Electronic Journal of Differential Equations, Vol. 2004(2004), No. 109, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

ISOPERIMETRIC INEQUALITY FOR AN INTERIOR FREE BOUNDARY PROBLEM WITH P-LAPLACIAN OPERATOR

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Abstract. By considering the p-Laplacian operator, we establish an existence and regularity result for a shape optimization problem. From a monotony result, we show the existence of a solution to the interior problem with a free surface for a family of Bernoulli constants. We also give an optimal estimation for the upper bound of the Bernoulli constant.

1. Introduction

Let us study the following interior Bernoulli problem: Given K, a $\mathcal{C}^2$-regular and bounded domain in $\mathbb{R}^N$, and a constant $c > 0$, find a domain Ω and a function u_Ω such that

$$\begin{gathered} -\Delta_p u_\Omega = 0 \quad \text{in } K\backslash\bar{\Omega},\ 1 < p < \infty \\ u_\Omega = 1 \quad \text{on } \partial\Omega \\ u_\Omega = 0 \quad \text{on } \partial K \\ \frac{\partial u_\Omega}{\partial \nu} = c \quad \text{on } \partial\Omega\,. \end{gathered} \tag{1.1}$$

Here Δ_p denotes the p-Laplace operator, i.e. $\Delta_p u := \operatorname{div}(\|\nabla u\|^{p-2}\nabla u)$ and ν is the interior unit normal of Ω. In this paper we give an optimal estimation for the upper bound of the Bernoulli constant c. This problem arises in various nonlinear flow laws, and other physical situations, e.g. electrochemical machining and potential flow in fluid mechanics. In the linear case a classical approach for such a problem consists in considering a variational formulation [1].

Inspired by the pioneering work of Beurling, where the notion of sub and super-solutions in geometrical case is used, Henrot and Shahgohlian studied this problem in [6]. They proved that when $K \subset \mathbb{R}^N$ is a bounded and convex domain:

- There exists a classical convex solution to (1.1) if only if $c \geq c_K$.
- The constant c_K is underestimated by $\frac{1}{R_K}$, where $R_K = \sup\{R > 0 : B(o,R) \subset K\}$.

2000 *Mathematics Subject Classification.* 35R35.
Key words and phrases. Bernoulli free boundary problem; starshaped domain; shape optimization; shape derivative; monotony.

Submitted April 7, 2004. Published September 14, 2004.

- For $N = 2$ and $p = 2$, the minimal value c_K, depending on K and p for which problem (1.1) has a solution is estimated from above by
$$c_K \leq \frac{6.252}{R_K}. \tag{1.2}$$
But this inequality is not optimal, since K is a disk of radius R.

In [5, p. 202], by considering the Laplacian, Flucher and Rumpf set the following problem:

> Let K be a connected domain and K^* a ball such that $\mathrm{vol}(K) = \mathrm{vol}(K^*)$. Let c_K(respectively c_{K^*}) be the minimal value of c for which the interior Bernoulli problem (1.1) admits a solution. Does c_K satisfy the isoperimetric inequality $c_K \geq c_{K^*}$?

In [3], Cardaliaguet and Tahraoui gave an estimate from above for the Bernoulli constant, by using the harmonic radius. But they didn't give an answer to the question posed by Flucher and Rumpf.

Now, by combining a variational approach and a sequential method, we establish an existence result for non-necessary convex domains. Then we show that c_K satisfies the isoperimetric inequality in the sense that $\max\{c_K : \mathrm{vol}(K) = \mathrm{vol}(K^*)\} \geq c_{K^*}$. This comparison answers the question posed by Flucher and Rumpf.

The structure of this paper is as follows: In the first part, we present the main result. In the second section, we give auxiliary results. The third part deals with the study of the shape optimization problem and the existence of Lagrange multiplier λ_Ω. Namely, we study at first the existence result for the shape optimization problem: Find
$$\min\{J(w), w \in \mathcal{O}_\epsilon\},$$
where
$$\mathcal{O}_\epsilon = \{w \subset K : w \text{ is an open set satisfying the } \epsilon\text{-cone property and } \mathrm{vol}(w) = m_0\}$$
where vol denotes the volume, m_0 is a fixed value in $\mathbb{R}_+^*$. The functional J is
$$J(w) := \frac{1}{p}\int_{K\setminus\bar{w}} \|\nabla u_w\|^p dx,$$
where u_w is a solution to the Dirichlet problem
$$\begin{gathered}
-\Delta_p u_w = 0 \quad \text{in } K\setminus\bar{w},\ 1 < p < \infty \\
u_w = 1 \quad \text{on} \quad \partial w \\
u_w = 0 \quad \text{on} \quad \partial K.
\end{gathered} \tag{1.3}$$
Next, we obtain an optimality condition:
$$\frac{\partial u}{\partial \nu} = (\frac{p}{p-1}\lambda_\Omega)^{1/p} \quad \text{on } \partial\Omega.$$
Then we conclude this section with a monotony result. The last part is devoted to the proof of the main result.

2. Main result

Let K be a $\mathcal{C}^2$-regular, star-shaped and bounded domain and K^* a ball of radius R_1 centered at the origin such that $\operatorname{vol}(K) = \operatorname{vol}(K^*)$. Let

$$\alpha(R_K, p, N) := \begin{cases} e/R_K & \text{if } p = N \\ \frac{|\frac{p-N}{p-1}|}{\left|(\frac{p-1}{N-1})^{\frac{N-1}{N-p}} - (\frac{p-1}{N-1})^{\frac{p-1}{N-p}}\right|} \frac{1}{R_K} & \text{if } p \neq N. \end{cases}$$

where $R_K = \sup\{R > 0 : B(o, R) \subset K\}$. Let

$$\mathcal{E} := \{c_K : \operatorname{vol}(K) = \operatorname{vol}(K^*)\},$$

where c_K is the minimal value for which the interior Bernoulli problem (1.1) admits a solution.

Theorem 2.1. *If the solution Ω of the shape optimization problem $\min\{J(w), w \in \mathcal{O}_\epsilon\}$ is $\mathcal{C}^2$-regular, then for all constant $c > 0$ satisfying $c \geq \alpha(R_K, p, N)$, Ω is the classical solution of the free-boundary problem (1.1). Moreover:*

(i) *The constant c_K satisfies $0 < c_K \leq \alpha(R_K, p, N)$.*

(ii) *Replacing K by K^*, the constant c_{K^*} which is the minimal value for which (1.1) admits a solution, satisfies*

$$c_{K^*} = \alpha(R_1, p, N),$$
$$0 < c_{K^*} \leq \alpha(R_K, p, N).$$

We have also $\alpha(R_K, p, N) = \max \mathcal{E}$.

To prove this theorem we need some auxiliary results.

3. Auxiliary results

For the rest of this article, we consider a fixed, closed domain D which contains all the open subsets used.

Let ζ be an unitary vector of $\mathbb{R}^N$, ϵ be a real number strictly positive and y be in $\mathbb{R}^N$. We call a cone with vertex y, of direction ζ and angle to the vertex and height ϵ, the set defined by

$$\mathcal{C}(y, \zeta, \epsilon, \epsilon) = \{x \in \mathbb{R}^N : |x - y| \leq \epsilon \text{ and } |(x - y)\zeta| \geq |x - y| \cos \epsilon\}.$$

Let Ω be an open set of $\mathbb{R}^N$, Ω is said to have the ϵ-cone property if for all $x \in \partial\Omega$ then there exists a direction ζ and a strictly positive real number ϵ such that

$$\mathcal{C}(y, \zeta, \epsilon, \epsilon) \subset \Omega, \text{for all } y \in B(x, \epsilon) \cap \bar{\Omega}.$$

Let K_1 and K_2 be two compact subsets of D. Let

$$d(x, K_1) = \inf_{y \in K_2} d(x, y), \quad d(x, K_2) = \inf_{y \in K_1} d(x, y).$$

Note that

$$\rho(K_1, K_2) = \sup_{x \in K_2} d(x, K_1), \quad \rho(K_2, K_1) = \sup_{x \in K_1} d(x, K_2).$$

Let

$$d_H(K_1, K_2) = \max[\rho(K_1, K_2), \rho(K_2, K_1)],$$

we call Hausdorff distance of K_1 and K_2, the following positive number, denoted $d_H(K_1, K_2)$.

Let (Ω_n) be a sequence of open subsets of D and Ω be an open subset of D. We say that the sequence (Ω_n) converges on Ω in the Hausdorff sense and we denote by $\Omega_n \xrightarrow{H} \Omega$ if $\lim_{n\to+\infty} d_H(\bar{D}\backslash\Omega_n, \bar{D}\backslash\Omega) = 0$.

Let (Ω_n) be a sequence of open sets of $\mathbb{R}^N$ and Ω be an open set of $\mathbb{R}^N$. We say that the sequence (Ω_n) converges on Ω in the sense of L^p, $1 \le p < \infty$ if χ_{Ω_n} converges on χ_Ω in $L^p_{\rm loc}(\mathbb{R}^N)$, χ_Ω being the characteristic functions of Ω.

Let (Ω_n) be a sequence of open subsets of D and Ω be an open subset of D. We say that the sequence (Ω_n) converges on Ω in the compact sense if

(1) Every compact G subset of Ω, is included in Ω_n for n large enough.
(2) Every compact Q subset of $\bar{\Omega}^c$, is included in $\bar{\Omega}_n^c$ for n large enough.

Lemma 3.1. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions of $L^p(\Omega)$, $1 \le p < \infty$ and $f \in L^p(\Omega)$. We suppose f_n converges on f a.e. and $\lim_{n\to\infty} \|f_n\|_p = \|f\|_p$. Then we have $\lim_{n\to\infty} \|f_n - f\|_p = 0$.*

For the proof of the above lemma see for example [7].

Lemma 3.2 (Brezis-Lieb)**.** *Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence of functions of $L^p(\Omega)$, $1 \le p < \infty$. We suppose that f_n converges on f a.e., then $f \in L^p(\Omega)$ and $\|f\|_p = \lim_{n\to\infty}(\|f_n - f\|_p + \|f_n\|_p)$.*

For the proof of the above lemma, see for example [7].

Lemma 3.3. *Let $(\Omega_n)_{n\in\mathbb{N}}$ be a sequence of open sets in $\mathbb{R}^N$ having the ϵ-cone property, with $\bar{\Omega}_n \subset F \subset D$, F a compact set and D a ball, then, there exists an open set Ω, included in F, which satisfies the $\frac{\epsilon}{2}$-cone property and a subsequence $(\Omega_{n_k})_{k\in\mathbb{N}}$ such that*

$$\chi_{\Omega_{n_k}} \xrightarrow{L^1} \chi_\Omega, \quad \Omega_{n_k} \xrightarrow{H} \Omega$$
$$\partial\Omega_{n_k} \xrightarrow{H} \partial\Omega, \quad \bar{\Omega}_{n_k} \xrightarrow{H} \bar{\Omega}.$$

The above lemma is a well known result in functional analysis related to shape optimization. But let us present the proof again.

Proof. It is known that the Hausdorff topology is compact, then there exists a subsequence $(\Omega_{n_k})_{k\in\mathbb{N}}, \Omega$ and an open set Ω such that $\Omega_{n_k} \xrightarrow{H} \Omega$, $\chi_{\Omega_{n_k}} \to f$ with $\sigma(L^\infty, L^1)$, $0 \le f \le 1$ and $\chi_\Omega \le f$ a.e. To obtain $\chi_\Omega = f$ a.e., we have only to show that f is identically equal to zero on $D\backslash\Omega$. Let us take $x \in \partial\Omega$ since $\bar{D}\backslash\Omega_{n_k} \xrightarrow{H} \bar{D}\backslash\Omega$ and denoting again n_k by n, there exists $x_n \in \bar{D}\backslash\Omega_n$ such that x_n converges on x.

Let $\hat{x}_n \in \partial\Omega_n$ such that $\|x_n - \hat{x}_n\| = d(x_n, \partial\Omega_n)$, we claim that $\hat{x}_n$ converges on x if not there exists n_i and $\eta > 0$ such that $d(x_{n_i}, \partial\Omega_{n_i}) \ge \eta$. This implies that $B(x_{n_i}, \eta) \subset \bar{D}\backslash\Omega_{n_i}$ and by the continuity of the inclusion for the Hausdorff convergence, we have $\bar{B}(x, \eta) \subset \bar{D}\backslash\Omega$. This is impossible, because $x \in \partial\Omega$. Since by assumption Ω_n satisfies the ϵ-cone property we have $C(\hat{x}_n, \zeta(\hat{x}_n), \epsilon, \epsilon) \subset \bar{D}\backslash\Omega_n$. There also exists a subsequence of $\zeta(\hat{x}_n)$ which converges on $\zeta = \zeta(x)$. By passing to the limit, we have $C(x, \zeta(x), \epsilon, \epsilon) \subset \bar{D}\backslash\Omega$, and then $C(x, \zeta(x), \frac{\epsilon}{2}, \frac{\epsilon}{2}) \subset \bar{D}\backslash\Omega$. Let us take $y \in B(x, \epsilon) \cap \bar{D}\backslash\Omega$, then there exists $y_n \in \bar{D}\backslash\Omega_n$ such that y_n converges on y and we have $\|y_n - x_n\|$ converges on $\|y - x\| < \epsilon$ and $\|x_n - \hat{x}_n\|$ converges on 0. Then, for n big enough, we have $\|y_n - \hat{x}_n\| < \epsilon$.

The ϵ-cone property implies that $C(y_n, \zeta(\hat{x}_n), \epsilon, \epsilon) \subset D\backslash\bar{\Omega}_n$ and by the continuity of the inclusion for the Hausdorff convergence, we obtain $C(y, \zeta(\bar{x}), \epsilon, \epsilon) \subset \bar{D}\backslash\Omega$ then

$C(y,\zeta(x),\frac{\epsilon}{2},\frac{\epsilon}{2})\subset\bar{D}\backslash\Omega$. This means that the $\frac{\epsilon}{2}-$ cone property is satisfied by $\bar{D}\backslash\Omega$ and then by Ω too. Let us take $\phi\in L^1(D)$, then,

$$\begin{aligned}\int_{C(y,\zeta(x),\epsilon,\epsilon)}\phi dx &= \lim_{n\to\infty}\int_{C(y_n,\zeta(\hat{x}_n),\epsilon,\epsilon)}\phi dx\\ &= \lim_{n\to\infty}\int_{C(y_n,\zeta(\hat{x}_n),\epsilon,\epsilon)}\chi_{D\backslash\Omega_n}\phi dx\\ &= \int_{C(y,\zeta(x),\epsilon,\epsilon)}\phi(\chi_D-f)dx\\ &= \int_{C(y,\zeta(x),\epsilon,\epsilon)}\phi dx-\int_{C(y,\zeta(x),\epsilon,\epsilon)}\phi f dx.\end{aligned}$$

We obtain that $\int_{C(y,\zeta(x),\epsilon,\epsilon)}\phi f dx=0$, for all $\phi\in L^1(D)$ and then $f=0$ on $C(y,\zeta(x),\epsilon,\epsilon)$ a.e.

By varying $y\in B(x,\epsilon)\cap\bar{D}\backslash\Omega$ and next $x\in\partial\Omega$, we obtain $f=0$ on the set $\{x\in D\backslash\Omega; d(x,\partial\Omega)<\epsilon\}$.

By the same reasoning for $y\in D\backslash\Omega$ such that $d(y,\partial\Omega)\geq\epsilon$, we show that $f=0$ on $\{y\in D\backslash\Omega; d(y,\partial\Omega)\geq\epsilon\}$. We also have just showed that $\chi_{\Omega_{n_k}}$ converges on χ_Ω a.e. and in $L^1(D)$ sense.

Now we show that $\bar{\Omega}_{n_k}\xrightarrow{H}\bar{\Omega}$: for a subsequence Ω_{n_k} such that $\bar{\Omega}_{n_k}\xrightarrow{H}G$ and it is sufficient to show that $G=\bar{\Omega}$. Let $\bar{B}(x,\eta)\subset\Omega$ then $\bar{B}(x,\eta)\subset\Omega_{n_k}$ for n large enough, then, $\bar{B}(x,\eta)\subset\bar{\Omega}_{n_k}$. By the continuity of the inclusion for the Hausdorff convergence, we have $\bar{B}(x,\eta)\subset G$ for any ball in Ω. This imply that $\Omega\subset G$ then $\bar{\Omega}\subset G$. Let $F=\bar{D}\backslash\Omega$ and $x\in G\cap F$, we have to show that $x\in\bar{\Omega}$. We remark that, there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}\subset\bar{\Omega}_{nk}$ such that x_{n_k} converges on x and $y_{n_k}\in\bar{D}\backslash\Omega_{n_k}$ such that y_{n_k} converges on x. The sequence $\hat{x}_{n_k}$ belongs to $[x_{n_k},y_{n_k}]\cap\partial\Omega_{n_k}$, then we have $\hat{x}_{n_k}$ which converges on x.

It is interesting to remark that

$$\begin{gathered}C(\hat{x_{n_k}},\zeta(\hat{x}_{n_k}),\epsilon,\epsilon)\subset\Omega_{n_k},\quad \Omega_{n_k}\subset\bar{\Omega}_{n_k},\\ C(\hat{x}_{n_k},-\zeta(\hat{x}_{n_k}),\epsilon,\epsilon)\subset D\backslash\bar{\Omega}_{n_k},\quad D\backslash\bar{\Omega}_{n_k}\subset\bar{D}\backslash\Omega_{n_k}.\end{gathered}$$

We can assume that $\zeta(\hat{x}_{n_k})$ converges on $\zeta(x)$ then

$$\begin{gathered}C(x,\zeta(x),\epsilon,\epsilon)\subset G\\ C(x,-\zeta(x),\epsilon,\epsilon)\subset\bar{D}\backslash\Omega.\end{gathered}$$

Let $\eta>0$ and set

$$C_{n_k}(\eta)=\{z\in C(\hat{x}_{n_k},\zeta(\hat{x}_{n_k}),\epsilon,\epsilon),d(z,\partial C(\hat{x}_{n_k},\zeta(\hat{x}_{n_k}),\epsilon,\epsilon)\geq\eta\}.$$

We remark that $\rho(C_{n_k}(\eta),F_{n_k})\geq\eta$, and by passing to the limit $\rho(C(\eta),F)\geq\eta$, then $C(\eta)\subset\Omega$. This implies that $\bar{C}(x,\zeta(x),\epsilon,\epsilon)\subset\bar{\Omega}$, then $G\cap F\subset\bar{\Omega}$ and $G\cap F\subset\partial\Omega$. It is easy to see that by an absurdity reasoning, we have $G\backslash\bar{\Omega}=\emptyset$, and then $G\subset\bar{\Omega}$. □

4. Shape optimization and monotony results

Theorem 4.1. *The problem "Find $\Omega\in\mathcal{O}_\epsilon$ such that $J(\Omega)=\min\{J(w),w\in\mathcal{O}_\epsilon\}$" admits a solution.*

Proof. Consider the function $\tilde{u}$ defined by

$$\tilde{u} = \begin{cases} u & \text{if } x \in K\backslash\bar{\Omega} \\ 1 & \text{if } x \in \bar{\Omega} \end{cases}$$

$$\nabla\tilde{u} = \begin{cases} \nabla u & \text{if } x \in K\backslash\bar{\Omega} \\ 0 & \text{if } x \in \bar{\Omega} \end{cases}$$

Let E be a functional defined on $W_0^{1,p}(K)$ by

$$E(\tilde{u}_w) = \frac{1}{p}\int_K \|\nabla\tilde{u}_w\|^p dx, \quad 1 < p < \infty$$

where $\tilde{u}_w$ is the extension by 1 in $\bar{\Omega}$ of u_w solution of the problem

$$\begin{gathered} -\Delta_p u_w = 0 \quad \text{in } w\backslash K \\ u_w = 1 \quad \text{on } \partial w \\ u_w = 0 \quad \text{on } \partial K \end{gathered}$$

Let $J(w) := E(\tilde{u}_w)$. Then $J(w) > 0$ this implies that $\inf\{J(w), w \in \mathcal{O}_\epsilon\} > -\infty$. Let $\alpha = \inf\{J(w), w \in \mathcal{O}_\epsilon\}$. Then, there exists a minimizing sequence $(\Omega_n)_{n\in\mathbb{N}} \subset \mathcal{O}_\epsilon$ such that $J(\Omega_n)$ converges on α.

Since the sequence $(\Omega_n)_{n\in\mathbb{N}}$ is bounded, there exists a compact set F such that $\bar{\Omega}_n \subset F \subset K$. By lemma (3.3), there is a subsequence $(\Omega_{n_k})_{k\in\mathbb{N}}$, and Ω verifying the ϵ-cone property such that $\Omega_{n_k} \xrightarrow{H} \Omega$ and $\chi_{\Omega_{n_k}} \to \chi_\Omega$ a.e. Let us set $u_{\Omega_n} = u_n$ and show that the sequence $(\tilde{u}_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(K)$. If not, for all s there exists a subsequence denoted $\tilde{u}_n^s \in W_0^{1,p}(K)$ such that $\int_K \|\nabla\tilde{u}_n\|^p dx > s$ and

$$\begin{gathered} \int_K \|\nabla\tilde{u}_n^s\|^p dx = \int_{K\backslash\bar{\Omega}_n} \|\nabla\tilde{u}_n^s\|^p dx + \int_{\bar{\Omega}_n} \|\nabla\tilde{u}_n^s\|^p dx\,, \\ \int_K \|\nabla\tilde{u}_n^s\|^p dx = \int_{K\backslash\bar{\Omega}_n} \|\nabla\tilde{u}_n^s\|^p dx. \end{gathered}$$

That is, $J(\Omega_n)$ converges on $+\infty$. Then, $\inf\{J(w), w \in \mathcal{O}_\epsilon\} = +\infty$ is a contradiction. Since $W^{1,p}(K)$ is a reflexive space, there exists a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and u^* such that u_{n_k} converges weakly on u^* in $W^{1,p}(K)$ and

$$\int_{K\backslash\bar{\Omega}} \|\nabla u^*\|^p dx \le \liminf \int_{K\backslash\bar{\Omega}_{n_k}} \|\nabla u_{n_k}\|^p dx.$$

From the above we get $J(\Omega) \le J(\Omega_{n_k})$ and $J(\Omega) \le \inf\{J(w), w \in \mathcal{O}_\epsilon\}$. Finally, we have $J(\Omega) = \min\{J(w), w \in \mathcal{O}_\epsilon\}$. □

Remark 4.2. On the one hand, it is easy to verify that u^* equals u_Ω and satisfies

$$\begin{gathered} -\Delta_p u^* = 0 \quad \text{in } K\backslash\bar{\Omega} \\ u^* = 1 \quad \text{on } \partial\Omega \\ u^* = 0 \text{on } \partial K \end{gathered}$$

On the other hand, we have a regularity of u_Ω solution to the problem (1.3); see [4, 9, 18].

For the rest of this article, we assume that Ω is $\mathcal{C}^2$-regular in order to use the shape derivatives. This hypothesis is possible because if we work with a class of domains which are $\mathcal{C}^3$-regular and verifying the geometric normal property, we can show that Ω solution to the shape optimization problem is $\mathcal{C}^2$-regular.

Theorem 4.3. *Let L be a compact set of $\mathbb{R}^N$. Let $(f_n)_{(n\in\mathbb{N})}$ be a sequence of functions, $f_n \in \mathcal{C}^3(L)$ with*

$$|\frac{\partial f_n}{\partial x_i}| \leq M, \quad |\frac{\partial^2 f_n}{\partial x_i \partial x_j}| \leq M, \quad |\frac{\partial^3 f_n}{\partial x_i \partial x_j \partial x_k}| \leq M,$$

where M is a positive constant independent of n. We define a sequence $(\Omega_n)_{(n\in\mathbb{N})}$, by $\Omega_n = \{x \in L : f_n(x) > 0\}$. We assume that there exists $\alpha > 0$ such that $|f_n(x)| + |\nabla f_n(x)| \geq \alpha$ for all x belonging to L. We assume in addition that Ω_n has the geometric normal property. Then there exists, Ω a $\mathcal{C}^2$-regular domain and a subsequence of $(\Omega_n)_{(n\in\mathbb{N})}$ denoted $(\Omega_{n_k})_{(k\in\mathbb{N})}$ such that Ω_{n_k} converges in the compact sense on Ω and $J(\Omega) = \min\{J(w) : w \in \mathcal{O}_\epsilon\}$.

We remark thatΩ_n and Ω as above belong to $\mathcal{O}_\epsilon$. For this theorem, we need the following lemma. Then the proof of Theorem 4.3 can be found in [13].

Lemma 4.4. *Let $(f_n)_{(n\in\mathbb{N})}$ be a sequence functions defined as in theorem 4.3. One supposes that Ω is an open set defined by*

$$\Omega = \{x \in L : h(x) > 0\} \quad \text{with} \quad \partial\Omega = \{x \in L : h(x) = 0\}$$

where h is a continuous function defined on L which is a compact set of $\mathbb{R}^N$. If f_n converges uniformly on h, then we have Ω_n converges in the compact sense to Ω.

Proof. Let K_1 be a compact set included in Ω, and let $\alpha = \inf_{K_1} h$, we have $\alpha > 0$. There exists n_0 belonging to $\mathbb{N}$, such that for all $n \geq n_0$ we get $|f_n - h|_{L^\infty(K)} < \alpha$. Then for all x belonging to K_1 we have $f_n(x) > h(x) - \alpha \geq 0$ for $n \geq n_0$. This implies that K_1 is contained in Ω_n.

Let L_0 be a compact subset of $\bar{\Omega}^c$ by hypothesis we have $\bar{\Omega} = \Omega \cup \partial\Omega = \{x \in L : h(x) \geq 0\}$ then $\beta := \max_{L_0} h < 0$. Therefore, there exists n_1 belonging to $\mathbb{N}$ such that for all $n \geq n_1$ implies that $|f_n - h|_{L^\infty(L_0)} < -\beta$. One has $f_n(x) \leq h(x) - \beta$ for all x belonging to L_0. This implies that $f_n(x) \leq 0$ and then L_0 is contained in $\bar{\Omega}_n^c$ because $\{x \in L : h(x) < 0\} \subset \bar{\Omega}_n^c$. □

The next theorem gives necessary conditions of optimality.

Theorem 4.5. *If Ω is the solution of the shape optimization problem* $\min\{J(w) : w \in \mathcal{O}_\epsilon\}$*, then there exists a Lagrange multiplier $\lambda_\Omega > 0$ such that $\frac{\partial u}{\partial \nu} = (\frac{p}{p-1}\lambda_\Omega)^{1/p}$ on $\partial\Omega$.*

Proof. The main technique used to prove this result is the shape derivatives as used in [16, 15]. For the computations, we refer to [10, page 42-52], □

Remark 4.6. A consequence of the Theorems (4.1) and (4.5) is that (Ω, u_Ω) satisfies

$$\begin{gathered}
-\Delta_p u_\Omega = 0 \quad \text{in } K\backslash\bar{\Omega}, 1 < p < \infty \\
u_\Omega = 1 \quad \text{on } \partial\Omega \\
u_\Omega = 0 \quad \text{on } \partial K \\
\frac{\partial u_\Omega}{\partial \nu} = \big(\frac{p}{p-1}\lambda_\Omega\big)^{1/p} \quad \text{on } \partial\Omega\,.
\end{gathered}$$

To conclude this section, we state a monotony result, in the following sense.

Theorem 4.7. *Suppose that K is star-shaped with respect to the origin. Let Ω_1 and Ω_2 be two different solutions to the shape optimization problem $\min\{J(w), w \in \mathcal{O}_\epsilon\}$, star-shaped with respect to the origin such that $\Omega_1 \subset \Omega_2$ and $\partial\Omega_1 \cap \partial\Omega_2 \neq \emptyset$, then $\lambda_{\Omega_1} \geq \lambda_{\Omega_2}$.*

Proof. For any $i \in \{1, 2\}$, if Ω_i is the solution of the shape optimization problem $\min\{J(w), w \in \mathcal{O}_\epsilon\}$, we have u_i which satisfies that

$$\begin{gathered}
-\Delta_p u_i = 0 \quad \text{in} K\backslash\bar{\Omega}_i,\ 1 < p < \infty \\
u_i = 1 \quad \text{on } \partial\Omega_i \\
u_i = 0 \quad \text{on } \partial K\,.
\end{gathered}$$

On the one hand, consider the problem

$$\begin{gathered}
-\Delta_p z = 0 \quad \text{in} K\backslash\bar{\Omega}_2,\ 1 < p < \infty \\
z = u_1 \quad \text{on } \partial\Omega_2 \\
z = 0 \quad \text{on } \partial K.
\end{gathered} \tag{4.1}$$

It is easy to see that $z = u_1$ is a solution to problem (4.1). We have $0 \leq u_1 \leq 1$, $0 \leq u_2 \leq 1$, and $u_2 \geq u_1$ on $\partial(K\backslash\bar{\Omega}_2)$. By the comparison principle [17], we obtain $u_2 \geq u_1$ in $K\backslash\bar{\Omega}_2$. Let $x_0 \in \partial\Omega_1 \cap \partial\Omega_2$, then

$$\frac{u_2(x_0 - \nu h) - u_2(x_0)}{h} \geq \frac{u_1(x_0 - \nu h) - u_1(x_0)}{h}.$$

Passing to the limit,

$$\lim_{h\to 0} \frac{u_2(x_0 - \nu h) - u_2(x_0)}{h} \geq \lim_{h\to 0} \frac{u_1(x_0 - \nu h) - u_1(x_0)}{h},$$

this implies

$$-\frac{\partial u_2}{\partial \nu}(x_0) \geq -\frac{\partial u_1}{\partial \nu}(x_0)\,;$$

hence, $\frac{\partial u_2}{\partial \nu}(x_0) \leq \frac{\partial u_1}{\partial \nu}(x_0)$.

On the other hand u_1 and u_2 are solutions to the shape optimisation problem , then there exists λ_{Ω_1} and λ_{Ω_2} such that

$$\begin{gathered}
\frac{\partial u_1}{\partial \nu} = (\frac{p}{p-1}\lambda_{\Omega_1})^{1/p} \quad \text{on } \partial\Omega_1\,, \\
\frac{\partial u_2}{\partial \nu} = (\frac{p}{p-1}\lambda_{\Omega_2})^{1/p} \quad \text{on } \partial\Omega_2.
\end{gathered}$$

Then $\frac{\partial u_1}{\partial \nu}(x_0) \geq \frac{\partial u_2}{\partial \nu}(x_0)$ is equivalent to

$$(\frac{p}{p-1}\lambda_{\Omega_1})^{1/p} \geq (\frac{p}{p-1}\lambda_{\Omega_2})^{1/p}$$

and therefore $\lambda_{\Omega_1} \geq \lambda_{\Omega_2}$. $\square$

5. Proof of the main result

We use the preceding theorems to prove the main result.

Proof of Theorem 2.1. Let $R_K = \sup\{R > 0 : B(o,R) \subset K\}$. Let $r > 0$ such that $B(o,r) \subset B(o,R_K)$. First, we have to look for a solution u_0 to the problem

$$\begin{gathered} -\Delta_p u = 0 \quad \text{in } B_{R_K} \backslash B_r \\ u = 0 \quad \text{on } \partial B_{R_K} \\ u = 1 \quad \text{on } \partial B_r \,. \end{gathered} \tag{5.1}$$

The solution u_0 is explicitly determined by

$$u_0(x) = \begin{cases} \frac{\ln \|x\| - \ln R_K}{\ln r - \ln R_K} & \text{if } p = N \\ \frac{-\|x\|^{\frac{p-N}{p-1}} + R_K^{\frac{p-N}{p-1}}}{R_K^{\frac{p-N}{p-1}} - r^{\frac{p-N}{p-1}}} & \text{if } p \neq N, \end{cases} \tag{5.2}$$

and

$$\|\nabla u_0(x)\| = \begin{cases} \frac{1}{r(\ln R_K - \ln r)} & \text{if } p = N \\ \frac{|\frac{p-N}{p-1}| \|x\|^{\frac{-N+1}{p-1}}}{|r^{\frac{p-N}{p-1}} - R^{\frac{p-N}{p-1}}|} & \text{if } p \neq N. \end{cases}$$

In particular $\|\nabla u_0\| > c$ on ∂B_r for r small enough.
Now consider the following problem

$$\begin{gathered} -\Delta_p u = 0 \quad \text{in } K \backslash B_r \\ u = 1 \quad \text{on } \partial B_r \\ u = 0 \quad \text{on } \partial K. \end{gathered} \tag{5.3}$$

The problem (5.3) admits a solution denoted by u_r. This solution is obtained by minimizing the functional J defined on the Sobolev space

$$V' = \{v \in W^{1,p}(K \backslash B_r), v = 1 \text{on } \partial B_r \text{ and} v = 0 \text{ on } \partial K\}$$

and $J(v) = \frac{1}{p} \int_{K \backslash B_r} \|\nabla v\|^p dx$.

Consider the problem

$$\begin{gathered} -\Delta_p v = 0 \quad \text{in } B_{R_K} \backslash B_r \\ v = 1 \quad \text{on } \partial B_r \\ v = u_r \quad \text{on } \partial B_{R_K}. \end{gathered} \tag{5.4}$$

It is easy to see that $v = u_r$ is a solution to problem (5.4). By the comparison principle [17], we obtain $0 \leq u_0 \leq 1$ and $0 \leq u_r \leq 1$. On $\partial(B_{R_K} \backslash B_r)$, we obtain $u_r \geq u_0$ and then, $u_r \geq u_0$ in $B_{R_K} \backslash B_r$. Finally, we have $\|\nabla u_r\| \leq \|\nabla u_0\| \text{on} \partial B_r$.

Case where $p = N$.

$$\|\nabla u_0\|_{|\partial B_r} = \frac{1}{r(\ln R_K - \ln r)} = g(r), \quad \forall r \in]0, R_K[.$$

It is easy to see that $g(r)$ is a strictly decreasing function on $]0, \frac{R_K}{e}[$ and a strictly increasing function on $]\frac{R_K}{e}, R_K[$. Then for all $r \in]0, R_K[$, $\|\nabla u_0\|_{|\partial B_r} \geq g(\frac{R_K}{e}) = \frac{e}{R_K}$.

(1) For $c = e/R_K$, let $\delta > 0$ be a fixed and sufficiently small number. To initialize we choose $r_0 \in]0, \frac{R_K}{e}[\cup]\frac{R_K}{e}, R_K[$ such that $\big|\|\nabla u_0\|_{|\partial B_{r_0}} - c\big| > \delta$. To fix ideas let us consider $r_0 \in]0, \frac{R_K}{e}[$. The process will be identical if $r_0 \in]\frac{R_K}{e}, R_K[$.

By varying r in the increasing sense, we will achieve a step denoted n such that

$$r_n \in]0, \frac{R_K}{e}[\text{and } \big|\|\nabla u_0\|_{|\partial B_{r_n}} - c\big| < \delta.$$

Consider $\mathcal{O}_n$ the class of admissible domains defined as follows

$$\mathcal{O}_n = \{w \in \mathcal{O}_\epsilon, B_{r_n} \subset w, \partial B_{r_n} \cap \partial w \neq \emptyset, \text{ and } \operatorname{vol}(w) = V_0\},$$

where V_0 denotes a fixed positive constant. We look for $\Omega \in \mathcal{O}_n$ such that

$$\begin{gathered} -\Delta_p u = 0 \quad \text{in } K\backslash\bar{\Omega} \\ u = 1 \quad \text{on } \partial\Omega \\ u = 0 \quad \text{on} \partial K \\ \frac{\partial u}{\partial \nu} = c_\Omega \quad \text{on } \partial\Omega \end{gathered} \tag{5.5}$$

where $c_\Omega = (\frac{p}{p-1}\lambda_\Omega)^{1/p}$. Applying the theorem (4.1), the shape optimization problem $\min\{J(w), w \in \mathcal{O}_n\}$ admits a solution and by theorem (4.5), Ω satisfies the overdetermined boundary condition $\frac{\partial u}{\partial \nu} = c_\Omega$. Then problem (5.5) admits a solution .

Since $\Omega \in \mathcal{O}_n$, we have $B_{r_n} \subset \Omega$, $\partial B_{r_n} \cap \partial\Omega \neq \emptyset$ and u_{r_n} satisfies

$$\begin{gathered} -\Delta_p u_{r_n} = 0 \quad \text{in } K\backslash B_{r_n} \\ u_{r_n} = 1 \quad \text{on } \partial B_{r_n} \\ u_{r_n} = 0 \quad \text{on } \partial K. \end{gathered} \tag{5.6}$$

Let us consider the problem

$$\begin{gathered} -\Delta_p z = 0 \quad \text{in } K\backslash\bar{\Omega} \\ z = u_{r_n} \quad \text{on } \partial\Omega \\ z = 0 \quad \text{on } \partial K. \end{gathered} \tag{5.7}$$

It is easy to see that $z = u_{r_n}$ is a solution to the problem (5.7), and we get $0 \leq u_{r_n} \leq 1$ and $0 \leq u \leq 1$. On $\partial(K\backslash\bar{\Omega})$, we have $u_{r_n} \leq u$. Since $\partial\Omega \cap \partial B_{r_n} \neq \emptyset$, let $x_0 \in \partial\Omega \cap \partial B_{r_n}$, we have

$$\lim_{h\to 0} \frac{u_{r_n}(x_0 - \nu h) - u_{r_n}(x_0)}{h} \leq \lim_{h\to 0} \frac{u(x_0 - \nu h) - u(x_0)}{h},$$

This is equivalent to

$$\frac{\partial u_{r_n}}{\partial \nu}(x_0) \geq \frac{\partial u}{\partial \nu}(x_0) = c_\Omega.$$

Let $\Omega = \Omega_0$ as the first iteration. We iterate by looking for $\Omega_1 \in \mathcal{O}_n^1$ such that

$$\begin{gathered} -\Delta_p u_1 = 0 \quad \text{in } K\backslash\bar{\Omega}_1 \\ u_1 = 1 \quad \text{on } \partial\Omega_1 \\ u_1 = 0 \quad \text{on } \partial K \\ \frac{\partial u_1}{\partial \nu} = c_{\Omega_1} \quad \text{on } \partial\Omega_1. \end{gathered} \tag{5.8}$$

where $c_{\Omega_1} = (\frac{p}{p-1}\lambda_{\Omega_1})^{1/p}$, and

$$\mathcal{O}_n^1 = \{w, w \in \mathcal{O}_\epsilon, \Omega_0 \subset w, \text{and} \partial w \cap \partial B_{r_n} \neq \emptyset \, \text{vol}(w) = V_1\}, whereV_1$$

is a strictly positive constant and $V_0 < V_1$. By the same reasoning as above, we conclude that

$$\frac{\partial u_{r_n}}{\partial \nu}(x_1) \geq \frac{\partial u_1}{\partial \nu}(x_1) = c_{\Omega_1}$$

where $x_1 \in \partial\Omega_1 \cap \partial B_{r_n}$. We can continue the process until a step denoted by k which we will determine and we have

$$\frac{\partial u_{r_n}}{\partial \nu}(x_k) \geq \frac{\partial u_k}{\partial \nu}(x_k) = c_{\Omega_k} \quad \text{and} \quad x_k \in \partial\Omega_k \cap \partial B_{r_n}.$$

Finally, we have constructed an increasing sequence of domain solutions: $\Omega_0 \subset \Omega_1 \subset \Omega_2 \dots \subset \Omega_k$. By the monotony result, we have $c_{\Omega_0} \geq c_{\Omega_1} \geq c_{\Omega_2} \dots \geq c_{\Omega_k}$.

Since $\|\nabla u_{r_n}\| \leq \|\nabla u_0\|$ on ∂B_{r_n}, k is chosen as follows: At each point $s_0 \in \partial B_{r_n}$, we have

$$c_{\Omega_k} \leq \frac{\partial u_0}{\partial \nu}(s_0) \leq c_{\Omega_{k-1}}.$$

Then we obtain the inequality

$$c_{\Omega_k} - \frac{e}{R_K} \leq \frac{\partial u_0}{\partial \nu}(s_0) - \frac{e}{R_K} \leq c_{\Omega_{k-1}} - \frac{e}{R_K}. \tag{5.9}$$

The sequence $(c_{\Omega_j})_{(0\leq j\leq k)}$ is decreasing and strictly positive, then it converges on l. Passing to the limit in (5.9), we obtain that $l = \frac{e}{R_K}$ and there exists Ω solution to problem (1.1). The sequence $(\Omega_j)_{(0\leq j\leq k)}$ gives a good approximation to Ω. The uniqueness of the solution Ω is given by the monotony result.

(2) For $c > \frac{e}{R_K}$ and $r \in]0, \frac{R_K}{e}[\cup]\frac{R_K}{e}, R_K[$. We have the same reasoning and we show that the problem (1.1) admits a solution.

Case where $p \neq N$. Here the reasoning is identical to the case $p = N$. We note that

$$\|\nabla u_0\|_{|\partial B_{r_n}} = |\frac{p-N}{p-1}| \frac{1}{1 - (\frac{r}{R_K})^{\frac{N-p}{p-1}}} \frac{1}{r} = g(r)$$

and g is strictly increasing on $](\frac{p-1}{N-1})^{\frac{p-1}{N-p}} R_K, R_K[$ and a strictly decreasing on $]0, (\frac{p-1}{N-1})^{\frac{p-1}{N-p}} R_K[$. For all

$$c \geq |\frac{p-N}{p-1}| \frac{1}{|(\frac{p-1}{N-1})^{\frac{N-1}{N-p}} - (\frac{p-1}{N-1})^{\frac{p-1}{N-p}}|} \frac{1}{R_K} = g((\frac{p-1}{N-1})^{\frac{p-1}{N-p}} R_K),$$

problem (1.1) admits a solution.

Let us now prove the assertions (i) and (ii) of theorem (2.1). It is easy to have, $0 < c_K \leq \alpha(R_K, p, N)$. If K is a ball of radius R, an explicit computation gives $c_K = \alpha(R, p, N)$ and for all $0 < c < c_K$ problem (1.1) has no solution.

To prove the assertion (ii), let K^* be a ball of radius R_1 and $K \subset \mathbb{R}^N$ be star-shaped with respect to the origin such that $\text{vol}(K) = \text{vol}(K^*)$. We remark that $R_K \leq R_1$ and this implies

$$\alpha(R_1, p, N) \leq \alpha(R_K, p, N), c_{K^*} = \alpha(R_1, p, N).$$

The sequence $(\alpha(R_K, p, N))_K$ is reduced by c_{K^*} and decreasing in the following sense: For all K, K' : $\text{vol}(K) = \text{vol}(K^*) = \text{vol}(K')$ if $R_K \leq R_{K'}$ then

$\alpha(R_{K'}, p, N) \leq \alpha(R_K, p, N)$ this implies that the sequence $\alpha(R_K, p, N)$ converges on c_{K^*}. $\square$

Commentary. If there is no K_1 different from the ball K^* such that $\mathrm{vol}(K^*) = \mathrm{vol}(K_1)$ and $c_{K^*} > c_{K_1}$ then for all K such that $\mathrm{vol}(K) = \mathrm{vol}(K^*)$, we have $c_{K^*} \leq c_K$. If there exists K_1 such that $\mathrm{vol}(K_1) = \mathrm{vol}(K^*)$ and $c_{K^*} > c_{K_1}$ then K_1 can't be a ball and $R_{K_1} < R_1$.

References

[1] Acker, A.; *Interior free boundary problem for the Laplace equation,* Arch.Rat.Mech.Anal , **75**(1981), 157-168.

[2] Acker, A, Meyer, R.A.; *A free boundary problem for the p-Laplacian: uniqueness,convexity, and sucessive approximation of solutions,* Electronic Journal of Differential Equations, **1995**(1995), No. ??, 1-20.

[3] Cardaliaguet, P. and Tahraoui R.; *Some uniqueness result for Bernoulli interior free boundary problems in convex domains,* Electronic Journal of Differential Equations, **2002**(2002), No. 102, pp 1-16.

[4] DiBenedetto, E.; *$\mathcal{C}^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations,* Nonlinear Analysis, **7**(1983), 827-850.

[5] Flucher, M. and Rumpf, M.; *Bernoulli's free boundary problem, qualitative theory and numerical approximation,* J. Reine Angew. Math, **486**(1997), 165-204.

[6] Henrot, A, Shahgholian, H.; *Existence of classical solutions to a free boundary problem for the p-Laplace operator II: the interior convex case,* Indiana Univ. Math. Journal, **49** (2000), No.1, 301-323.

[7] Kavian, O.; *Introduction à la théorie des points critiques et applications aux problèmes elliptiques,* Springer-Verlag, France, Paris, 1993.

[8] Lavrent'ev, M. A.; *Variational methods for boundary value problems for systems of elliptic equations,* Noordhoff, 1963.

[9] Lewis, J. L.; *Regularity of the derivatives of solutions to certain degenerate elliptic equations,* Indiana Univ . Math. J., **32**(1983), 849-858.

[10] Ly, I.; *Thèse de doctorat de troisième cycle: Résultat d'existence en optimisation de forme et étude d'un problème extérieur à frontière libre: cas du p-laplacien,* Soutenue le 20 juin 2002 , UGB, Sénégal.

[11] Ly,I., Seck ,D.; *Etude d'un problème à frontière libre pour le p-laplacien,* CRAS, Paris **332**(2001), Série I, 899-902.

[12] Ly,I., Seck ,D.; *Estimation de la constante de Bernoulli dans le problème intérieur cas du p-laplacien,* CRAS, Paris **337**(2003), Série I, 393-396.

[13] Ly,I., Seck ,D.; *Optimisation de forme et problème à frontière libre pour le p-laplacien,* Ann. Fac. Sci. Toulouse Math. (6) 12 (2003), no.1, 103-126.

[14] Ly, I., Seck ,D.; *About free boundary problem with quasi linear operator and non constant Bernoulli boundary condition,* to appear.

[15] Simon, J.; *Differential with respect to the domain in boundary value problems,* Numer. Funct. Anal. and Optimiz. **2** (7 - 8), 649-687(1980).

[16] Sokolowski, J., Zolesio, J. P.; *Introduction to Shape Optimization,* Springer-Verlag, Paris, 1992.

[17] Tolksdorf, P.; *On the Dirichlet problem for quasilinear equations in domains with conical boundary points,* Comm. Partial Differential Equations, **8**(1983), 773-817.

[18] Tolksdorf, P.; *Regularity for a more general class of quasilinear elliptic equations,* J. Differential Equations, **52**(1984), 126-150.

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