ON THE SECOND EIGENVALUE OF A HARDY-SOBOLEV OPERATOR

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Abstract. In this note, we study the variational characterization and some properties of the second smallest eigenvalue of the Hardy-Sobolev operator

\[ L_\mu := -\Delta_p - \frac{\mu}{|x|^p} \] with respect to an indefinite weight \( V(x) \).

1. Introduction

Let \( \Omega \) be a domain in \( \mathbb{R}^N \) containing 0. We recall the classical Hardy-Sobolev inequality which states that, for \( 1 < p < N \),

\[ \int_\Omega |\nabla u|^p dx \geq \left( \frac{N-p}{p} \right)^p \int \Omega |u|^p |x|^{-p} dx, \quad \forall u \in C_c^\infty(\Omega). \tag{1.1} \]

Let \( D_{1,p}^0(\Omega) \) be the closure of \( C_c^\infty(\Omega) \) with respect to the norm \( \|u\|_{1,p} := \|\nabla u\|_{L_p(\Omega)} \).

The Hardy-Sobolev operator \( L_\mu \) on \( D_{1,p}^0(\Omega) \) is defined as

\[ L_\mu u := -\Delta_p u - \frac{\mu}{|x|^p} |u|^{p-2} u, \quad 0 < \mu < \left( \frac{N-p}{p} \right)^p, \]

where \( \Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u) \), is the \( p \)-Laplacian.

We are interested in the variational characterization and some properties of the second smallest eigenvalue of the problem

\[ L_\mu u = \lambda V(x)|u|^{p-2} u \quad \text{in} \quad \Omega \]
\[ u = 0 \quad \text{on} \quad \partial \Omega. \tag{1.2} \]

On the weight on \( V(x) \), we assume the following:

(H1) \( V \in L_{\text{loc}}^1(\Omega), V^+ = V_1 + V_2 \neq 0 \) with \( V_1 \in L^{N/p}(\Omega) \) and \( V_2 \) is such that \( \lim_{x \to y, x \in \Omega} |x-y|^p V_2(x) = 0 \) for all \( y \in \Omega \), \( \lim_{|x| \to \infty, x \in \Omega} |x|^p V_2(x) = 0 \), where \( V^+(x) = \max\{V(x), 0\} \).

(H2) There exists \( r > N/p \) and a closed subset \( S \) of measure zero in \( \mathbb{R}^N \) such that \( \Omega \setminus S \) is connected and \( V \in L_r^\text{loc}(\Omega \setminus S) \).

Here we note that there is no global integrability condition assumed on \( V^- \).

This work is motivated by the work in [8]. The eigenvalue problem with indefinite weights has been studied for the case \( \mu = 0 \) by Szulkin-Willem [8]. However, some important properties, of the smallest eigenvalue \( \lambda_1 \), such as simplicity and...
being isolated were shown only for \( p = 2 \). Recently the author in [6] proved the simplicity of \( \lambda_1 \) and sign changing nature of eigenfunctions corresponding to other eigenvalues when \( \Omega \) is bounded. Infact in [6] the author studied these properties for \( L_\mu \). Following the same arguments, one can prove these results in the present case. However, showing that \( \lambda_1 \) is isolated and characterization of the second smallest eigenvalue, were open questions. To prove these properties, we follow the ideas in [5] and in [3]. Here we should mention that our results are new even for the case \( \mu = 0 \). We use the following results in later sections.

**Proposition 1.1** (Boccardo-Murat [1]). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) and let \( u_n \in W^{1,p}(\Omega) \) satisfy

\[
-\Delta_p u_n = f_n + g_n \quad \text{in} \quad \mathcal{D}'(\Omega)
\]

and

\[
\begin{align*}
(i) & \quad u_n \rightharpoonup u \quad \text{weakly in} \quad W^{1,p}(\Omega) \\
(ii) & \quad u_n \rightarrow u \quad \text{in} \quad L^p(\Omega) \\
(iii) & \quad f_n \rightarrow f \quad \text{in} \quad W^{-1,p'} \\
(iv) & \quad g_n \text{ is a bounded sequence of Radon measures.}
\end{align*}
\]

Then there exists a subsequence \( \{u_n\} \) of \( \{u_n\} \) such that \( \nabla u_n \rightarrow \nabla u \) a.e. in \( \Omega \).

**Proposition 1.2** (Brezis-Lieb [2]). Let \( f_n \rightarrow f \) a.e in \( \Omega \) as \( n \rightarrow \infty \) and \( f_n \) be bounded in \( L^p(\Omega) \), for some \( p > 1 \). Then

\[
\lim_{n \to \infty} \left\{ \|f_n\|_p - \|f_n - f\|_p \right\} = \|f\|_p.
\]

Let \( X \) be a Banach space and let \( M = \{u \in X | g(u) = 0\} \) with \( g \in C^1 \). Also let \( f : X \rightarrow \mathbb{R} \) be a \( C^1 \) functional and let \( \bar{f} \) be the restriction of \( f \) to \( M \). Then we have the following form of the Mountain pass Theorem [7].

**Proposition 1.3.** Let \( u, v \in M \) with \( u \not\equiv v \) and suppose that

\[
c := \inf_{h \in \Gamma} \max_{w \in h(t)} f(w) > \max\{f(u), f(v)\}
\]

where

\[
\Gamma := \{h \in C([-1,1], M) | h(-1) = u \quad \text{and} \quad h(1) = v \} \neq \emptyset
\]

Also suppose that \( \bar{f} \) satisfies Palais-Smale (PS) condition on \( M \). Then \( c \) is a critical value of \( \bar{f} \).

We define the norm

\[
\|\bar{f}\|_* = \inf \{\|f'(u) - t g'(u)\|_{X^*} : t \in \mathbb{R}\}.
\]

The variational characterization of the smallest eigenvalue is given by

\[
\lambda_1 = \inf_{0 \neq u \in W^{1,p}_0(\Omega)} \frac{\int_\Omega |\nabla u|^p dx - \int_\Omega |u|^p |x|^\mu dx}{\int_\Omega |u|^p V(x) dx}
\]

and the corresponding eigenfunction is denoted by \( \phi_1 \), which is unique under the condition \( \int_\Omega |\phi|^p V(x) dx = 1 \) (see [6]). We will prove the following property.

**Theorem 1.4.** The eigenvalue \( \lambda_1 \) is isolated in the spectrum of \( L_\mu \).
We will establish the following variational characterisation of the second smallest eigenvalue:
\[
\lambda_2 = \inf_{\gamma \in \Gamma} \sup_{u \in \gamma} \frac{\int_{\Omega} |\nabla u|^p dx - \int_{\Omega} |u|^p dx}{\int_{\Omega} |u|^p V(x) dx},
\]
where \( \Gamma = \{ \gamma \in C_0^\infty([-1,1] : M) \mid \gamma(-1) = -\phi_1, \gamma(1) = \phi_1 \} \) and \( M \) is defined as in the next section. We show also the following property of \( \lambda_2 \).

**Theorem 1.5.** If \( V_a \leq V_b \), then \( \lambda_2(V_a) \geq \lambda_2(V_b) \).

2. Proofs of results

In this section we show that \( \lambda_1 \) is isolated and give a variational characterization for second smallest eigenvalue of \( L_\mu \).

**Lemma 2.1.** The mapping \( u \mapsto \int_{\Omega} V^+ |u|^p dx \) is weakly continuous.

The proof of this lemma follows from (1.1) and (H1). We refer the reader to [8] for more details.

Now, we consider the set
\[
M = \left\{ u \in D_0^{1,p}(\Omega) \mid \int_{\Omega} |u|^p V(x) = 1 \right\}.
\]
Since \( M \) is not a manifold in \( D_0^{1,p}(\Omega) \), we define \( X = \{ u \in D_0^{1,p}(\Omega) \mid \|u\|_X < \infty \} \), where
\[
\|u\|_X := \int_{\Omega} |\nabla u|^p dx + \int_{\Omega} |u|^p V^- dx.
\]
Then \( M \) is a \( C^1 \)-manifold as a subset of the space \( X \). On this space, we define the functional
\[
J_\mu(u) = \frac{\int_{\Omega} |\nabla u|^p dx - \int_{\Omega} \frac{1}{p} |u|^p dx}{\int_{\Omega} |u|^p V dx}.
\]
Let \( \tilde{J}_\mu \) denote the restriction of \( J_\mu \) to \( M \). and let \( \|u\|_{L^p(V)}^p = \int_{\Omega} |u|^p V(x) dx \).

**Lemma 2.2.** The functional \( \tilde{J}_\mu \) satisfies the Palais-Smale condition at any positive level.

**Proof.** Let \( \{u_n\} \) be a sequence in \( M \) such that \( J_\mu(u_n) \to \lambda > 0 \) and
\[
\langle J_\mu'(u_n), \phi \rangle - J_\mu(u_n) \int_{\Omega} |u_n|^{p-2} u_n \phi V dx = o(1). \tag{2.1}
\]
Using Hardy-Sobolev inequality and \( u_n \in M \), it follows that \( u_n \) is bounded in \( X \) which gives the existence of a subsequence \( \{u_n\} \) of \( \{u_n\} \) and \( u \) such that \( u_n \rightharpoonup u \) weakly in \( D_0^{1,p}(\Omega) \). Since \( \lambda > 0 \) we may assume that \( J_\mu(u_n) \geq 0 \). Using Lemma 2.1 and (2.1), we get
\[
\langle J_\mu'(u_n) - J_\mu'(u), u_n - u \rangle + J_\mu(u_n) \int_{\Omega} \left[ |u_n|^{p-2} u_n - |u|^{p-2} u \right] (u_n - u) V^- dx = o(1).
\]
By Fatou’s Lemma,
\[
0 = \lim_{n \to \infty} \int_{\Omega} \left[ |u_n|^{p-2} u_n - |u|^{p-2} u \right] [u_n - u] V^- \\
\leq \liminf_{n \to \infty} \int_{\Omega} \left[ |u_n|^{p-2} u_n - |u|^{p-2} u \right] [u_n - u] V^- dx.
\]
Also, \( u_n \) satisfies
\[
-\Delta_p u_n - \frac{\mu}{|x|^p} |u_n|^{p-2} u_n - J_\mu(u_n)|u_n|^{p-2} u_n V(x) = o(1) \quad \text{in} \ D'(\Omega_m),
\]
where \( \Omega_m \) is a bounded domain such that \( \Omega = \bigcup_{n=1}^{\infty} \Omega_m \). By Proposition 1.1, noting that \( \frac{\mu}{|x|^p} |u_n|^{p-2} u_n + J_\mu(u_n)|u_n|^{p-2} u_n V^- \) is a bounded sequence of Radon measures, there exists a subsequence \( \{u_n^m\} \) of \( \{u_n\} \) an \( u \) such that \( \nabla u_n^m \to \nabla u \) a.e., in \( \Omega_m \). By the process of diagonalization we can choose a subsequence \( \{u_n\} \) such that \( \nabla u_n \to \nabla u \) a.e. in \( \Omega \). By Proposition 1.2, we have
\[
\|u_n - u\|_{1,p}^p = \|u_n\|_{1,p}^p - \|u\|_{1,p}^p + o(1)
\]
(2.2)
\[
\frac{\|u_n - u\|_{L^p(1)}}{|x|} = \frac{\|u_n - u\|_{L^p(1)}}{|x|} - \frac{\|u\|_{L^p(1)}}{|x|} + o(1).
\]
(2.3)

We also have, by Fatou's lemma,
\[
\int_{\Omega} V^- |u_n|^p + |u|^p - |u_n|^{p-2} u_n u - |u|^{p-2} u_n dx
\]
\[
\geq \int_{\Omega} V^- |u_n|^p + |u|^p - \left( \int_{\Omega} V^- |u_n|^p \right)^{(p-1)/p} \left( \int_{\Omega} V^- |u|^p \right)^{1/p}
\]
\[
- \left( \int_{\Omega} V^- |u|^p \right)^{(p-1)/p} \left( \int_{\Omega} V^- |u_n|^p \right)^{1/p}
\]
\[
= \left[ \left( \int_{\Omega} V^- |u_n|^p \right)^{(p-1)/p} - \left( \int_{\Omega} V^- |u|^p \right)^{(p-1)/p} \right]
\]
\[
\times \left[ \left( \int_{\Omega} V^- |u_n|^p \right)^{1/p} - \left( \int_{\Omega} V^- |u|^p \right)^{1/p} \right] \geq 0.
\]
Now using (2.2) and (2.3),
\[
o(1) = \langle J_\mu(u_n) - J_\mu(u), (u_n - u) \rangle + J_\mu(u_n) \int_{\Omega} [\|u_n|^{p-2} u_n - |u|^{p-2} u](u_n - u) V^- dx
\]
\[
\geq \int_{\Omega} |\nabla u_n - \nabla u|^p - \int_{\Omega} \frac{\mu}{|x|^p} |u_n - u|^p + o(1)
\]
\[
\geq (1 - \frac{\mu}{\lambda_N})\|u_n - u\|_{1,p} + o(1).
\]
i.e., \( u_n \to u \) in \( D_0^{1,p}(\Omega) \). Notice that
\[
o(1) = \langle J_\mu(u_n) - J_\mu(u), u_n - u \rangle
\]
\[
= \int_{\Omega} V^- \left( |u_n|^{p-2} u_n - |u|^{p-2} u \right) (u_n - u) dx + o(1) \geq 0.
\]
Therefore, \( \int_{\Omega} V^- |u_n|^p dx \to \int_{\Omega} V^- |u|^p dx \) and hence \( \|u_n\|_X \to \|u\|_X \).

Observe that \( \tilde{J}_\mu(u) \geq \lambda_1 \) and \( \tilde{J}_\mu(\pm \phi_1) = \lambda_1 \). So \( +\phi_1 \) and \(-\phi_1 \) are two global minima of \( \tilde{J}_\mu \). Now consider
\[
\Gamma = \{ \gamma \in C([-1,1]; M) \mid \gamma(-1) = -\phi_1, \gamma(1) = \phi_1 \}.
\]
By Proposition 1.3, there exists \( u \in X \) such that \( \tilde{J}_\mu(u) = 0 \) and \( J_\mu(u) = C \), where
\[
C = \inf_{\gamma \in \Gamma} \sup_{u \in \gamma} \tilde{J}_\mu(u).
\]
(2.4)

**Lemma 2.3.**

(i) \( M \) is locally arc wise connected
(ii) Any connected open subset $B$ of $M$ is arcwise connected
(iii) if $B'$ is a component of an open set $A$, then $\partial B' \cap B$ is empty.

The proof of this lemma follows from the fact that $M$ is a Banach Manifold. For a
proof we refer the reader to [3]. Define $\mathcal{O} = \{ u \in M \big| \tilde{J}_\mu(u) < r \}$

**Lemma 2.4.** Each component of $\mathcal{O}$ contains a critical point of $\tilde{J}_\mu$.

**Proof.** Let $\mathcal{O}_1$ be a component of $\mathcal{O}$ and let $d = \inf \{ \tilde{J}_\mu(u), u \in \mathcal{O}_1 \}$, where $\overline{\mathcal{O}_1}$ is $X$-closure of $\mathcal{O}$. Suppose this infimum is achieved by $v \in \overline{\mathcal{O}_1}$. Then by Lemma 2.3 this cannot be in $\partial \mathcal{O}_1$ and hence $v$ is in $\mathcal{O}_1$ and is a critical point of $\tilde{J}_\mu$.

Now we show that $d$ is achieved. Let $u_n \in \mathcal{O}_1$ be a minimizing sequence with $\tilde{J}_\mu(u_n) \leq d + \frac{1}{n^2}$. By Ekeland Variational Principle, we get $v_n \in \mathcal{O}_1$ such that

$$
\tilde{J}_\mu(v_n) \leq \tilde{J}_\mu(u_n),
$$

$$
\|v_n - u_n\|_X \leq \frac{1}{n},
$$

$$
\tilde{J}_\mu(v_n) \leq \tilde{J}_\mu(v) + \frac{1}{n}\|v_n - v\|_X, \quad \forall v \in \mathcal{O}_1.
$$

From (2.5) it follows that $\tilde{J}_\mu(v_n)$ is bounded. Now we claim that $\|\tilde{J}_\mu'(v_n)\|_s \to 0$. We fix $n$ and choose $w \in X$ tangent to $M$ at $v_n$, i.e., $\int_{\Omega} |v_n|^{p-2}v_n wV = 0$. Now we consider the path

$$
u_t = \frac{v_n + tw}{\|v_n + tw\|_{L^p(V)}}.
$$

Since $\tilde{J}_\mu(v_n) \leq d + \frac{1}{n} < r$ for $n$ large, we have $v_n \in \overline{\mathcal{O}_1}$ and by Lemma 2.3 (iii), $v_n \notin \partial \mathcal{O}_1$. So $u_t \in \mathcal{O}_1$ for $|t|$ small. Taking $v = u_t$ in (2.7) we obtain

$$
\frac{\tilde{J}_\mu(v_n) - \tilde{J}_\mu(v_n + tw)}{t} \leq \frac{1}{mt}\|v_n\|_X^2 + \frac{1}{n}\|w\| + \frac{1}{t}\left(\frac{1}{r(t)} - 1\right)\tilde{J}_\mu(v_n + tw),
$$

where $r(t) = \|v_n + tw\|_{L^p(V)}$. The last term in (2.8) involves $\frac{r(t)^{p-1}}{t}$ which can be calculated as

$$
\frac{d}{dt}r(s)^{p}|_{s=0} = p \int_{\Omega} |v_n|^{p-2}v_n wV(x)dx = 0.
$$

On the other hand since $w$ is tangent to $M$ at $v_n$,

$$
\frac{d}{dt}r(s)^{p}|_{s=0} = p \int_{\Omega} |v_n|^{p-2}v_n wV(x)dx = 0.
$$

Therefore, we have $\frac{r(t)^{p-1}}{t} \to 0$ as $t \to 0$ and that the second term goes to 0. Similarly, the first term also goes to zero as $t \to 0$. Taking limit $t \to 0$ in (2.8) we get

$$
\langle \tilde{J}_\mu'(v_n), w \rangle \leq \frac{1}{n}\|w\|_X, \quad \forall w \in X \text{ tangent to } M \text{ at } v_n.
$$

Now if $w$ is arbitrary in $X$. We choose $\alpha_n$ so that $(w - \alpha_n v_n)$ is tangent to $M$ at $v_n$, i.e., $\alpha_n = \int_{\Omega} |v_n|^{p-2}v_n wV(x)dx$. So (2.8) gives,

$$
|\langle \tilde{J}_\mu'(v_n), w \rangle - \langle \tilde{J}_\mu'(v_n), v_n \rangle| \int_{\Omega} |v_n|^{p-2}v_n w| \leq \frac{1}{n}\|w - \alpha_n v_n\|_X.
$$
Since \( \|\alpha_n v_n\|_X \leq C \|w\|_X \), we have
\[
|(J'_\mu(v_n), w) - t_n \int_{\Omega} |v_n|^{p-2} v_n w V(x)dx| \leq \epsilon_n \|w\|_X
\]
where \( t_n = \langle J'_\mu(v_n), v_n \rangle \) and \( \epsilon_n \to 0 \). Therefore, \( \|J'_\mu(v_n)\|_* \to 0 \) and \( v_n \) is a Palais-Smale sequence. Hence by Lemma 2.2, \( \{v_n\} \) has a convergent subsequence with limit, say, \( v \). Then \( d \) is achieved at \( v \).

**Lemma 2.5.** The number \( C \) defined by (2.4) is the second smallest eigenvalue of \( L_\mu \)

**Proof.** We follow the proof in [3]. Assume by contradiction that there exists an eigenvalue \( \delta \) such that \( \lambda_1 < \delta < C \). In other words, \( \hat{J}_\mu \) has a critical value \( \delta \) with \( \lambda_1 < \delta < C \). We will construct a path in \( \Gamma \) on which \( \hat{J}_\mu \) remains \( \leq \delta \), which yields a contradiction with the definition of \( C \). Let \( u \in \mathcal{M} \) satisfies the equation
\[
-\Delta_p u - \frac{\mu}{|x|^p} |u|^{p-2} u = \delta V(x)|u|^{p-2} u \quad \text{in} \quad \mathcal{D}'(\Omega),
\]
and \( u \) changes sign in \( \Omega \). Taking \( u^+ \) and \( u^- \) as test function we get
\[
\int_{\Omega} |\nabla u^+|^p \, dx - \int_{\Omega} \frac{\mu}{|x|^p} |u^+|^p \, dx = \delta \int_{\Omega} (u^+)^p V(x) \, dx
\]
\[
\int_{\Omega} |\nabla u^-|^p \, dx - \int_{\Omega} \frac{\mu}{|x|^p} |u^-|^p \, dx = \delta \int_{\Omega} (u^-)^p V(x) \, dx.
\]
Consequently
\[
\hat{J}_\mu(u) = \hat{J}_\mu\left( \frac{u^+}{\|u^+\|_{L^p(V)}} \right) = \hat{J}_\mu\left( \frac{-u^-}{\|u^-\|_{L^p(V)}} \right) = \hat{J}_\mu\left( \frac{u^-}{\|u^-\|_{L^p(V)}} \right) = \delta.
\]
We will consider the following three paths in \( \mathcal{M} \), which go respectively from \( u \) to \( \frac{u^+}{\|u^+\|_{L^p(V)}} \), from \( \frac{u^+}{\|u^+\|_{L^p(V)}} \) to \( \frac{-u^-}{\|u^-\|_{L^p(V)}} \) and \( \frac{u^-}{\|u^-\|_{L^p(V)}} \) to \( u \):
\[
u_1(t) = \frac{tu + (1-t)u^+}{\|tu + (1-t)u^+\|_{L^p(V)}},
\]
\[
u_2(t) = \frac{tu^+ + (1-t)u^-}{\|tu + (1-t)u^-\|_{L^p(V)}},
\]
\[
u_3(t) = \frac{-tu^- + (1-t)u}{\|-tu^- + (1-t)u\|_{L^p(V)}}.
\]
Also we have
\[
\hat{J}_\mu(u_1(t)) = \hat{J}_\mu(u_2(t)) = \hat{J}_\mu(u_3(t)) = \delta.
\]
By joining the paths \( u_1(t) \) and \( u_2(t) \) we get a new path which connects \( u \) and \( \frac{u^+}{\|u^+\|_{L^p(V)}} \) and stays at levels \( \leq \delta \). Call this path as \( u_4(t) \). Now we define \( O = \{v \in \mathcal{M} \mid \hat{J}_\mu(v) < \delta \} \). Clearly \( \phi_1, -\phi_1 \in O \). Since \( \frac{u^-}{\|u^-\|_{L^p(V)}} \) does not change sign and vanishes on a set of positive measure it is not a critical point of \( \hat{J}_\mu \). So \( \frac{u^-}{\|u^-\|_{L^p(V)}} \) is a regular value of \( \hat{J}_\mu \), and consequently there exists a \( C^1 \) path \( \eta : [-\epsilon, \epsilon] \to \mathcal{M} \) with \( \eta(0) = \frac{u^-}{\|u^-\|_{L^p(V)}} \) and \( \frac{d}{dt}(\hat{J}_\mu(\eta(t)))|_{t=0} \neq 0 \). Choose a point \( v \in O \) on this path (this is possible because \( \hat{J}_\mu'(\eta(t))|_{t=0} \neq 0 \)) we can thus move from \( \frac{u^-}{\|u^-\|_{L^p(V)}} \) to \( v \) through this path which lies at levels \( < \delta \). Taking the component of \( O \) which
contains \( v \) and applying Lemma 2.3 together with Lemma 2.4, we can connect \( v \) to \( +\phi_1 \) (or to \(-\phi_1 \)) with a path in \( M \) at levels \( \delta \). Let us assume that this is \( +\phi_1 \) which is reached in this way. Now call this path connecting \( {u-\|u\|_{L^p(V)}} \) and \( \phi_1 \) as \( u_5(t) \), and consider the symmetric path \(-u_5(t)\), which goes from \( -{u-\|u\|_{L^p(V)}} \) to \(-\phi_1 \). We evaluate the functional \( \tilde{J}_\mu \) along \(-u_5(t)\). Since \( \tilde{J}_\mu \) is even,

\[
\tilde{J}_\mu(-u_5(t)) = \tilde{J}_\mu(u_5(t)) \leq \delta.
\]

Finally with \( u_5(t) \) we can connect \(-u \) with \( u \) by a path which stays at level \( \delta \). Putting every thing together we get a path connecting \(-\phi_1 \) and \( \phi_1 \) staying at levels \( \leq \delta \). This concludes the proof. \( \square \)

Note that Theorem 1.4 is an immediate consequence of Lemma 2.5. So we have the following characterization of \( \lambda_2 \), the second smallest eigenvalue of \( L_\mu \),

\[
\lambda_2 = \inf_{\gamma \in \Gamma} \sup_{u \in \gamma} \int_\Omega |\nabla u|^p dx - \int_\Omega |{u-\|u\|_{L^p(V)}}|^p dx.
\]

Now let \( \mu_k \) be the sequence of eigenvalues obtained in [6] which are characterized as

\[
\mu_k = \inf_{A \in \mathcal{F}} \sup_{u \in A} \int_\Omega |\nabla u|^p dx - \int_\Omega \frac{\mu}{|x|^p} |u|^p dx,
\]

where \( \mathcal{F} = \{ A \subset M \mid \text{the genus of } A \geq k \} \).

**Corollary 2.6.** With the notation above, \( \mu_2 = \lambda_2 \).

**Proof.** Let \( \gamma \) be a curve in \( \Gamma \). By joining this with its symmetric path \(-\gamma(t)\) we can get a set of genus \( \geq 2 \) where \( J_\mu \) does not increase its values. Therefore, \( \lambda_2 \geq \mu_2 \). But by Theorem 1.4, there is no eigenvalue between \( \lambda_1 \) and \( \lambda_2 \). Hence \( \lambda_2 = \mu_2 \). \( \square \)

**Lemma 2.7.** Let \( u \in X \) be a solution of (1.2) and let \( \mathcal{O} \) be a component of \( \{ x \in \Omega \mid u(x) > 0 \} \). Then \( u|_\mathcal{O} \in D_0^{1,p}(\mathcal{O}) \)

**Proof.** Let \( u_n \in C_c(\Omega) \cap D_0^{1,p}(\Omega) \) such that \( u_n \rightarrow u \) in \( D_0^{1,p}(\Omega) \). Then \( u_n^+ \rightarrow u^+ \) in \( D_0^{1,p}(\Omega) \). Let \( v_n = \min(u_n, u) \) and let \( \phi : \mathbb{R} \rightarrow \mathbb{R} \) be a \( C^1 \) function such that

\[
\phi(t) = \begin{cases} 
0 & \text{for } t \leq 1/2 \\
1 & \text{for } t \geq 1 
\end{cases}
\]

and \( |\phi'| \leq 1 \). Let \( \psi_r(x) = \phi(d(x, S)/r) \) where \( d(x, S) = \text{dist}(x, S) \). Then

\[
\psi_r(x) = \begin{cases} 
0 & \text{for } d(x, S) \leq r/2 \\
1 & \text{for } d(x, S) \geq r 
\end{cases}
\]

and \( \|\nabla \psi_r(x)\| \leq C/r \) for some constant \( C \). Now we define \( w_n(x) = \psi_r v_n(x) \mid_{\mathcal{O}} \). Since \( \psi_r v_n \in C(\overline{\mathcal{O}}) \), we have \( w_n \in C(\overline{\mathcal{O}}) \) and vanishes on the boundary \( \partial \mathcal{O} \). Indeed for \( x \in \partial \mathcal{O} \cap S \) then \( \psi_r(x) = 0 \) and so \( w_n(x) = 0 \). If \( x \in \partial \mathcal{O} \cap \Omega \) and \( x \notin S \) then \( u(x) = 0 \) (since \( u \) is continuous except at 0) and so \( v_n(x) = 0 \). If \( x \in \partial \mathcal{O} \) then \( u_n(x) = 0 \) and hence \( v_n(x) = 0 \). So in all the cases \( w_n(x) = 0 \) for \( x \in \partial \mathcal{O} \).
Therefore, \( w_{n,r} \in D_0^{1,p}(\Omega) \).

\[
\int_{\Omega} |\nabla (w_{n,r}) - \nabla (\psi_r u)|^p dx = \int_{\Omega} |(\nabla \psi_r) v_n + \psi_r \nabla v_n - (\nabla \psi_r) u - \psi_r \nabla u|^p dx \\
\leq \|\nabla \psi_r v_n - \nabla \psi_r u\|_{L^p(\Omega)}^p + \|\psi_r \nabla v_n - \psi_r \nabla u\|_{L^p(\Omega)}^p
\]

which goes to 0 as \( n \to \infty \), i.e., \( w_{n,r} \to \psi_r u \) in \( D_0^{1,p}(\Omega) \). Now

\[
\int_{\Omega} |\nabla \psi_r u + \psi_r \nabla u - u|^p dx \leq \int_{\Omega} |\psi_r \nabla u - \nabla u|^p dx + \int_{\Omega \cap \{|r/2<|x|<r\}} |\nabla \psi_r|^p u dx 
\]

\( \to 0 \) as \( r \to 0 \) by (1.1). Therefore, \( u|_{\Omega} \in D_0^{1,p}(\Omega) \).

\[\Box\]

**Proof of Theorem 1.5.** We denote \( J_\mu \) corresponding to \( V_\delta \) with \( J_{\mu,b} \). Let \( u_a \) be a solution to

\[-\Delta_\mu u - \frac{\mu}{|x|^2}|u|^{p-2}u = \lambda_2 V_a(x)|u|^{p-2}u \quad \text{in } D'(\Omega).\]

Assuming that the claim below is true, we have

\[
J_{\mu,b} \left( -\frac{v^+}{\|v^+\|_{L^p(V_a)}} \right) < \lambda_2(V_a), \quad J_{\mu,b} \left( -\frac{v^-}{\|v^-\|_{L^p(V_a)}} \right) < \lambda_2(V_a), \quad J_{\mu,b}(v) < \lambda_2(V_a).
\]

Define \( O_b = \{ u \in X, \int_{\Omega} |u|^p V_b = 1, \ J_{\mu,b}(v) < \lambda_2(V_a) \} \). Now we proceed as in Lemma 2.5, to define the paths \( v_i(t), i = 1, \ldots, 5 \) on which \( J_{\mu,b} < \lambda_2(V_a) \). We join these paths in a way described in Lemma 2.5 to obtain a path \( \gamma(t) \in \Gamma_b \) (the family of paths corresponds to \( V_b \)) such that \( J_{\mu,b}(\gamma(t)) < \lambda_2(V_a) \). This completes the proof.

**Claim:** There exists \( v \in X \), which changes sign and

\[
\int_{\Omega} |\nabla v^+|^p dx - \int_{\Omega} \frac{\mu}{|x|^2} |v^+|^p dx < \lambda_2(V_a), \quad \int_{\Omega} |\nabla v^-|^p dx - \int_{\Omega} \frac{\mu}{|x|^2} |v^-|^p dx < \lambda_2(V_a).
\]

(2.9)

**Proof of Claim:** Since \( u_a \) is an eigenfunction corresponding to \( \lambda_2 > \lambda_1 \), it has to change sign in \( \Omega \) (see [6]). Let \( O_1 \) and \( O_2 \) be positive and negative nodal domains of \( u_a \) respectively such that

\[
\int_{O_1} V_a (u_a^+)^p dx < \int_{O_1} V_b (u_a^+)^p dx \quad \text{and} \quad \int_{O_2} V_a (u_a^-)^p dx \leq \int_{O_2} V_b (u_a^-)^p dx.
\]

By Lemma 2.7, \( u_a|_{O_1} \in D_0^{1,p}(O_1) \) and also in \( L^p(O_1,V^-) \). We have

\[
\lambda_1(O_1,V_b) \leq \frac{\int_{O_1} |\nabla u_a|^p - \frac{\mu}{|x|^2}|u_a|^p dx}{\int_{O_1} |u_a|^p V_b} < \lambda_2(V_a).
\]

Therefore, \( \lambda_1(O_1,V_b) < \lambda_2(V_a) \). Similarily \( \lambda_1(O_2,V_b) \leq \lambda_2(V_a) \). Now we modify \( O_1 \) and \( O_2 \) to get \( \tilde{O}_1 \) and \( \tilde{O}_2 \) with empty intersection and \( \lambda_1(\tilde{O}_1,V_b) < \lambda_2(V_a) \) and \( \lambda_1(\tilde{O}_2,V_b) < \lambda_2 \). For \( \eta > 0 \), let \( O_1(\eta) = \{ x \in O_1 | \text{dist}(x,\tilde{O}_1) > \eta \} \). Then \( \lambda_1(O_1(\eta),V_b) \geq \lambda_1(O_1,V_b) \) and \( \lambda_1(O_1(\eta),V_b) \to \lambda_1(O_1,V_b) \) as \( \eta \to 0 \). Therefore, there exists \( \eta_0 > 0 \) such that \( \lambda_1(O_1(\eta),V_b) < \lambda_2(V_a) \) for \( 0 < \eta < \eta_0 \). Let \( x \in \partial O_2 \cap \Omega \) and \( 0 < \eta < \min\{\eta_0, \text{dist}(x_0,\Omega^c)\} \). Now define \( \tilde{O}_2 = O_2 \cup B(x_0,\eta/2) \). Then \( \tilde{O}_2 \cap O_1(\eta) = \emptyset \), \( \lambda_1(O_2,V_b) < \lambda_1(O_2,V_b) < \lambda_2(V_a) \). Now we consider the function
\( v = v_1 - v_2 \), where \( v_i \) are the extensions by zero outside \( \tilde{O}_i \) of the eigenfunctions associated to \( \lambda_1(\tilde{O}_i, V_b) \). Then \( v \) satisfies (2.9).

\[\square\]

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