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ASYMPTOTIC PROPERTIES, NONOSCILLATION, AND STABILITY FOR SCALAR FIRST ORDER LINEAR AUTONOMOUS NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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Abstract. We study scalar first order linear autonomous neutral delay differential equations with distributed type delays. This article presents some new results on the asymptotic behavior, the nonoscillation and the stability. These results are obtained via a real root (with an appropriate property) of the characteristic equation. Applications to the special cases such as (non-neutral) delay differential equations are also presented.

1. Introduction

Neutral delay differential equations are differential equations depending on past and present values, which involve derivatives with delays as well as the unknown function itself. Besides its theoretical interest, the study of such equations has great importance in various applications in natural sciences and technology. For the basic theory of neutral delay differential equations, the reader is referred to the books by Diekmann *et al.* [2], Hale [10], and Hale and Verduyn Lunel [11].

Driver, Sasser and Slater [6] have obtained some significant results on the asymptotic behavior, the nonoscillation and the stability for a first order linear delay differential equation with constant coefficients and one constant delay. These results have been improved and extended by Philos [13] for first order linear delay differential equations in which the coefficients are periodic functions with a common period and the delays are constants and multiples of this period. The results in [6] have also been improved and extended by Kordonis, Niyianni and Philos [12] for first order linear neutral delay differential equations with constant coefficients and constant delays. Philos and Purnaras [14] have studied the more general case of first order linear neutral delay differential equations with periodic coefficients and constant delays, where the coefficients have a common period and the delays are multiples of this period. The results in [14] contain especially those in [13] (in an improved version) as well as the ones given in [12]. Moreover, the results obtained by Graef and Qian [8] are also motivated by those in [6] and are closely related.

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For some related results we refer to the papers by Arino and Pituk [1], Driver [4], and Győri [9].

In [3], Driver studied first order linear autonomous delay differential equations with infinitely many distributed delays and obtained some important results on the asymptotic behavior, the nonoscillation and the stability. For previous related results we refer to the references cited in [3]. The results given in this paper are essentially motivated by the corresponding ones in [3] and the techniques applied in the present paper are originated in some of the methods used in [3].

This paper deals with the asymptotic behavior, the nonoscillation and the stability for scalar first order linear autonomous neutral delay differential equations with distributed type delays. A basic asymptotic criterion is established. Also, a nonoscillation result is given. Moreover, a useful estimate of the solutions is obtained and a stability criterion is derived. Our results are obtained by the use of a real root (with an appropriate property) of the corresponding characteristic equation. The results given here can be applied to the corresponding *non-neutral* equations. An application of our results to the special case of (non-neutral) delay differential equations leads to an improved version of some of the results given by Driver in [3].

Recently, a very interesting article has been published by Frasson and Verduyn Lunel [7] concerning the large time behaviour of linear functional differential equations. It is shown there that the spectral theory for linear autonomous as well as periodic functional differential equations yields explicit formulas for the large time behaviour of solutions. The results in [7] are based on resolvent computations and Dunford calculus. Some known results (see [6, 12]) can be obtained as applications of the general results given in [7]. The work in [7] may be viewed as a generalization of previous works for first order scalar linear autonomous and periodic functional differential equations (see [3, 6, 12, 13, 14]). It must be noted that, in [3, 6, 12, 13, 14] as well as in the present paper, the method used in obtaining the results is very simple and is essentially based on elementary calculus.

Consider the neutral delay differential equation

$$\Big[x(t) + \int_{-\sigma}^{0} x(t+s)d\zeta(s)\Big]' = \int_{-\tau}^{0} x(t+s)d\eta(s), \tag{1.1}$$

where σ *and* τ *are positive constants,* ζ *and* η *are real-valued functions of bounded variation on the intervals* $[-\sigma, 0]$ *and* $[-\tau, 0]$ *respectively, and the integrals are Riemann-Stieltjes integrals. It will be supposed that* η *is not constant on* $[-\tau, 0]$.

Set

$$r = \max\{\sigma, \tau\}.$$

Clearly, r is a positive constant.

As usual, a continuous real-valued function x defined on the interval $[-r, \infty)$ is said to be a *solution* of the neutral delay differential equation (1.1) if the function $x(t) + \int_{-\sigma}^{0} x(t+s)d\zeta(s)$ is continuously differentiable for $t \geq 0$ and x satisfies (1.1) for all $t \geq 0$.

In the sequel, by $C([-r, 0], \mathbb{R})$ we will denote the set of all continuous real-valued functions on the interval $[-r, 0]$. This set is a Banach space endowed with the sup-norm $\|\phi\| = \sup_{t\in[-r,0]} |\phi(t)|$.

It is well-known (see, for example, Diekmann *et al.* [2], Hale [10], or Hale and Verduyn Lunel [11]) that, for any given *initial function* ϕ in $C([-r, 0], \mathbb{R})$, there exists a unique solution x of the differential equation (1.1) which satisfies the *initial condition*

$$x(t) = \phi(t) \quad \text{for } t \in [-r, 0]; \tag{1.2}$$

this function x will be called the solution of the *initial problem* (1.1)-(1.2) or, more briefly, the solution of (1.1)-(1.2).

The *characteristic equation* of (1.1) is

$$\lambda\Big[1 + \int_{-\sigma}^{0} e^{\lambda s} d\zeta(s)\Big] = \int_{-\tau}^{0} e^{\lambda s} d\eta(s). \tag{1.3}$$

Throughout this paper, by $V(\zeta)$ we will denote the *total variation function* of ζ, which is defined on the interval $[-\sigma, 0]$ as follows: $V(\zeta)(-\sigma) = 0$, and $V(\zeta)(s)$ is the total variation of ζ on $[-\sigma, s]$ for each s in $(-\sigma, 0]$. Also, $V(\eta)$ will stand for the total variation function of η defined on the interval $[-\tau, 0]$ by an analogous way: $V(\eta)(-\tau) = 0$, and $V(\eta)(s)$ is equal to the total variation of η on $[-\tau, s]$ for each $s \in (-\tau, 0]$. Note that the functions $V(\zeta)$ and $V(\eta)$ are nonnegative and increasing on the intervals $[-\sigma, 0]$ and $[-\tau, 0]$ respectively. Moreover, it must be noted that $V(\zeta)$ is identically zero on $[-\sigma, 0]$ if ζ is constant on this interval, and that $V(\eta)$ is not identically zero on the interval $[-\tau, 0]$ (and so it is always not constant on $[-\tau, 0]$). It will be considered that the reader is familiar with the theory of functions of bounded variation and the theory of Riemann-Stieltjes integration.

To obtain the main results of this paper, we will make use of a real root λ_0 of the characteristic equation (1.3) with the property

$$\int_{-\sigma}^{0} [1 + |\lambda_0| (-s)]\, e^{\lambda_0 s} dV(\zeta)(s) + \int_{-\tau}^{0} (-s) e^{\lambda_0 s} dV(\eta)(s) < 1. \tag{1.4}$$

Let us consider the special case of the (*non-neutral*) delay differential equation

$$x'(t) = \int_{-\tau}^{0} x(t+s) d\eta(s). \tag{1.5}$$

This equation can be obtained (as a special case) from the differential equation (1.1), by choosing σ to be an arbitrary positive constant with $\sigma \le \tau$ and considering ζ to be any constant real-valued function on $[-\sigma, 0]$.

As it concerns the (non-neutral) delay differential equation (1.5), we have the constant τ in place of r.

By a *solution* of (1.5), we mean a continuous real-valued function x defined on the interval $[-\tau, \infty)$, which is continuously differentiable on $[0, \infty)$ and satisfies (1.5) for $t \ge 0$. In the special case of (1.5), the Initial Condition (1.2) becomes

$$x(t) = \phi(t) \quad \text{for } t \in [-\tau, 0]. \tag{1.6}$$

The *characteristic equation* of (1.5) is

$$\lambda = \int_{-\tau}^{0} e^{\lambda s} d\eta(s). \tag{1.7}$$

With respect to the (non-neutral) delay differential equation (1.5), we need a real root λ_0 of the characteristic equation (1.7) with the property

$$\int_{-\tau}^{0} (-s) e^{\lambda_0 s} dV(\eta)(s) < 1. \tag{1.8}$$

The notions of the *stability*, *instability*, *uniform stability*, *asymptotic stability* and *uniform asymptotic stability* of the *trivial solution* of a neutral (or non-neutral) delay differential equation will be considered in the usual sense (see, for example, Diekmann *et al.* [2], Hale [10], or Hale and Verduyn Lunel [11]; for the non-neutral case, see also Driver [5]). Note that, since the differential equation (1.1) (and, in particular, the differential equation (1.5)) is autonomous, the trivial solution of (1.1) (and, in particular, of (1.5)) is uniformly stable or uniformly asymptotically stable if and only if it is stable (at 0) or asymptotically stable (at 0) respectively.

Our main results are two theorems and two corollaries of the first of these theorems. The main results of the paper are stated in Section 2. The proof of the first theorem is given in Section 3, while the proof of the second theorem is presented in Section 4. Section 5 is devoted to the application of the main results to the special case of the (non-neutral) delay differential equation (1.5). Sufficient conditions for the characteristic equation (1.3) (and, in particular, for (1.7)) to have a real root λ_0 with the property (1.4) (and, in particular, with the property (1.8)) are obtained in Section 6.

2. Statement of the main results

Theorem 2.1 below is a basic asymptotic criterion for the solutions of the neutral delay differential equation (1.1).

Theorem 2.1. *Let λ_0 be a real root of the characteristic equation* (1.3) *with the property* (1.4) *and set*

$$\gamma(\lambda_0) = \int_{-\sigma}^{0} [1 - \lambda_0(-s)]\, e^{\lambda_0 s} d\zeta(s) + \int_{-\tau}^{0} (-s) e^{\lambda_0 s} d\eta(s).$$

Then, for every $\phi \in C([-r,0],\mathbb{R})$, the solution x of (1.1)-(1.2) *satisfies*

$$\lim_{t\to\infty} \left[e^{-\lambda_0 t} x(t)\right] = \frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)},$$

where

$$\begin{aligned} L(\lambda_0;\phi) = \phi(0) &+ \int_{-\sigma}^{0} \Big[\phi(s) - \lambda_0 e^{\lambda_0 s} \int_{s}^{0} e^{-\lambda_0 u}\phi(u)du\Big] d\zeta(s) \\ &+ \int_{-\tau}^{0} e^{\lambda_0 s}\Big[\int_{s}^{0} e^{-\lambda_0 u}\phi(u)du\Big] d\eta(s). \end{aligned}$$

Note: Property (1.4) guarantees that $1+\gamma(\lambda_0) > 0$.

We immediately see that $\lambda_0 = 0$ is a root of the characteristic equation (1.3) with the property (1.4) if and only if

$$\int_{-\tau}^{0} d\eta(s) = 0 \quad \text{and} \quad \int_{-\sigma}^{0} dV(\zeta)(s) + \int_{-\tau}^{0} (-s) dV(\eta)(s) < 1\,,$$

i.e. if and only if the following condition holds:

$$\eta(-\tau) = \eta(0) \quad \text{and} \quad V(\zeta)(0) + \int_{-\tau}^{0} (-s) dV(\eta)(s) < 1\,. \tag{2.1}$$

Note that $V(\zeta)(0)$ is the total variation of ζ on the interval $[-\sigma, 0]$. Thus, an application of Theorem 2.1 with $\lambda_0 = 0$ leads to the following corollary.

Corollary 2.2. *Let Condition* (2.1) *be satisfied. Then, for* $\phi \in C([-r,0],\mathbb{R})$*, the solution* x *of* (1.1)-(1.2) *satisfies*

$$\lim_{t\to\infty} x(t) = \frac{\phi(0) + \int_{-\sigma}^{0} \phi(s)d\zeta(s) + \int_{-\tau}^{0} \big[\int_{s}^{0} \phi(u)du\big] d\eta(s)}{1 + [\zeta(0) - \zeta(-\sigma)] + \int_{-\tau}^{0}(-s)d\eta(s)}.$$

Note: The second assumption of (2.1) ensures that

$$1 + [\zeta(0) - \zeta(-\sigma)] + \int_{-\tau}^{0} (-s)d\eta(s) > 0.$$

Another immediate consequence of Theorem 2.1 is the following result. As customary, a solution of (1.1) is said to be *nonoscillatory* if it is either eventually positive or eventually negative.

Corollary 2.3. *Let* λ_0 *be a real root of the characteristic equation* (1.3) *with the property* (1.4)*. Then, for any* $\phi \in C([-r,0],\mathbb{R})$*, the solution* x *of* (1.1)-(1.2) *will be nonoscillatory, except possibly if* ϕ *is such that* $L(\lambda_0;\phi) = 0$*, where* $L(\lambda_0;\phi)$ *is defined as in Theorem 2.1.*

Consider a real root λ_0 of (1.3) with the property (1.4) and, for any $\phi \in C([-r,0],\mathbb{R})$, let $L(\lambda_0;\phi)$ be defined as in Theorem 2.1. Clearly, the operator $L(\lambda_0;\cdot)$ is linear. Moreover, there exists a function $\phi_0 \in C([-r,0],\mathbb{R})$ such that $L(\lambda_0;\phi_0) \neq 0$. Indeed, if we set

$$\phi_0(t) = e^{\lambda_0 t} \quad \text{for } t \in [-r,0],$$

then $\phi_0 \in C([-r,0],\mathbb{R})$ and we have

$$\begin{aligned}
L(\lambda_0;\phi_0) &\equiv \phi_0(0) + \int_{-\sigma}^{0} \Big[\phi_0(s) - \lambda_0 e^{\lambda_0 s} \int_{s}^{0} e^{-\lambda_0 u}\phi_0(u)du\Big] d\zeta(s) \\
&\quad + \int_{-\tau}^{0} e^{\lambda_0 s}\Big[\int_{s}^{0} e^{-\lambda_0 u}\phi_0(u)du\Big] d\eta(s) \\
&= 1 + \int_{-\sigma}^{0} \big[e^{\lambda_0 s} - \lambda_0 e^{\lambda_0 s}(-s)\big]\, d\zeta(s) + \int_{-\tau}^{0} e^{\lambda_0 s}(-s)d\eta(s) \\
&= 1 + \int_{-\sigma}^{0} [1 - \lambda_0(-s)]\, e^{\lambda_0 s} d\zeta(s) + \int_{-\tau}^{0} (-s)e^{\lambda_0 s} d\eta(s) \\
&= 1 + \gamma(\lambda_0) > 0,
\end{aligned}$$

where $\gamma(\lambda_0)$ is defined as in Theorem 2.1. So, by the same method with the one that was used by Driver in [3] (see, also, Philos [13]), one can prove the following result, which can be considered as a complement of Corollary 2.3.

Let λ_0 *be a real root of the characteristic equation* (1.3) *with the property* (1.4)*. Moreover, for any* $\phi \in C([-r,0],\mathbb{R})$*, let* $L(\lambda_0;\phi)$ *be defined as in Theorem 2.1. Then the set of all functions* $\phi \in C([-r,0],\mathbb{R})$ *which satisfy* $L(\lambda_0;\phi) = 0$ *is a nowhere dense subset of the Banach space* $C([-r,0],\mathbb{R})$ *(with the sup-norm).*

The following theorem establishes an estimate for the solutions of the neutral delay differential equation (1.1) and, also, a stability criterion for the trivial solution of (1.1).

Theorem 2.4. *Let λ_0 be a real root of the characteristic equation* (1.3) *with the property* (1.4)*. Consider $\gamma(\lambda_0)$ as in Theorem 2.1 and set*

$$\mu(\lambda_0) = \int_{-\sigma}^{0} [1 + |\lambda_0| (-s)] e^{\lambda_0 s} dV(\zeta)(s) + \int_{-\tau}^{0} (-s) e^{\lambda_0 s} dV(\eta)(s).$$

Then, for any $\phi \in C([-r,0],\mathbb{R})$, the solution x of (1.1)-(1.2) *satisfies*

$$|x(t)| \le N(\lambda_0) \|\phi\| e^{\lambda_0 t} \quad \text{for all } t \ge 0,$$

where

$$N(\lambda_0) = \frac{1+\mu(\lambda_0)}{1+\gamma(\lambda_0)} + \Big[1 + \frac{1+\mu(\lambda_0)}{1+\gamma(\lambda_0)}\Big] \mu(\lambda_0) \max\{1, e^{\lambda_0 r}\}.$$

Here the constant $N(\lambda_0)$ is greater than 1. Moreover, the trivial solution of (1.1) *is uniformly stable if $\lambda_0 = 0$, uniformly asymptotically stable if $\lambda_0 < 0$, and unstable if $\lambda_0 > 0$.*

Note that the criterion for the uniform stability stated in Theorem 2.4 can equivalently be formulated as follows:

The trivial solution of (1.1) *is uniformly stable if Condition* (2.1) *holds.*

3. Proof of Theorem 2.1

First of all, let us define $\mu(\lambda_0)$ as in Theorem 2.4, i.e.

$$\mu(\lambda_0) = \int_{-\sigma}^{0} [1 + |\lambda_0| (-s)] e^{\lambda_0 s} dV(\zeta)(s) + \int_{-\tau}^{0} (-s) e^{\lambda_0 s} dV(\eta)(s).$$

Property (1.4) implies

$$0 < \mu(\lambda_0) < 1. \tag{3.1}$$

We have

$$\begin{aligned} |\gamma(\lambda_0)| &\le \Big| \int_{-\sigma}^{0} [1 - \lambda_0(-s)] e^{\lambda_0 s} d\zeta(s) \Big| + \Big| \int_{-\tau}^{0} (-s) e^{\lambda_0 s} d\eta(s) \Big| \\ &\le \int_{-\sigma}^{0} |1 - \lambda_0(-s)| e^{\lambda_0 s} dV(\zeta)(s) + \int_{-\tau}^{0} (-s) e^{\lambda_0 s} dV(\eta)(s) \\ &\le \int_{-\sigma}^{0} [1 + |\lambda_0| (-s)] e^{\lambda_0 s} dV(\zeta)(s) + \int_{-\tau}^{0} (-s) e^{\lambda_0 s} dV(\eta)(s), \end{aligned}$$

that is $|\gamma(\lambda_0)| \le \mu(\lambda_0)$. So, in view of (3.1), it holds $|\gamma(\lambda_0)| < 1$. This, in particular, implies that $1 + \gamma(\lambda_0) > 0$.

Consider now an arbitrary initial function ϕ in $C([-r,0],\mathbb{R})$ and let x be the solution of (1.1)-(1.2). Define

$$y(t) = e^{-\lambda_0 t} x(t) \quad \text{for } t \ge -r.$$

Then, using the fact that λ_0 is a (real) root of the characteristic equation (1.3), we obtain for every $t \ge 0$

$$\begin{aligned} &\Big[x(t) + \int_{-\sigma}^{0} x(t+s) d\zeta(s)\Big]' - \int_{-\tau}^{0} x(t+s) d\eta(s) \\ &= e^{\lambda_0 t} \Big\{ \Big[y(t) + \int_{-\sigma}^{0} e^{\lambda_0 s} y(t+s) d\zeta(s)\Big]' + \lambda_0 \Big[y(t) + \int_{-\sigma}^{0} e^{\lambda_0 s} y(t+s) d\zeta(s)\Big] \\ &\quad - \int_{-\tau}^{0} e^{\lambda_0 s} y(t+s) d\eta(s) \Big\} \end{aligned}$$

$$\begin{aligned}
&= e^{\lambda_0 t}\Big\{\Big[y(t)+\int_{-\sigma}^{0} e^{\lambda_0 s}y(t+s)d\zeta(s)\Big]' \\
&\quad + \Big[-\lambda_0\int_{-\sigma}^{0} e^{\lambda_0 s}d\zeta(s)+\int_{-\tau}^{0} e^{\lambda_0 s}d\eta(s)\Big]y(t) \\
&\quad + \lambda_0\int_{-\sigma}^{0} e^{\lambda_0 s}y(t+s)d\zeta(s)-\int_{-\tau}^{0} e^{\lambda_0 s}y(t+s)d\eta(s)\Big\} \\
&= e^{\lambda_0 t}\Big\{\Big[y(t)+\int_{-\sigma}^{0} e^{\lambda_0 s}y(t+s)d\zeta(s)\Big]'-\lambda_0\int_{-\sigma}^{0} e^{\lambda_0 s}\big[y(t)-y(t+s)\big]d\zeta(s) \\
&\quad + \int_{-\tau}^{0} e^{\lambda_0 s}\big[y(t)-y(t+s)\big]d\eta(s)\Big\}.
\end{aligned}$$

Thus, since x satisfies (1.1) for all $t \ge 0$, it follows that y satisfies

$$\begin{aligned}
&\Big[y(t)+\int_{-\sigma}^{0} e^{\lambda_0 s}y(t+s)d\zeta(s)\Big]' \\
&= \lambda_0\int_{-\sigma}^{0} e^{\lambda_0 s}\big[y(t)-y(t+s)\big]d\zeta(s)-\int_{-\tau}^{0} e^{\lambda_0 s}\big[y(t)-y(t+s)\big]d\eta(s), t\ge 0.
\end{aligned} \tag{3.2}$$

On the other hand, the Initial Condition (1.2) becomes

$$y(t)=e^{-\lambda_0 t}\phi(t) \quad \text{for } t\in[-r,0]. \tag{3.3}$$

Furthermore, we can see that (3.2) is equivalently written as

$$\begin{aligned}
&y(t)+\int_{-\sigma}^{0} e^{\lambda_0 s}y(t+s)d\zeta(s) \\
&= \lambda_0\int_{-\sigma}^{0} e^{\lambda_0 s}\Big[\int_{t+s}^{t} y(u)du\Big]d\zeta(s)-\int_{-\tau}^{0} e^{\lambda_0 s}\Big[\int_{t+s}^{t} y(u)du\Big]d\eta(s)+K \quad \text{for } t\ge 0
\end{aligned}$$

for some real constant K. But, by taking into account (3.3) and the definition of $L(\lambda_0;\phi)$, we have

$$\begin{aligned}
K &= y(0)+\int_{-\sigma}^{0} e^{\lambda_0 s}y(s)d\zeta(s)-\lambda_0\int_{-\sigma}^{0} e^{\lambda_0 s}\Big[\int_{s}^{0} y(u)du\Big]d\zeta(s) \\
&\quad + \int_{-\tau}^{0} e^{\lambda_0 s}\Big[\int_{s}^{0} y(u)du\Big]d\eta(s) \\
&= \phi(0)+\int_{-\sigma}^{0}\phi(s)d\zeta(s)-\lambda_0\int_{-\sigma}^{0} e^{\lambda_0 s}\Big[\int_{s}^{0} e^{-\lambda_0 u}\phi(u)du\Big]d\zeta(s) \\
&\quad + \int_{-\tau}^{0} e^{\lambda_0 s}\Big[\int_{s}^{0} e^{-\lambda_0 u}\phi(u)du\Big]d\eta(s) \\
&= \phi(0)+\int_{-\sigma}^{0}\Big[\phi(s)-\lambda_0 e^{\lambda_0 s}\int_{s}^{0} e^{-\lambda_0 u}\phi(u)du\Big]\,d\zeta(s) \\
&\quad + \int_{-\tau}^{0} e^{\lambda_0 s}\Big[\int_{s}^{0} e^{-\lambda_0 u}\phi(u)du\Big]d\eta(s) \\
&\equiv L(\lambda_0;\phi).
\end{aligned}$$

So, (3.2) is equivalent to

$$\begin{aligned}&y(t)+\int_{-\sigma}^{0} e^{\lambda_0 s} y(t+s)d\zeta(s)\\&=\lambda_0\int_{-\sigma}^{0} e^{\lambda_0 s}\Big[\int_{t+s}^{t} y(u)du\Big]d\zeta(s)-\int_{-\tau}^{0} e^{\lambda_0 s}\Big[\int_{t+s}^{t} y(u)du\Big]d\eta(s)+L(\lambda_0;\phi)\end{aligned}\tag{3.4}$$

for $t \geq 0$.

Next, we set

$$z(t)=y(t)-\frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)}\quad \text{for } t\geq -r.$$

Then, using the definition of $\gamma(\lambda_0)$, it is easy to check that (3.4) takes the following equivalent form

$$\begin{aligned}&z(t)+\int_{-\sigma}^{0} e^{\lambda_0 s} z(t+s)d\zeta(s)\\&=\lambda_0\int_{-\sigma}^{0} e^{\lambda_0 s}\Big[\int_{t+s}^{t} z(u)du\Big]d\zeta(s)-\int_{-\tau}^{0} e^{\lambda_0 s}\Big[\int_{t+s}^{t} z(u)du\Big]d\eta(s)\quad \text{for } t\geq 0.\end{aligned}\tag{3.5}$$

Moreover, (3.3) is written as

$$z(t)=e^{-\lambda_0 t}\phi(t)-\frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)}\quad \text{for } t\in[-r,0].\tag{3.6}$$

By the definitions of y and z, what we have to prove is that

$$\lim_{t\to\infty} z(t)=0.\tag{3.7}$$

In the rest of the proof we will establish (3.7). Put

$$M(\lambda_0;\phi)=\max_{t\in[-r,0]}\Big|e^{-\lambda_0 t}\phi(t)-\frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)}\Big|.$$

Then, in view of (3.6), we have

$$|z(t)|\leq M(\lambda_0;\phi)\quad \text{for } -r\leq t\leq 0.\tag{3.8}$$

We will show that $M(\lambda_0;\phi)$ is a bound of z on the whole interval $[-r,\infty)$, namely that

$$|z(t)|\leq M(\lambda_0;\phi)\quad \text{for all } t\geq -r.\tag{3.9}$$

To this end, let us consider an arbitrary number $\epsilon>0$. We claim that

$$|z(t)|<M(\lambda_0;\phi)+\epsilon\quad \text{for every } t\geq -r.\tag{3.10}$$

Otherwise, because of (3.8), there exists a point $t_0>0$ such that

$$|z(t)|<M(\lambda_0;\phi)+\epsilon\quad \text{for } -r\leq t<t_0,\quad \text{and}\quad |z(t_0)|=M(\lambda_0;\phi)+\epsilon.$$

Then, by taking into account the definition of $\mu(\lambda_0)$ and using (3.1), from (3.5) we obtain

$$\begin{aligned}&M(\lambda_0;\phi)+\epsilon=|z(t_0)|\\&=\Big|-\int_{-\sigma}^{0} e^{\lambda_0 s} z(t_0+s)d\zeta(s)+\lambda_0\int_{-\sigma}^{0} e^{\lambda_0 s}\Big[\int_{t_0+s}^{t_0} z(u)du\Big]d\zeta(s)\\&\quad-\int_{-\tau}^{0} e^{\lambda_0 s}\Big[\int_{t_0+s}^{t_0} z(u)du\Big]d\eta(s)\Big|\end{aligned}$$

$$
\begin{aligned}
&\le \Big|\int_{-\sigma}^{0} e^{\lambda_0 s} z(t_0+s)d\zeta(s)\Big| + |\lambda_0|\Big|\int_{-\sigma}^{0} e^{\lambda_0 s}\Big[\int_{t_0+s}^{t_0} z(u)du\Big]d\zeta(s)\Big| \\
&\quad + \Big|\int_{-\tau}^{0} e^{\lambda_0 s}\Big[\int_{t_0+s}^{t_0} z(u)du\Big]d\eta(s)\Big| \\
&\le \int_{-\sigma}^{0} e^{\lambda_0 s}|z(t_0+s)|dV(\zeta)(s) + |\lambda_0|\int_{-\sigma}^{0} e^{\lambda_0 s}\Big|\int_{t_0+s}^{t_0} z(u)du\Big|dV(\zeta)(s) \\
&\quad + \int_{-\tau}^{0} e^{\lambda_0 s}\Big|\int_{t_0+s}^{t_0} z(u)du\Big|dV(\eta)(s) \\
&\le \int_{-\sigma}^{0} e^{\lambda_0 s}|z(t_0+s)|dV(\zeta)(s) + |\lambda_0|\int_{-\sigma}^{0} e^{\lambda_0 s}\Big[\int_{t_0+s}^{t_0} |z(u)|du\Big]dV(\zeta)(s) \\
&\quad + \int_{-\tau}^{0} e^{\lambda_0 s}\Big[\int_{t_0+s}^{t_0} |z(u)|du\Big]dV(\eta)(s) \\
&\le \Big[\int_{-\sigma}^{0} e^{\lambda_0 s}dV(\zeta)(s) + |\lambda_0|\int_{-\sigma}^{0} e^{\lambda_0 s}(-s)dV(\zeta)(s) \\
&\quad + \int_{-\tau}^{0} e^{\lambda_0 s}(-s)dV(\eta)(s)\Big]\big[M(\lambda_0;\phi)+\epsilon\big] \\
&= \Big\{\int_{-\sigma}^{0}[1+|\lambda_0|(-s)]e^{\lambda_0 s}dV(\zeta)(s) + \int_{-\tau}^{0}(-s)e^{\lambda_0 s}dV(\eta)(s)\Big\}[M(\lambda_0;\phi)+\epsilon] \\
&\equiv \mu(\lambda_0)[M(\lambda_0;\phi)+\epsilon] < M(\lambda_0;\phi)+\epsilon.
\end{aligned}
$$

This is a contradiction and so our claim is true, i.e. (3.10) holds. We have thus proved that (3.10) is fulfilled for all numbers $\epsilon > 0$. Hence, (3.9) is satisfied. Now, by virtue of (3.9), from (3.5) we derive for $t \ge 0$,

$$
\begin{aligned}
|z(t)| &\le \Big|\int_{-\sigma}^{0} e^{\lambda_0 s} z(t+s)d\zeta(s)\Big| + |\lambda_0|\Big|\int_{-\sigma}^{0} e^{\lambda_0 s}\Big[\int_{t+s}^{t} z(u)du\Big]d\zeta(s)\Big| \\
&\quad + \Big|\int_{-\tau}^{0} e^{\lambda_0 s}\Big[\int_{t+s}^{t} z(u)du\Big]d\eta(s)\Big| \\
&\le \int_{-\sigma}^{0} e^{\lambda_0 s}|z(t+s)|\,dV(\zeta)(s) + |\lambda_0|\int_{-\sigma}^{0} e^{\lambda_0 s}\Big|\int_{t+s}^{t} z(u)du\Big|\,dV(\zeta)(s) \\
&\quad + \int_{-\tau}^{0} e^{\lambda_0 s}\Big|\int_{t+s}^{t} z(u)du\Big|dV(\eta)(s) \\
&\le \int_{-\sigma}^{0} e^{\lambda_0 s}|z(t+s)|\,dV(\zeta)(s) + |\lambda_0|\int_{-\sigma}^{0} e^{\lambda_0 s}\Big[\int_{t+s}^{t} |z(u)|\,du\Big]dV(\zeta)(s) \\
&\quad + \int_{-\tau}^{0} e^{\lambda_0 s}\Big[\int_{t+s}^{t} |z(u)|\,du\Big]dV(\eta)(s) \\
&\le \Big[\int_{-\sigma}^{0} e^{\lambda_0 s}dV(\zeta)(s) + |\lambda_0|\int_{-\sigma}^{0} e^{\lambda_0 s}(-s)dV(\zeta)(s) \\
&\quad + \int_{-\tau}^{0} e^{\lambda_0 s}(-s)dV(\eta)(s)\Big]M(\lambda_0;\phi) \\
&= \Big\{\int_{-\sigma}^{0}[1+|\lambda_0|(-s)]\,e^{\lambda_0 s}dV(\zeta)(s) + \int_{-\tau}^{0} e^{\lambda_0 s}(-s)dV(\eta)(s)\Big\}M(\lambda_0;\phi)\,.
\end{aligned}
$$

Consequently, by the definition of $\mu(\lambda_0)$, we have

$$|z(t)| \le \mu(\lambda_0) M(\lambda_0;\phi) \quad \text{for every } t \ge 0. \tag{3.11}$$

Using (3.5) and taking into account the definition of $\mu(\lambda_0)$ as well as (3.9) and (3.11), one can show, by an easy induction, that z satisfies

$$|z(t)| \le [\mu(\lambda_0)]^{\nu} M(\lambda_0;\phi) \quad \text{for all } t \ge \nu r - r \quad (\nu = 0,1,2,\dots). \tag{3.12}$$

Because of (3.1), we have $\lim_{\nu\to\infty} [\mu(\lambda_0)]^{\nu} = 0$. Thus, from (3.12) it follows that $\lim_{t\to\infty} z(t) = 0$, i.e. (3.7) holds. The proof of Theorem 2.1 is complete.

4. Proof or Theorem 2.4

We first notice that, as in the proof of Theorem 2.1, we have $0 < \mu(\lambda_0) < 1$, $|\gamma(\lambda_0)| \le \mu(\lambda_0)$ and $1+\gamma(\lambda_0) > 0$. It follows immediately that $N(\lambda_0) > 1$. Consider an arbitrary function ϕ in $C([-r,0],\mathbb{R})$ and let x be the solution of (1.1)-(1.2). Let y and z be defined as in the proof of Theorem 2.1, i.e.

$$y(t) = e^{-\lambda_0 t} x(t) \quad \text{for } t \ge -r, \quad \text{and} \quad z(t) = y(t) - \frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)} \quad \text{for } t \ge -r,$$

where $L(\lambda_0;\phi)$ is defined as in Theorem 2.1. Moreover, let $M(\lambda_0;\phi)$ be defined as in the proof of Theorem 2.1, i.e.

$$M(\lambda_0;\phi) = \max_{t\in[-r,0]} \Big| e^{-\lambda_0 t}\phi(t) - \frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)} \Big|.$$

Then, as in the proof of Theorem 2.1, we can show that z satisfies (3.11), namely

$$|z(t)| \le \mu(\lambda_0) M(\lambda_0;\phi) \quad \text{for every } t \ge 0.$$

By the definition of z, from the last inequality it follows that

$$|y(t)| \le \frac{|L(\lambda_0;\phi)|}{1+\gamma(\lambda_0)} + \mu(\lambda_0) M(\lambda_0;\phi) \quad \text{for } t \ge 0. \tag{4.1}$$

On the other hand, from the definition of $M(\lambda_0;\phi)$ we get

$$M(\lambda_0;\phi) \le \|\phi\| \max\{1, e^{\lambda_0 r}\} + \frac{|L(\lambda_0;\phi)|}{1+\gamma(\lambda_0)}.$$

So, (4.1) gives

$$|y(t)| \le \frac{1+\mu(\lambda_0)}{1+\gamma(\lambda_0)} |L(\lambda_0;\phi)| + \|\phi\|\, \mu(\lambda_0) \max\{1, e^{\lambda_0 r}\}, \quad t \ge 0. \tag{4.2}$$

Furthermore, by the definition of $L(\lambda_0;\phi)$, we obtain

$$\begin{aligned}
|L(\lambda_0;\phi)| &\le |\phi(0)| + \Big| \int_{-\sigma}^{0} \Big[\phi(s) - \lambda_0 e^{\lambda_0 s} \int_s^0 e^{-\lambda_0 u}\phi(u)du\Big] d\zeta(s) \Big| \\
&\quad + \Big| \int_{-\tau}^{0} e^{\lambda_0 s} \Big[\int_s^0 e^{-\lambda_0 u}\phi(u)du\Big] d\eta(s) \Big| \\
&= |\phi(0)| + \Big| \int_{-\sigma}^{0} \Big[e^{-\lambda_0 s}\phi(s) - \lambda_0 \int_s^0 e^{-\lambda_0 u}\phi(u)du\Big] e^{\lambda_0 s} d\zeta(s) \Big| \\
&\quad + \Big| \int_{-\tau}^{0} \Big[\int_s^0 e^{-\lambda_0 u}\phi(u)du\Big] e^{\lambda_0 s} d\eta(s) \Big|
\end{aligned}$$

$$
\begin{aligned}
&\le |\phi(0)| + \int_{-\sigma}^{0} \Big| e^{-\lambda_0 s}\phi(s) - \lambda_0 \int_s^0 e^{-\lambda_0 u}\phi(u)du \Big| e^{\lambda_0 s} dV(\zeta)(s) \\
&\quad + \int_{-\tau}^{0} \Big| \int_s^0 e^{-\lambda_0 u}\phi(u)du \Big| e^{\lambda_0 s} dV(\eta)(s) \\
&\le |\phi(0)| + \int_{-\sigma}^{0} \Big[e^{-\lambda_0 s}|\phi(s)| + |\lambda_0| \int_s^0 e^{-\lambda_0 u}\,|\phi(u)|\,du \Big] e^{\lambda_0 s} dV(\zeta)(s) \\
&\quad + \int_{-\tau}^{0} \Big[\int_s^0 e^{-\lambda_0 u}\,|\phi(u)|\,du \Big] e^{\lambda_0 s} dV(\eta)(s)\,.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
|L(\lambda_0;\phi)| \le \|\phi\| \Big[1 &+ \int_{-\sigma}^{0} \Big(e^{-\lambda_0 s} + |\lambda_0| \int_s^0 e^{-\lambda_0 u} du \Big) e^{\lambda_0 s} dV(\zeta)(s) \\
&+ \int_{-\tau}^{0} \Big(\int_s^0 e^{-\lambda_0 u} du \Big) e^{\lambda_0 s} dV(\eta)(s) \Big].
\end{aligned}
\tag{4.3}
$$

We have previously used the elementary inequality $e^{-\lambda_0 t} \le \max\{1, e^{\lambda_0 r}\}$ for each $t \in [-r, 0]$. Therefore,

$$
\begin{gathered}
e^{-\lambda_0 s} \le \max\{1, e^{\lambda_0 r}\} \quad \text{for } s \in [-\sigma, 0], \\
\int_s^0 e^{-\lambda_0 u} du \le (-s) \max\{1, e^{\lambda_0 r}\} \quad \text{for } s \in [-\sigma, 0], \\
\int_s^0 e^{-\lambda_0 u} du \le (-s) \max\{1, e^{\lambda_0 r}\} \quad \text{for } s \in [-\tau, 0].
\end{gathered}
$$

Thus, (4.3) leads to

$$
\begin{aligned}
|L(\lambda_0;\phi)| \le \|\phi\| \Big\{ 1 + \Big(&\int_{-\sigma}^{0} [1 + |\lambda_0|\,(-s)]\, e^{\lambda_0 s} dV(\zeta)(s) \\
&+ \int_{-\tau}^{0} (-s) e^{\lambda_0 s} dV(\eta)(s) \Big) \max\{1, e^{\lambda_0 r}\} \Big\},
\end{aligned}
$$

which, in view of the definition of $\mu(\lambda_0)$, can be written as

$$
|L(\lambda_0;\phi)| \le \|\phi\| \left[1 + \mu(\lambda_0) \max\{1, e^{\lambda_0 r}\} \right].
$$

Hence, for $t \ge 0$, (4.2) gives

$$
\begin{aligned}
|y(t)| &\le \Big\{ \frac{1+\mu(\lambda_0)}{1+\gamma(\lambda_0)} \left[1 + \mu(\lambda_0) \max\{1, e^{\lambda_0 r}\} \right] + \mu(\lambda_0) \max\{1, e^{\lambda_0 r}\} \Big\} \|\phi\| \\
&= \Big\{ \frac{1+\mu(\lambda_0)}{1+\gamma(\lambda_0)} + \Big[1 + \frac{1+\mu(\lambda_0)}{1+\gamma(\lambda_0)} \Big] \mu(\lambda_0) \max\{1, e^{\lambda_0 r}\} \Big\} \|\phi\|
\end{aligned}
$$

and so, because of the definition of $N(\lambda_0)$, we have

$$
|y(t)| \le N(\lambda_0)\,\|\phi\| \quad \text{for every } t \ge 0.
$$

Finally, in view of the definition of y, we obtain

$$
|x(t)| \le N(\lambda_0)\,\|\phi\|\, e^{\lambda_0 t} \quad \text{for all } t \ge 0. \tag{4.4}
$$

This completes the proof of the first part of the theorem. It remains to show the stability criterion contained in the theorem.

Let us suppose that $\lambda_0 \le 0$. Let $\phi \in C([-r,0],\mathbb{R})$ be an arbitrary initial function and let x be the solution of (1.1)-(1.2). Then (4.4) holds and hence

$$|x(t)| \le N(\lambda_0)\,\|\phi\| \quad \text{for every } t \ge 0.$$

Since $N(\lambda_0) > 1$, it follows that $|x(t)| \le N(\lambda_0)\,\|\phi\|$ for all $t \ge -r$. Using this inequality, we can immediately verify that the trivial solution of (1.1) is stable (at 0). Moreover, if $\lambda_0 < 0$, then (4.4) guarantees that

$$\lim_{t\to\infty} x(t) = 0.$$

Thus, for $\lambda_0 < 0$ the trivial solution of (1.1) is asymptotically stable (at 0). Because of the autonomous character of (1.1), the trivial solution of (1.1) is uniformly stable if $\lambda_0 = 0$ and it is uniformly asymptotically stable if $\lambda_0 < 0$.

Finally, we assume that $\lambda_0 > 0$ and we will show that the trivial solution of (1.1) is unstable. Suppose, for the sake of contradiction, that the trivial solution of (1.1) is stable (at 0). Then we can choose a number $\delta > 0$ such that, for each $\phi \in C([-r,0],\mathbb{R})$ with $\|\phi\| < \delta$, the solution x of (1.1)-(1.2) satisfies

$$|x(t)| < 1 \quad \text{for all } t \ge -r. \tag{4.5}$$

Set

$$\phi_0(t) = e^{\lambda_0 t} \quad \text{for } t \in [-r,0].$$

We see that $\phi_0 \in C([-r,0],\mathbb{R})$ and, as in Section 2, we can verify that

$$L(\lambda_0;\phi_0) = 1 + \gamma(\lambda_0) > 0, \tag{4.6}$$

where $\gamma(\lambda_0)$ and, for any $\phi \in C([-r,0],\mathbb{R})$, $L(\lambda_0;\phi)$ are defined as in Theorem 2.1. Next, we consider a number δ_0 with $0 < \delta_0 < \delta$ and we put

$$\phi = \frac{\delta_0}{\|\phi_0\|}\phi_0.$$

Clearly, ϕ belongs to $C([-r,0],\mathbb{R})$ and $\|\phi\| = \delta_0 < \delta$. Hence, for this initial function, the solution x of (1.1)-(1.2) satisfies (4.5). On the other hand, by applying Theorem 2.1 and taking into account (4.6) as well as the linearity of the operator $L(\lambda_0;\cdot)$, we obtain

$$\lim_{t\to\infty}\left[e^{-\lambda_0 t}x(t)\right] = \frac{L(\lambda_0;\phi)}{1+\gamma(\lambda_0)} = \frac{(\delta_0/\|\phi_0\|)\,L(\lambda_0;\phi_0)}{1+\gamma(\lambda_0)} = \frac{\delta_0}{\|\phi_0\|} > 0.$$

But, since $\lambda_0 > 0$, from (4.5) it follows that

$$\lim_{t\to\infty}\left[e^{-\lambda_0 t}x(t)\right] = 0.$$

We have thus arrived at a contradiction. The proof of Theorem 2.4 is now complete.

5. Application of the main results to the special case of non-neutral equations

In this section, we will concentrate on the (non-neutral) delay differential equation (1.5) and we shall apply our main results to this equation. For the delay differential equation (1.5), the following results hold.

Theorem 5.1. *Let λ_0 be a real root of the characteristic equation* (1.7) *with the property* (1.8). *Then, for any $\phi \in C([-\tau,0],\mathbb{R})$, the solution x of* (1.5)-(1.6) *satisfies*

$$\lim_{t\to\infty}\left[e^{-\lambda_0 t}x(t)\right] = \frac{\ell(\lambda_0;\phi)}{1+\int_{-\tau}^{0}(-s)e^{\lambda_0 s}d\eta(s)},$$

where

$$\ell(\lambda_0;\phi) = \phi(0) + \int_{-\tau}^{0} e^{\lambda_0 s}\Big[\int_{s}^{0} e^{-\lambda_0 u}\phi(u)du\Big]d\eta(s).$$

Note that Property (1.8) guarantees that $1+\int_{-\tau}^{0}(-s)e^{\lambda_0 s}d\eta(s) > 0$.

Corollary 5.2. *Assume that*

$$\eta(-\tau) = \eta(0) \quad \textit{and} \quad \int_{-\tau}^{0}(-s)dV(\eta)(s) < 1. \tag{5.1}$$

Then, for any $\phi \in C([-\tau,0],\mathbb{R})$, the solution x of (1.5)-(1.6) *satisfies*

$$\lim_{t\to\infty} x(t) = \frac{\phi(0)+\int_{-\tau}^{0}\big[\int_{s}^{0}\phi(u)du\big]d\eta(s)}{1+\int_{-\tau}^{0}(-s)d\eta(s)}.$$

Note that the second assumption of (5.1) ensures that $1+\int_{-\tau}^{0}(-s)d\eta(s) > 0$.

Corollary 5.3. *Let λ_0 be a real root of the characteristic equation* (1.7) *with the property* (1.8). *Then, for any $\phi \in C([-\tau,0],\mathbb{R})$, the solution x of* (1.5)-(1.6) *will be nonoscillatory, except possibly if ϕ satisfies $\ell(\lambda_0;\phi) = 0$, where $\ell(\lambda_0;\phi)$ is defined as in Theorem 5.1.*

As a complement to Corollary 5.3, we have: *Let λ_0 be a real root of the characteristic equation* (1.7) *with the property* (1.8). *Moreover, for any $\phi \in C([-\tau,0],\mathbb{R})$, let $\ell(\lambda_0;\phi)$ be defined as in Theorem 5.1. Then the set of all functions $\phi \in C([-\tau,0],\mathbb{R})$ which satisfy $\ell(\lambda_0;\phi) = 0$ is a nowhere dense subset of the Banach space $C([-\tau,0],\mathbb{R})$ (with the sup-norm).*

Theorem 5.4. *Let λ_0 be a real root of the characteristic equation* (1.7) *with the property* (1.8). *Then, for any $\phi \in C([-\tau,0],\mathbb{R})$, the solution x of* (1.5)-(1.6) *satisfies*

$$|x(t)| \le n(\lambda_0)\|\phi\|e^{\lambda_0 t} \quad \textit{for all } t \ge 0,$$

where

$$\begin{aligned} n(\lambda_0) = {}& \frac{1+\int_{-\tau}^{0}(-s)e^{\lambda_0 s}dV(\eta)(s)}{1+\int_{-\tau}^{0}(-s)e^{\lambda_0 s}d\eta(s)} \\ &+ \Big[1+\frac{1+\int_{-\tau}^{0}(-s)e^{\lambda_0 s}dV(\eta)(s)}{1+\int_{-\tau}^{0}(-s)e^{\lambda_0 s}d\eta(s)}\Big]\Big[\int_{-\tau}^{0}(-s)e^{\lambda_0 s}dV(\eta)(s)\Big]\max\{1,e^{\lambda_0\tau}\} \end{aligned}$$

with the constant $n(\lambda_0)$ being greater than 1. Moreover, the trivial solution of (1.5) *is uniformly stable if $\lambda_0 = 0$, uniformly asymptotically stable if $\lambda_0 < 0$, and unstable if $\lambda_0 > 0$.*

We observe that, concerning the uniform stability, the corresponding result in Theorem 5.4 can be equivalently stated as: *The trivial solution of* (1.5) *is uniformly stable if Condition* (5.1) *holds.*

6. Sufficient conditions for the characteristic equation to have a real root with the property required

In this section, we give some conditions, under which the characteristic equation (1.3) (and, in particular, the characteristic equation (1.7)) has a real root λ_0 with the property (1.4) (and, in particular, with the property (1.8)).

Lemma 6.1. *Assume that*

$$\int_{-\sigma}^{0} e^{-s/r} d\zeta(s) + r\int_{-\tau}^{0} e^{-s/r} d\eta(s) > -1, \tag{6.1}$$

$$-\int_{-\sigma}^{0} e^{s/r} d\zeta(s) + r\int_{-\tau}^{0} e^{s/r} d\eta(s) < 1, \tag{6.2}$$

$$\int_{-\sigma}^{0} [1 + (-s)/r]\, e^{-s/r} dV(\zeta)(s) + \int_{-\tau}^{0} (-s)e^{-s/r} dV(\eta)(s) \le 1. \tag{6.3}$$

Then, in the interval $(-1/r, 1/r)$*, the characteristic equation* (1.3) *has a unique root* λ_0*, and this root satisfies the property* (1.4)*.*

Proof. Define

$$F(\lambda) = \lambda\Big[1 + \int_{-\sigma}^{0} e^{\lambda s} d\zeta(s)\Big] - \int_{-\tau}^{0} e^{\lambda s} d\eta(s) \quad \text{for } \lambda \in [-1/r, 1/r].$$

We have

$$\begin{aligned} F(-1/r) &= -\frac{1}{r}\Big[1 + \int_{-\sigma}^{0} e^{-s/r} d\zeta(s)\Big] - \int_{-\tau}^{0} e^{-s/r} d\eta(s) \\ &= -\frac{1}{r}\Big[1 + \int_{-\sigma}^{0} e^{-s/r} d\zeta(s) + r\int_{-\tau}^{0} e^{-s/r} d\eta(s)\Big] \end{aligned}$$

and so, by (6.1), we get $F(-1/r) < 0$. Moreover,

$$\begin{aligned} F(1/r) &= \frac{1}{r}\Big[1 + \int_{-\sigma}^{0} e^{s/r} d\zeta(s)\Big] - \int_{-\tau}^{0} e^{s/r} d\eta(s) \\ &= -\frac{1}{r}\Big[-1 - \int_{-\sigma}^{0} e^{s/r} d\zeta(s) + r\int_{-\tau}^{0} e^{s/r} d\eta(s)\Big] \end{aligned}$$

and hence from (6.2) it follows that $F(1/r) > 0$. Furthermore, by taking into account (6.3), for $\lambda \in (-1/r, 1/r)$, we obtain

$$\begin{aligned} F'(\lambda) &= 1 + \int_{-\sigma}^{0} [1 - \lambda(-s)]\, e^{\lambda s} d\zeta(s) + \int_{-\tau}^{0} (-s)e^{\lambda s} d\eta(s) \\ &\ge 1 - \Big|\int_{-\sigma}^{0} [1 - \lambda(-s)]\, e^{\lambda s} d\zeta(s)\Big| - \Big|\int_{-\tau}^{0} (-s)e^{\lambda s} d\eta(s)\Big| \\ &\ge 1 - \int_{-\sigma}^{0} |1 - \lambda(-s)|\, e^{\lambda s} dV(\zeta)(s) - \int_{-\tau}^{0} (-s)e^{\lambda s} dV(\eta)(s) \\ &\ge 1 - \int_{-\sigma}^{0} [1 + |\lambda|\,(-s)]\, e^{\lambda s} dV(\zeta)(s) - \int_{-\tau}^{0} (-s)e^{\lambda s} dV(\eta)(s) \\ &> 1 - \int_{-\sigma}^{0} [1 + (-s)/r]\, e^{-s/r} dV(\zeta)(s) - \int_{-\tau}^{0} (-s)e^{-s/r} dV(\eta)(s) \\ &\ge 0\,. \end{aligned}$$

Therefore, F is strictly increasing on the interval $(-1/r, 1/r)$. So, in the interval $(-1/r, 1/r)$, the equation $F(\lambda) = 0$ (which coincides with (1.3)) has a unique root λ_0. This root satisfies (1.4). Indeed, by using again (6.3), we have

$$\begin{aligned}
&\int_{-\sigma}^{0} [1 + |\lambda_0| (-s)]\, e^{\lambda_0 s} dV(\zeta)(s) + \int_{-\tau}^{0} (-s)e^{\lambda_0 s} dV(\eta)(s) \\
&< \int_{-\sigma}^{0} [1 + (-s)/r]\, e^{-s/r} dV(\zeta)(s) + \int_{-\tau}^{0} (-s)e^{-s/r} dV(\eta)(s) \le 1.
\end{aligned}$$

This completes the proof. □

Now, we will confine our attention to the special case of the (non-neutral) delay differential equation (1.5), for which the characteristic equation is (1.7). In this case, Conditions (6.1), (6.2), (6.3) take the form

$$\tau \int_{-\tau}^{0} e^{-s/\tau} d\eta(s) > -1, \tag{6.4}$$

$$\tau \int_{-\tau}^{0} e^{s/\tau} d\eta(s) < 1, \tag{6.5}$$

$$\int_{-\tau}^{0} (-s)e^{-s/\tau} dV(\eta)(s) \le 1\,. \tag{6.6}$$

Lemma 6.1 can be applied to the case of the characteristic equation (1.7) with the assumptions (6.4)–(6.6) instead of (6.1)–(6.3). However, we have the following result which is slightly better.

Lemma 6.2. *Let* (6.4) *and* (6.6) *be satisfied. Then, in the interval* $(-1/\tau, \infty)$*, the characteristic equation* (1.7) *has a unique root* λ_0*; this root has the property* (1.8) *and, provided that* (6.5) *holds, the root* λ_0 *is less than* $1/\tau$*.*

Proof. Set

$$F_0(\lambda) = \lambda - \int_{-\tau}^{0} e^{\lambda s} d\eta(s) \quad \text{for } \lambda \ge -1/\tau.$$

From (6.4), it follows immediately that $F_0(-1/\tau) < 0$. Next, for every $\lambda \ge -1/\tau$, we obtain

$$F_0(\lambda) \ge \lambda - \Big| \int_{-\tau}^{0} e^{\lambda s} d\eta(s) \Big| \ge \lambda - \int_{-\tau}^{0} e^{\lambda s} dV(\eta)(s) \ge \lambda - \int_{-\tau}^{0} e^{-s/\tau} dV(\eta)(s)$$

and consequently $F_0(\infty) = \infty$. Moreover, for $\lambda > -1/\tau$, we have

$$\begin{aligned}
F_0'(\lambda) &= 1 + \int_{-\tau}^{0} (-s)e^{\lambda s} d\eta(s) \ge 1 - \Big| \int_{-\tau}^{0} (-s)e^{\lambda s} d\eta(s) \Big| \\
&\ge 1 - \int_{-\tau}^{0} (-s)e^{\lambda s} dV(\eta)(s) > 1 - \int_{-\tau}^{0} (-s)e^{-s/\tau} dV(\eta)(s)
\end{aligned}$$

and so, by (6.6), it follows that F_0 is strictly increasing on $(-1/\tau, \infty)$. Hence, in the interval $(-1/\tau, \infty)$, there exists a unique root λ_0 of the equation $F_0(\lambda) = 0$ (or, equivalently, of (1.7)). By using again (6.6), we get

$$\int_{-\tau}^{0} (-s)e^{\lambda_0 s} dV(\eta)(s) < \int_{-\tau}^{0} (-s)e^{-s/\tau} dV(\eta)(s) \le 1\,.$$

Consequently the root λ_0 satisfies (1.8). Now assume that (6.5) is also satisfied. This assumption implies that $F_0(1/\tau) > 0$. Thus, we can immediately conclude that the root λ_0 is always less than $1/\tau$. The proof is now complete. $\square$

We remark that *Conditions* (6.4)–(6.6) *are satisfied if the following stronger condition holds:*

$$\tau \int_{-\tau}^{0} e^{-s/\tau} dV(\eta)(s) < 1. \tag{6.7}$$

In fact, we have

$$\tau \int_{-\tau}^{0} e^{-s/\tau} d\eta(s) \geq -\tau \Big| \int_{-\tau}^{0} e^{-s/\tau} d\eta(s) \Big| \geq -\tau \int_{-\tau}^{0} e^{-s/\tau} dV(\eta)(s),$$

$$\begin{aligned} \tau \int_{-\tau}^{0} e^{s/\tau} d\eta(s) &\leq \tau \Big| \int_{-\tau}^{0} e^{s/\tau} d\eta(s) \Big| \leq \tau \int_{-\tau}^{0} e^{s/\tau} dV(\eta)(s) \\ &\leq \tau \int_{-\tau}^{0} dV(\eta)(s) \leq \tau \int_{-\tau}^{0} e^{-s/\tau} dV(\eta)(s) \end{aligned}$$

and

$$\int_{-\tau}^{0} (-s) e^{-s/\tau} dV(\eta)(s) \leq \tau \int_{-\tau}^{0} e^{-s/\tau} dV(\eta)(s)$$

and so our assertion is true. Furthermore, since

$$\tau \int_{-\tau}^{0} e^{-s/\tau} dV(\eta)(s) \leq \tau e \int_{-\tau}^{0} dV(\eta)(s) = \tau e V(\eta)(0),$$

we conclude that *Condition* (6.7) *holds if*

$$\tau e V(\eta)(0) < 1. \tag{6.8}$$

Note that $V(\eta)(0)$ is the total variation of η on the interval $[-\tau, 0]$. Condition (6.7) and, in particular, Condition (6.8) were used by Driver [3].

Note that it is an interesting question to find other conditions on σ and τ and on the integrators ζ and η, which are sufficient for the characteristic equation (1.3) to have a real root λ_0 with the property (1.4). This problem remains interesting still in the special case of the characteristic equation (1.7).

Before closing this section and the paper, we will use Lemma 6.1 (and, in particular, Lemma 6.2) to find some explicit conditions in terms of σ, τ and ζ, η (and, in particular, in terms of τ and η), under which the trivial solution of (1.1) (and, in particular, of (1.5)) is uniformly asymptotically stable or unstable. Note that analogous conditions for the uniform stability of the trivial solution of (1.1) (and, in particular, of (1.5)) have already been given in previous sections.

Let us assume that (6.1)–(6.3) hold. Then Lemma 6.1 guarantees that, in the interval $(-1/r, 1/r)$, the characteristic equation (1.3) has a unique root λ_0; this root satisfies the property (1.4). Let F be defined as in the proof of Lemma 6.1. For this function, as in the proof of Lemma 6.1, we have

$$F(-1/r) < 0 \quad \text{and} \quad F(1/r) > 0.$$

Clearly, λ_0 is negative if $F(0) > 0$, and λ_0 is positive if $F(0) < 0$. On the other hand,

$$F(0) = -\int_{-\tau}^{0} d\eta(s) = -[\eta(0) - \eta(-\tau)].$$

So, $\lambda_0 < 0$ if $\eta(0) < \eta(-\tau)$, and $\lambda_0 > 0$ if $\eta(0) > \eta(-\tau)$. Hence, from the stability criterion contained in Theorem 2.4 we can obtain the following result.

Corollary 6.3. *Assume that* (6.1)–(6.3) *are satisfied. Then the trivial solution of* (1.1) *is uniformly asymptotically stable if* $\eta(0) < \eta(-\tau)$ *and it is unstable if* $\eta(0) > \eta(-\tau)$.

By an analogous way, we can use Lemma 6.2 and the stability criterion contained in Theorem 5.4 to derive the following result.

Corollary 6.4. *Assume that* (6.4) *and* (6.6) *are satisfied. Then the trivial solution of* (1.5) *is uniformly asymptotically stable if* $\eta(0) < \eta(-\tau)$ *and it is unstable if* $\eta(0) > \eta(-\tau)$.

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