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# DOUBLE SOLUTIONS OF THREE-POINT BOUNDARY-VALUE PROBLEMS FOR SECOND-ORDER DIFFERENTIAL EQUATIONS

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Abstract. A double fixed point theorem is applied to yield the existence of at least two nonnegative solutions for the three-point boundary-value problem for a second-order differential equation,

$$y'' + f(y) = 0, \quad 0 \le t \le 1,$$
$$y(0) = 0, \quad y(p) - y(1) = 0,$$

where $0 < p < 1$ is fixed, and $f : \mathbb{R} \to [0, \infty)$ is continuous.

## 1. Introduction

This paper fits in the rapidly growing literature devoted to applications of multiple fixed point theorems for boundary value problems for each of ordinary differential equations, finite difference equations, and dynamic equations on time scales. Some of these applications can be found in, to mention a few, the papers [2] - [5], [11] - [13] and [18]. These applications involve in some cases multiple uses of a Guo-Krasnosel'skii [19] fixed point theorem or uses of the Leggett-Williams [14] triple fixed point theorem. Other applications have used functional-type cone expansion-compression fixed point theorems such as found in the above cited papers [2] - [5]. In this paper, we apply the Avery-Henderson [4] double fixed point theorem to obtain at least two positive solutions of the three-point boundary value problem for the second order differential equation,

$$y'' + f(y) = 0, \quad 0 \le t \le 1, \tag{1.1}$$

$$y(0) = 0, \quad y(p) - y(1) = 0, \tag{1.2}$$

where $0 < p < 1$ is fixed throughout, and $f : \mathbb{R} \to [0, \infty)$ is continuous. Multipoint problems such as (1.1), (1.2) have received considerable attention, often with (1.2) replaced by $u(1) - \sum_{i=1}^{n} \alpha_i u(t_i) = 0$, $a < t_1 < \dots < t_n < 1$, and $0 < \sum_{i=1}^{n} \alpha_i < 1$. For a few such papers, see [1], [6] - [10], [16] and [17]. In Section 2, we provide some background results and we state the double fixed point theorem. Then, in Section 3, we impose growth conditions on $f$ which allow us to apply the fixed point theorem in obtaining double positive solutions of (1.1), (1.2). We remark that Liu and Ge [15] recently obtained a double fixed point theorem which would be considered as a dual

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theorem to the Avery-Henderson double fixed point theorem. At the conclusion of this paper, we state a theorem establishing double solutions of (1.1), (1.2) which arise from an application of the Liu-Ge double fixed point theorem. In addition, we mention that the term "nonnegative" may better describe than "positive" one of the solutions of (1.1), (1.2). Yet, if conditions such as $f(0) > 0$ are satisfied, then our double solutions are indeed positive.

## 2. Background preliminaries and a double fixed point theorem

In this section, we provide some background from the theory of cones in Banach spaces, and we then state a double fixed point theorem for a cone preserving operator.

**Definition 2.1.** Let $(\mathcal{B}, \|\cdot\|)$ be a real Banach space. A nonempty, closed, convex set $\mathcal{P} \subset \mathcal{B}$ is said to be a *cone* provided the following are satisfied:

(a) If $y \in \mathcal{P}$ and $\lambda \geq 0$ , then $\lambda y \in \mathcal{P}$;
(b) If $y \in \mathcal{P}$ and $-y \in \mathcal{P}$ , then $y = 0$.

Every cone $\mathcal{P} \subset \mathcal{B}$ induces a partial ordering, $\leq$, on $\mathcal{B}$ defined by

$$x \leq y \quad \text{if and only if} \quad y - x \in \mathcal{P}.$$

**Definition 2.2.** Given a cone $\mathcal{P}$ in a real Banach space $\mathcal{B}$, a functional $\psi : \mathcal{P} \to R$ is said to be *increasing* on $\mathcal{P}$, provided $\psi(x) \leq \psi(y)$, for all $x, y \in \mathcal{P}$ with $x \leq y$.

**Definition 2.3.** Given a nonnegative continuous functional $\gamma$ on a cone $\mathcal{P}$ of a real Banach space $\mathcal{B}$, (*i.e.*, $\gamma : \mathcal{P} \to [0, \infty)$ continuous), we define, for each $d > 0$, the convex set

$$\mathcal{P}(\gamma, d) = \{x \in \mathcal{P} : \gamma(x) < d\}.$$

Our main results concerning multiple positive solutions of (1.1), (1.2) will arise as applications of the following fixed point theorem due to Avery and Henderson [4].

**Theorem 2.4.** *Let $\mathcal{P}$ be a cone in a real Banach space $\mathcal{B}$. Let $\alpha$ and $\gamma$ be increasing, nonnegative, continuous functionals on $\mathcal{P}$, and let $\theta$ be a nonnegative continuous functional on $\mathcal{P}$ with $\theta(0) = 0$ such that, for some $c > 0$ and $M > 0$,*

$$\gamma(x) \leq \theta(x) \leq \alpha(x) \quad \textit{and} \quad \|x\| \leq M\gamma(x),$$

*for all $x \in \overline{\mathcal{P}(\gamma, c)}$. Suppose there exist a completely continuous operator $A : \overline{\mathcal{P}(\gamma, c)} \to \mathcal{P}$ and $0 < a < b < c$ such that*

$$\theta(\lambda x) \leq \lambda\theta(x), \quad \textit{for } 0 \leq \lambda \leq 1 \textit{ and } x \in \partial\mathcal{P}(\theta, b),$$

*and*

(i) *$\gamma(Ax) > c$, for all $x \in \partial\mathcal{P}(\gamma, c)$;*
(ii) *$\theta(Ax) < b$, for all $x \in \partial\mathcal{P}(\theta, b)$;*
(iii) *$\mathcal{P}(\alpha, a) \neq \emptyset$, and $\alpha(Ax) > a$, for all $x \in \partial\mathcal{P}(\alpha, a)$.*

*Then $A$ has at least two fixed points, $x_1$ and $x_2$ belonging to $\overline{\mathcal{P}(\gamma, c)}$ such that*

$$a < \alpha(x_1), \quad \textit{with } \theta(x_1) < b,$$

*and*

$$b < \theta(x_2), \quad \textit{with } \gamma(x_2) < c.$$

## 3. Double positive solutions of (1.1), (1.2)

In this section, we impose growth conditions of $f$ and then apply Theorem 2.4 to establish the existence of double positive solutions of (1.1), (1.2). We note that from the nonnegativity of $f$, a solution $y$ of (1.1), (1.2) is both nonnegative and concave on $[0,1]$, and in addition, assumes its maximum in the interval $(p,1)$. We will apply Theorem 2.4 to a completely continuous operator whose kernel, $G(t,s)$, is the Green's function for

$$-y'' = 0, \tag{3.1}$$

satisfying (1.2). In this instance,

$$G(t,s) = \begin{cases} t, & t \le s \le p, \\ s, & s \le t \text{ and } s \le p, \\ \frac{1-s}{1-p}t, & t \le s \text{ and } s \ge p, \\ s + \frac{p-s}{1-p}t, & p \le s \le t. \end{cases} \tag{3.2}$$

Properties of $G(t,s)$ for which we will make use include

$$G(t,s) \le G(s,s), \quad 0 \le t, s \le 1, \tag{3.3}$$

$$G(t,s) \ge G(p,s), \quad p \le t \le 1,\ 0 \le s \le 1. \tag{3.4}$$

Let the Banach space $\mathcal{B} = C[0,1]$ be equipped with the norm $\|y\| = \max_{0\le t\le 1}|y(t)|$, and choose the cone $\mathcal{P} \subset \mathcal{B}$ defined by

$$\mathcal{P} = \{y \in \mathcal{B} : y \text{ is concave and nonnegative-valued on } [0,1], \text{ and } y(p) = y(1)\}.$$

For the remainder of the paper, fix $r \in (p,1)$, and define the nonnegative, increasing functionals, $\gamma, \theta$ and $\alpha$, on $\mathcal{P}$ by

$$\gamma(y) = \min_{p\le t\le r} y(t) = y(p) = y(1),$$

$$\theta(y) = \max_{0\le t\le p} y(t) = y(p),$$

$$\alpha(y) = \max_{0\le t\le r} y(t).$$

We observe that, for each $y \in \mathcal{P}$,

$$\gamma(y) = \theta(y) \le \alpha(y). \tag{3.5}$$

In addition, for each $y \in \mathcal{P}$,

$$\|y\| \le \frac{1}{p}y(p) \le \frac{1}{p}\gamma(y). \tag{3.6}$$

Finally, we note that

$$\theta(\lambda y) = \lambda\theta(y), \quad 0 \le \lambda \le 1, \quad \text{and} \quad y \in \partial\mathcal{P}(\theta, b). \tag{3.7}$$

We now state growth conditions on $f$ so that (1.1), (1.2) has at least two positive solutions.

**Theorem 3.1.** *Let*

$$0 < a < \frac{r[r(1-r)+p(r-p)]}{p(1-p)}b < \frac{r[r(1-r)+p(r-p)]}{(1-p)}c\,,$$

*and suppose that $f$ satisfies the following conditions:*

(A) $f(w) > \frac{2c}{p(1-p)}$, *if* $c \le w \le \frac{c}{p}$,

(B) $f(w) < \frac{2b}{p}$, *if* $0 \le w \le \frac{b}{p}$,

(C) $f(w) > \frac{2(1-p)a}{r[r(1-r)+p(r-p)]}$, *if* $0 \le w \le a$.

*Then, the three point boundary value problem (1.1), (1.2) has at least two positive solutions,* $x_1$ *and* $x_2$*, such that*

$$a < \max_{0 \le t \le r} x_1(t), \quad \textit{with} \max_{0 \le t \le p} x_1(t) < b,$$

*and*

$$b < \max_{0 \le t \le p} x_2(t), \quad \textit{with} \min_{p \le t \le r} x_2(t) < c.$$

*Proof.* We begin by defining a completely continuous integral operator $A : \mathcal{B} \to \mathcal{B}$ by

$$Ax(t) = \int_0^1 G(t,s) f(x(s)) ds, \ x \in \mathcal{B}, \quad 0 \le t \le 1.$$

Solutions of (1.1), (1.2) are fixed points of $A$ and conversely. Our proof consists of showing the conditions of Theorem 2.4 are satisfied. First, we choose $x \in \overline{\mathcal{P}(\gamma, c)}$. By the nonnegativity of $f$ and $G$, for $0 \le t \le 1$,

$$Ax(t) = \int_0^1 G(t,s) f(x(s)) ds \ge 0.$$

Moreover, $(Ax)''(t) = -f(x(t)) \le 0$, and so $(Ax)(t)$ is concave on $[0,1]$. Since $G(t,s)$ satisfies the boundary conditions (1.2) as a function of $t$, we have $(Ax)(p) = (Ax)(1)$. Thus, $Ax \in \mathcal{P}$ and $A : \overline{\mathcal{P}(\gamma, c)} \to \mathcal{P}$. We now consider property (i) of Theorem 2.4. If we choose $x \in \partial\mathcal{P}(\gamma, c)$, then $\gamma(x) = \min_{p \le t \le r} x(t) = x(p) = c$. Since $x \in \mathcal{P}$, $x(t) \ge c$, $p \le t \le 1$. By recalling $\|x\| \le \frac{1}{p}\gamma(x) = \frac{1}{p}x(p) = \frac{c}{p}$, we have

$$c \le x(t) \le \frac{c}{p}, \ p \le t \le 1.$$

As a consequence of (A),

$$f(x(s)) > \frac{2c}{p(1-p)}, \ p \le s \le 1.$$

Also, $Ax \in \mathcal{P}$, and so

$$\begin{aligned}
\gamma(Ax) &= (Ax)(p) \\
&= \int_0^1 G(p,s) f(x(s)) ds \\
&\ge \int_p^1 G(p,s) f(x(s)) ds \\
&= \int_p^1 (\frac{1-s}{1-p}) p f(x(s)) ds \\
&> \frac{2c}{p(1-p)} \int_p^1 (\frac{1-s}{1-p}) p ds \\
&= c.
\end{aligned}$$

We conclude that (i) of Theorem 2.4 is satisfied. We next address (ii) of Theorem 2.4. We choose $x \in \partial\mathcal{P}(\theta, b)$. Then $\theta(x) = \max_{0 \le t \le p} x(t) = x(p) = b$. Then

$0 \le x(t) \le b$, $0 \le t \le p$, and since $x \in \mathcal{P}$, we also have $b \le x(t) \le \|x\|$, $p \le t \le 1$. Moreover, $\|x\| \le \frac{1}{p}\gamma(x) \le \frac{1}{p}\theta(x) = \frac{b}{p}$. So,

$$0 \le x(t) \le \frac{b}{p}, \quad 0 \le t \le 1.$$

From (B),

$$f(x(s)) < \frac{2b}{p}, \quad 0 \le s \le 1.$$

$Ax \in \mathcal{P}$, and so

$$\begin{aligned}
\theta(Ax) &= (Ax)(p) \\
&= \int_0^1 G(p,s) f(x(s)) ds \\
&< \frac{2b}{p} \int_0^1 G(p,s) ds \\
&= \frac{2b}{p} \Big[ \int_0^p G(p,s) ds + \int_p^1 G(p,s) ds \Big] \\
&= \frac{2b}{p} \Big[ \int_0^p s ds + \int_p^1 \big(\frac{1-s}{1-p}\big) p ds \Big] \\
&= b.
\end{aligned}$$

In particular, (ii) of Theorem 2.4 holds. For the final part, we turn to (iii) of Theorem 2.4. If we first define $y(t) = \frac{a}{2}, 0 \le t \le 1$, then $\alpha(y) = \frac{a}{2} < a$, and $\mathcal{P}(\alpha, a) \neq \emptyset$. Now, let us choose $x \in \partial\mathcal{P}(\alpha, a)$. Then, for some $r_0 \in (p,1)$, $\alpha(x) = \max_{0 \le t \le r} x(t) = x(r_0) = a$. So, in particular

$$0 \le x(t) \le a, \quad 0 \le t \le r.$$

From assumption (C),

$$f(x(s)) > \frac{2(1-p)a}{r[r(1-r) + p(r-p)]}, \quad 0 \le s \le r.$$

As before, $Ax \in \mathcal{P}$, and so for some $\rho_0 \in (p,1)$,

$$\begin{aligned}
\alpha(Ax) &= (Ax)(\rho_0) \\
&\ge (Ax)(r) \\
&= \int_0^1 G(r,s) f(x(s)) ds \\
&\ge \int_0^r G(r,s) f(x(s)) ds \\
&> \frac{2(1-p)a}{r[r(1-r) + p(r-p)]} \Big[ \int_0^p s ds + \int_p^r s + \big(\frac{p-s}{1-p}\big) r ds \Big] \\
&= a.
\end{aligned}$$

Thus, (iii) of Theorem 2.4 is also satisfied. Hence, there exist at least two fixed points of $A$ which are positive solutions $x_1$ and $x_2$, belonging to $\overline{\mathcal{P}(\gamma, c)}$, of the boundary value problem (1.1), (1.2) such that

$$a < \alpha(x_1), \quad \text{with } \theta(x_1) < b,$$

and

$$b < \theta(x_2), \quad \text{with } \gamma(x_2) < c.$$

The proof is complete. $\square$

**Example.** For $0 < p < r < 1$ fixed and

$$0 < a < \frac{r[r(1-r)+p(r-p)]}{p(1-p)} b < \frac{r[r(1-r)+p(r-p)]}{(1-p)} c,$$

if $f : \mathbb{R} \to [0, \infty)$ is defined by

$$f(w) = \begin{cases} \frac{b}{p} + \frac{(1-p)a}{r[r(1-r)+p(r-p)]}, & w \leq \frac{b}{p}, \\ \ell(w), & \frac{b}{p} \leq w \leq c, \\ \frac{2c}{p(1-p)} + 1, & c \leq w, \end{cases}$$

where $\ell(w)$ satisfies $\ell'' = 0$, $\ell(\frac{b}{p}) = \frac{b}{p} + \frac{(1-p)a}{r[r(1-r)+p(r-p)]}$ and $\ell(c) = \frac{2c}{p(1-p)} + 1$, then by Theorem 3.1, the boundary value problem (1.1), (1.2) has at least two positive solutions.

**Remark.** Liu and Ge recently obtained a double fixed point theorem [15, Lemma 2, p. 553] which could be considered as a type of dual to Theorem 2.4 in that, conditions are given for the existence of double fixed points when inequalities (i), (ii) and (iii) of Theorem 2.4 are reversed. We provide in this remark, as an application of the Liu-Ge double fixed point, a dual result to Theorem 3.1 for double positive solutions of (1.1), (1.2). Because of close similarity of its proof to that of Theorem 3.1, we will omit the proof. For convenience of notation, we will define

$$\lambda = \max_{0 \leq t \leq r} \int_0^1 G(t,s) ds.$$

**Theorem 3.2.** *Let $0 < a < b < c$ be such that $0 < a < \min\{pb, \frac{2\lambda b}{p(1-p)}\} < \frac{2\lambda c}{p}$, and suppose that $f$ satisfies the following conditions:*

(A) $f(w) < \frac{2c}{p}$, *if* $0 \leq w \leq \frac{c}{p}$,
(B) $f(w) > \frac{2b}{p(1-p)}$, *if* $b \leq w \leq \frac{b}{p}$,
(C) $f(w) < \frac{a}{\lambda}$, *if* $0 \leq w \leq \frac{a}{p}$.

*Then, the three point boundary value problem (1.1), (1.2) has at least two positive solutions, $x_1$ and $x_2$, such that*

$$a < \max_{0 \leq t \leq r} x_1(t), \quad \text{with } \max_{0 \leq t \leq p} x_1(t) < b,$$

*and*

$$b < \max_{0 \leq t \leq p} x_2(t), \quad \text{with } \min_{p \leq t \leq r} x_2(t) < c.$$

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