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STABILITY PROPERTIES OF NON-NEGATIVE SOLUTIONS OF SEMILINEAR SYMMETRIC COOPERATIVE SYSTEMS

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ABSTRACT. We investigate the stability of non-negative stationary solutions of symmetric cooperative semilinear systems with some convex (resp. concave) nonlinearity condition, namely all second-order partial derivatives of each coordinate being non-negative (resp. non-positive). In these cases, we will show following [8], extending its results, that this along with some sign condition on the non-linearity at the origin yields instability (resp. stability).

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be some bounded domain of smooth boundary. In this paper, we study the stability of positive stationary solutions of the coupled system of semilinear partial differential equations

$$\partial_t u(t,x) = \Delta u(t,x) + f(u(t,x)) \quad t > 0, x \in \Omega \subset \mathbb{R}^n \tag{1.1}$$

subject to the boundary condition

$$hu\big|_{\partial\Omega} + g\partial_\nu u\big|_{\partial\Omega} = 0, \tag{1.2}$$

where

$$u : \mathbb{R}_+ \times \Omega \to \mathbb{R}^m, \quad f \in \mathsf{C}^2(\mathbb{R}^m;\mathbb{R}^m), \quad g, h : \partial\Omega \to \mathbb{R}^m_+, \tag{1.3}$$

and g and h are nowhere simultaneously vanishing non-negative smooth functions.

This problem concerning a single equation ($m = 1$) was studied by several authors even for equations with delay. Shivaji and co-authors have shown that every non-trivial solution of (1.1) with Dirichlet boundary

$$u\big|_{\partial\Omega} = 0 \tag{1.4}$$

is unstable if $f'' > 0$ and $f(0) \le 0$. They first considered the monotone case, i.e. $f' > 0$ in [2]. The statement in the non-monotone case was first proved by Tertikas [15] using sub- and supersolutions. The first simplification was given by Maya and Shivaji in [12] by reducing the problem to the monotone case via decomposition of f to a monotone and a linear function.

Karátson and Simon gave a direct proof of the result in [8]. Moreover, this proof showed the stability of the concave counterpart at the same time, and could be

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easily extended to the general elliptic operator $\operatorname{div}(A\nabla u)$, where $A:\Omega\to\mathbb{R}^{n\times n}$, This can be summed up in the theorem below.

Theorem 1.1. *Let $m=1$ and $f:\mathbb{R}\to\mathbb{R}$ be a twice continuously differentiable function. Then*

(i) *if $f''>0$ and $f(0)\le 0$, then every nontrivial nonnegative solution of (1.1)–(1.3) is unstable, while*
(ii) *if $f''<0$ and $f(0)\ge 0$, then every nontrivial nonnegative solution of (1.1)–(1.3) is stable.*

The solution is not unique even in case of $\Omega=B(0,R)\subset\mathbb{R}^n$, and has typically two solutions, in the case of the different sign condition on $f(0)$, This problem was studied by several authors. In the concave case for general nonlinearities and general domains it was investigated in [1]. In [6, 10] the one dimensional case for general nonlinearity, and in [9, 11, 14, 17] the multidimensional case with special nonlinearities are studied. The authors of [8] pointed out that the sign condition of f at the origin in some sense is necessary, since the small solution now has different stability from the one claimed in the theorem in both cases. Concerning systems, both cooperative and competitive, Castro, Chhetri and Shivaji established sufficient conditions on the nonlinearity for the solution to be stable and unstable in [3].

The technique introduced in [8] can be applied to more general cases. In [4] equations with delay are considered, in [7] the corresponding equation with p-Laplacian is studied.

In this paper, we will show a generalisation of Theorem 1 to a system of equations given in (1.1)–(1.3). Our main result is formulated in Theorem 2.4, and the possible extension to general elliptic operator as in [8] is noted in the following remark.

2. The Case of Systems

The equilibria of (1.1)–(1.3) are defined by the elliptic system

$$0=\Delta u+f(u), \tag{2.1}$$

the linearisation of which at the stationary solution u is

$$\Delta v+f'(u)v=0.$$

The corresponding eigenvalue problem is

$$\Delta v+f'(u)v=\lambda v. \tag{2.2}$$

Since the boundary condition is linear, it is the same for the eigenvalue problem as that for the original one, that is

$$hv\big|_{\partial\Omega}+g\partial_\nu v\big|_{\partial\Omega}=0. \tag{2.3}$$

Proposition 2.1. *Assume that $f'(u)$ is cooperative, i.e. the off-diagonal elements are non-negative. Then the dominant eigenfunction v of the eigenvalue problem (1.1)–(1.3) is positive, i.e. $v_i\ge 0$ $(i=1,\dots,m)$ and not all coordinates are constant zero.*

Proof. Introducing $M : \Omega \to \mathbb{R}^{m\times m}$ given by $M(x) = -f'(u(x)) - \mu I$ let us transform the eigenvalue problem as follows.

$$\begin{gathered}(\Delta + f'(u))v = \lambda v\\ (-\Delta + M)v = (-\lambda - \mu)v\\ (-\Delta + M)^{-1}v = -\frac{1}{\lambda+\mu}v.\end{gathered}$$

The matrix $M(x)$ has non-positive off-diagonal elements in each $x \in \Omega$, and μ can be chosen such that M is uniformly positive definite, since $f'(u)$ is continuous on the closure of the bounded domain Ω. Moreover, a dominant eigenfunction of $(-\Delta + M)^{-1}$ is a dominant eigenfunction of $(\Delta + f'(u))$, which can be shown similarly. According to Krein–Rutman's theorem [16] every positive compact operator on a Banach lattice has a positive dominant eigenvector. Hence we only have to show that $(-\Delta+M)^{-1} : \mathsf{L}^2(\Omega)^m \to \mathsf{L}^2(\Omega)^m$ is a positive compact operator, where $\mathsf{L}^2(\Omega)^m$ denotes product of m copies of $\mathsf{L}^2(\Omega)$, that is $\mathsf{L}^2(\Omega) \times \cdots \times \mathsf{L}^2(\Omega)$. The proof of the compactness is similar to that of the case of a single equation ($m = 1$) [5, 13]. Thus it remains to prove positivity.

To this end, let

$$-\Delta v + Mv \ge 0 \tag{2.4}$$

Let us introduce $v^{\pm} = \max\{\pm v, 0\}$, and multiply in the inner product sense this latter equation by $v^- \ge 0$, that is multiply the ith row by v_i^- and sum them up over all i,

$$-\int_\Omega \langle v^-, \Delta v\rangle + \int_\Omega \langle v^-, Mv\rangle \ge 0$$

and by Gauss-Ostogradskii theorem

$$-\int_{\partial\Omega} \langle v^-, \partial_\nu v\rangle + \sum_i \int_\Omega \langle \nabla v_i^-, \nabla v_i\rangle + \int_\Omega \langle v^-, Mv\rangle \ge 0 \tag{2.5}$$

Let

$$\gamma : \partial\Omega \to \mathbb{R}^m, \quad \gamma_i(x) = \begin{cases} \frac{h_i(x)}{g_i(x)} & \text{if } g_i(x) > 0\\ 0 & \text{if } g_i(x) = 0\end{cases} \quad (i = 1, \dots, m)$$

Now $v_i^-(x)\partial_\nu v_i(x) = \gamma_i v_i^-(x)^2$ on $\partial\Omega$, since

- $v_i(x) = 0$, where $g_i(x) = 0$,
- $v_i^-(x)$, where $v_i(x) > 0$,
- everywhere else $v_i(x) = -v_i^-(x)$, and $\partial_\nu v_i(x) = -\frac{h_i(x)}{g_i(x)} v_i(x)$,

and similarly $\langle \nabla v_i^-, \nabla v_i\rangle = -\langle \nabla v_i^-, \nabla v_i^-\rangle$ on Ω. Substitute this into (2.5), use $v = v^+ - v^-$, and multiply it by -1 to get

$$\sum_i \int_{\partial\Omega} \gamma_i (v_i^-)^2 + \sum_i \int_\Omega |\nabla v_i^-|^2 + \int_\Omega \langle v^-, Mv^-\rangle - \int_\Omega \langle v^-, Mv^+\rangle \le 0. \tag{2.6}$$

Since M is positive definite, $\langle v^-, Mv^-\rangle \ge 0$, and since $v_i^- v_i^+ = 0$ and M has non-positive off-diagonal elements,

$$-\langle v^-, Mv^+\rangle = -\sum_i m_{ii} v_i^- v_i^+ + \sum_{i\ne j} (-m_{ij}) v_i^- v_j^+ \ge 0.$$

Therefore each term of (2.6) is non-negative, hence the left-hand side of (2.6) is constant zero, thus $v_i^- = 0$ from the third term due to M being uniformly elliptic, i.e. $v \geq 0$. This shows the desired positivity, and concludes the proof. $\square$

Remark 2.2. Consider the one dimensional ($n = 1$) case with $f(u(x)) = Mu(x)$, where $M \in \mathbb{R}^{m\times m}$; this way Ω is a real interval. Then the dominant eigenfunction of (2.2) with Dirichlet boundary condition is positive if and only if M has a positive eigenvector, which is in harmony with the previous proposition via the well-known Perron-Frobeinus theorem.

Remark 2.3. This proof also shows the monotone case, i.e. if f' is cooperative negative definite, then there is no need for the translation, thus via Krein-Rutman's theorem the dominant eigenvalue is negative, hence the equation (1.1)–(1.3) is stable.

Now set v to this dominant eigenvectorfunction, the existence of which was proved in Proposition 2.1. Multiply (2.1) by v, and (2.2) by u, and integrate them over Ω

$$\int_\Omega \langle v, \Delta u\rangle + \int_\Omega \langle v, f(u)\rangle = 0$$

$$\int_\Omega \langle u, \Delta v\rangle + \int_\Omega \langle u, f'(u)v\rangle = \lambda \int_\Omega \langle u, v\rangle$$

Use that $\langle u, f'(u)v\rangle = \langle v, f'(u)^T u\rangle$, and subtract the first one from the second one, and obtain

$$\int_\Omega \langle v, f'(u)^T u - f(u)\rangle = \lambda \int_\Omega \langle v, u\rangle, \tag{2.7}$$

since the first term of one equation eliminates against that of the other by Green's second formula

$$\int_\Omega (u_i \Delta v_i - v_i \Delta u_i) = 0$$

because of the common boundary condition.

Since both u and v are positive, guaranteeing the sign of $f'(u)^T u - f(u)$ provides the same sign of λ by (2.7). So denote $l : u \mapsto f'(u)^T u - f(u)$, i.e.

$$l_j(u) = \langle \partial_j f(u), u\rangle - f_j(u),$$

thus

$$\partial_k l_j(u) = \langle \partial_{jk} f(u), u\rangle + \partial_j f_k(u) - \partial_k f_j(u).$$

To have a constant sign of l for all positive values of u, it suffices to have $l(0)$ and l' of the same sign, for which it suffices to assume f' symmetric, and that f_i'' are entry-wise of the same sign for each i and f_i are of the opposite sign. On these conditions we have shown the sign of the dominant eigenvalue, which proves the following theorem.

Theorem 2.4. *Let f be such that f' symmetric and cooperative. Then*

(i) *if $f_i'' \geq 0$ entry-wise and $f_i(0) \leq 0$ with at least one of the inequalities being strict at zero for each coordinate i, then every nontrivial nonnegative solution of* (1.1)–(1.3) *is unstable, while*

(ii) *if $f_i'' \leq 0$ entry-wise and $f_i(0) \geq 0$ with at least one of the inequalities being strict at zero for each coordinate i, then every nontrivial nonnegative solution of* (1.1)–(1.3) *is stable.*

Remark 2.5. This result can be extended to the general elliptic operator of the coordinate-wise form $Lu := \operatorname{div}(A\nabla u)$ as in [8], that is $(Lu)_i = \operatorname{div}(A_i \nabla u_i)$, where all A_i is strongly elliptic, and to more general nonlinearities $f : \Omega \times \mathbb{R}^m \to \mathbb{R}^m$, Hence our equation reads now

$$\partial_t u(t,x) = Lu(t,x) + f(x, u(t,x)) \tag{2.8}$$

subject to the boundary

$$hu\big|_{\partial\Omega} + g\partial_{A\nu} u\big|_{\partial\Omega} = 0, \tag{2.9}$$

where ν is the outer unit normal vector of Ω, and

$$(\partial_{A\nu} u)_i = \partial_{A_i\nu} u_i = \langle A_i \nu, \nabla u_i\rangle. \tag{2.10}$$

We only indicate the differences in the proof. The linearised eigenvalue problem reads as

$$\begin{gathered} Lv + f'(u)v = \lambda v \\ hv\big|_{\partial\Omega} + g\partial_{A\nu} v\big|_{\partial\Omega} = 0. \end{gathered} \tag{2.11}$$

In the proof of Proposition 2.1, now we show the positivity of $(-L+M)^{-1}$. Hence let (cf. (2.4))

$$-Lv + Mv \geq 0,$$

from which the inequality which yields positivity is (cf. (2.6))

$$\int_{\partial\Omega} \langle v^-, \gamma v^-\rangle + \sum_i \int_\Omega |\nabla v_i^-|^2 + \int_\Omega \langle v^-, Mv^-\rangle - \int_\Omega \langle v^-, Mv^+\rangle \leq 0,$$

for which the same argument holds, where M and γ is defined similarly using the conormal derivatives implied by (2.9).

Now the combination of (2.8)–(2.10) and (2.11) will be

$$\int_\Omega \langle v, (\partial_u f)^T u - f\rangle = \lambda \int_\Omega \langle v, u\rangle.$$

Then l is to be defined as

$$l(u) := (\partial_u f)^T u - f,$$

for which the rest literally holds with the appropriate changes in the notation. Hence the necessary condition for stability translates to sign conditions of $f_i(\cdot, 0)$ and entry-wise $\partial_u^2 f_i$ for all $i = 1, \dots, m$ and $x \in \Omega$,

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