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MULTIPLE POSITIVE SOLUTIONS TO FOURTH-ORDER SINGULAR BOUNDARY-VALUE PROBLEMS IN ABSTRACT SPACES

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Abstract. We prove the existence of multiple positive solutions to singular boundary-value problems for fourth-order equations in abstract spaces. Our results improve and extend that obtained in [14, 15, 16], even in the scalar case.

1. Introduction

In this paper, we consider the following singular boundary-value problem (BVP) for fourth-order differential equations in a Banach space E:

$$x^{(4)}(t) = f(t, x(t)), \quad 0 < t < 1 \tag{1.1}$$

subject to the boundary conditions

$$x(0) = x(1) = x''(0) = x''(1) = 0, \tag{1.2}$$

where $f \in C[(0,1) \times P \setminus \{\theta\}, P]$ which may be singular at $t = 0$, $t = 1$, and $x = \theta$; P is a cone of Banach space E, which will be stated in detail in section 2; θ is the zero element of E.

Boundary-value problems arise from applied mathematical sciences, and they have received a great deal of attention in the literature. Problems of the form (1.1) subject to (1.2), for example, are used to model such phenomena as the deflection of an elastic beam supported at the endpoints. Most of the available literature on fourth order boundary value problems, for instance [1, 2, 4, 5, 8, 9, 11], discuss the case where f is either continuous or a Caratheodory function. Recently some papers such as [14, 15, 16], by using approximating method or upper-lower solution approach, investigate (1.1) with suitable boundary conditions in the case where f may be singular at $t = 0$, $t = 1$, or $x = 0$. However, all of the above-mentioned references consider (1.1) only in scalar space; and especially in the singular case, [14, 15, 16] concentrate on the solvability of (1.1) subject to some suitable boundary conditions, not the existence of multiple solutions for such problems. On the other hand, the theory of ordinary differential equations (ODE) in abstract spaces is

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becoming an important branch of mathematics in last thirty years because of its application in partial differential equations and ODE's in appropriately infinite dimensional spaces (see, for example [3, 10, 12]). As a result the goal of this paper is to fill up the gap in this area, that is, to investigate the existence of multiple positive solutions of (1.1) with (1.2) in a Banach space E.

The technique used in this paper are the well-known Krein-Rutman theorem, a specially constructed cone, and the fixed point theorem of cone expansion and compression. It is remarkable that neither approximating method nor upper-lower solution approach is applied in present paper.

This paper is organized as follows. Section 2 gives some preliminaries and some lemmas. Section 3 is devoted to the main result and its proof. Some examples are worked out in Section 4 to indicate the application of our main result.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$, and $P \subset E$ be a cone which defines a partial order relation $\leq$ by $x \leq y$ if and only if $y - x \in P$, $x < y$ if and only if $x \leq y$ and $x \neq y$, where $x, y \in E$. Also suppose P is a normal cone, that is, there exists $N > 0$ such that $\|u\| \leq N\|v\|$ if $\theta \leq u \leq v$, where θ denotes the zero element of E.

Evidently, $C[I, E]$ is a Banach space with norm $\|x\|_c := \max_{t\in I} \|x(t)\|$. Moreover,

$$C[I, P] := \{x \in C[I, E] : \ x(t) \in P, \ t \in I\}$$

is a normal cone of $C[I, E]$ with the same normal constant N as P in E.

A function x is said to be a solution of (1.1) subject to (1.2) if $x \in C^2[I, E] \cap C^4[(0, 1), E]$ satisfies (1.1) and boundary conditions (1.2); in addition, x is said to be a positive solution if $x(t) > \theta$ for $t \in (0, 1)$ and x is a solution of (1.1) with (1.2).

Let $u : (0, 1] \to E$ be continuous. The abstract generalized integral $\int_0^1 u(t)dt$ is said to be convergent if the limit $\lim_{\epsilon\to 0^+} \int_\epsilon^1 u(t)dt$ exists. The convergency or divergency of other kinds of generalized integrals can be defined similarly.

For a bounded subset V of Banach space E, by $\alpha(V)$ we denote the Kuratowskii measure of noncompactness of V(for details, see [3, 10]). In this paper, the Kuratowskii measure of noncompactness of bounded set in E and $C[I, E]$ are denoted by $\alpha(\cdot)$ and $\alpha_c(\cdot)$, respectively.

To conclude this section, we list three lemmas which will be used in Section 3.

Lemma 2.1 ([7])**.** *If $V \subset C[J, E]$ is bounded and equicontinuous, then $\alpha(V(t))$ is continuous on J and $\alpha_c(V) = \max\{\alpha(V(t))|t \in J\}$, where $V(t) = \{x(t)|x \in V\}$.*

Lemma 2.2 ([13])**.** *Let K be a cone of a real Banach space E and B: $K \to K$ a completely continuous operator. Assume that B is order-preserving and positively homogeneous of degree 1 and that there exist $v \in K \setminus \{\theta\}$, $\lambda > 0$ such that $Bv \geq \lambda v$. Then $r(B) \geq \lambda$, where $r(B)$ denotes the spectral radius of B.*

Lemma 2.3 (Fixed point theorem of cone expansion and compression [7])**.** *Let P be a cone of a real Banach space E and $P_{r,s} = \{x \in P : r \leq \|x\| \leq s\}$ with $s > r > 0$. Suppose that A: $P_{r,s} \to P$ is a strict contraction such that one of the following two conditions is satisfied:*

(i) *$Ax \not\leq x$ for $x \in P$, $\|x\| = r$ and $Ax \not\geq x$ for $x \in P$, $\|x\| = s$.*
(ii) *$Ax \not\geq x$ for $x \in P$, $\|x\| = r$ and $Ax \not\leq x$ for $x \in P$, $\|x\| = s$.*

Then, the operator A has a fixed point $x \in P$ such that $r < \|x\| < s$.

3. Main Result

For convenience, we list the following assumptions:

(H0) $f \in C[(0,1) \times P \setminus \{\theta\}, P]$ and satisfies
$$0 < \int_0^1 t(1-t) f_{r,R}(t)dt < +\infty, \quad \forall R \geq r > 0,$$
where for $t \in (0,1)$, $z(t) := \min\{t, 1-t\}$ and
$$f_{r,R}(t) := \sup\{\|f(t,x)\| : \ \frac{z(t)}{N} r \leq \|x\| \leq R, \ x \in P\}.$$

(H1) For every $[c,d] \subset (0,1)$, and positive numbers $R_2 > R_1 > 0$, $f(t,x)$ is uniformly continuous on $[c,d] \times \overline{P_{R_2}} \setminus P_{R_1}$ with respect to t. Here $P_r =: \{x \in P : \|x\| < r\}$ for each $r > 0$.

(H2) For every $t \in (0,1)$ and every bounded subset $D \subset \overline{P_{R_2}} \setminus P_{R_1}$ ($R_2 > R_1 > 0$), we have $\alpha(f(t,D)) \leq l\alpha(D)$, where l is a constant with $l < 15$.

(H3) There exist $\varphi \in L[I, R^+]$, $[c,d] \subset [0,1]$, and $\varphi^* \in P^*$ (here P^* denotes the dual cone of P) with $\|\varphi^*\| = 1$ such that
$$\liminf_{x \to \theta, \ x \in P} \varphi^*(f(t,x)) \geq \varphi(t)$$
uniformly with respect to $t \in [c,d]$ and $\int_c^d s(1-s)\varphi(s)ds > 0$.

(H4) There exist functions $a \in C[I, R^+]$ and $b \in C[I, P]$ with $a(t) \not\equiv 0$ on every subinterval of I such that
$$f(t,x) \geq a(t)x - b(t) \quad \textit{for } t \in (0,1) \textit{ and } x \in P \setminus \{\theta\}.$$

(H5) There exists a positive number R such that
$$N \int_0^1 s(1-s) f_{R,R}(s)ds < 8R,$$
where $f_{R,R}(s)$ is the same as in (H0).

Note that assumption (H3) is reasonable since $f(t,x)$ is singular at $x = \theta$.

We assume that (H0) holds throughout the remainder of the paper. To overcome the difficulties arising from singularities, we define
$$Q := \{x \in C[I, P] : \ x(t) \geq z(t)x(s) \geq \theta, \ \forall t, s \in I\}. \tag{3.1}$$
It is easy to see that Q is a nonempty (notice $t(1-t) \in Q$), convex, and closed subset of $C[I, E]$. Furthermore, Q is a cone of the Banach space $C[I, E]$ and for every $x \in Q \setminus \{\theta\}$, by (3.1) and the normality of the cone P, we have
$$\|x(t)\| \geq \frac{z(t)}{N} \|x\|_c > 0 \qquad \text{for } t \in (0,1). \tag{3.2}$$
Therefore, x is a positive solution of (1.1)-(1.2) provided that $x \in Q \setminus \{\theta\}$ is a solution of (1.1)-(1.2).

Define the operator A on $Q \setminus \{\theta\}$ by
$$(Ax)(t) := \int_0^1 J(t,\tau) f(\tau, x(\tau))d\tau, \quad \forall x \in Q \setminus \{\theta\}, \tag{3.3}$$

where $J(t,\tau)=\int_0^1 G(t,s)G(s,\tau)ds$ and

$$G(t,s)=\begin{cases} t(1-s), & 0\le t\le s\le 1;\\ s(1-t), & 0\le s\le t\le 1.\end{cases} \tag{3.4}$$

Now we show the operator A is well defined on $Q\setminus\{\theta\}$. First we claim that for each $x\in Q\setminus\{\theta\}$, $\int_0^1 G(s,\tau)f(\tau,x(\tau))d\tau$ is convergent. In fact, since $x\in Q\setminus\{\theta\}$, we can see by (3.2) that $\|x\|_c\neq 0$ and

$$\frac{z(t)}{N}\|x\|_c\le\|x(\tau)\|\le\|x\|_c\quad\text{for each }\tau\in(0,1).$$

This together with $G(s,\tau)\le\tau(1-\tau)$ for all $s,\tau\in I$ and (H0) implies that $\int_0^1 G(s,\tau)f(\tau,x(\tau))d\tau$ is convergent and

$$\int_0^1\tau(1-\tau)\|f(\tau,x(\tau))\|d\tau<+\infty\quad\text{for each }x\in Q\setminus\{\theta\}.$$

The Lebesgue dominated convergence theorem yields that for every $x\in Q\setminus\{\theta\}$, $\int_0^1 G(s,\tau)f(\tau,x(\tau))d\tau$ is continuous in s on I. Therefore, by (3.3) we obtain that $Ax\in C^2[I,P]$ and

$$\begin{gathered}(Ax)^{(4)}(t)=f(t,x(t)),\quad 0<t<1;\\ (Ax)(0)=(Ax)(1)=(Ax)''(0)=(Ax)''(1)=0.\end{gathered}\tag{3.5}$$

Now we are in position to state the following lemma.

Lemma 3.1. *The boundary-value problem* (1.1)-(1.2) *has a positive solution in* $C^2[I,P]\cap C^4[(0,1),P]$ *if and only if A has a fixed point x in $Q\setminus\{\theta\}$.*

Proof. From the above, sufficiency is evident. In what follows, we prove only necessity. Suppose x is a positive solution of (1.1)-(1.2), that is, $x(t)>\theta$ for $t\in(0,1)$ and satisfies (1.1)-(1.2). Now we show $x\in Q\setminus\{\theta\}$. To see this notice that

$$x(t)=\int_0^1 G(t,s)\int_0^1 G(s,\tau)f(\tau,x(\tau))d\tau ds\quad\text{for }t\in I. \tag{3.6}$$

This and

$$\frac{G(t,s)}{G(\xi,s)}=\begin{cases}\frac{t(1-s)}{\xi(1-s)}\ge t\ge z(t), & t,\xi\le s;\\ \frac{s(1-t)}{s(1-\xi)}\ge 1-t\ge z(t), & t,\xi\ge s;\\ \frac{t(1-s)}{s(1-\xi)}\ge t\ge z(t), & t<s<\xi;\\ \frac{s(1-t)}{\xi(1-s)}\ge 1-t\ge z(t), & t>s>\xi\end{cases}\tag{3.7}$$

yield

$$x(t)\ge z(t)\int_0^1 G(\xi,s)\int_0^1 G(s,\tau)f(\tau,x(\tau))d\tau\quad\text{for }t,\xi\in I,$$

which implies $x\in Q\setminus\{\theta\}$. On the other hand, it is easy to see by (3.6) that $Ax=x$. This completes the proof. □

Consequently, the existence of positive solution for (1.2) is equivalent to that of fixed point of A in $Q\setminus\{\theta\}$. By (3.5) and the process similar to the proof of Lemma 3.1, we also obtain the following Lemma.

Lemma 3.2. $A(Q\setminus\{\theta\})\subset Q$.

Lemma 3.3. *For every pair of positive numbers R_2 and R_1 with $R_2 > R_1 > 0$, $A : \overline{Q_{R_2}} \setminus Q_{R_1} \to Q$ is a strict set contraction, where $Q_r := \{x \in Q : \|x\|_c < r\} (r > 0)$.*

Proof. First, under the assumptions for R_2 and R_1, (H0) guarantees that for each $x \in \overline{Q_{R_2}} \setminus Q_{R_1}$,

$$\int_0^1 \tau(1-\tau)\|f(\tau, x(\tau))\| d\tau \le \int_0^1 \tau(1-\tau) f_{R_1,R_2}(\tau) d\tau < +\infty \tag{3.8}$$

which implies A: $\overline{Q_{R_2}} \setminus Q_{R_1} \to Q$ is bounded.

Next we show A: $\overline{Q_{R_2}} \setminus Q_{R_1} \to Q$ is continuous. To see this from (3.3) it follows that for $x \in \overline{Q_{R_2}} \setminus Q_{R_1}$ and $t_1, t_2 \in I$,

$$\|(Ax)(t_1) - (Ax)(t_2)\| \le \int_0^1 |G(t_1, s) - G(t_2, s)| ds \int_0^1 G(s, \tau)\|f(\tau, x(\tau))\| d\tau. \tag{3.9}$$

This and (3.8) yield that for every subset $V \subset \overline{Q_{R_2}} \setminus Q_{R_1}$, $(AV)(t)$ is equicontinuous on I, where $(AV)(t) = \{(Ax)(t) : x \in V\}$, $t \in I$.

Let $x_n, x \in \overline{Q_{R_2}} \setminus Q_{R_1}$ with $\|x_n - x\|_c \to 0$ as $n \to +\infty$. This implies

$$\|x_n(t) - x(t)\| \to 0 \quad \text{as } n \to +\infty \quad \text{for } t \in I.$$

From Lebesgue dominated convergence theorem and (3.8), it follows that

$$\|(Ax_n)(t) - (Ax)(t)\| \to 0 \quad \text{as } n \to +\infty.$$

Thus, $\{(Ax_n)(t)\}$ is relatively compact for every $t \in I$. From this and the equicontinuity of $\{Ax_n(t)\}$ by the Ascoli-Arzela theorem, we obtain that $\{Ax_n\}$ is a relatively compact subset of Q.

Now it remains to show $\|Ax_n - Ax\|_c \to 0$ as $n \to +\infty$. In fact, if this is not true, then there is a constant $\epsilon_0 > 0$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\|Ax_{n_i} - Ax\|_c \ge \epsilon_0$ $(i = 1, 2, \dots)$. However, the relative compactness of $\{Ax_n\}$ implies that $\{Ax_{n_i}\}$ contains a subsequence which converges in $C[I, P]$. Without loss of generality, we may assume that $\{Ax_{n_i}\}$ itself converges to y, that is, $\|Ax_{n_i} - y\|_c \to 0$ as $i \to +\infty$. So we have $y = Ax$. This is a contradiction. Therefore, A is continuous.

Finally, we show A: $\overline{Q_{R_2}} \setminus Q_{R_1} \to Q$ is a strict set contraction, that is, there exists $k \in (0, 1)$ such that $\alpha_c(AV) \le k\alpha_c(V)$ for each $V \subset \overline{Q_{R_2}} \setminus Q_{R_1}$. Fix $V \subset \overline{Q_{R_2}} \setminus Q_{R_1}$. Let

$$(A_n x)(t) := \int_{1/n}^{1-(1/n)} J(t, s) f(s, x(s)) ds \quad \text{for } x \in V. \tag{3.10}$$

By (3.8) we know

$$(A_n x)(t) \to (Ax)(t) \quad \text{as } n \to +\infty \quad \text{for each } x \in V \text{ and } t \in I. \tag{3.11}$$

This implies

$$d_H((A_n V)(t), (AV)(t)) \to 0 \quad \text{as } n \to +\infty \quad \text{for each } t \in I,$$

where $d_H(\cdot, \cdot)$ denotes the Hausdorff metric. Thus, by the property of noncompactness measure,

$$\alpha((A_n V)(t)) \to \alpha((AV)(t)) \qquad \text{for } t \in I. \tag{3.12}$$

In what follows, we estimate $\alpha((A_nV)(t))$ for each $t \in I$. Note that $x(s) \geq z(s)x(\tau) \geq \theta$ for $s, \tau \in I$ and $x \in V$. Thus,

$$\frac{R_1}{nN} \leq \frac{\|x\|_c}{nN} \leq \|x(s)\| \leq R_2 \qquad \text{for } s \in [\frac{1}{n}, 1 - \frac{1}{n}].$$

By the definition of integration and (3.4), respectively, we have

$$\int_{1/n}^{1-(1/n)} J(t,s)f(s,x(s))ds \in (1 - \frac{2}{n})\overline{co}(\{J(t,s)f(s,x(s)) : s \in [\frac{1}{n}, 1 - \frac{1}{n}]\}$$

and

$$J(t,s) = \int_0^1 G(t,\tau)G(\tau,s)d\tau \leq \int_0^1 \tau^2(1-\tau)^2 d\tau = \frac{1}{30} \quad \text{for all } t, s \in I.$$

These, (H1), (H2), and Lemma 2.1 guarantee that

$$\begin{aligned} \alpha((A_nV)(t)) &= \alpha(\{\int_{1/n}^{1-(1/n)} J(t,s)f(s,x(s))ds |\ x \in V\}) \\ &\leq (1 - \frac{2}{n})\alpha(\overline{co}\{J(t,s)f(s,x(s))|\ s \in [\frac{1}{n},\ 1 - \frac{1}{n}],\ x \in V\}) \\ &\leq \alpha(\{J(t,s)f(s,x(s))|\ s \in [\frac{1}{n},\ 1 - \frac{1}{n}],\ x \in V\}) \\ &\leq \frac{1}{30}\alpha(\{f(s,x(s))|\ s \in [\frac{1}{n},\ 1 - \frac{1}{n}],\ x \in V\}) \\ &\leq \frac{1}{30}\alpha(f(I_n \times V(I_n)) \leq \frac{1}{30} l \cdot \alpha V(I_n)) \leq \frac{1}{15} l \alpha_c(V), \end{aligned} \tag{3.13}$$

where $I_n = [\frac{1}{n}, 1 - \frac{1}{n}]$ and $V(I_n) = \{x(s)\colon\ x \in V,\ s \in I_n\}$. Combining Lemma 2.1 with (3.9), (3.12), and (3.13) again, one can obtain

$$\alpha_c(AV) = \max_{t \in I} \alpha((AV)(t)) \leq \frac{1}{15} l \cdot \alpha_c(V).$$

This implies A is a strict set contraction with $k = \frac{1}{15} l < 1$ from $\overline{Q_{R_2}} \backslash Q_{R_1}$ to Q. $\square$

Using (H4), we define an operator L on $C[I, R]$ by

$$(Lu)(t) := \int_0^1 J(s,t)a(s)u(s)ds = \int_0^1 G(\tau,t) \int_0^1 G(s,\tau)a(s)u(s)dsd\tau$$

for $u \in C[I, R]$ where J is given by (3.4), and a is the same as in (H4).

It is easy to see L: $C[I, R] \to C[I, R]$ is a completely continuous positive operator. Note that if $v(t) = t(1-t)$ on I, then $\|v\|_c = \frac{1}{4}$. By $G(t,\tau) \geq t\tau(1-t)(1-\tau)$ for $t, \tau \in I$, we know

$$(Lv)(t) \geq \int_0^1 \tau^2(1-\tau)^2 d\tau \int_0^1 s^2(1-s)^2 a(s)ds \cdot v(t) = \delta_0 v(t) \qquad \text{for } t \in I,$$

where $\delta_0 = \frac{1}{30}\int_0^1 s^2(1-s)^2 a(s)ds > 0$. From Lemma 2.2 it follows that the spectral radius $r(L) \geq \delta_0 > 0$. So the well-known Krein-Rutman theorem [13] guarantees that there exists an $p \in C[I, R^+]$ with $p(t) \not\equiv 0$ on I such that

$$(Lp)(t) = \int_0^1 J(s,t)a(s)p(s)ds = r(L)p(t) \qquad \text{for } t \in I. \tag{3.14}$$

From (3.7) one deduces that

$$\begin{aligned} p(t) &= \frac{1}{r(L)} \int_0^1 (\int_0^1 G(s,\tau)G(\tau,t)d\tau)a(s)p(s)ds \\ &\geq \frac{1}{r(L)} \int_0^1 z(s) \int_0^1 G(\xi,\tau)G(\tau,t)d\tau)a(s)p(s)ds \\ &\geq \frac{1}{r(L)} \int_0^1 z(s)a(s)p(s)ds \cdot J(\xi,t) \quad \text{for all } t,\xi \in I. \end{aligned}$$

Therefore,

$$p(t) \geq \delta J(\xi,t) \quad \text{for all } t,\xi \in I, \tag{3.15}$$

where $\delta := \frac{1}{r(L)} \int_0^1 z(s)a(s)p(s)ds$. This and (3.14) guarantees that

$$\int_0^1 p(t)a(t)dt \geq \delta r(L). \tag{3.16}$$

Now we are ready to give the main result of the present paper.

Theorem 3.4. *Assume that (H0)–(H5) hold and* $r(L) > 1$. *Then* (1.1) *subject to* (1.2) *has at least two positive solutions.*

Proof. Set

$$K := \{x \in Q : \quad \int_0^1 p(t)a(t)x(t)dt \geq \delta r(L)x(s), \quad \forall s \in I\}, \tag{3.17}$$

where Q is given by (3.1), $a(t)$ is given by (H4), $p(t)$, $r(L)$, and δ are given by (3.14) and (3.15).

By (3.16) it is easy to see $K \setminus \{\theta\} \neq \emptyset$ and K is also a cone of $C[I,E]$. We now prove that the operator A defined by (3.3) maps $Q \setminus \{\theta\}$ into K. In fact, for $x \in Q \setminus \{\theta\}$, it follows from (3.3), (3.15), and (3.2) that

$$\begin{aligned} \int_0^1 p(t)a(t)(Ax)(t)dt &= \int_0^1 p(t)a(t) \int_0^1 J(t,s)f(s,x(s))dsdt \\ &= \lim_{n\to+\infty} \int_0^1 p(t)a(t) \int_{1/n}^{1-(1/n)} J(t,s)f(s,x(s))dsdt \\ &= \lim_{n\to+\infty} \int_{1/n}^{1-(1/n)} f(s,x(s))ds \int_0^1 J(t,s)a(t)p(t)dt \\ &= r(L) \lim_{n\to+\infty} \int_{1/n}^{1-(1/n)} p(s)f(s,x(s))ds \\ &\geq \delta r(L) \lim_{n\to+\infty} \int_{1/n}^{1-(1/n)} J(\tau,s)f(s,x(s))ds \\ &= \delta r(L) \lim_{n\to+\infty} \int_0^1 J(\tau,s)f(s,x(s))ds \\ &= \delta r(L)(Ax)(\tau) \quad \text{for all } \tau \in I, \end{aligned}$$

which implies $A(Q \setminus \{\theta\}) \subset K$. Consequently, we obtain by Lemma 3.2 and Lemma 3.3 that $A : \overline{K_{R_2}} \setminus K_{R_1} \to K$ is a strict set contraction for every pair of positive numbers R_2 and R_1 with $R_2 > R_1 > 0$, where $K_{R_1} = \{x \in K : \|x\|_c < R_1\}$.

Choose a positive number R_0 with $R_0 > R$ such that

$$R_0 > \frac{N\|b\|_c}{30\delta(r(L)-1)} \int_0^1 a(t)p(t)dt.$$

We proceed to prove

$$Ax \nleq x \quad \text{for all } x \in \partial K_{R_0}. \tag{3.18}$$

Suppose, on the contrary, there exists an $x_0 \in \partial K_{R_0}$ such that $Ax_0 \leq x_0$. Therefore,

$$x_0(t) \geq (Ax_0)(t) = \int_0^1 J(t,s)f(s,x_0(s))ds \quad \text{for all } t \in I.$$

Multiply by $p(t)a(t)$ and integrate from 0 to 1 to obtain

$$\begin{aligned}
&\int_0^1 p(t)a(t)x_0(t)dt \geq \int_0^1 p(t)a(t)\int_0^1 J(t,s)f(s,x_0(s))dsdt \\
&\geq \int_0^1 p(t)a(t)\int_0^1 J(t,s)a(s)x_0(s)dsdt - \int_0^1\int_0^1 J(t,s)a(t)p(t)b(s)dsdt \\
&= \int_0^1 (\int_0^1 J(t,s)a(t)p(t)dt)a(s)x_0(s)ds - \int_0^1\int_0^1 J(t,s)a(t)p(t)b(s)dsdt \\
&= r(L)\int_0^1 p(s)a(s)x_0(s)ds - \int_0^1\int_0^1 J(t,s)a(t)p(t)b(s)dsdt.
\end{aligned}$$

This and (3.17) yield

$$\begin{aligned}
\int_0^1\int_0^1 J(t,s)a(t)p(t)b(s)dsdt &\geq (r(L)-1)\int_0^1 p(s)a(s)x_0(s)ds \\
&\geq (r(L)-1)\delta x_0(\tau) \geq \theta \quad \text{for all } \tau \in I.
\end{aligned}$$

The normality of the cone P and $|J(t,s)| \leq \frac{1}{30}$ for all $t,s \in I$ guarantee that

$$\frac{N\|b\|_c}{30}\int_0^1 a(t)p(t)dt \geq \delta(r(L)-1)R_0.$$

This is a contradiction with the selection of R_0. Consequently, (3.18) holds.

In what follows we show

$$Ax \ngeq x \quad \text{for all } x \in \partial K_R. \tag{3.19}$$

If this is false, then there exists an $x_1 \in \partial K_R$ such that $x_1 \leq Ax_1$, that is,

$$\theta \leq x_1(t) \leq (Ax_1)(t) \qquad \text{for } t \in I.$$

Since $x_1 \in K \subset Q$, we get

$$x_1(t) \geq z(t)x_1(\tau) \geq \theta \qquad \text{for } t,\tau \in I.$$

As a result,

$$\frac{z(t)}{N}R = \frac{z(t)}{N}\|x_1\|_c \leq \|x_1(t)\| \leq R \qquad \text{for } t \in I.$$

This implies

$$\|f(t,x_1(t))\| \leq f_{R,R}(t) \qquad \text{for } t \in (0,1).$$

Combining the above with (H5) we know

$$\begin{aligned}\|x_1(t)\| &\le N\|(Ax_1)(t)\| \le N\|\int_0^1 J(t,s)f(s,x_1(s))ds\| \\ &\le N\int_0^1 J(t,s)\|f(s,x_1(s))\|ds \\ &\le N\int_0^1 J(t,s)f_{R,R}(s)ds \\ &\le N\int_0^1 G(t,\tau)(\int_0^1 s(1-s)f_{R,R}(s)ds)d\tau \\ &\le \frac{N}{8}\int_0^1 s(1-s)f_{R,R}(s)ds < R \qquad \text{for } t\in I.\end{aligned}$$

This is a contradiction. Then (3.19) follows. Finally, we prove that there exists a positive number R' with $R' < R$ such that

$$Ax \not\le x \qquad \text{for } x \in \partial K_{R'}. \tag{3.20}$$

In fact, by (H3), given $\epsilon \in (0, \int_c^d J(\frac{1}{2}, s)\phi(s)ds)$, there exists an $R'' > 0$ such that

$$\phi^*(f(t,x(t)) \ge \phi(t) - \epsilon \qquad \text{for } t\in[c,d] \text{ and } x\in P_{R''}\setminus\{\theta\}. \tag{3.21}$$

Choose

$$R' := \min\big\{\frac{R}{2},\ R'',\ \int_c^d J(\frac{1}{2},s)\phi(s)ds - \epsilon\big\}. \tag{3.22}$$

Now we are ready to prove (3.20) holds. Suppose, on the contrary, there exists an $x_2 \in \partial K_{R'}$ such that $Ax_2 \le x_2$. Then by (3.2) we know

$$\frac{z(t)}{N}R' \le \|x_2(t)\| \le R' \le R'' \qquad \text{for } t\in(0,1).$$

This and (3.21) guarantee that

$$\begin{aligned}\phi^*(x_2(t)) &\ge \int_0^1 J(t,s)\phi^*(f(s,x_2(s))ds \\ &\ge \int_c^d J(t,s)[\phi(s)-\epsilon]ds \\ &> \int_c^d J(t,s)\phi(s)ds - \frac{\epsilon}{2} \qquad \text{for } t\in(0,1).\end{aligned}$$

Consequently,

$$\|x_2\|_c \ge \phi^*(x_2(\frac{1}{2})) \ge \int_c^d J(\frac{1}{2},s)\phi(s)ds - \frac{\epsilon}{2} > R'.$$

This is a contradiction with $x_2 \in \partial K_{R'}$. Then the conclusion follows from Lemma 2.3. □

Remark 3.5. If $r(L) > 1$ is replaced with $\int_0^1 s^2(1-s)^2 a(s)ds > 30$ in Theorem 3.4, the conclusion of Theorem 3.4 also holds. In fact, by $v(t) = t(1-t) \in Q$ and $G(t,s) \ge ts(1-t)(1-s)$ for $t,s \in I$, and using the definition of the operator L, one can obtain

$$Lv \ge \frac{1}{30}\int_0^1 s^2(1-s)^2 a(s)ds \cdot v.$$

Then by Lemma 2.2, $r(L) > 1$ follows.

For the next theorem we replace (H4) by

(H'4) There exists an $\psi^* \in P^*$ with $\|\psi^*\| = 1$ and a subinterval $[c', d'] \subset (0, 1)$ such that
$$\lim_{\|x\|\to+\infty,\ x\in P} \frac{\psi^*(f(t,x))}{\|x\|} > \mu$$
uniformly with respect to $t \in [c', d']$, where
$$\mu = N\Big(\min\{c', 1-d'\} \cdot \int_{c'}^{d'} J(\frac{1}{2}, s)ds\Big)^{-1}.$$

Theorem 3.6. *Assume that (H0)-(H3), (H'4), and (H5) hold. Then* (1.1) *subject to* (1.2) *has at least two positive solutions.*

Proof. Consider the operator A in $Q \setminus \{\theta\}$, where Q is defined by (3.1). From the proof of Theorem 3.4, it is not difficult to see that (3.19) and (3.20) hold for $x \in \partial Q_{R'}$ and $x \in \partial Q_R$, respectively; where R' is given by (3.22) and R is given by (H5).

It remains to show that there exists a positive number R_0 with $R_0 > R$ such that (3.18) holds for $x \in \partial Q_{R_0}$. By (H'4), there exist an $\epsilon > 0$ and an $R_1 > 0$ such that
$$\psi^*(f(t,x)) \ge (\mu + \epsilon)\|x\| \qquad \text{for } t \in [c', d'],\ x \in P,\ \text{and } \|x\| \ge R_1. \tag{3.23}$$
Choose
$$R_0 \ge \max\big\{R + 1,\ \frac{NR_1}{\min\{c', 1-d'\}}\big\}.$$
We proceed to prove (3.18) holds for $x \in \partial Q_{R_0}$. If this is false, then there exists an $x_0 \in \partial Q_{R_0}$ such that $Ax_0 \le x_0$. By (3.1) we know
$$x_0(t) \ge z(t)x_0(s) \ge \min\{c', 1-d'\} \cdot x_0(s) \ge \theta \qquad \text{for } t \in [c', d'] \text{ and } s \in I.$$
¿From the normality of the cone P it follows that
$$N\|x_0(t)\| \ge \min\{c',\ 1-d'\} \cdot \|x_0\|_c = \min\{c',\ 1-d'\} \cdot R_0 \qquad \text{for } t \in [c',\ d'],$$
that is, $\|x_0(t)\| \ge R_1$ for $t \in [c,\ d']$. This, (3.23), and (3.3) guarantee that
$$\begin{aligned}
R_0 &\ge \psi^*(x_0(\frac{1}{2}))\\
&\ge \int_0^1 J(\frac{1}{2}, s)\psi^*(f(s, x_0(s))ds\\
&\ge \int_{c'}^{d'} J(\frac{1}{2}, s)(\mu + \epsilon)\|x_0(s)\|ds\\
&\ge \frac{\min\{c', 1-d'\}}{N} \int_{c'}^{d'} J(\frac{1}{2}, s)(\mu + \epsilon)R_0 ds > R_0,
\end{aligned}$$
which is a contradiction. Then the statement of Theorem 3.6 follows. □

Corollary 3.7. *Suppose (H0), (H1), (H2), and one of the following conditions are satisfied:*

(i) *(H3) and (H5).*
(ii) *(H4), (H5), and* $r(L) > 1$.
(iii) *(H'4) and (H5).*

Then (1.1) *subject to* (1.2) *has at least one positive solution.*

Remark 3.8. By the same method used above, we can study the existence of multiple positive solutions of second order nonlinear singular boundary-value problems in scalar or in abstract space.

4. Examples

Example 4.1. Consider the boundary-value problem consisting of a finite system of fourth-order scalar nonlinear differential equations.

$$\begin{gathered} x_n^{(4)}(t) = \frac{1}{\sqrt{t(1-t)}}\big(x_n^{\frac{3}{2}} + \frac{1}{\max_{1\le i\le m}|x_i|}\big), \quad 0 < t < 1; \\ x_n(0) = x_n(1) = x_n''(0) = x_n''(1) = 0, \quad (n = 1, 2, \dots m). \end{gathered} \tag{4.1}$$

Claim: (4.1) has at least two positive solutions $x^*(t) = (x_1^*(t), x_2^*(t), \dots x_m^*(t))$ and $x^{**}(t) = (x_1^{**}(t), x_2^{**}(t), \dots x_m^{**}(t))$ such that

$$0 < \max_{1\le i\le m,\ t\in[0,1]} |x_i^*(t)| < 1 < \max_{1\le i\le m,\ t\in[0,1]} |x_i^{**}(t)|.$$

Proof. Let E be the m-dimensional Euclidean space $R^m = \{x = (x_1, x_2, \dots x_m)\}$ with norm $\|x\| = \max_{1\le i\le m}|x_i|$ and

$$P = \{x = (x_1, x_2, \dots x_m) : x_n \ge 0 \text{ for } n = 1, 2, \dots m\}.$$

Then P is a normal cone in E, $P^* = P$ and the normal constant is $N = 1$. System (4.1) can be regarded as a boundary-value problem of form (1.1) subject to(1.2), where $x = (x_1, x_2, \dots x_m)$,

$$f(t, x) = (f_1(t, x_1, \dots x_m), \dots, f_n(t, x_1, \dots x_m), \dots, f_m(t, x_1, \dots x_m))$$

with

$$f_n(t, x_1, \dots x_m) := \frac{1}{\sqrt{t(1-t)}}\big(x_n^{3/2} + \frac{1}{\max_{1\le i\le m}|x_i|}\big).$$

Evidently, $f \in C[(0,1) \times P \setminus \{\theta\}, P]$ and is singular at $t = 0$, $t = 1$, and $x = \theta = (0, 0, \dots, 0)$. Notice that

$$f_{r,R}(t) \le \frac{1}{\sqrt{t(1-t)}}\big(R^{\frac{3}{2}} + \frac{1}{rt(1-t)}\big) \qquad \text{for } t \in (0,1) \text{ and } R \ge r > 0$$

and

$$\int_0^1 t(1-t)f_{1,1}(t)dt \le \int_0^1 [\sqrt{t(1-t)} + \frac{1}{\sqrt{t(1-t)}}]dt < (\frac{1}{8} + \pi) < 8.$$

Therefore, by Theorem 3.4 or Theorem 3.6, our conclusion follows. ☐

Example 4.2. Consider the boundary-value problem consisting of an infinite system of fourth order scalar nonlinear differential equations.

$$\begin{gathered} x_n^{(4)}(t) = f_n(t, x(t)), \quad t \in (0,1); \\ x_n(0) = x_n(1) = x_n''(0) = x_n''(1) = 0, \quad (n = 1, 2, \dots). \end{gathered} \tag{4.2}$$

where

$$f_1(t,x) = \frac{1}{\sqrt{t(1-t)}}\big(x_1 + \frac{x_2}{2} + (\sum_{i\geq 1}|x_i|)^2 + b_1\big),$$
$$f_2(t,x) = \frac{1}{\sqrt{t(1-t)}}\big(\frac{x_2}{2} + \frac{x_3}{3} + \frac{1}{\sum_{i\geq 1}|x_i|} + b_2\big),$$
$$f_n(t,x) = \frac{1}{\sqrt{t(1-t)}}\big(\frac{x_n}{n} + \frac{x_{n+1}}{n+1} + b_n\big), \quad n = 3,4,\dots;$$
$$x = (x_1, x_2, \dots), \quad b_i \geq 0 \quad (i = 1,2,\dots), \quad \sum_{i\geq 1} b_i \leq 1.$$

Claim: System (4.2) has at least two positive solutions $x^*(t) = (x_1^*(t), x_2^*(t), \dots)$ and $x^{**}(t) = (x_1^{**}(t), x_2^{**}(t), \dots)$ such that

$$0 < \sum_{1\leq i,\ t\in[0,1]} |x_i^*(t)| < 1 < \sum_{1\leq i,\ t\in[0,1]} |x_i^{**}(t)|.$$

Proof. Let $E = l^1$ with norm $\|x\| = \sum_{i\geq 1}|x_i|$ and

$$P = \{x = (x_1, x_2, \dots) : x_n \geq 0 \quad \text{for } n = 1,2,\dots\}.$$

Then P is a normal cone in E and the normal constant is $N = 1$. System (4.2) can be regarded as a boundary-value problem form (1.1) with (1.2), where $x = (x_1, x_2, \dots)$,

$$f(t,x) = (f_1(t,x_1,\dots),\dots,f_n(t,x_1,\dots),\dots,).$$

Evidently, $f \in C[(0,1)\times P\setminus\{\theta\}, P]$ and is singular at $t = 0$, $t = 1$, and $x = \theta = (0,0,\dots,)$. Note that for $t \in (0,1)$ and $R \geq r > 0$,

$$f_{r,R}(t) \leq \frac{1}{\sqrt{t(1-t)}}\big(2R + R^2 + \frac{1}{rt(1-t)} + \|b\|\big)$$

So, (H0) is satisfied. In addition, (H1) is obvious. As in [7, Example 2.1.2], one can see (H2) is satisfied with $l = 0$. Choosing $\phi^* = \psi^* = (1,1,0,\dots,0,\dots)$, we know that (H3) and (H'4) holds for (4.1). From

$$\int_0^1 s(1-s)f_{1,1}(s)ds \leq \int_0^1 ((3+\|b\|)\sqrt{s(1-s)} + \frac{1}{\sqrt{s(1-s)}})ds \leq (\frac{4}{8} + \pi) < 8,$$

it follows that (H5) is satisfied. By Theorem 3.6, our conclusion follows. □

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