( m \geq 3 \).

Let us introduce some notation used throughout this paper. For any sequence \( \{s_1, s_2, \ldots, s_m\} \), we assume that \( s_{m+j} = s_j, j = 1, 2, \ldots \); that is, the suffixes should be taken in the sense of \( \mathbb{Z}/m\mathbb{Z} \). Denote

\[
A = \alpha_1 \alpha_2 \cdots \alpha_m.
\]

For real constants \( \lambda_1, \lambda_2, \ldots, \lambda_m \), we put

\[
\Lambda_i = \lambda_i - 2 + (\lambda_{i+1} - 2)\alpha_i + (\lambda_{i+2} - 2)\alpha_i \alpha_{i+1} + \ldots + (\lambda_{i+m-1} - 2)\alpha_i \alpha_{i+1} \alpha_{i+2} \cdots \alpha_{i+m-2} \]

\[
= \lambda_i - 2 + \sum_{j=1}^{m-1} \left\{ (\lambda_{i+j} - 2) \prod_{k=0}^{j-1} \alpha_{i+k} \right\}, \quad i = 1, 2, \ldots, m,
\]

and

\[
\beta_i = \frac{\Lambda_i}{A-1}, \quad i = 1, 2, \ldots, m.
\]

Since our assumptions imposed on \( P_i \), \( 1 \leq i \leq m \), essentially take the forms

\[
\liminf_{|x| \to \infty} |x|^\lambda_i P_i(x) > 0 \quad \text{or} \quad \limsup_{|x| \to \infty} |x|^\lambda_i P_i(x) < \infty,
\]

all our results are formulated by means of the numbers \( \lambda_i, \Lambda_i, \beta_i, 1 \leq i \leq m \).

This paper is organized as follows. In Section 2, we give nonexistence criteria of nonnegative nontrivial entire solutions of (1.1). In Section 3, to show the sharpness of our nonexistence criteria we give existence theorems of positive entire solutions for (1.1) under the assumption that \( P_i \) have radial symmetry. In the final section (Section 4), we prove a Liouville type theorem for nonnegative entire solutions.
2. A PRIORI ESTIMATE AND NONEXISTENCE RESULTS

2.1. Growth estimate of nonnegative entire solutions. In this subsection, we study the estimate for nonnegative entire solutions of (1.1) which will play an important role to prove nonexistence theorems for nonnegative nontrivial entire solutions.

For a nonnegative function \( v \) defined on \( \mathbb{R}^N \), we denote its spherical mean over the sphere \( |x| = r, \ r > 0 \), \( \bar{v}(r) \) by

\[
\bar{v}(r) = \frac{1}{\omega_{N-1}} \int_{|x|=r} v(x) \, dS,
\]

where \( dS \) denotes the volume element in the surface integral, \( \omega_N \) is the surface area of the unit sphere in \( \mathbb{R}^N \). Moreover we introduce the function \( \hat{P}(r) \), \( r \geq 0 \), by

\[
\begin{align*}
\hat{P}(r) &= \begin{cases}
\frac{1}{\omega_{N-1}} \int_{|x|=r} P(x) r^{-\alpha'} dS, & \alpha > 1, \\
\min_{|x|=r} P(x), & \alpha = 1,
\end{cases} \\
&= 0 \quad \text{if} \int_{|x|=r} P(x) r^{-\alpha'/\alpha} dS = \infty.
\end{align*}
\]

We note that \( \hat{P} = P \) when \( P \) has radial symmetry. We have the following well-known result (see [2, p.654], [9, p.508] and [10, p.70]).

**Lemma 2.1.** Let \( \alpha_i \geq 1, \ i = 1, 2, \ldots, m \), and \((u_1, u_2, \ldots, u_m)\) be a nonnegative entire solution of (1.1). Then its spherical mean \((\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m)\) satisfies system of ordinary differential inequalities

\[
\begin{align*}
(r^{N-1} u_i'(r))^r &\geq r^{N-1} \hat{P}_i(r) u_i(r)^\alpha, \quad r > 0, \\
u_i'(0) &= 0,
\end{align*}
\]

where \( i = 1, 2, \ldots, m \).

Our main result is as follows.

**Theorem 2.2.** Let \( N \geq 3, \ \alpha_i \geq 1, \ i = 1, 2, \ldots, m, \) and \( A > 1 \). Suppose that \( P_i, \ i = 1, 2, \ldots, m, \) satisfy

\[
\liminf_{|x| \to \infty} \frac{1}{|x|^{\lambda_i}} P_i(x) > 0,
\]

where \( \lambda_i, i = 1, 2, \ldots, m, \) are constants. Let \((u_1, u_2, \ldots, u_m)\) be a nonnegative entire solution of (1.1). Then \( u_i, \ i = 1, 2, \ldots, m, \) satisfy

\[
u_i(x) \leq C_i |x|^\beta_i \quad \text{at } \infty,
\]

where \( C_i > 0 \) are constants and \( \beta_i \) are defined by (1.5).

Assume that (2.3) holds. Then there are constants \( C_i > 0, \ i = 1, 2, \ldots, m, \) and \( R_0 > 0 \) such that

\[
P_i(x) \geq \frac{C_i}{|x|^\lambda_i}, \quad |x| \geq R_0, \quad i = 1, 2, \ldots, m.
\]

So we can see that \( \hat{P}_i, \ i = 1, 2, \ldots, m, \) defined by (2.1) satisfy

\[
\hat{P}_i(r) \geq \frac{C_i}{r^{\lambda_i}}, \quad r \geq R_0.
\]
Proof of Theorem 2.2. Let \((u_1, u_2, \ldots, u_m)\) be a nonnegative entire solution of (1.1). We may assume that \((u_1, u_2, \ldots, u_m) \neq (0, 0, \ldots, 0)\). Then, by Lemma 2.1, its spherical mean \((\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m)\) satisfies the system of ordinary differential inequalities (2.2).

Integrating (2.2) over \([0, r]\), we have
\[
\bar{u}'(r) \geq r^{1-N} \int_0^r s^{N-1} \bar{P}(s) \bar{u}_{i+1}(s)^{\alpha} ds, \quad i = 1, 2, \ldots, m.
\]
Hence, we see that \(\bar{u}'(r) \geq 0\) for \(r \geq 0\). Integrating (2.2) twice over \([r, R]\), \(R \geq 0\) and \(i = 1, 2, \ldots, m\), we have
\[
\bar{u}_i(r) \geq \bar{u}_i(R) + \frac{1}{N-2} \int_r^R s [1 - \left(\frac{s}{r}\right)^{N-2}] \bar{P}(s) \bar{u}_{i+1}(s)^{\alpha} ds.
\]
(2.5)

Since \((u_1, u_2, \ldots, u_m)\) is nonnegative and nontrivial, there exists a point \(x_0 \in \mathbb{R}^N\) such that \(u_{i_0}(x_0) > 0\) for some \(i_0 \in \{1, 2, \ldots, m\}\); that is, \(\bar{u}_{i_0}(r_*) > 0\), \(r_* = |x_0|\). We may assume that \(r_* \geq R_0\). Therefore, we see from (2.5) with \(R = r_*\) that \(\bar{u}_i(r) > 0\) for \(r > r_*, \; i = 1, 2, \ldots, m\).

First, we will show that
\[
\bar{u}_i(r) = O(r^{\bar{\alpha}_i}) \quad \text{as} \quad r \to \infty, \quad i = 1, 2, \ldots, m.
\]
(2.6)

Let us fix \(R > r_*\) arbitrarily. Using (2.4) and the inequality
\[
s [1 - \left(\frac{s}{r}\right)^{N-2}] \geq \frac{N-2}{3N-2} (r-s) \quad \text{for} \quad R \leq r \leq 3R,
\]
in (2.5), we have
\[
\bar{u}_i(r) \geq \bar{u}_i(R) + \bar{C}_i \int_R^r \left(s^{-\bar{\lambda}_i}(r-s)\bar{u}_{i+1}(s)^{\alpha}\right) ds
\]
\[
\geq \bar{C}_i R^{-\bar{\lambda}_i} \int_R^r (r-s) \bar{u}_{i+1}(s)^{\alpha} ds,
\]
where \(R \leq r \leq 3R\) and \(\bar{C}_i\) are some positive constants independent of \(r\) and \(R\). We put
\[
f_i(r; R) = \bar{C}_i R^{-\bar{\lambda}_i} \int_R^r (r-s) \bar{u}_{i+1}(s)^{\alpha} ds, \quad R \leq r \leq 3R.
\]
(2.7)

For simplicity of notation we write \(f_i(r) = f_i(r; R)\) if there is no ambiguity. Clearly, \(f_i(r), \; i = 1, 2, \ldots, m\), satisfy
\[
\bar{u}_i(r) \geq f_i(r), \quad f_i(R) = 0,
\]
\[
f'_i(r) \geq 0, \quad f'_i(R) = 0,
\]
and
\[
f''_i(r) = \bar{C}_i R^{-\bar{\lambda}_i} \bar{u}_{i+1}(r)^{\alpha_i} \geq \bar{C}_i R^{-\bar{\lambda}_i} f_{i+1}(r)^{\alpha_i}, \quad R \leq r \leq 3R.
\]
(2.8)

From (2.7) and the monotonicity of \(\bar{u}_i\), we see that
\[
f_i(r; R) \geq \frac{\bar{C}_i}{2} R^{-\bar{\lambda}_i} \bar{u}_{i+1}(R)^{\alpha_i} (r-R)^2, \quad R \leq r \leq 3R.
\]
(2.9)

Let us fix \(i \in \{1, 2, \ldots, m\}\). Multiplying (2.8) by \(f'_{i+1}(r) \geq 0\) and integrating by parts of the resulting inequality on \([R, r]\), we have
\[
f'_{i+1}(r) f'_i(r) \geq C R^{-\bar{\lambda}_i} f_{i+1}(r)^{\alpha_i+1}, \quad R \leq r \leq 3R,
\]
where $C = \tilde{C}_i/(\alpha_i + 1)$. For the rest of this article, $C$ denotes various positive constants independent of $r$ and $R$. Multiplying this inequality by $f'_{i+1}(r) \geq 0$ and integrating by parts, we obtain

$$f'_{i+1}(r)^2 f_i(r) \geq CR^{-\lambda_i} f_{i+1}(r)^{\alpha_{i-1} + 2}, \quad R \leq r \leq 3R.$$  

From (2.8), we see that

$$f'_{i+1}(r)^{2\alpha_{i-1}} f''_{i-1}(r) \geq CR^{-\lambda_i \alpha_{i-1} - \lambda_{i-1}} f_{i+1}(r)^{\alpha_{i-1} + 2} f_{i-1}(r), \quad R \leq r \leq 3R.$$  

Again multiplying this relation by $f'_{i+1}(r) \geq 0$ and integrating by parts on $[R, r]$ twice, we have

$$f'_{i+1}(r)^{2\alpha_{i-1} + 2} f_{i-1}(r) \geq CR^{-\lambda_i \alpha_{i-1} - \lambda_{i-1}} f_{i+1}(r)^{\alpha_{i-1} + 2} f_{i-1}(r), \quad R \leq r \leq 3R.$$  

From (2.8), we see that for $R \leq r \leq 3R$,

$$f'_{i+1}(r)^{2\alpha_{i-1} \alpha_{i-2} + 2\alpha_{i-2}} f''_{i-2}(r) \geq CR^{-\lambda_i \alpha_{i-1} - \lambda_{i-1} \alpha_{i-2} - \lambda_{i-2}} f_{i+1}(r)^{\alpha_{i-1} \alpha_{i-2} + 2\alpha_{i-1} \alpha_{i-2} + 2\alpha_{i-2}}.$$  

By repeating this procedure, we obtain

$$f'_{i+1}(r)^K f''_{i-(m-1)}(r) = f'_{i+1}(r)^K f''_{i+1}(r) \geq CR^{-L_i} f_{i+1}(r)^{M_i}, \quad (2.10)$$  

where

$$K_i = 2 \sum_{j=1}^{m-1} \prod_{k=j}^{m-1} \alpha_{i-k},$$

$$L_i = \sum_{j=1}^{m-1} \left\{ \lambda_{i-(j-1)} \prod_{k=j}^{m-1} \alpha_{i-k} \right\} + \lambda_{i+1},$$

$$M_i = \prod_{k=0}^{m-1} \alpha_{i-k} + 2 \sum_{j=1}^{m-1} \prod_{k=j}^{m-1} \alpha_{i-k} = A + K_i.$$  

Multiplying the inequality (2.10) by $f'_{i+1}(r) \geq 0$ and integrating on $[R, r]$, we obtain

$$f'_{i+1}(r) f_{i+1}(r)^{\frac{M_i+1}{\alpha_{i+1} \delta_i}} \geq CR^{-\frac{L_i}{\alpha_{i+1} \delta_i}}, \quad R < r \leq 3R.$$  

Since $(M_i + 1)/(K_i + 2) > 1$, we may set $(M_i + 1)/(K_i + 2) = \delta_i + 1$, $\delta_i = (A - 1)/(K_i + 2)$. Integrating this inequality on $[2R, 3R]$ we get

$$f_{i+1}(2R)^{-\delta_i} \geq CR^{-\frac{L_i}{\alpha_{i+1} \delta_i} + 1}.$$  

From (2.9) with $r = 2R$ and this inequality, we have $\bar{u}_{i+2}(R) \leq CR^\tau_i$, where

$$\tau_i = \frac{1}{\alpha_{i+1} \delta_i} \left\{ \frac{L_i}{K_i + 2} - 1 + (\lambda_{i+1} - 2)\delta_i \right\}.$$
From the definitions of $K_i, L_i, \text{and } \delta_i$, we see that

$$
\tau_i = \frac{1}{\alpha_i + \delta_i (K_i + 2)} \left[ \sum_{j=1}^{m-1} (\lambda_{i-j+1} \prod_{k=j}^{m-1} \alpha_{i-k}) - 2 \sum_{j=1}^{m-1} \prod_{k=j}^{m-1} \alpha_{i-k} \right]
+ (\lambda_{i+1} - 2) \prod_{k=0}^{m-1} \alpha_{i-k}
$$

$$
= \frac{1}{\alpha_i + 1 (A-1)} \left[ \sum_{j=0}^{m-2} \{ (\lambda_{i-j+1} - 2) \prod_{k=j}^{m-2} \alpha_{i-k} \} + (\lambda_{i+2} - 2) \alpha_{i+1} \right]
+ (\lambda_{i+1} - 2) \prod_{k=0}^{m-1} \alpha_{i-k}
$$

$$
= \frac{1}{A-1} \left[ \sum_{j=0}^{m-2} \{ (\lambda_{i-j+1} - 2) \prod_{k=j}^{m-2} \alpha_{i-k} \} + \lambda_{i+2} - 2 \right]
+ (\lambda_{i-m+3} - 2) \alpha_{i-m+2} + \lambda_{i+2} - 2
+ (\lambda_{i-m+3} - 2) \alpha_{i-m+2} + \lambda_{i+2} - 2
$$

$$
= \frac{1}{A-1} \left[ \lambda_{i+2} - 2 + \sum_{j=1}^{m-1} \{ (\lambda_{i+2+j} - 2) \prod_{k=0}^{j-1} \alpha_{i+2+k} \} \right] = \frac{\Lambda_{i+2}}{A-1}.
$$

Therefore, we obtain (2.6) by the definition of $\beta_i$.

Put $B_\rho(x) = \{ y \in \mathbb{R}^N : |y - x| \leq \rho \}$. Since $u_i, i = 1, 2, \ldots, m$, are subharmonic functions in $\mathbb{R}^N$, we have

$$
u_i(x) \leq \frac{1}{|B_{|x|/2}(x)|} \int_{B_{|x|/2}(x)} u_i(y) dy
\leq C \frac{1}{|x|^N} \int_{B_{|x|/2}(0) \setminus B_{|x|/2}(0)} u_i(y) dy
= C \frac{1}{|x|^N} \int_{|x|/2}^{3|x|/2} \int_{|y|=r} u_i(y) dS d\rho
= C \frac{1}{|x|^N} \int_{|x|/2}^{3|x|/2} r^{N-1} \tilde{u}_i(r) d\rho
\leq C \frac{1}{|x|^N} \int_{|x|/2}^{3|x|/2} r^{N-1 + \beta} d\rho
= C \frac{1}{|x|^N} \left( \frac{3|x|^N}{2} \right)^{N+\beta} - \left( \frac{|x|^N}{2} \right)^{N+\beta}.
$$

where $C > 0$ is a constant. Thus the proof is complete. \qed
Remark 2.3. In [1], M-F. Bidaut-Veron and P. Grillot have obtained important estimates of solutions on singularities for the case \( m = 2 \). In the case \( m = 2 \), by using Kelvin transformation, the estimates which they obtained become the same as those which we got in Theorem 2.2. Furthermore, it is important that these estimates hold without assumptions \( \alpha_1 \geq 1 \) and \( \alpha_2 \geq 1 \).

2.2. Radially symmetric system. In this subsection we study the nonexistence of nonnegative nontrivial radial entire solutions of (1.1). Through this subsection we always assume that \( P_i, i = 1, 2, \ldots, m \), have radial symmetry.

Theorem 2.4. Let \( N \geq 3 \). Suppose that \( P_i, i = 1, 2, \ldots, m \), satisfy

\[
P_i(r) \geq \frac{C_i}{r^{\lambda_i}}, \quad r \geq R_0 > 0,
\]  

where \( C_i > 0 \) and \( \lambda_i \) are constants. Moreover, \( \Lambda_i \) defined by (1.4) satisfy

\[
\Lambda_i \leq 0 \quad \text{for some } i \in \{1, 2, \ldots, m\}.
\]

If \( (u_1, u_2, \ldots, u_m) \) is a nonnegative radial entire solution of (1.1), then

\[
(u_1, u_2, \ldots, u_m) \equiv (0, 0, \ldots, 0).
\]

Theorem 2.5. Let \( N = 2 \). Suppose that \( P_i, i = 1, 2, \ldots, m \), satisfy

\[
P_i(r) \geq \frac{C_i}{r^{\lambda_i} \log r^M}, \quad r \geq R_0 > 1,
\]  

where \( C_i > 0 \) and \( \lambda_i, i = 1, 2, \ldots, m \), are constants. Moreover

\[
\Lambda_i \leq A - 1 \quad \text{for some } i \in \{1, 2, \ldots, m\}.
\]

If \( (u_1, u_2, \ldots, u_m) \) is a nonnegative radial entire solution of (1.1), then

\[
(u_1, u_2, \ldots, u_m) \equiv (0, 0, \ldots, 0).
\]

Theorem 2.6. Let \( N = 1 \). Suppose that \( P_i, i = 1, 2, \ldots, m \), satisfy (2.11) with some constants \( C_i > 0 \) and \( \lambda_i, i = 1, 2, \ldots, m \). Moreover

\[
\Lambda_i \leq A - 1 \quad \text{for some } i \in \{1, 2, \ldots, m\}.
\]

If \( (u_1, u_2, \ldots, u_m) \) is a nonnegative radial entire solution of (1.1), then

\[
(u_1, u_2, \ldots, u_m) \equiv (0, 0, \ldots, 0).
\]

Proof of Theorem 2.4. Let \( (u_1, u_2, \ldots, u_m) \) be a nonnegative nontrivial radial entire solution of (1.1). Then \( (u_1, u_2, \ldots, u_m) \) satisfies the system of ordinary differential equations

\[
(r^{N-1}u_i'(r))' = r^{N-1}P_i(r)u_{i+1}(r)^{\alpha_i}, \quad r > 0,
\]

\[
u_i'(0) = 0,
\]

(2.15)

Integrating (2.15) over \([0, r]\), we have

\[
u_i'(r) = r^{1-N} \int_0^r s^{N-1}P_i(s)u_{i+1}(s)^{\alpha_i}ds, \quad i = 1, 2, \ldots, m.
\]

Hence, we see that \( u_i, i = 1, 2, \ldots, m \), are nondecreasing on \( r \geq 0 \). Integrating (2.15) twice over \([R, r]\), for \( R \geq 0 \) and \( i = 1, 2, \ldots, m \), we have

\[
u_i(r) \geq \nu_i(R) + \frac{1}{N-2} \int_R^r s[1 - \left(\frac{s}{r}\right)^{N-2}]P_i(s)u_{i+1}(s)^{\alpha_i}ds,
\]

(2.16)
Since \( u_i, i = 1, 2, \ldots, m \), are nonnegative, nontrivial and nondecreasing functions, there exists an \( r_* > 0 \) such that \( u_{i_0}(r_*) > 0 \) for some \( i_0 \in \{1, 2, \ldots, m\} \). We may assume that \( r_* \geq R_0 \). We see from (2.16) with \( R = r_* \) that \( u_i(r) > 0 \) for \( r > r_*, i = 1, 2, \ldots, m \).

Using similar arguments as in the proof of Theorem 2.2, we obtain

\[
\frac{u_i(r)}{r^{\beta_i}} \leq C_i r^{\beta_i} \quad \text{at } \infty, \quad i = 1, 2, \ldots, m, \tag{2.17}
\]

where \( C_i > 0 \) are constants and \( \beta_i \) are defined by (1.5). Note that our assumption (2.12) shows \( \beta_i \leq 0 \) for some \( i \in \{1, 2, \ldots, m\} \).

If there exists an \( i_0 \in \{1, 2, \ldots, m\} \) such that \( \Lambda_{i_0} < 0 \), then we see that \( \beta_{i_0} < 0 \) in (2.17). This shows that \( u_{i_0} \) tends to 0 as \( r \to \infty \). On the other hand, from (2.16) with \( R = r_* \) we see that

\[
u_{i_0}(r) > u_{i_0}(r^*) > 0, \quad r > r_* + 1.
\]

This is a contradiction. It remains only to discuss the case that \( \Lambda_i \geq 0, i = 1, 2, \ldots, m \). From the assumption of \( \Lambda_i \), there exists an \( i_0 \in \{1, 2, \ldots, m\} \) such that \( \Lambda_{i_0} = 0 \). Without loss of generality we may assume that \( i_0 = m \), that is,

\[
\Lambda_i \geq 0, \quad i = 1, 2, \ldots, m - 1 \quad \text{and} \quad \Lambda_m = 0.
\]

From the definition of \( \beta_i \) it follows that \( \beta_i \geq 0 \) and \( \beta_m = 0 \).

We first observe that

\[
\lambda_{m-1} \leq 2 \tag{2.18}
\]

and

\[
\lambda_i \geq - \sum_{j=1}^{m-i-1} \left\{ (\lambda_i + j - 2) \prod_{k=0}^{j-1} \alpha_{i+k} \right\} + 1, \quad i = 1, 2, \ldots, m - 2. \tag{2.19}
\]

In fact, from the definition of \( \Lambda_i \), we obtain

\[
\begin{align*}
\lambda_i & \geq - \sum_{j=1}^{m-i-1} \left\{ (\lambda_{i+j} - 2) \prod_{k=0}^{j-1} \alpha_{i+k} \right\} + 2 \\
& = - \left( \sum_{j=1}^{m-i-1} + \sum_{j=m-i+1}^{m-1} \right) \left\{ (\lambda_{i+j} - 2) \prod_{k=0}^{j-1} \alpha_{i+k} \right\} - (\lambda_m - 2) \prod_{k=0}^{m-i-1} \alpha_{i+k} + 2 \\
& \equiv -S_1 - S_2 - S_3 + 2.
\end{align*}
\]

From the assumption on \( \Lambda_m \), we have

\[
\lambda_m - 2 = - \sum_{j=1}^{m-i-1} \left\{ (\lambda_{m+j} - 2) \prod_{k=1}^{j-1} \alpha_{m+k} \right\}.
\]
Substituting this relation to $S_3$ we have

$$S_3 = - \sum_{j=1}^{m-1} \left\{ (\lambda_{m+j} - 2) \prod_{k=0}^{j-1} \alpha_{m+k} \right\} \prod_{k=0}^{m-i-1} \alpha_{i+k}$$

$$= - \sum_{j=1}^{m-1} \left\{ (\lambda_{m+j} - 2) \prod_{k=0}^{j+m-i-1} \alpha_{i+k} \right\}$$

$$= - \left( \sum_{j=m-i+1}^{m-1} + \sum_{j=m}^{2m-i-1} \right) \left\{ (\lambda_{i+j} - 2) \prod_{k=0}^{j-1} \alpha_{i+k} \right\}$$

$$= - S_2 - \sum_{j=0}^{m-i-1} \left\{ (\lambda_{i+m+j} - 2) \prod_{k=0}^{j+m-1} \alpha_{i+k} \right\}$$

$$= - S_2 - S_1 \Lambda - (\lambda_i - 2) A.$$

Thus we obtain $\lambda_i \geq S_1 (A - 1) + (\lambda_i - 2) A + 2$, namely

$$0 \geq (A - 1)(\lambda_i - 2 + S_1)$$

$$= (A - 1) \left[ \lambda_i - 2 + \sum_{j=1}^{m-i-1} \left\{ (\lambda_{i+j} - 2) \prod_{k=0}^{j-1} \alpha_{i+k} \right\} \right].$$

Since $A > 1$, we see that (2.19) holds. Similarly we can get (2.18). From the above computation we see that

$$\lambda_i < - \sum_{j=1}^{m-i-1} \left\{ (\lambda_{i+j} - 2) \prod_{k=0}^{j-1} \alpha_{i+k} \right\} + 2 \quad \text{if } \Lambda_i > 0$$

and

$$\lambda_i = - \sum_{j=1}^{m-i-1} \left\{ (\lambda_{i+j} - 2) \prod_{k=0}^{j-1} \alpha_{i+k} \right\} + 2 \quad \text{if } \Lambda_i = 0.$$
Next we consider the case that $\Lambda_{m-2} = 0$. Then we see from (2.18) and (2.19) with $i = m - 2$ that
\[ \lambda_{m-1} < 2 \quad \text{and} \quad \lambda_{m-2} = -(\lambda_{m-1} - 2)\alpha_{m-2} + 2. \]
From (2.20) with $i = m - 1$ we have
\[ u_{m-1}(r) \geq C u_m(r_\ast)^{\alpha_{m-1}} \int_{r_\ast}^{r/2} s^{1-\lambda_{m-1}} ds \geq C r^{2-\lambda_{m-1}}, \quad r \geq r_1 > 2r_\ast. \]
From this estimate and (2.20) with $i = m - 2$ we obtain
\[ u_{m-2}(r) \geq C \int_{r_1}^{r/2} s^{1-\lambda_{m-2} + (2-\lambda_{m-1})\alpha_{m-2}} ds \]
\[ = C \int_{r_1}^{r/2} s^{-1} ds \geq C \log r, \quad r \geq r_2 > 2r_1. \]
On the other hand, we can see that $\beta_{m-2} = 0$ in (2.17); that is, $u_{m-2}$ is bounded near infinity. This is a contradiction.

Similarly, suppose that there exists an $i_0 \in \{1, 2, \ldots, m\}$ such that $\Lambda_{i_0} = 0$ and $\Lambda_i > 0$, $i = i_0 + 1, \ldots, m - 1$. Then we see from (2.18) and (2.20) with $i = m - 1$ that
\[ u_{m-1}(r) \geq C r^{2-\lambda_{m-1}}, \quad r \geq r_1 > 2r_\ast. \]
From this estimate, (2.19) with $i = m - 2$, (2.20) with $i = m - 2$, we have
\[ u_{m-2}(r) \geq C \int_{r_1}^{r/2} s^{1-\lambda_{m-2} + \alpha_{m-2}(2-\lambda_{m-1})} ds \]
\[ \geq C r^{2-\lambda_{m-2} + \alpha_{m-2}(2-\lambda_{m-1})}, \quad r \geq r_2 > 2r_1. \]
By repeating the above procedure, we get a sequence $\{r_j\}_{j=2}^{m-i_0-1}$ such that
\[ u_i(r) \geq C r^{\tau_i}, \quad r \geq r_j > 2r_{j-1}, \quad i = m - 2, m - 3, \ldots, i_0 + 1, \]
where
\[ \tau_i = 2 - \lambda_i + \alpha_i \tau_{i+1} \]
\[ = 2 - \lambda_i + \sum_{j=1}^{m-i-1} \left\{ (2 - \lambda_{i+j}) \prod_{k=0}^{j-1} \alpha_{i+k} \right\} > 0. \]
From (2.19) with $i = i_0$ and (2.20) with $i = i_0$, we have
\[ u_{i_0}(r) \geq C \int_{r_{m-i_0-1}}^{r/2} s^{1-\lambda_{i_0} + \alpha_{i_0} \tau_{i_0+1}} ds \]
\[ = C \int_{r_{m-i_0-1}}^{r/2} s^{-1} ds \geq C \log r, \quad r \geq r_{m-i_0} > 2r_{m-i_0-1}. \]
On the other hand, since $\Lambda_{i_0} = 0$, we have $\beta_{i_0} = 0$ in (2.17). This yields a contradiction. Thus the proof of Theorem 2.4 is complete. \qed
Proof of Theorem 2.5. Suppose to the contrary that (1.1) has a nonnegative nontrivial radial entire solution \((u_1, u_2, \ldots, u_m)\). Then \((u_1, u_2, \ldots, u_m)\) satisfies (2.15). Integrating (2.15) twice over \([0, r]\), we have

\[
u_i(r) = u_i(0) + \int_0^r s \log\left(\frac{r}{s}\right) P_i(s) u_{i+1}(s)^{\alpha_i} ds, \quad i = 1, 2, \ldots, m. \tag{2.21}
\]

Let \(r \geq e\). Then from (2.21), we have

\[
u_i(r) = u_i(0) + \int_0^e s \log\left(\frac{r}{s}\right) P_i(s) u_{i+1}(s)^{\alpha_i} ds + \int_e^r s \log\left(\frac{r}{s}\right) P_i(s) u_{i+1}(s)^{\alpha_i} ds
\]

\[
\geq u_i(0) + u_{i+1}(0)^{\alpha_i} \int_0^1 s P_i(s) ds \log r + \int_e^r s \log\left(\frac{r}{s}\right) P_i(s) u_{i+1}(s)^{\alpha_i} ds
\]

\[
\geq \tilde{C}_i \log r + \int_e^r s \log\left(\frac{r}{s}\right) P_i(s) u_{i+1}(s)^{\alpha_i} ds, \quad r \geq e,
\]

where \(i = 1, 2, \ldots, m\) and \(\tilde{C}_i \geq 0\) are constants.

Let \(u_i(r) = v_i(r) \log r\). Then from (2.22), we have

\[
v_i(r) \geq \tilde{C}_i + \int_e^r s \left(1 - \frac{\log s}{\log r}\right) P_i(s) (\log s)^{\alpha_i} v_{i+1}(s)^{\alpha_i} ds. \tag{2.23}
\]

Let \(t = \log s, \eta = \log r\), and \(v_i(r) = v_i(e^\eta) = \tilde{v}_i(\eta)\). Then (2.23) becomes

\[
\tilde{v}_i(\eta) \geq \tilde{C}_i + \int_1^e t \left(1 - \frac{t}{\eta}\right) \tilde{P}_i(t) \tilde{v}_i(t)^{\alpha_i} dt, \quad i = 1, 2, \ldots, m,
\]

where \(\tilde{P}_i, i = 1, 2, \ldots, m\), are given by \(\tilde{P}_i(t) = e^t P_i(e^t)^{\alpha_i-1}\). From (2.13), we have

\[
\tilde{P}_i(t) \geq e^t \frac{C_i}{e^{2t} (\log e^t)^{\lambda_i}} t^{\lambda_i-\alpha_i+1}, \quad t \geq \log R_0, \quad i = 1, 2, \ldots, m.
\]

From (2.14) and the definition of \(\Lambda_i\),

\[
\lambda_i - \alpha_i + 1 = \Lambda_i + 2 - \sum_{j=1}^{m-1} \left\{ (\lambda_{i+j} - 2) \prod_{k=0}^{j-1} \alpha_{i+k} \right\} - \alpha_i + 1
\]

\[
\leq 2 - \sum_{j=1}^{m-1} \left\{ (\lambda_{i+j} - \alpha_{i+j} + 1) - 2 \prod_{k=0}^{j-1} \alpha_{i+k} \right\}
\]

\[
+ A - \alpha_i - \sum_{j=1}^{m-1} \left( \alpha_{i+j} - \frac{1}{2} \prod_{k=0}^{j-1} \alpha_{i+k} \right)
\]

\[
= 2 - \sum_{j=1}^{m-1} \left( (\lambda_{i+j} - \alpha_{i+j} + 1) - 2 \prod_{k=0}^{j-1} \alpha_{i+k} \right),
\]

namely, for some \(i \in \{1, 2, \ldots, m\},

\[
(\lambda_i - \alpha_i + 1) - 2 + \sum_{j=1}^{m-1} \left( (\lambda_{i+j} - \alpha_{i+j} + 1) - 2 \prod_{k=0}^{j-1} \alpha_{i+k} \right) \leq 0.
\]

Using similar arguments as in the proof of Theorem 2.4, we obtain a contradiction. Thus the proof is complete. \(\square\)
Proof of Theorem 2.6. Let \((u_1, u_2, \ldots, u_m)\) be a nonnegative nontrivial radial entire solution of (1.1). Then by integrating (1.1) over \([0, r]\), we have
\[
u_i(r) = u_i(0) + \int_0^1 (r - s)P_i(s)u_{i+1}(s)^{\alpha_i}ds + \int_1^r (r - s)P_i(s)u_{i+1}(s)^{\alpha_i}ds
\geq u_i(0) + u_{i+1}(0)^{\alpha_i} \int_0^1 r \left(1 - \frac{s}{r}\right)P_i(s)ds + \int_1^r (r - s)P_i(s)u_{i+1}(s)^{\alpha_i}ds
\geq \bar{C}_ir + \int_1^r (r - s)P_i(s)u_{i+1}(s)^{\alpha_i}ds,
\]
where \(i = 1, 2, \ldots, m, \ r \geq 2, \) and \(\bar{C}_i \geq 0\) are constants.

Setting \(u_i(r) = rv_i(r)\) for \(r \geq 2\) and \(i = 1, 2, \ldots, m,\) we obtain
\[
v_i(r) \geq \bar{C}_i + \int_1^r s \left(1 - \frac{s}{r}\right)\tilde{P}_i(s)v_{i+1}(s)^{\alpha_i}ds,
\]
where \(\tilde{P}_i(s) = P_i(s)s^{\alpha_i-1}\). From (2.11), we have
\[
\tilde{P}_i(s) \geq \frac{C_i}{s^{\lambda_i - \alpha_i + 1}}, \quad s \geq R_0, \quad i = 1, 2, \ldots, m.
\]
Using the same computation as in the proof of Theorem 2.5, we can see that for some \(i \in \{1, 2, \ldots, m\},\)
\[
(\lambda_i - \alpha_i + 1) - 2 + \sum_{j=1}^{m-1} \left\{((\lambda_i - \alpha_i + 1) - 2) \prod_{k=0}^{j-1} \alpha_{i+k}\right\} \leq 0.
\]
From the proof of Theorem 2.4, we get a contradiction. Thus the proof is complete. \(\square\)

2.3. System (1.1) without radial symmetry. In this subsection we consider the nonexistence of nonnegative nontrivial entire solutions of (1.1) without radial symmetry. Through this subsection we always assume that \(\alpha_i \geq 1, \ i = 1, 2, \ldots, m,\) and \(A > 1.\)

Theorem 2.7. Let \(N \geq 3.\) Suppose that \(P_i, \ i = 1, 2, \ldots, m,\) satisfy
\[
\liminf_{|x| \to \infty} |x|^\lambda_i P_i(x) > 0, \quad (2.24)
\]
where \(\lambda_i, \ i = 1, 2, \ldots, m,\) are constants. Also \(\Lambda_i \leq 0\) for some \(i \in \{1, 2, \ldots, m\}.\) If \((u_1, u_2, \ldots, u_m)\) is nonnegative entire solution of (1.1), then
\[
(u_1, u_2, \ldots, u_m) \equiv (0, 0, \ldots, 0).
\]

Theorem 2.8. Let \(N = 2.\) Suppose that \(P_i, \ i = 1, 2, \ldots, m,\) satisfy
\[
\liminf_{|x| \to \infty} |x|^2 (\log |x|)^\lambda_i P_i(x) > 0, \quad (2.25)
\]
where \(\lambda_i\) are constants. Moreover \(\Lambda_i \leq A - 1\) for some \(i \in \{1, 2, \ldots, m\}.\) If \((u_1, u_2, \ldots, u_m)\) is nonnegative entire solution of (1.1), then
\[
(u_1, u_2, \ldots, u_m) \equiv (0, 0, \ldots, 0).
\]

Theorem 2.9. Let \(N = 1.\) Suppose that \(P_i,\) satisfy (2.24) with some constants \(\lambda_i, \ i = 1, 2, \ldots, m.\) Moreover \(\Lambda_i \leq A - 1\) for some \(i \in \{1, 2, \ldots, m\}.\) If \((u_1, u_2, \ldots, u_m)\) is nonnegative entire solution of (1.1), then
\[
(u_1, u_2, \ldots, u_m) \equiv (0, 0, \ldots, 0).
\]
Suppose that (2.24) holds. Then there exist some constants $C_i > 0$ and $R_0 > 0$ such that

$$P_i(x) \geq \frac{C_i}{|x|^{\lambda_i}}, \quad |x| \geq R_0, \quad i = 1, 2, \ldots, m.$$  

So we can see that $\hat{P}_i$ defined by (2.1) satisfy

$$\hat{P}_i(r) \geq \frac{C_i}{r^{\lambda_i}}, \quad r \geq R_0.$$  

Similarly, suppose that (2.25) holds. Then $\hat{P}_i$ satisfy

$$\hat{P}_i(r) \geq \frac{C_i}{r^2 (\log r)^{\lambda_i}}, \quad r \geq R_0 > 1,$$

where $i = 1, 2, \ldots, m$, and $C_i > 0$ are some constants.

The proof of Theorem 2.7 follows from Lemma 2.1, Theorem 2.2 and the proof of Theorem 2.4. Similarly, the proofs of Theorems 2.8 and 2.9 follow from Lemma 2.1 and the proofs of Theorems 2.5 and 2.6, respectively.

**Remark 2.10.** When $m = 2$, our nonexistence results (Theorems 2.7–2.9) reduce to those obtained in [12]. However, the proofs presented here are simpler than in [12].  

3. Existence results

In this section we consider existence of positive radial entire solutions of the semilinear elliptic system

$$\begin{align*}
\Delta u_1 &= P_1(|x|)u_2^{\alpha_1}, \\
\Delta u_2 &= P_2(|x|)u_3^{\alpha_2}, \\
& \quad \vdots \\
\Delta u_m &= P_m(|x|)u_{m+1}^{\alpha_m}, \quad u_{m+1} = u_1.
\end{align*}$$  

(3.1)

Through this section, we assume that $P_i(r), r = |x|, i = 1, 2, \ldots, m$, are nonnegative continuous functions and $\alpha_i > 0$ are constants satisfying $A > 1$.

**Theorem 3.1.** Let $N \geq 3$. Suppose that $P_i$ satisfy

$$P_i(r) \leq \frac{C_i}{r^{\lambda_i}}, \quad r \geq R_0 > 0,$$

(3.2)

where $i = 1, 2, \ldots, m$, and $C_i > 0$, $\lambda_i$ are constants. Moreover $\Lambda_i > 0$, $i = 1, 2, \ldots, m$. Then (3.1) has infinitely many positive radial entire solutions.

**Theorem 3.2.** Let $N = 2$. Suppose that $P_i$ satisfy

$$P_i(r) \leq \frac{C_i}{r^2 (\log r)^{\lambda_i}}, \quad r \geq R_0 > 1,$$

(3.3)

where $i = 1, 2, \ldots, m$, and $C_i > 0$ and $\lambda_i$ are constants. Moreover

$$\Lambda_i > A - 1, \quad i = 1, 2, \ldots, m.$$  

(3.4)

Then (3.1) has infinitely many positive radial entire solutions.

**Theorem 3.3.** Let $N = 1$. Suppose that $P_i$ satisfy (3.2) with some constants $C_i > 0$ and $\lambda_i$, $i = 1, 2, \ldots, m$. Moreover $\Lambda_i > A - 1$, $i = 1, 2, \ldots, m$. Then (3.1) has infinitely many positive entire solutions.

We give an example that shows the sharpness of our results.
Example. Let us consider the elliptic system

\[
\Delta u_1 = \frac{1}{(1 + |x|)^{\lambda_1}} u_2^\alpha_1,
\]
\[
\Delta u_2 = \frac{1}{(1 + |x|)^{\lambda_2}} u_3^\alpha_2,
\]
\[
\vdots
\]
\[
\Delta u_m = \frac{1}{(1 + |x|)^{\lambda_m}} u_1^\alpha_m,
\]

where \( x \in \mathbb{R}^N, N \geq 3 \), and \( \alpha_i > 0, i = 1, 2, \ldots, m \), are constants satisfying \( \alpha_1 \alpha_2 \cdots \alpha_m > 1 \). We can completely characterize the existence of positive radial entire solutions of this system in terms of \( \alpha_i \) and \( \lambda_i, i = 1, 2, \ldots, m \). In fact, we can see that the inequalities

\[
\frac{C_i}{|x|^\lambda_i} \leq \frac{1}{(1 + |x|)^{\lambda_i}} \leq \tilde{C}_i |x|^{-\lambda_i}, \quad |x| \geq 1, \quad i = 1, 2, \ldots, m
\]

hold for some constants \( C_i > 0 \) and \( \tilde{C}_i > 0, i = 1, 2, \ldots, m \). Then, from Theorem 2.4 and Theorem 3.1, a necessary and sufficient condition for (3.5) to have positive radial entire solution is

\[
\Lambda_i > 0, \quad i = 1, 2, \ldots, m.
\]

Proof of Theorem 3.1. Without loss of generality, we assume that \( R_0 = 1 \) in (3.2). We first observe that \((u_1, u_2, \ldots, u_m)\) is a positive radial entire solution of (3.1) if and only if the function \((v_1(r), v_2(r), \ldots, v_m(r)) = (u_1(x), u_2(x), \ldots, u_m(x))\), \( r = |x| \), satisfies the system of second order ordinary differential equations

\[
v^{1-N(r^{N-1}v_i')'} = P_i(r)v_i^{\alpha_i}, \quad r > 0,
\]
\[
v_i'(0) = 0,
\]

where \( i = 1, 2, \ldots, m \), and \( ' = d/dr \). Integrating (3.6) twice, we obtain the following system of integral equations equivalent to (3.6):

\[
v_i(r) = a_i + \frac{1}{N-2} \int_0^r s [1 - \left(\frac{2}{r}\right)^{N-2}] P_i(s)v_i^{\alpha_i}(s)ds,
\]

where \( r \geq 0, i = 1, 2, \ldots, m \), and \( a_i = v_i(0) \). Therefore, it suffices to solve (3.7). Choose constants \( a_i > 0, i = 1, 2, \ldots, m \), so that

\[
\frac{(2a_{i+1})^{\alpha_i}}{N-2} \int_0^1 sP_i(s)ds \leq \frac{a_i}{2},
\]
\[
\frac{C_i(2a_{i+1})^{\alpha_i}}{(N-2)(2 - \lambda_i + \alpha_i\beta_{i+1})} \leq \frac{a_i}{2}
\]

where \( \beta_i, i = 1, 2, \ldots, m \), are defined by (1.5). It is possible to choose such \( a_i \)'s by the assumption \( A > 1 \). We note that \( 2 - \lambda_i + \alpha_i\beta_{i+1} = \beta_i \) by the definitions of \( \Lambda_i \) and \( \beta_i \). Define the functions \( F_i, i = 1, 2, \ldots, m \), by

\[
F_i(r) = \begin{cases} 2a_i & \text{for } 0 \leq r \leq 1, \\ 2a_i r^{\beta_i} & \text{for } r \geq 1. \end{cases}
\]
We regard the space $(C[0, \infty))^m$ as a Fréchet space equipped with the topology of uniform convergence of functions on each compact subinterval of $[0, \infty)$. Let \( X \subset (C[0, \infty))^m \) denote the subset defined by
\[
X = \{(v_1, v_2, \ldots, v_m) \in (C[0, \infty))^m : a_i \leq v_i(r) \leq F_i(r), \ r \geq 0, \ 1 \leq i \leq m \}.
\]
Clearly, \( X \) is a non-empty closed convex subset of \((C[0, \infty))^m\). Define the mapping \( \mathcal{F} : X \to (C[0, \infty))^m \) by \( \mathcal{F}(v_1, v_2, \ldots, v_m) = (\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_m) \), where
\[
\tilde{v}_i(r) = a_i + \frac{1}{N - 2} \int_0^r s \left[ 1 - \left( \frac{s}{r} \right)^{N - 2} \right] P_i(s) v_{i+1}(s)^{\alpha_i} \, ds, \quad r \geq 0.
\]
To apply the Schauder-Tychonoff fixed point theorem, we show that \( \mathcal{F} \) is a continuous mapping from \( X \) into itself such that \( \mathcal{F}(X) \) is relatively compact.

(I) \( \mathcal{F} \) maps \(X\) into itself. Let \((v_1, v_2, \ldots, v_m) \in X\). Clearly, \( \tilde{v}_i \geq a_i, \ i = 1, 2, \ldots, m \).

For \(0 \leq r \leq 1\), we have
\[
\tilde{v}_i(r) \leq a_i + \frac{1}{N - 2} \int_0^r s P_i(s) v_{i+1}(s)^{\alpha_i} \, ds \leq a_i + \frac{(2a_{i+1})^{\alpha_i}}{N - 2} \int_0^1 s P_i(s) \, ds \leq a_i + \frac{a_i}{2} < 2a_i, \quad i = 1, 2, \ldots, m.
\]

For \(r \geq 1\), from (3.2), we have
\[
\tilde{v}_i(r) \leq a_i + \frac{1}{N - 2} \int_0^1 s P_i(s) v_{i+1}(s)^{\alpha_i} \, ds + \frac{1}{N - 2} \int_1^r s P_i(s) v_{i+1}(s)^{\alpha_i} \, ds \\
\leq \frac{3a_i}{2} + \frac{(2a_{i+1})^{\alpha_i} C_i}{N - 2} \int_1^r s^{-\lambda_i + \alpha_i \beta_{i+1}} \, ds \\
\leq \frac{3a_i}{2} + \frac{(2a_{i+1})^{\alpha_i} C_i}{(N - 2)(2 - \lambda_i + \alpha_i \beta_{i+1})} r^{2 - \lambda_i + \alpha_i \beta_{i+1}} \\
\leq \frac{3a_i}{2} + a_i r^{\beta_i} \leq 2a_i r^{\beta_i}, \quad i = 1, 2, \ldots, m.
\]

Therefore, \( \mathcal{F}(X) \subset X \).

(II) \( \mathcal{F} \) is continuous. Let \( \{(v_{1,l}, v_{2,l}, \ldots, v_{m,l})\}_{l=1}^\infty \) be a sequence in \(X\) which converges to \((v_1, v_2, \ldots, v_m) \in X\) uniformly on each compact subinterval of \([0, \infty)\). Then
\[
|\tilde{v}_{i,l}(r) - \tilde{v}_i(r)| \leq \frac{1}{N - 2} \int_0^r s \left[ 1 - \left( \frac{s}{r} \right)^{N - 2} \right] P_i(s) |v_{i+1,l}(s)^{\alpha_i} - v_{i+1}(s)^{\alpha_i}| \, ds \\
\leq \frac{1}{N - 2} \int_0^r s P_i(s) |v_{i+1,l}(s)^{\alpha_i} - v_{i+1}(s)^{\alpha_i}| \, ds, \quad i = 1, 2, \ldots, m.
\]

Since the functions \( h_{i,l}(s) = s P_i(s) |v_{i+1,l}(s)^{\alpha_i} - v_{i+1}(s)^{\alpha_i}|, \ l \in \mathbb{N}, \ 1 \leq i \leq m \), satisfy \( h_{i,l}(s) \leq 2s P_i(s) F_{i+1}(s)^{\alpha_i}, \ s \geq 0 \), and \( \{h_{i,l}(s)\}_{l=1}^\infty, \ i = 1, 2, \ldots, m \), converge to 0 at every point \( s \), the Lebesgue dominated convergence theorem implies that \( \{\tilde{v}_{i,l}\}_{l=1}^\infty, \ i = 1, 2, \ldots, m \), converge to \( \tilde{v}_i \) uniformly on each compact subinterval of \([0, \infty)\). These imply the continuity of \( \mathcal{F} \).

(III) \( \mathcal{F}(X) \) is relatively compact. It suffices to show the local equicontinuity of \( \mathcal{F}(X) \), since \( \mathcal{F}(X) \) is locally uniformly bounded by the fact that \( \mathcal{F}(X) \subset X \). Let
where $\beta$ functions $F$ are relatively compact. We will verify that 
\[
\{v_1, v_2, \ldots, v_m\} \in X \). Hence the relative compactness of $\mathcal{F}(X)$ is shown by the Ascoli-Arzelà theorem.

Therefore, applying the Schauder-Tychonoff fixed point theorem, there exists an element $(v_1, v_2, \ldots, v_m) \in X$ such that $(v_1, v_2, \ldots, v_m) = \mathcal{F}(v_1, v_2, \ldots, v_m)$, that is, $(v_1, v_2, \ldots, v_m)$ satisfies the system of integral equations (3.7). The function $(u_1(x), u_2(x), \ldots, u_m(x)) = (v_1(|x|), v_2(|x|), \ldots, v_m(|x|))$ then gives a solution of (3.6). Since infinitely many $(a_1, a_2, \ldots, a_m)$ satisfy (3.8), we can construct an infinitude of positive radial entire solutions of (3.1). This completes the proof. \[
\]

**Proof of Theorem 3.2.** Without loss of generality, we may assume that $R_0 = e$ in (3.3). As before, it suffices to solve the following system of integral equations:

\[
\begin{align*}
v_i(r) &= a_i + \int_0^r s \log\left(\frac{r}{s}\right) P_i(s) v_{i+1}(s) \alpha_i ds, \quad r \geq 0, \quad i = 1, 2, \ldots, m,
\end{align*}
\]

where $a_i = v_i(0)$. Choose constants $a_i > 0$ so that

\[
\begin{align*}
(2a_{i+1})^{\alpha_i} e \int_0^r P_i(s) ds & \leq \frac{a_i}{2}, \\
\frac{(2a_{i+1})^{\alpha_i} C_i}{1 - \lambda_i + \alpha_i \beta_i} & \leq \frac{a_i}{2},
\end{align*}
\]

where $\beta_i, i = 1, 2, \ldots, m$, are defined by (1.5). It is possible to choose such $a_i$’s by the assumption $A > 1$. We notice that $\beta_i > 1$ by the assumption (3.4). Define the functions $F_i, i = 1, 2, \ldots, m$, by

\[
F_i(r) = \begin{cases} 2a_i & \text{for } 0 \leq r \leq e, \\ 2a_i (\log r)^{\beta_i} & \text{for } r \geq e. \end{cases}
\]

Consider the set

\[
Y = \{(v_1, v_2, \ldots, v_m) \in (C[0, \infty))^m : a \leq v_i(r) \leq F_i(r), r \geq 0, 1 \leq i \leq m\},
\]

which is a closed convex subset of $(C[0, \infty))^m$. Define the mapping $\mathcal{F} : Y \rightarrow (C[0, \infty))^m$ by $\mathcal{F}(v_1, v_2, \ldots, v_m) = (\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_m)$, where

\[
\tilde{v}_i(r) = a_i + \int_0^r s \log\left(\frac{r}{s}\right) P_i(s) v_{i+1}(s) \alpha_i ds, \quad r \geq 0, \quad i = 1, 2, \ldots, m.
\]

We will verify that $\mathcal{F}$ is a continuous mapping from $Y$ into itself such that $\mathcal{F}(Y)$ is relatively compact.

We first show that $\mathcal{F}$ maps $Y$ into itself. Let $(v_1, v_2, \ldots, v_m) \in Y$. It is clear that $\tilde{v}_i \geq a_i, i = 1, 2, \ldots, m$. Let $0 \leq r < e$. Then, using the inequality $0 \leq s \log(r/s) \leq r/e$ for $0 \leq s \leq r$, we have

\[
\tilde{v}_i(r) \leq a_i + \frac{r}{e} \int_0^r P_i(s) v_i(s) \alpha_i ds \\
\leq a_i + (2a_{i+1})^{\alpha_i} \int_0^e P_i(s) ds \\
\leq a_i + \frac{a_i}{2} < 2a_i, \quad i = 1, 2, \ldots, m.
\]
Let \( r \geq e \). Then we write
\[
\tilde{v}_i(r) = a_i + \left( \int_1^e + \int_e^r \right) s \log(r/s) P_i(s) v_{i+1}(s)^{\alpha_i} ds = a_i + I_1 + I_2 + I_3.
\]
The inequality \( 0 \leq s \log(r/s) \leq \log r \) for \( 0 \leq s \leq 1 \) implies that
\[
I_1 \leq \int_0^1 P_i(s)v_{i+1}(s)^{\alpha_i} ds \log r \leq (2a_{i+1})^{\alpha_i} e \int_0^1 P_i(s)ds \log r.
\]
The integrals \( I_2 \) and \( I_3 \) are estimated as follows:
\[
I_2 \leq \int_1^e s P_i(s)v_{i+1}(s)^{\alpha_i} ds \log r
\]
\[
\leq (2a_{i+1})^{\alpha_i} \int_1^e s P_i(s)ds \log r
\]
\[
\leq (2a_{i+1})^{\alpha_i} e \int_1^e P_i(s)ds \log r;
\]
\[
I_3 \leq \int_e^r s P_i(s)v_{i+1}(s)^{\alpha_i} ds \log r
\]
\[
\leq (2a_{i+1})^{\alpha_i} C_i \int_1^r s^{-1} (\log s)^{-\lambda_i + \alpha_i \beta_{i+1}} ds \log r
\]
\[
= (2a_{i+1})^{\alpha_i} C_i \int_1^{\log r} t^{-\lambda_i + \alpha_i \beta_{i+1}} dt \log r
\]
\[
\leq \frac{(2a_{i+1})^{\alpha_i} C_i}{1 - \lambda_i + \alpha_i \beta_{i+1}} (\log r)^{2 - \lambda_i + \alpha_i \beta_{i+1}}
\]
\[
\leq \frac{a_i}{2} (\log r)^{\beta_i}.
\]
From (3.9) and (3.10), we have
\[
I_1 + I_2 \leq (2a_{i+1})^{\alpha_i} e \int_0^1 P_i(s)ds \log r \leq \frac{a_i}{2} (\log r)^{\beta_i}.
\]
Thus by (3.11) and (3.12) we obtain \( \tilde{v}_i(r) \leq 2a_i (\log r)^{\beta_i}, \) \( i = 1, 2, \ldots, m. \) Therefore, \( F(v_1, v_2, \ldots, v_m) \in Y. \)

The continuity of \( F \) and the relative compactness of \( F(Y) \) can be verified in a routine manner. Thus there exists an element \( (v_1, v_2, \ldots, v_m) \in Y \) such that \( (v_1, v_2, \ldots, v_m) = F(v_1, v_2, \ldots, v_m) \) by the Schauder-Tychonoff fixed point theorem. It is clear that this \( (v_1, v_2, \ldots, v_m) \) gives rise to a positive radial entire solution \( (u_1(x), u_2(x), \ldots, u_m(x)) = (v_1(|x|), v_2(|x|), \ldots, v_m(|x|)) \) of (3.1). \( \square \)

The proof of Theorem 3.3 is the same as that of Theorem 3.1. So we leave the proof to the reader.

4. Liouville type theorem

Consider the semilinear elliptic system
\[
\Delta u_1 = P_1(x) u_2^{\alpha_1},
\]
\[
\Delta u_2 = P_2(x) u_3^{\alpha_2},
\]
\[
\vdots
\]
\[
\Delta u_m = P_m(x) u_{m+1}^{\alpha_m}, \quad u_{m+1} = u_1,
\]
\[
(4.1)
\]
where \( x \in \mathbb{R}^N \), \( N \geq 3 \) and \( m \geq 2 \) are integers and \( \alpha_i > 0 \), \( i = 1, 2, \ldots, m \), are constants satisfying \( \alpha_1 \alpha_2 \cdots \alpha_m > 1 \). Suppose that

\[
P_i(x) \geq \frac{C_i}{|x|^{\lambda_i}}, \quad |x| \geq x_0 > 0, \quad i = 1, 2, \ldots, m,
\]

hold for some constants \( C_i > 0 \) and \( \lambda_i \in \mathbb{R} \), satisfying \( \Lambda_i \leq 0 \) for some \( i \in \{1, 2, \ldots, m\} \). If, in addition, \( \alpha_i \geq 1 \), \( i = 1, 2, \ldots, m \), then as studied in Sections 2.2 and 2.3 one can conclude from Theorems 2.4 and 2.7 that system (4.1) has no nonnegative nontrivial entire solutions. However, if at least one of \( \alpha_i \), \( i \in \{1, 2, \ldots, m\} \), is less than 1, then one cannot derive any information about the nonnegative nontrivial entire solutions without radial symmetry. When \( \alpha_1 \alpha_2 \cdots \alpha_m > 1 \) and the same hypothesis of Theorem 2.7 hold, does not (4.1) possess a nonnegative nontrivial entire solutions? To give a partial answer this question we prove a Liouville type theorem for nonnegative entire solutions of (4.1). Our result is as follows:

**Theorem 4.1.** Let \( N \geq 3 \). Suppose that

\[
\liminf_{|x| \to \infty} |x|^N P_i(x) > 0, \quad i = 1, 2, \ldots, m, \tag{4.2}
\]

hold for some constants \( \lambda_i, i = 1, 2, \ldots, m \), and there exists an \( i_0 \in \{1, 2, \ldots, m\} \) such that \( \Lambda_{i_0} \leq 0 \). If \( (u_1, u_2, \ldots, u_m) \) is a nonnegative entire solution of (4.1) satisfying

\[
u_i(x) = O(\exp |x|^\rho) \text{ as } |x| \to \infty \quad \text{for some } \rho > 0,
\]

then \( (u_1, u_2, \ldots, u_m) \equiv (0, 0, \ldots, 0) \).

The next lemma is needed in proving Theorem 4.1.

**Lemma 4.2.** Let \( (u_1, u_2, \ldots, u_m) \) be a nonnegative entire solution of (4.1), and \( b \in (0, 1) \) be a constant. Then its spherical mean \((\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m)\) satisfies the ordinary differential inequalities

\[
\bar{u}_i'(r) \geq C_i r P_i(r) \bar{u}_{i+1}(br)^{\alpha_i}, \quad r > 0, \quad i = 1, 2, \ldots, m, \tag{4.4}
\]

\[
\bar{u}_i'(0) = 0,
\]

where \( C_i = C_i(N, \alpha, b) > 0 \), \( i = 1, 2, \ldots, m \), are constants and

\[
P_i(r) = \min_{|x| \leq r} P_i(x), \quad r \geq 0, \quad i = 1, 2, \ldots, m.
\]

To prove this lemma, we present the following lemma; see [4, p.244] or [11, p.225].

**Lemma 4.3.** Let \( D \) be a domain in \( \mathbb{R}^N \). Suppose that \( \sigma > 0 \) is a constant, and \( x_0 \in D \) and \( r > 0 \) satisfy \( B_{2r}(x_0) \equiv \{x \in \mathbb{R}^N; |x - x_0| \leq 2r\} \subset D \). Then, we can find a constant \( C = C(N, \sigma) > 0 \) satisfying

\[
\left(\max_{B_{\sigma r}(x_0)} u\right)^\sigma \leq \frac{C}{r^N} \int_{B_{2r}(x_0)} u^\sigma \, dx,
\]

for any function \( u \in C^2(D) \) satisfying \( u \geq 0, \Delta u \geq 0 \) in \( D \).

**Proof of Lemma 4.2.** Let \( (u_1, u_2, \ldots, u_m) \) be a nonnegative entire solution of (4.1). By taking the mean value of (4.1), we have

\[
(r^N \bar{u}_i'(r))' = \frac{1}{\omega_N} \int_{|z|=r} P_i(x) u_{i+1}(x)^{\alpha_i} \, dS, \quad r \geq 0, \quad i = 1, 2, \ldots, m. \tag{4.5}
\]
Since an integration of (4.5) shows that \(\bar{u}_i(r)\), \(i = 1, 2, \ldots, m\), are nondecreasing on \([0, \infty)\), we may assume that \(b > 1/2\) in (4.4). Put \(b = 1 - a\), \(a \in (0, 1/2)\). Integrating (4.5) over \([0, r]\), we have

\[
\bar{u}_i'(r) = \frac{1}{\omega_{N-1} r \alpha} \int_{|x| \leq r} P_i(x) u_{i+1}(x)^{\alpha} \, dx \geq \frac{P_i(r)}{\omega_{N-1} \alpha} \int_{|x| \leq r} u_{i+1}(x)^{\alpha} \, dx. \tag{4.6}
\]

Let \(r > 0\) be fixed. We take \(y_i + 1 \in \mathbb{R}^N\), \(i = 1, 2, \ldots, m\), such that

\[
u_{i+1}(y_i + 1) = \max_{|x| = (1 - a)r} u_{i+1}(x) \quad (= \max_{|x| \leq (1 - a)r} u_{i+1}(x),
\]

and take \(z_{i+1} \in \mathbb{R}^N\), \(i = 1, 2, \ldots, m\), such that \(z_{i+1} = My_{i+1}, 0 < M < 1\), and \(|y_{i+1} - z_{i+1}| = ar\). Then we can see that

\[
\int_{|x| \leq r} u_{i+1}(x)^{\alpha} \, dx \geq \int_{|x - z_{i+1}| \leq 2ar} u_{i+1}(x)^{\alpha} \, dx, \quad i = 1, 2, \ldots, m,
\]

and using Lemma 4.3, we obtain

\[
\int_{|x - z_{i+1}| \leq 2ar} u_{i+1}(x)^{\alpha} \, dx \geq C_i r^N \left( \max_{|x - z_{i+1}| \leq ar} u_{i+1}(x) \right)^{\alpha_i}
\]

\[
= C_i r^N (u_{i+1}(y_i + 1))^{\alpha_i}
\]

\[
= C_i r^N \left( \max_{|x| = (1 - a)r} u_{i+1}(x) \right)^{\alpha_i}
\]

\[
\geq C_i r^N \bar{u}_{i+1}((1 - a)r)^{\alpha_i},
\]

where \(i = 1, 2, \ldots, m\) and \(C_i = C_i(N, \alpha_i, a) > 0\) are constants. From this estimate and (4.6) we obtain (4.4). Thus the proof is complete. \(\square\)

**Proof of Theorem 4.1.** Assume that (4.2) holds. Then there exist positive constants \(C_i > 0\), \(i = 1, 2, \ldots, m\), and \(R_0 > 0\) such that

\[
P_i(x) \geq \frac{C_i}{|x|^{\alpha_i}} \quad \text{for } |x| \geq R_0.
\]

So that

\[
P_{i+1}(r) \geq \frac{C_i}{r^N} \quad \text{for } r \geq R_0. \tag{4.7}
\]

Without loss of generality we may assume that \(i_0 = 1\). Suppose to the contrary that (4.1) has a nonnegative nontrivial entire solution \((u_1, u_2, \ldots, u_m)\) satisfying (4.3) with \(i_0 = 1\). Then, by Lemma 4.2, its spherical mean \((\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m)\) satisfies (4.4).

We choose the constant \(b < 1\) in (4.4) such that \(1 < b^{-2m} < A^{1/\rho}\), where \(\rho\) is the number appearing in (4.3). We first show that

\[
\lim_{r \to \infty} \bar{u}_1(r) = \infty, \tag{4.8}
\]

and

\[
\bar{u}_1(lr) \geq L \bar{u}_1(r)^A \quad \text{near } + \infty \tag{4.9}
\]

where \(L > 0\) is some constant and \(l = b^{-2m}\).

Integrating (4.4) on \([0, r]\), we have

\[
\bar{u}_i(r) \geq \bar{u}_i(0) + C_i \int_0^r s P_{i+1}(s)^{\alpha_i} \, ds, \quad r \geq 0, \quad i = 1, 2, \ldots, m. \tag{4.10}
\]
Since \((u_1, u_2, \ldots, u_m)\) is nonnegative and nontrivial, for some point \(x_* \in \mathbb{R}^N\) we have \(u_i(x_*) > 0\) for some \(i \in \{1, 2, \ldots, m\}\); that is \(\bar{u}_i(r_* > 0, \ r_* = |x_*|.\) We may assume that \(r_* \geq R_0\). Therefore, we see from (4.10) that \(\bar{u}_i(r) > 0\) for \(r > r_*\).

Let \(r \geq r_*/b\) be large enough. Integrating (4.4) over \([br, r]\), from (4.7) and the monotonicity of \(u_i\) we have

\[
\bar{u}_i(r) - \bar{u}_i(br) \geq \tilde{C}_i \int_{br}^{r} sP(r)(bs)^{a_i}ds
\]

\[
\geq \tilde{C}_i \bar{u}_i(b^2r)^{a_i} \int_{br}^{r} s^{1-\lambda_i}ds
\]

\[
= \tilde{C}_i \frac{1 - b^{2-\lambda_i}}{2 - \lambda_i} \bar{u}_i(b^2r)^{\alpha_i r^{2-\lambda_i}},
\]

namely,

\[
\bar{u}_i(r) \geq C r^{2-\lambda_i} \bar{u}_i(b^2r)^{\alpha_i}, \quad i = 1, 2, \ldots, m,
\]

(4.11)

where \(C\) is some positive constant. Notice that (4.11) is still valid even though \(\lambda_i = 2\) (with \(C = \tilde{C}_i \log b^{-1}\)).

From (4.11), by iteration, it follows that

\[
\bar{u}_1(r) \geq C r^{-\Lambda_1} \bar{u}_1(b^{2m}r)^{A}, \quad r > \frac{r_*}{b^{2m}},
\]

where \(C > 0\) is some constant. From the assumption \(\Lambda_1 \leq 0\), we obtain (4.9).

The inequality (4.4) with \(i = 1\) and (4.11) imply

\[
\bar{u}_1'(r) \geq C r^\tau P_1(r) \bar{u}_1(b^{2(m-1)}r+1)^{A},
\]

(4.12)

where

\[
\tau = 1 + \sum_{j=1}^{m-1} \left\{ (2 - \lambda_{1+j}) \prod_{k=0}^{j-1} \alpha_{1+k} \right\} = \lambda_1 - 1 - \Lambda_1.
\]

Integrating (4.12) over \([r_1, r]\), \(b^{2(m-1)+1}r_1 > r_*\), we have

\[
\bar{u}_1(r) \geq \bar{u}_1(r_1) + C \int_{r_1}^{r} s^{\tau} P(r)(bs)^{a_1(b^2r)^{\alpha_1 s^{1-\lambda_1}}ds
\]

\[
\geq \bar{u}_1(r_1) + C \bar{u}_1(b^{2(m-1)+1}r_1)^{A} \int_{r_1}^{r} s^{\tau-\lambda_1}ds.
\]

From the assumption \(\Lambda_1 \leq 0\), we can see that \(\tau - \lambda_1 \geq -1\), which implies that (4.8) holds. Let \(\tilde{r}\) be large so that

\[
L \int\limits_{\tilde{r}}^{r} \bar{u}_1(\tilde{r}) \geq e,
\]

(4.13)

and

\[
\bar{u}_1(r) \geq L \bar{u}_1(r)^{A}, \quad r \geq \tilde{r},
\]

(4.14)
where $L > 0$ is the constant appearing in (4.9). It is possible to choose such an $\tilde{r}$ by (4.8) and (4.9). For $k \in \mathbb{N}$, from (4.14) we obtain
\[
\bar{u}_1(l^k \tilde{r}) \geq L^{1+A} \bar{u}_1(l^{k-2} \tilde{r}) A^2 \\
\geq \ldots \\
\geq L^{1+A+\ldots+A^{k-1}} \bar{u}_1(\tilde{r}) A^k \\
= L^{-\frac{1}{1+\rho}} \left[ L^{\frac{1}{1+\rho}} \bar{u}_1(\tilde{r}) \right]^A.
\]
Hence we see from (4.13) that
\[
\bar{u}_1(l^k \tilde{r}) \geq L^{-\frac{1}{1+\rho}} \exp A^k. \tag{4.15}
\]
Let $r \geq l \tilde{r}$. Then we can find that there exists a unique positive integer $k = k(r)$ such that $l^k \tilde{r} \leq r < l^{k+1} \tilde{r}$. Thus $k$ satisfies
\[
k > \frac{\log r - \log \tilde{r}}{\log l} - 1.
\]
It follows therefore from (4.15) that
\[
\bar{u}_1(r) \geq \bar{u}_1(l^k \tilde{r}) \geq L^{-\frac{1}{1+\rho}} \exp A^k \\
\geq L^{-\frac{1}{1+\rho}} \exp \left\{ A^{\frac{\log r - \log \tilde{r}}{\log l}} \right\} \exp A^{\frac{\log r - \log \tilde{r}}{\log l}} \\
= L^{-\frac{1}{1+\rho}} \exp \left\{ A^{\frac{\log r - \log \tilde{r}}{\log l}} \right\}. \tag{4.16}
\]
On the other hand, because $u_1(x) = O(\exp |x|^\rho)$ as $|x| \to \infty$, we obviously have
\[
\bar{u}_1(r) = O(\exp r^\rho) \text{ as } r \to \infty.
\]
Since $\log A/\log l = \log A/\log b^{-2m} > \rho$ from our choice of $b$, (4.16) gives a contradiction. The proof is complete. \(\square\)

**Remark 4.4.** (i) When $m = 2$, Theorem 4.1 reduces to [13, Theorem 1]. However, the proof given here is simpler than in [13].

(ii) As described in Remark 2.3, in the case $m = 2$, the nonnegative entire solution $(u_1, u_2)$ of (4.1) satisfies
\[
u_1(x) \leq C|x|^{\alpha_1} \text{ and } u_2(x) \leq C|x|^{\beta_2} \text{ at } \infty
\]
without the assumptions $\alpha_1 \geq 1$ and $\alpha_2 \geq 1$ under the condition (4.2). From this fact and (4.8), we can see that if $(\lambda_1, \lambda_2)$ satisfies $\Lambda_1 \leq 0$, then the system (4.1) does not have nonnegative nontrivial entire solutions. Therefore, we find that Theorem 2.7 holds without the assumptions $\alpha_i \geq 1$, $i = 1, 2, \ldots, m$.

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References


Tomomitsu Teramoto
Faculty of Economics, Management & Information Science, Onomichi University, 1600 Hisayamada, Onomichi Hiroshima, 722-8506, Japan
E-mail address: teramoto@onomichi-u.ac.jp