Electronic Journal of Differential Equations, Vol. 2004(2004), No. 137, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

EXISTENCE OF MULTIPLE SOLUTIONS FOR A CLASS OF SECOND-ORDER ORDINARY DIFFERENTIAL EQUATIONS

XIAO-BAO SHU, YUAN-TONG XU

Abstract. By means of variational structure and Z_2 group index theory, we obtain multiple solutions for the second-order differential equation

$$\frac{d}{dt}(p(t)\frac{du}{dt}) + q(t)u + f(t,u) = 0, \quad 0 < t < 1,$$

subject to one of the following two sets of boundary conditions:

$$u'(0) = u(1) + u'(1) = 0 \quad \text{or} \quad u(0) = u(1) = 0\,.$$

1. Introduction

Erbe and Mathsen [6] study the boundary-value problem

$$-(ru')' + qu = \lambda f(t,u), \quad 0 < t < 1,$$
$$\alpha u(0) - \beta u'(0) = 0 = \gamma u(1) + \delta u'(1),$$

where $\lambda > 0$ is a parameter, $\alpha, \beta, \gamma, \delta \geq 0$ and $\alpha\delta + \alpha\gamma + \beta\gamma > 0, f \in C((0,1) \times R, R), r \in C([0,1],(0,\infty))$ and $q \in C([0,1],[0,\infty))$.

In this paper we are interested in the study of second-order ordinary differential equation

$$\frac{d}{dt}(p(t)\frac{du}{dt}) + q(t)u + f(t,u) = 0, \quad 0 < t < 1, \tag{1.1}$$

subject to one of the following two sets of boundary conditions

$$u'(0) = 0 = \gamma u(1) + u'(1) \tag{1.2}$$

or

$$u(0) = u(1) = 0 \tag{1.3}$$

By means of variational structure and Z_2 group index theory, we obtain multiple solutions of boundary-value problems for (1.1) and lower bound estimate of number for the solutions.

2000 *Mathematics Subject Classification.* 34B15, 34B05, 65K10, 34B24.
Key words and phrases. Variational structure; Z_2 group index theory; critical points; boundary value problems.

Submitted October 5, 2004. Published November 25, 2004.
Supported by grant 10471155 from NNSF of China, grant 031608 from the Foundation of the Guangdong province Natural Science Committee, and grant 20020558092 from Foundation for PhD Specialities of Educational Department of China.

A critical point of f is a point x_0 where $f'(x_0)=\theta$ and a critical value is a number c such that $f(x_0)=c$ for some critical point x_0. Next, we recall the definition of the Palais-Smale condition.

Definition Let E be real Banach space and $f\in C^1(E,R)$. We say that f satisfies the Palais-Smale condition if every sequence $\{x_n\}\subset E$ such that $\{f(x_n)\}$ is bounded and $f'(x_n)\to\theta$ as $n\to\infty$ has a converging subsequence.

Let $K=\{x\in E : f'(x)=\theta\}$, $K_c=\{x\in E : f'(x)=\theta,\ f(x)=c\}$ and $f_c=\{x\in E : f(x)\le c\}$. The class of subsets of $X\setminus\{\theta\}\subset E$ closed and symmetric with respect to the origin will be denoted by $\sum$. Next, we recall the concept of genus.

Definition Let E be a real Banach space, and $\Sigma=\{A : A\subset E\setminus\{\theta\}$ is a closed, symmetric set$\}$. Define $\gamma:\Sigma\to Z^+\cup\{+\infty\}$ as follows

$$\gamma(A)=\begin{cases}\min\{n\in Z: \text{ there exists an odd continuous map } \varphi:A\to\mathbb{R}^n\setminus\{\theta\}\};\\ 0 \quad \text{If } A=\emptyset;\\ +\infty \quad \text{If there is no odd continuous map } \varphi:A\to\mathbb{R}^n\setminus\{\theta\} \text{ for } n\in Z.\end{cases}$$

Then we say γ is the genus of $\sum$. Denote $i_1(f)=\lim_{a\to -0}\gamma(f_a)$ and $i_2(f)=\lim_{a\to-\infty}\gamma(f_a)$.

We know that if $A\in\sum$ and if there exists an odd homeomorphism of $n-$sphere onto A then $\gamma(A)=n+1$; If X is a Hilbert space, and E is an $n-$dimensional subspace of X, and $A\in\sum$ is such that $A\cap E^\perp=\emptyset$ then $\gamma(A)\le n$.

The following Lemmas play an important role in proving our main results.

Lemma 1.1 ([5]). *Let $f\in C^1(E,\mathbb{R})$ be an even functional which satisfies the Palais-Smale condition and $f(\theta)=0$. Then*

(P1) *If there exists an m-dimensional subspace X of E and $\rho>0$ such that*
$$\sup_{x\in X\cap S_\rho} f(x)<0,$$
then we have $i_1(f)\ge m$

(P2) *If there exists a j-dimensional subspace $\widetilde{X}$ of E such that*
$$\inf_{x\in\widetilde{X}^\perp} f(x)>-\infty,$$
we have $i_2(f)\le j$

If $m\ge j$, (P1) and (P2) hold, then f has at least $2(m-j)$ distinct critical points.

Lemma 1.2 ([9]). *Let $f\in C^1(X,\mathbb{R})$ be an even functional which satisfies the Palais-Smale condition and $f(\theta)=0$. If*

(F1) *There exists $\rho>0$, $\alpha>0$ and a finite dimensional subspace E of X, such that $f|_{E^\perp\cap S_\rho}\ge\alpha$*

(F2) *For all finite dimensional subspace $\widetilde{E}$ of X, there is a $r=r(\widetilde{E})>0$, such that $f(x)\le 0$ for $x\in\widetilde{E}\backslash B_r$*

Then f possesses an unbounded sequence of critical values.

2. Main Results

In this paper, we use Lemma 1.1 and 1.2 to study the boundary-value problems (1.1)-(1.2) and (1.1)-(1.3)

Theorem 2.1. *Let f, $p(t)$ and $q(t)$ satisfy the following conditions:*

(i) $p(t) \in C[0,1]$ *and* $0 < m \le p(t) \le M$ *for* $t \in [0,1]$
(ii) $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$
(iii) $\lim_{u\to 0} \frac{f(t,u)}{u} = \xi(t) > 0$ *uniformly for* $t \in [0,1]$, $\lambda = \min_{0\le t\le 1} \xi(t)$
(iv) *There exists* $\alpha > 0$ *such that* $f(t,\alpha) \le 0$
(v) $f(t,u)$ *is odd in* u
(vi) $-\frac{\lambda}{2} < q(t) + p(t) \le 0$, *for all* $0 \le t \le 1$.

Then (1.1)-(1.2), *has at least* $2n$ *nontrivial solutions in* $C^2[0,1]$ *whenever*

$$2n^2(M + p(1)|\gamma|)(1+\pi^2) < \lambda \le 2(n+1)^2(M + p(1)|\gamma|)(1+\pi^2)$$

and $\gamma > -\frac{m}{2p(1)}$

Proof. Set $h : [0,1] \times \mathbb{R} \to \mathbb{R}$,

$$h(t,u) = \begin{cases} f(t,\alpha) & \text{if } u > \alpha, \\ f(t,u) & \text{if } |u| \le \alpha, \\ f(t,-\alpha) & \text{if } u < -\alpha \end{cases}$$

Let us consider the functional defined on $H_0^1(0,1)$ by

$$I(u) = \int_0^1 [\frac{1}{2}p(t)|u'(t)|^2 - \frac{1}{2}q(t)|u(t)|^2 - G(t,u(t))]dt + \frac{p(1)}{2}\gamma u^2(1), \tag{2.1}$$

Where $G(t,u) = \int_0^u h(t,v)dv$. The norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$ can be defined respectively by

$$\|u\| = (\int_0^1 (|u'(t)|^2 + |u(t)|^2)dt)^{1/2}; \quad (u,v) = \int_0^1 (u'(t)v'(t) + u(t)v(t))dt\,.$$

Thus $H_0^1(0,1) = W_0^{1,2}(0,1)$ will be a Hilbert space.

Let $E = H_0^1(0,1)$, since $h(t,u)$ is an odd continuous map in u, we know that $I \in C^1(E,R)$ is even in u and $I(\theta) = 0$.

First, we will show that the critical points of the $I(u)$ are the solutions of (1.1)-(1.2) in $C^2[0,1]$. Since

$$\begin{aligned} I(u+sv) = I(u) + s\{&\int_0^1 [p(t)u'(t)v'(t) - q(t)u(t)v(t) - h(t,u+\theta(t)sv)v(t)]dt \\ &+ p(1)\gamma u(1)v(1)\} + \frac{s^2}{2}\{\int_0^1 (p(t)|v'(t)|^2 \\ &- q(t)|v(t)|^2)dt + p(1)\gamma v^2(1)\} \quad \forall u,v \in E, 0 < \theta(t) < 1 \end{aligned} \tag{2.2}$$

We have, for all $u, v \in E$,

$$(I'(u),v) = \int_0^1 [p(t)u'(t)v'(t) - q(t)u(t)v(t) - h(t,u(t))v]dt + p(1)\gamma u(1)v(1)\,. \tag{2.3}$$

By $I'(u) = 0$, one gets

$$\int_0^1 [p(t)u'(t)v'(t) - q(t)u(t)v(t) - h(t,u(t))v]dt + p(1)\gamma u(1)v(1) = 0 \tag{2.4}$$

for all $v \in E$. On the other hand,

$$\begin{aligned} &\int_0^1 p(t)u'(t)v'(t)dt + \int_0^1 \frac{d}{dt}(p(t)\frac{du}{dt})vdt \\ &= \int_0^1 p(t)u'(t)v'(t)dt + \int_0^1 p(t)u''(t)v(t)dt + \int_0^1 p'(t)u'(t)v(t)dt \\ &= p(1)v(1)u'(1) - p(0)u'(0)v(0) = 0 \end{aligned} \tag{2.5}$$

So, it is easy to see that

$$\begin{aligned} &\int_0^1 v[\frac{d}{dt}(p(t)\frac{du}{dt}) + q(t)u(t) + h(t,u(t))]dt \\ &= p(1)v(1)(u'(1) + \gamma u(1)) - p(0)u'(0)v(0) = 0 \end{aligned}$$

Hence we obtain

$$\frac{d}{dt}(p(t)\frac{du}{dt}) + q(t)u(t) + h(t,u(t)) = 0$$

Thus the critical points of $I(u)$ are the solutions of (1.1)-(1.2) in $C^2[0,1]$.

For convenience, we transform (2.1) into

$$\begin{aligned} I(u) = \int_0^1 [&\frac{1}{2}p(t)(|u'(t)|^2 + |u(t)|^2) \\ &- \frac{1}{2}(q(t)+p(t))|u(t)|^2 - G(t,u(t))]dt + \frac{p(1)}{2}\gamma u^2(1)\,. \end{aligned}$$

By condition (iv) of Theorem 2.1, we have $uh(u) \le 0$ when $|u(t)| \ge \alpha$. So

$$\int_0^1 G(t,u) = \int_0^1 \int_0^u h(t,v)dvdt \le \int_0^1 \int_{-\alpha}^{\alpha} |h(t,v)|dv\,dt\,.$$

Denote by c the value of $\int_0^1 \int_{-\alpha}^{\alpha} |h(t,v)|dvdt$. On the other hand, $q(t)+p(t) \le 0$, then $-\int_0^1 (q(t)+p(t))|u(t)|^2 dt \ge 0$. So, we have

$$I(u) \ge \frac{m}{2}\|u\|^2 - c + \frac{p(1)}{2}\gamma u^2(1) \quad \forall u \in E.$$

Next, we show that $I(u)$ has lower bound. We divide our proof into two parts
(I) When $\gamma \ge 0$, it is easy to see

$$I(u) \ge \frac{m}{2}\|u\|^2 - c \quad \forall u \in E \tag{2.6}$$

(II) When $-\frac{m}{2p(1)} < \gamma < 0$, we divide again our proof into two parts in order to show $I(u)$ has lower bound: (a) If there exist $t_0 \in [0,1]$ such that $u(t_0) = 0$, then

$$|u(1)| = |\int_{t_0}^1 u'(s)ds| \le \int_0^1 |u'(s)|ds \le \sqrt{2}\|u\|\,.$$

So, we get

$$I(u) \ge \frac{1}{2}(m + 2\gamma p(1))\|u\|^2 - c \quad \forall u \in E, \tag{2.7}$$

(b) If does not exist $t_1 \in [0,1]$ such that $u(t_1) = 0$, then $u(t) > 0$ or $u(t) < 0$, for all $t \in [0,1]$. We might as well let $u(t) > 0$ for all $t \in [0,1]$.

When $\max_{0\le t\le 1} u(t) \le 1$,we have $u(1) \le 1$ and

$$I(u) \ge \frac{m}{2}\|u\|^2 - c + \frac{1}{2}\gamma p(1),$$

i.e., $I(u)$ has lower bound.

When $\max_{0\le t\le 1}|u(t)|>1$, since $u\in C^2[0,1]$, there exist $t_2\in[0,1]$ such that $u(t_2)=\min_{0\le t\le 1}u(t)$. So

$$u(1)-u(t_2)=|\int_{t_2}^1 u'(s)ds|\le\int_0^1|u'(s)|ds$$

i.e.

$$\begin{aligned}u(1)&\le u(t_2)+\int_0^1|u'(s)|ds\le(\int_0^1 u^2(t)dt)^{\frac12}+(\int_0^1|u'(t)|^2)^{\frac12}\\&\le\sqrt{2}(\int_0^1 u^2(t)dt+\int_0^1|u'(t)|^2)^{1/2}=\sqrt{2}\|u\|\end{aligned}$$

As in the proof of (I), we have

$$I(u)\ge\frac12(m+2\gamma p(1))\|u\|^2-c.\quad\forall u\in E. \tag{2.8}$$

By (a) and (b), it is easy to see $I(u)$ has lower bound when $-\frac{m}{2p(1)}<\gamma<0$. From (I) and (II), we get that $I(u)$ has lower bound for all $u\in H_0^1(0,1)$, i.e., $i_2(I)=0$.

Next, we verify that $I(u)$ satisfies the Palais-Smale condition. Suppose that $\{u_n\}\subset E$ with and

$$c_1\le I(u_n)\le c_2, \tag{2.9}$$

$$I'(u_n)\to 0\quad\text{as}\quad n\to\infty\,. \tag{2.10}$$

Then

$$\begin{gathered}\sup\{\int_0^1[p(t)u_n'v'-q(t)u_nv-h(t,u_n(t))v]dt+\gamma p(1)u_n(1)v(1)\}\to 0,\\ \text{as } n\to\infty,\quad\forall u,v\in E,\|v\|=1\end{gathered} \tag{2.11}$$

with $\|z_n\|=\|I'(x_n)\|$. Let us denote $\varepsilon_n=\|z_n\|$, then $\varepsilon_n\to 0$ as $n\to\infty$. Replace v by u_n in above equality. By (2.3) and (2.11), we have

$$\int_0^1[p(t)|u_n'(t)|^2-q(t)|u_n(t)|^2]dt=\int_0^1 h(t,u_n)u_n(t)dt+\langle z_n,u_n\rangle.$$

The above equality is equivalent to

$$\begin{aligned}&\int_0^1 p(t)[|u_n'(t)|^2+|u_n(t)|^2]dt\\&=\int_0^1[(q(t)+p(t))|u_n(t)|^2+h(t,u_n)u_n(t)]dt+\langle z_n,u_n\rangle\end{aligned}$$

So, there exist $\xi\in[0,1]$ such that

$$p(\xi)\|u_n\|^2=\int_0^1[(q(t)+p(t))|u_n(t)|^2+h(t,u_n)u_n(t)]dt+\langle z_n,u_n\rangle\,. \tag{2.12}$$

Next, we show that $\{u_n\}$ satisfying condition (2.9) and (2.10) is bounded. We divide again our proof into two parts.

(c) When $\gamma\ge 0$, by (2.6), one gets

$$\|u_n\|^2\le\frac{2}{m}(I(u_n)+c)\le\frac{2}{m}(c_2+c),$$

i.e., $\|u_n\|\le\sqrt{\frac{2}{m}(c_2+c)}$.

(d) When $-\frac{m}{2p(1)} < \gamma < 0$, by the above proof and (2.7) and (2.8), we have

$$\|u_n\|^2 \leq \frac{2}{m+2\gamma p(1)}(I(u_n)+c) \leq \frac{2}{1+2\gamma p(1)}(c_2+c)$$

or

$$\|u\|^2 \leq \frac{2}{m}[c_2+c-\frac{1}{2}\gamma p(1)]$$

i.e.,

$$\|u_n\| \leq \sqrt{\frac{2}{m+2\gamma p(1)}(c_2+c)} \quad \text{or} \quad \|u\| \leq \sqrt{\frac{2}{m}(c_2+c-\frac{1}{2}\gamma p(1))}.$$

By (c) and (d), it is easy to see $\{u_n\}$ is bounded in the space $H_0^1(0,1)$. Reflexivity of $H_0^1(0,1)$ implies that there exists a subsequence of $\{u_n\}$ which is weak convergent in $H_0^1(0,1)$. We still denote it by $\{u_n\}$ and suppose that $u_n \rightharpoonup u_0$ in $H_0^1(0,1)$ as $n \to \infty$. On the one hand, by boundedness of $\{u_n\}$ and (2.12), we have

$$p(\xi)\|u_n\|^2 - \int_0^1 [(q(t)+p(t))|u_n(t)|^2 + h(t,u_n)u_n(t)]dt \to 0 \quad \text{as } n \to \infty$$

Note that the weak convergent of $\{u_n\}$ in $H_0^1(0,1)$ implies the uniform convergence of $\{u_n\}$ in $C([0,1],R)$ [8, Proposition 1.2]. Hence

$$p(\xi)\|u_n\|^2 \to \int_0^1 [(q(t)+p(t))|u_0(t)|^2 + h(t,u_0)u_0(t)]dt \quad \text{as } n \to \infty$$

This means that $\{u_n\}$ converges in $H_0^1(0,1)$. So the P.S. condition holds.

Thirdly, we show that Theorem 2.1 holds by Lemma 1.1. Denote $\beta_k(t) = \frac{\sqrt{2}}{k\pi}\cos k\pi t$, $k = 1,2,3,\dots,n,\dots$, then

$$\int_0^1 |\beta_k(t)|^2 dt = \frac{1}{k^2\pi^2}, \quad \int_0^1 |\beta_k'(t)|^2 dt = 1$$

Definite n-dimensional linear space

$$E_n = \operatorname{span}\{\beta_1(t), \beta_2(t), \dots \beta_n(t)\}.$$

It is obvious that E_n is a symmetric set. Suppose $\rho > 0$, then

$$E_n \cap S_\rho = \{\sum_{k=0}^n b_k\beta_k : \sum_{k=0}^n b_k^2(1+\frac{1}{k^2\pi^2}) = \rho^2\}$$

Let $g(t,u) = \frac{1}{\xi(t)}h(t,u) - u$, by condition (iii) of Theorem 2.1, $\lim_{u\to 0}\frac{g(u)}{u} = 0$, uniformly for $t \in [0,1]$. We choose ε such that

$$0 < \varepsilon < \frac{1}{n^2} - \frac{2(M+p(1)|\gamma|)(1+\pi^2)}{\lambda}.$$

By condition (iii) of Theorem 2.1, there exist $\delta > 0$ such that $|g(t,u)| \leq \varepsilon|u|$ whenever $|u| \leq \delta$. We can choose ρ such that $0 < \rho < \min\{\alpha, \delta\}$, and have

$$\begin{aligned}\max_{0\leq t\leq 1} u(t) &\leq \sum_{k=0}^n \frac{\sqrt{2}}{k\pi}|b_k| \leq \|u\| = \|\sum_{k=0}^n b_k\beta_k\| \\ &= (\sum_{k=0}^n b_k^2(1+\frac{1}{k^2\pi^2}))^{1/2} = \rho < \min\{\alpha,\delta\}\end{aligned}$$

when $u \in E_n \cap S_\rho$. So

$$\begin{aligned}
G(t,u) &= \xi(t)\int_0^u [v+g(t,v))]dv \\
&= \frac{1}{2}\xi(t)|u(t)|^2 + \xi(t)\int_0^u g(t,v)dv \\
&\geq \frac{1}{2}\xi(t)|u(t)|^2 - \xi(t)\int_0^u \varepsilon v dv \\
&= \frac{1}{2}\xi(t)(1-\varepsilon)|u(t)|^2 \\
&\geq \lambda(1-\varepsilon)|u(t)|^2
\end{aligned}$$

From $q(t)+p(t) \geq -\frac{\lambda}{2}$, we get that

$$-\int_0^1 (q(t)+p(t))|u(t)|^2 dt \leq \frac{\lambda}{2}\sum_{k=0}^n \frac{b_k^2}{k^2\pi^2}$$

So

$$\begin{aligned}
I(u) &= \int_0^1 [\frac{1}{2}p(t)(|u'(t)|^2+|u(t)|^2) - \frac{1}{2}(q(t)+p(t))|u(t)|^2]dt \\
&\quad -\int_0^1 G(t,u)dt + \frac{p(1)}{2}\gamma u^2(1) \\
&\leq M\int_0^1 [\frac{1}{2}(|u'(t)|^2+|u(t)|^2) - \frac{1}{2}\lambda(1-\varepsilon)|u(t)|^2]dt \\
&\quad + \frac{\lambda}{4}\sum_{k=0}^n \frac{b_k^2}{k^2\pi^2} + \frac{p(1)}{2}|\gamma|\|u\|_C^2 \\
&\leq \frac{M}{2}\sum_{k=0}^n b_k^2(1+\frac{1}{k^2\pi^2}) - \frac{1}{4}\lambda(1-2\varepsilon)(\sum_{k=0}^n \frac{b_k^2}{k^2\pi^2}) + \frac{p(1)}{2}|\gamma|(\sum_{k=0}^n \frac{\sqrt{2}|b_k|}{k\pi})^2 \\
&\leq \frac{M+p(1)|\gamma|}{2}\sum_{k=0}^n b_k^2(1+\frac{1}{k^2\pi^2}) - \frac{1}{4}\lambda(1-2\varepsilon)\sum_{k=0}^n \frac{b_k^2}{k^2\pi^2} \\
&< \frac{M+p(1)|\gamma|}{2}\sum_{k=0}^n b_k^2(1+\frac{1}{\pi^2}) - \frac{1}{4}\lambda(1-\varepsilon)\sum_{k=0}^n \frac{b_k^2}{n^2\pi^2} \\
&\leq \frac{\lambda}{2}(\frac{M+p(1)|\gamma|}{\lambda}\frac{\pi^2+1}{\pi^2} - \frac{1}{2n^2\pi^2} + \varepsilon)\sum_{k=0}^n b_k^2 \\
&\leq \frac{\lambda}{2\pi^2}(\frac{(M+p(1)|\gamma|)(1+\pi^2)}{\lambda} - \frac{1}{2n^2} + \varepsilon)\sum_{k=0}^n b_k^2 \\
&\leq \frac{\lambda}{4\pi^2}(\frac{2(M+p(1)|\gamma|)(1+\pi^2)}{\lambda} - \frac{1}{n^2} + \varepsilon)\sum_{k=0}^n b_k^2 < 0
\end{aligned}$$

By Lemma 1.1 and the above result, we have $i_1(I) \geq n$ and I has $2n$ distinct critical points, i.e., boundary-value problem (1.1)-(1.2) has at least $2n$ nontrivial solutions in $C^2[0,1]$. □

Next we consider the boundary-value problem (1.1)-(1.3). Similar to Theorem 2.1, we have the following result.

Theorem 2.2. *Let f, $p(t)$ and $q(t)$ satisfy the following conditions:*

(i) $p(t) \in C[0,1]$ *and* $0 < m \le p(t) \le M$ *for* $t \in [0,1]$
(ii) $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$
(iii) $\lim_{u\to 0} \frac{f(t,u)}{u} = \xi(t) > 0$ *uniformly for* $t \in [0,1]$, $\lambda = \min_{0\le t\le 1} \xi(t)$
(iv) *There exists* $\alpha > 0$, *such that* $f(t,\alpha) \le 0$
(v) $f(t,u)$ *is odd in* u
(vi) $-\frac{\lambda}{2} < q(t) + p(t) \le 0$ *for all* $0 \le t \le 1$.

Then (1.1)-(1.3) *has at least* $2n$ *nontrivial solutions in* $C^2[0,1]$ *whenever*

$$2n^2 M(1+\pi^2) < \lambda \le 2(n+1)^2 M(1+\pi^2).$$

Next, we consider the boundary-value problem (1.1)-(1.2).

Theorem 2.3. *Let f, $p(t)$ and $q(t)$ satisfy the following conditions:*

(i) $p(t) \in C[0,1]$ *and* $0 < m \le p(t) \le M$ *for* $t \in [0,1]$
(ii) $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$
(iii) *There exists* T *such that* $\limsup_{u\to 0} \frac{f(t,u)}{u} \le T$
(iv) *There exists* $\theta > \frac{2M}{m} \ge 2$ *and* $\alpha > 0$ *such that*

$$0 < G(t,u) = \int_0^u f(t,v)dv \le \frac{1}{\theta} u f(t,u), \quad \forall |u| \ge \alpha$$

(v) $f(t,u)$ *is odd in* u
(vi) $q(t) \in C[0,1]$, $q(t) + p(t) \le 0$ *for all* $0 \le t \le 1$.

If $\gamma > -\frac{m\theta - 2M}{2(\theta-2)p(1)}$, *then* (1.1)-(1.2) *has infinite nontrivial solutions in* $C^2[0,1]$.

Proof. It is easy to see that for $u \in H_0^1(0,1)$, the functional

$$I(u) = \int_0^1 [\frac{1}{2}p(t)|u'(t)|^2 - \frac{1}{2}q(t)|u(t)|^2 G(t,u(t))]dt + \frac{p(1)}{2}\gamma u^2(1) \tag{2.13}$$

is well defined. The solutions of boundary-value problems (1.1)-(1.2) are the critical points of the functional $I(u)$. Note that $I(u)$ is equivalent to

$$I(u) = \int_0^1 [\frac{1}{2}p(t)(|u'(t)|^2 + |u(t)|^2) - \frac{1}{2}(q(t)+p(t))|u(t)|^2 - \lambda G(t,u)]dt + \frac{p(1)}{2}\gamma u^2(1)$$

We show that Theorem 2.3 holds by using Lemma 1.2. Since $f(t,u)$ is an odd continuous map in u, we know that $I \in C^1(E,R)$ is even in u and $I(\theta) = 0$. Moreover, As in the proof of Theorem 2.1, one gets

$$(I'(u),v) = \int_0^1 [p(t)u'(t)v'(t) - q(t)u(t)v(t)) - f(t,u)v(t)]dt + \gamma p(1)u(1)v(1),$$

for all $u, v \in E$. The above equality is equivalent to

$$\begin{aligned}\int_0^1 f(t,u)v(t)dt &= \int_0^1 p(t)(u'(t)v'(t) + u(t)v(t))dt \\ &\quad - \int_0^1 (p(t)+q(t))u(t)v(t)dt - (I'(u),v) + p(1)\gamma u(1)v(1).\end{aligned}$$

Next, we verify that $I(u)$ satisfies the Palais-Smale condition. Suppose that $u_n \subset E$ with

$$c_1 \le I(u_n) \le c_2, \tag{2.14}$$
$$I'(u_n) \to 0 \quad \text{as } n \to \infty \tag{2.15}$$

Now, we show that $\{u_n\}$ of satisfying (2.14) and (2.15) is bounded. Denote $E_1 = \{t \in [0,1] | |u_n(t)| \ge \alpha\}$, $E_2 = [0,1] \setminus E_1$. We divide our proof into two parts .
(A) When $-\frac{m\theta - 2M}{2(\theta-2)p(1)} < \gamma < 0$, by (iv), we have

$$\begin{aligned}
I(u_n) &= \int_0^1 \frac{p(t)}{2}(|u_n(t)|^2 + |u_n'(t)|^2)dt - \int_0^1 \frac{1}{2}(q(t)+p(t))|u_n(t)|^2 dt \\
&\quad - \int_0^1 G(t,u_n(t))dt + \frac{p(1)}{2}\gamma u_n^2(1) \\
&\ge \frac{m}{2}\|u_n\|^2 - \int_{E_1} G(t,u_n(t))dt - \int_{E_2} G(t,u_n(t))dt \\
&\quad + \frac{p(1)}{2}\gamma u_n^2(1) - \int_0^1 \frac{1}{2}(q(t)+p(t))|u_n(t)|^2 dt \\
&\ge \frac{m}{2}\|u_n\|^2 - \int_{E_1} \frac{1}{\theta} u_n(t) f(t,u_n(t))dt - c_3 \\
&\quad + \frac{p(1)}{2}\gamma u_n^2(1) - \int_0^1 \frac{1}{2}(q(t)+p(t))|u_n(t)|^2 dt \\
&\ge \frac{m}{2}\|u_n\|^2 - \int_0^1 \frac{1}{\theta} u_n(t) f(t,u_n(t))dt - c_4 \\
&\quad + \frac{p(1)}{2}\gamma u_n^2(1) - \int_0^1 \frac{1}{2}(q(t)+p(t))|u_n(t)|^2 dt \\
&= \frac{m}{2}\|u_n\|^2 - \frac{1}{\theta}(\int_0^1 p(t)(|u_n'(t)|^2 + |u_n(t)|^2)dt - (I'(u_n),u_n) \\
&\quad + p(1)\gamma u_n^2(1)) - c_4 + \frac{p(1)}{2}\gamma u_n^2(1) - (\frac{1}{2} - \frac{1}{\theta})\int_0^1 (q(t)+p(t))|u_n(t)|^2 dt \\
&\ge (\frac{m}{2} - \frac{M}{\theta})\|u_n\|^2 + \frac{1}{\theta}(I'(u_n),u_n) - c_4 + (\frac{1}{2} - \frac{1}{\theta})p(1)\gamma u_n^2(1) \\
&\ge (\frac{m}{2} - \frac{M}{\theta})\|u_n\|^2 + \frac{1}{\theta}\|I'(u_n)\|\|u_n\| - c_4 + (\frac{1}{2} - \frac{1}{\theta})p(1)\gamma u_n^2(1)
\end{aligned}$$

Remarks: (1) for the rest of this article, $c_i > 0$. (2) The above equality makes use of $-(\frac{1}{2} - \frac{1}{\theta})\int_0^1 (q(t)+p(t))|u_n(t)|^2 dt \ge 0$

To show that $\{u_n\}$ is bounded, we divide again our proof into two parts.
(I') When $\max_{0 \le t \le 1} |u(t)| \le 1$, as in the proof of Theorem 2.1, we have $u^2(1) \le 1$ and

$$\begin{gathered}
I(u_n) \ge (\frac{m}{2} - \frac{M}{\theta})\|u_n\|^2 + \frac{1}{\theta}\|I'(u_n)\|\|u_n\| - c_4 + (\frac{1}{2} - \frac{1}{\theta})\gamma \\
(\frac{m}{2} - \frac{M}{\theta})\|u_n\|^2 \le I(u_n) - \frac{1}{\theta}\|I'(u_n)\|\|u_n\| + c_4 - (\frac{1}{2} - \frac{1}{\theta})\gamma
\end{gathered}$$

Using $\theta > \frac{2M}{m}$, (2.14) and (2.15), it is not difficulty to see that $\{\|u_n\|\}$ is bounded.

(II') When $\max_{0\leq t\leq 1}|u(t)|>1$, as in the proof of Theorem 2.1, we have $u^2(1)\leq 2\|u\|^2$ and

$$[(\frac{m}{2}-\frac{M}{\theta})+(\frac{1}{2}-\frac{1}{\theta})2p(1)\gamma]\|u_n\|^2\leq I(u_n)-\frac{1}{\theta}\|I'(u_n)\|\|u_n\|+c_4\leq c_5\|u_n\|+c_5\,.$$

Since $\theta>\frac{2M}{m}\geq 2$ and $\gamma>-\frac{m\theta-2M}{2(\theta-2)p(1)}$, it follows that $\{\|u_n\|\}$ is bounded. From (I') and (II'), we get that $\{\|u_n\|\}$ is bounded when $-\frac{m\theta-2M}{2(\theta-2)p(1)}<\gamma<0$.

(B) when $\gamma>0$, as in the proof of (A), it is not difficult to see that

$$\begin{aligned}
I(u_n)&=\int_0^1\frac{p(t)}{2}(|u_n(t)|^2+|u_n'(t)|^2dt-\int_0^1 G(t,u_n(t))dt\\
&\quad+\frac{p(1)}{2}\gamma u_n^2(1)-\int_0^1\frac{1}{2}(q(t)+p(t))|u_n(t)|^2dt\\
&\geq\frac{m}{2}\|u_n\|^2-\int_0^1\frac{1}{\theta}u_n(t)f(t,u_n(t))dt\\
&\quad+\frac{p(1)}{2}\gamma u_n^2(1)-\int_0^1\frac{1}{2}(q(t)+p(t))|u_n(t)|^2dt\\
&\geq(\frac{m}{2}-\frac{M}{\theta})\|u_n\|^2+\frac{1}{\theta}\|I'(u_n)\|\|u_n\|-c_4+(\frac{1}{2}-\frac{1}{\theta})p(1)\gamma u_n^2(1)\\
&\geq(\frac{m}{2}-\frac{M}{\theta})\|u_n\|^2+\frac{1}{\theta}\|I'(u_n)\|\|u_n\|-c_4
\end{aligned}$$

(We remark that the above equality use $(\frac{1}{2}-\frac{1}{\theta})p(1)\gamma u^2(1)\geq 0$) and

$$(\frac{m}{2}-\frac{M}{\theta})\|u_n\|^2\leq I(u_n)-\frac{1}{\theta}\|I'(u_n)\|\|u_n\|+c_4\leq c_5\|u_n\|+c_5.$$

So, we get that $\{\|u_n\|\}$ is bounded when $\gamma>0$.

From (A) and (B), we obtain that $\{u_n\}$ satisfying (2.14) and (2.15) is bounded. So the P.S. condition holds.

Thirdly, we show that Theorem 2.3 holds by using Lemma 1.2. First, we verify condition (F1) of Lemma 1.2. Let $\beta_j(t)=\cos jt$, $j=1,2,\dots$. Consider the n-dimensional subspace

$$E_n=\operatorname{span}\{\beta_1(t),\beta_2(t),\dots,\beta_n(t)\}$$

and let $X=V^{\perp}$. By (ii), we have $\delta>0$ such that $|f(t,u(t))|\leq T|u|$, whenever $|u|\leq\delta$.

Let ρ with $\rho=\delta$. For any $u\in S_\rho\cap X$, we have $\|u\|_C\leq\|u\|=\rho=\delta$. From

$$\int_0^1|u(t)|^2dt\leq\frac{1}{n^2}\int_0^1|u'(t)|^2dt$$

it is easy to see

$$\int_0^1|u(t)|^2dt\leq\frac{\rho^2}{n^2+1}$$

So, when $\gamma>0$, we have

$$\begin{aligned}
I(u_n)&=\int_0^1\frac{p(t)}{2}(|u_n(t)|^2+|u_n'(t)|^2)dt-\int_0^1 G(t,u_n(t))dt\\
&\quad+\frac{p(1)}{2}\gamma u_n^2(1)-\int_0^1\frac{1}{2}(q(t)+p(t))|u_n(t)|^2dt
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{m}{2}\|u\|^2-\int_0^1 G(t,u(t))dt\\
&\geq \frac{m}{2}\rho^2-\int_0^1(\int_0^{|u(t)|}Tvdv)dt\\
&\geq \frac{m}{2}\rho^2-\frac{T}{2(n^2+1)}\rho^2=\frac{1}{2}(m-\frac{T}{n^2+1})\rho^2>0\,.
\end{aligned}
$$

Note that in the above equality, we use $n^2>\max\{\frac{T}{m},\frac{T}{m+p(1)\gamma}\}$ and

$$
-\int_0^1\frac{1}{2}(q(t)+p(t))|u_n(t)|^2dt>0.
$$

When $-\frac{m\theta-2M}{2(\theta-2)p(1)}<\gamma<0$, we get

$$
\begin{aligned}
I(u_n)&=\int_0^1\frac{p(t)}{2}(|u_n(t)|^2+|u_n'(t)|^2)dt-\int_0^1 G(t,u_n(t))dt\\
&\quad+\frac{p(1)}{2}\gamma u_n^2(1)-\int_0^1\frac{1}{2}(q(t)+p(t))|u_n(t)|^2dt\\
&\geq \frac{m}{2}\|u\|^2-\int_0^1 G(t,u(t))dt+\frac{p(1)}{2}\gamma u_n^2(1)\\
&\geq \frac{m+p(1)\gamma}{2}\rho^2-\int_0^1(\int_0^{|u(t)|}Tvdv)dt\\
&\geq \frac{m+p(1)\gamma}{2}\rho^2-\frac{T}{2(n^2+1)}\rho^2\\
&=\frac{1}{2}(m+p(1)\gamma-\frac{T}{n^2+1})\rho^2>0\,.
\end{aligned}
$$

Note that the above equality uses $n^2>\max\{\frac{T}{m},\frac{T}{m+p(1)\gamma}\}$.

We sum up the conclusions above to obtain that $I(u)>0$ for all $u\in S_\rho\cap X$, i.e., condition (F1) of Lemma 1.2 holds.

Finally, we verify condition (F2) of Lemma 1.2. By (iv), one gets

$$
G(t,u(t))\geq c_7|u|^\theta-c_8.
$$

For all finite dimensional subspace E_1 of E, there exist c_9 such that

$$
\Big(\int_0^1|u(t)|^\theta dt\Big)^{1/\theta}\geq c_9\|u\|,\quad \forall u\in E_1.
$$

On the other hand, since $p(t)\in C^1[0,1], q(t)\in C[0,1]$ and $p(t)+q(t)\leq 0$, there exist a positive number Q such that $-Q=\min_{0\leq t\leq 1}p(t)+q(t)$, so

$$
-\int_0^1(p(t)+q(t))|u(t)|^2dt\leq Q\int_0^1|u(t)|^2dt<Q\|u\|^2.
$$

When $u\in E_1$ and $-\frac{m\theta-2M}{2(\theta-2)p(1)}<\gamma<0$, from the above result, it is easy to obtain

$$
\begin{aligned}
I(u)&=\int_0^1[\frac{p(t)}{2}(|u'(t)|^2+|u(t)|^2)-G(t,u(t))]dt\\
&\quad+\frac{p(1)}{2}\gamma u^2(1)-\int_0^1\frac{1}{2}(q(t)+p(t))|u(t)|^2dt
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{M+Q}{2}\|u\|^2 - \int_0^1 G(t,u(t))dt \\
&\le \frac{M+Q}{2}\|u\|^2 - c_7\int_0^1 |u(t)|^\theta dt + c_8 \\
&\le \frac{M+Q}{2}\|u\|^2 - c_7 c_9^\theta \|u\|^\theta + c_8 \\
&= (\frac{M+Q}{2} - c_7 c_9^\theta \|u\|^{\theta-2})\|u\|^2 + c_8\,.
\end{aligned}
$$

When $u \in E_1$ and $\gamma \ge 0$, as in the proof of (I') and (II'), we have the following two results.

(1) when $\max_{0\le t\le 1}|u(t)| \le 1$, we have

$$
\begin{aligned}
I(u) &= \int_0^1 [\frac{p(t)}{2}(|u'(t)|^2 + |u(t)|^2) - G(t,u(t))]dt \\
&\quad + \frac{p(1)}{2}\gamma u^2(1) - \int_0^1 \frac{1}{2}(q(t)+p(t))|u(t)|^2 dt \\
&\le \frac{M+Q}{2}\|u\|^2 - c_7\int_0^1 |u(t)|^\theta dt + c_8 + \frac{p(1)}{2}\gamma \\
&\le \frac{M+Q}{2}\|u\|^2 - c_7 c_9^\theta\|u\|^\theta + c_8 + \frac{p(1)}{2}\gamma \\
&= (\frac{M+Q}{2} - c_7 c_9^\theta\|u\|^{\theta-2})\|u\|^2 + c_8 + \frac{p(1)}{2}\gamma
\end{aligned}
$$

(2) when $\max_{0\le t\le 1}|u(t)| > 1$, we have $u^2(1) \le 2\|u\|^2$ and

$$
\begin{aligned}
I(u) &= \int_0^1 [\frac{p(t)}{2}(|u'(t)|^2 + |u(t)|^2) - G(t,u(t))]dt \\
&\quad + \frac{p(1)}{2}\gamma u^2(1) - \int_0^1 \frac{1}{2}(q(t)+p(t))|u(t)|^2 dt \\
&\le \frac{M+Q+2\gamma}{2}\|u\|^2 - c_7\int_0^1 |u(t)|^\theta dt + c_8 \\
&\le \frac{M+Q+2\gamma}{2}\|u\|^2 - c_7 c_9^\theta\|u\|^\theta + c_8 \\
&= (\frac{M+Q+2\gamma}{2} - c_7 c_9^\theta\|u\|^{\theta-2})\|u\|^2 + c_8
\end{aligned}
$$

We sum up the conclusions above to obtain that $I(u) \le 0$, for all $u \in E_1\backslash B_R$ when $R = R(E_1)$ is adequately big, i.e., condition (F2) of Lemma 1.2 holds. So I possesses infinite critical point, i.e. the boundary-value problem (1.1)-(1.2) has infinitely many nontrivial solutions in $C^2[0,1]$. □

Using a technique similar to the one above, we can show that the following theorem.

Theorem 2.4. *Let f, $p(t)$ and $q(t)$ be the function satisfying the following conditions:*

(i) $p(t) \in C[0,1]$ *and* $0 < m \le p(t) \le M$ *for* $t \in [0,1]$
(ii) $f \in C([0,1]\times\mathbb{R},\mathbb{R})$
(iii) *There exists T such that* $\limsup_{u\to 0}\frac{f(t,u)}{u} \le T$

(iv) *There exists* $\theta > \frac{2M}{m} \geq 2$ *and* $\alpha > 0$ *such that*

$$0 < G(t,u) = \int_0^u f(t,v)dv \leq \frac{1}{\theta} u f(t,u), \quad \forall |u| \geq \alpha$$

(v) $f(t,u)$ *is odd in* u

(vi) $q(t) \in C[0,1]$, $q(t) + p(t) \leq 0$ *for* $0 \leq t \leq 1$.

Then boundary-value problem (1.1)-(1.3) *has infinitely solutions in* $C^2[0,1]$.

3. Examples

Example 3.1. For $0 < t < 1$, consider the boundary-value problem

$$\begin{gathered} \frac{d}{dt}((6+\sin t)\frac{du}{dt}) + (-100+\cos t)u + (1000(1+t^2)\sin u - 10u^3) = 0, \\ u'(0) = 0, \quad u(1) + u'(1) = 0 \end{gathered} \tag{3.1}$$

Note that

$$f(t,u) = 1000(1+t^2)\sin u - 10u^3, \quad p(t) = 6 + \sin t, \quad q(t) = -100 + \cos t$$

So $f(t,u)$ satisfy conditions (ii) and (v) of Theorem 2.1. In addition,

$$0 < 5 \leq p(t) = 6 + \sin t \leq 7 \quad \forall t \in [0,1].$$

then (i) and (vi) of Theorem 2.1 hold. When $|u(t)| = 4$, we have

$$f(t,4) = 1000(1+t^2)\sin 4 - 10 \times 4^3 < 0\,,$$

i.e., (iv) of Theorem 2.1 holds. On the other that $\lim_{u\to 0} \frac{f(t,u)}{u} = 1 + t^2$ uniformly for $t \in [0,1]$, $\lambda = \min_{0\leq t\leq 1} 1 + t^2 = 1000$, i.e., (iii) holds, and

$$2 \times 1^2 \times (13 + \sin 1)(1+\pi^2) < \lambda < 2 \times 2^2 \times (13 + \sin 1)(1+\pi^2)$$

By Theorem 2.1 we have (3.1) has at least 2 nontrivial solutions in $C^2[0,1]$.

Example 3.2. For $0 < t < 1$, consider boundary-value problem

$$\begin{gathered} \frac{d}{dt}((6+\sin t)\frac{du}{dt}) + (-100+\cos t)u + (1000(1+t^2)\sin u - 10u^3) = 0, \\ u(0) = u(1) = 0, \end{gathered} \tag{3.2}$$

As in Example 3.1, it is easy to verify all conditions of Theorem 2.2 hold and

$$2 \times 2^2 \times 7(1+\pi^2) < \lambda < 2 \times 3^2 \times 7(1+\pi^2)$$

By Theorem 2.2, we have (3.2) has at least 4 nontrivial solutions in $C^2[0,1]$.

Example 3.3. *Consider the boundary-value problem*

$$\begin{gathered} \frac{d}{dt}((6+\sin t)\frac{du}{dt}) + (-9+t^2)u(t) + t(u^3(t) + u(t)) = 0, \quad 0 < t < 1, \\ u'(0) = 0, \quad u(1) + u'(1) = 0 \end{gathered} \tag{3.3}$$

It is easy to see that

$$f(t,u) = t(u^3(t) + u(t)), \quad p(t) = 6 + \sin t \quad q(t) = -9 + t^2$$

So, $f(t,u)$ *satisfies conditions (ii) and (v) of Theorem 2.3. In addition*

$$\limsup_{u\to 0} \frac{f(t,u)}{u} = \limsup_{u\to 0} \frac{t(u^3+u)}{u} = 1$$

i.e., conditions (iii) of Theorem 2.3 hold. Moreover,

$$\int_0^u f(t,v)dv = \int_0^u t(v^3+v)dv = t(\frac{1}{4}u^4 + \frac{1}{2}u^2) \le \frac{1}{3}ut(u^3+u) \quad \forall |u(t)| > \sqrt{2}$$

$$0 < 5 \le p(t) \le 7, \quad \theta = 3 > \frac{2M}{m} = \frac{14}{5}$$

So conditions (iv) holds. On the other hand, it is easy to see that

$$p(t) + q(t) = 6 + \sin t + (-9 + t^2) = -3 + t^2 + \sin t \le 0 \quad \forall t \in [0,1]$$

So conditions (i) and (vi) of Theorem 2.3 hold. By Theorem 2.3, we obtain that (3.3) *has infinitely many nontrivial solutions in* $C^2[0,1]$.

Example 3.4. Consider boundary-value problem

$$\begin{gathered}\frac{d}{dt}((6+\sin t)\frac{du}{dt}) + (-9+t^2)u(t) + t(u^3(t)+u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = 0,\end{gathered} \tag{3.4}$$

As in Example 3.3, it is easy to verify that all conditions of Theorem 2.4 hold. By Theorem 2.4, we obtain that (3.4) has infinitely many nontrivial solutions in $C^2[0,1]$.

References

[1] I. Addou and Shin-Hwa Wang. *Exact multiplicity results for a p-Laplacian positone problem with concave-convex-concave nonlinearities*, Electron J. diff. Eqns., Vol. 2004(2004), No. 72, 1-24.

[2] R. P. Agarwal and D. O'Regan; *Existence theory for single and multiple solutions to singular positone boundary value problems.* J. Differential Equations 175 (2001), no. 2, 393-414.

[3] G. Anello; *A multiplicity theorem for critical points of functionals on reflexive Banach spaces.* Arch. Math. 82 (2004), no. 2, 172-179.

[4] L. E. Bobisud, J. E. Calvert, and W. D. Royalty; *Some existence results for singular boundary value problems*, Differential Integral Equations 6 (1993), 553-571.

[5] Chang Kung Ching; *Critical point theory and its applications*(Chinese), Shanghai Kexue Jishu Chubanshe, Shanghai, 1986.

[6] L. H. Erbe and R. M. Mathsen; *Positive solutions for singular nonlinear boundary value problems.* Nonlinear Anal. 46 (2001), no. 7, Ser. A: Theory Methods, 979-986.

[7] Guo Da Jun, V. Lakshmikantham; *Multiple solutions of two-point boundary value problems of ordinary differential equations in Banach spaces.* J. Math. Anal. Appl. 129 (1988), no. 1, 211-222.

[8] J. Mawhin and M. Willem; *Critical point theory and Hamiltonian systems.* Applied Mathematical Sciences, 74. Springer-Verlag, New York, 1989.

[9] P. H. Rabinowitz; *Minimax methods in critical point theory with applications to differential equations.* CBMS Regional Conference Series in Mathematics, 65. by the American Mathematical Society, Providence, RI, 1986.

[10] Xu Yuan Tong and Guo Zhi-Ming; *Applications of a Z_p index theory to periodic solutions for a class of functional differential equations.* J. Math. Anal. Appl. 257 (2001), no. 1, 189-205.

[11] Xu Yuan Tong; *Subharmonic solutions for convex nonautonomous Hamiltonian systems.* Nonlinear Anal. 28 (1997), 1359-1371.

Xiao-Bao Shu
Department of Mathematics, Sun Yat-Sen University, Guangzhou, 510275, China
E-mail address: sxb0221@163.com

Yuan-Tong Xu
Department of Mathematics, Sun Yat-Sen University, Guangzhou, 510275, China
E-mail address: xyt@zsu.edu.cn