Electronic Journal of Differential Equations, Vol. 2004(2004), No. 146, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
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RESOLVENT ESTIMATES FOR SCALAR FIELDS WITH ELECTROMAGNETIC PERTURBATION

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Abstract. In this note we prove some estimates for the resolvent of the operator $-\Delta$ perturbed by the differential operator
$$V(x,D) = ia(x)\cdot\nabla + V(x) \quad \text{in } \mathbb{R}^3 .$$
This differential operator is of short range type and a compact perturbation of the Laplacian on $\mathbb{R}^3$. Also we find estimates in the space-time norm for the solution of the wave equation with such perturbation.

1. Introduction

In this work, we study perturbations for the classical wave equation, the classical Schrödinger equation, and the classical Dirac equation. More precisely we consider the following three Cauchy problems:
$$\begin{gathered}\Box u + ia(x)\cdot\nabla u + V(x)u = F,\\ u(0)=0,\quad \partial_t u(0)=0,\end{gathered}\tag{1.1}$$
$$\begin{gathered}i\partial_t u - \Delta u + ia(x)\cdot\nabla u + V(x)u = F,\quad t\in\mathbb{R},\ x\in\mathbb{R}^3,\\ u(0,x)=0,\end{gathered}\tag{1.2}$$
and
$$\begin{gathered}i\gamma_\mu\partial_\mu u + ia(x)\cdot\nabla u + V(x)u = F,\quad t\in\mathbb{R},\ x\in\mathbb{R}^3,\\ u(0,x)=0\,.\end{gathered}\tag{1.3}$$
The solution of problem (1.3) is usually called spinor. Here the Dirac matrices γ_μ are
$$\gamma_0=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \gamma_k=\begin{pmatrix}0&\sigma_k\\-\sigma_k&0\end{pmatrix},\quad k=1,2,3.$$
and the Pauli matrices σ_k are
$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$
For the 1-form $a=\sum_{j=1}^3 a_j dx^j$ for the magnetic potential, by the Poincaré lemma, we know that if a', a are two magnetic potentials with $da = da'$, then $a = a' + d\phi$,

2000 *Mathematics Subject Classification.* 35L05, 35J10, 35P25, 35B25, 35B34, 35B40.
Key words and phrases. Perturbed wave equation; perturbed Schrödinger equation; perturbed Dirac equation; resolvent; short range perturbation; smoothing estimates.

Submitted July 12, 2004. Published December 7, 2004.

where $\phi \in C^\infty$. The operators $(-\Delta + ia' \cdot \nabla + V)$ and $(-\Delta + ia \cdot \nabla + \widetilde{V})$ are related by

$$(-\Delta + ia' \cdot \nabla + V) = e^{-i\phi}(-\Delta + ia \cdot \nabla + \widetilde{V})e^{i\phi}, \tag{1.4}$$

where $V = V_1 - i \cdot \nabla a' + (a')^2$ and $\widetilde{V} = \widetilde{V}_1 - i \cdot \nabla a - \Delta\phi + a^2 + \phi^2$. So we will assume that $a = (a_1, a_2, a_3)$ are measurable functions, such that ∇a_j exists (in distributional sense) and it is measurable, defined as $a_j = a'_j + \partial_j \phi$ for $j = 1, 2, 3$, where the functions a'_j and $\partial_j \phi$ satisfy the inequalities

$$\begin{gathered} |a'_j(x)| + ||x|\nabla a'_j(x)| \le \frac{C_0 \delta}{|x|\, W_{\epsilon_0}(x)}, \quad \text{a.e. } x \in \mathbb{R}^3, \delta > 0, \\ |\partial_j \phi(x)| + ||x|\nabla \partial_j \phi(x)| \le \frac{C_0}{|x|\, W_{\epsilon_0}(x)}, \quad \text{a.e. } x \in \mathbb{R}^3. \end{gathered} \tag{1.5}$$

The potential V (resp. V_1, $\widetilde{V}_1$) is a non-negative measurable function satisfying the inequality

$$|V(x)| \le \frac{C_1}{|x|^2\, W_{\epsilon_0}(x)}, \quad \text{a.e. } x \in \mathbb{R}^3, \tag{1.6}$$

where ϵ_0, $C_0 > 0$, $C_1 > 0$ are constants, and

$$W_\epsilon(|x|) := |x|^\epsilon + |x|^{-\epsilon}, \quad \forall x \in \mathbb{R}^3. \tag{1.7}$$

We see that the potential $a_j(x)$ is bounded from above by $C\delta|x|^{-1-\varepsilon_0}$ if $|x| \ge 1$, while $a_j(x) \le C\delta|x|^{-1+\varepsilon_0}$ if $|x| \le 1$, and the potential $V(x)$ is bounded from above by $C\delta|x|^{-2-\varepsilon_0}$ if $|x| \ge 1$, while $V(x) \le \frac{C}{|x|^{-2+\varepsilon_0}}$ if $|x| \le 1$. The last estimate shows that we admit singularities of a_j and V, such that a_j is in $L^2_{loc}(\mathbb{R}^3)$, while V is not in $L^2_{loc}(\mathbb{R}^3)$. In the papers [1], [2] Agmon showed how scattering theory could be developed for general elliptic operator with perturbations $O(|x|^{-1-\epsilon})$ at infinity and Agmon-Hörmander generalized the techniques required to study the perturbation of simply characteristic operators (see [21]). In [14] one can find a perturbation theory for potentials decaying as $|x|^{-2-\epsilon}$ at infinity.

In [35] the free wave equation and Schrödinger equation (i.e. $a = 0, V = 0$) are studied and for both the following estimate are obtained (in [35] some sharper estimates are proved):

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} \nabla u(x,t)\|_{L^2_t L^2_x} \le C \||x|^{\frac{1}{2}} W_\delta F(x,t)\|_{L^2_t L^2_x}. \tag{1.8}$$

Similar estimate leads for other dispersive equations of mathematical physics. The equation (1.8) is known as smoothing estimate for the Schrödinger equation.

In this work we shall establish the same estimate (1.8) for potential perturbation of the wave and the Schrödinger equations.

Theorem 1.1. *If $u(x,t)$ is the solution of* (1.1) *with $(-\Delta + ia \cdot \nabla + V)$ satisfying* (1.5) *and* (1.6) *then, for any $\delta, \delta' > 0$:*

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} \nabla u(x,t)\|_{L^2_t L^2_x} \le C \||x|^{\frac{1}{2}} W_\delta F(x,t)\|_{L^2_t L^2_x}, \tag{1.9}$$

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} u(x,t)\|_{L^2_t L^2_x} \le C \|F(x,t)\|_{L^2_t L^1_x}, \tag{1.10}$$

$$\||x|^{\frac{1}{2}} W_\delta V(x,D) u(x,t)\|_{L^2_t L^2_x} \le C \||x|^{\frac{1}{2}} W_\delta F(x,t)\|_{L^2_t L^2_x}. \tag{1.11}$$

For (1.2) we have the following statement

Theorem 1.2. *If $u(x,t)$ is the solution of* (1.2) *and* (1.3) *with* $(-\Delta + ia\cdot\nabla + V)$ *satisfying* (1.5) *and* (1.6) *then, for any $\delta, \delta' > 0$:*

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} \nabla u(x,t)\|_{L_t^2 L_x^2} \le C \||x|^{\frac{1}{2}} W_\delta F(x,t)\|_{L_t^2 L_x^2}, \tag{1.12}$$

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} u(x,t)\|_{L_t^2 L_x^2} \le C \|F(x,t)\|_{L_t^2 L_x^1}, \tag{1.13}$$

$$\||x|^{\frac{1}{2}} W_\delta V(x,D) u(x,t)\|_{L_t^2 L_x^2} \le C \||x|^{\frac{1}{2}} W_\delta F(x,t)\|_{L_t^2 L_x^2}. \tag{1.14}$$

For the corresponding homogeneous problem

$$i\partial_t u - \Delta u + ia(x)\cdot\nabla u + V(x) u = 0, \quad t \in \mathbb{R},\ x \in \mathbb{R}^3, u(0,x) = f, \tag{1.15}$$

we have the following result.

Theorem 1.3. *If $u(x,t)$ is the solution of* (1.15) *then, for any $\delta, \delta' > 0$:*

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} \nabla u(x,t)\|_{L_t^2 L_x^2} \le C \|f\|_{\dot H_V^{1/2}}, \tag{1.16}$$

where $\dot H_V^s(\mathbb{R}^3)$ is the perturbed homogeneous Sobolev space.

Recall that $\dot H_V^s(\mathbb{R}^3)$ is defined, for any $p, q \ge 1$ and for any $s \in \mathbb{R}$, as the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm:

$$\|f\|_{\dot H_V^s}^2 := \sum_{j\in\mathbb{Z}} 2^{2js} \|\varphi_j(\sqrt{-\Delta_V}) f\|_{L^2}^2, \forall f \in C_0^\infty(\mathbb{R}^3), \tag{1.17}$$

where $-\Delta_V$ is the operator

$$-\Delta_V := -\Delta + V(x,D), \tag{1.18}$$

with

$$V(x,D) = ia(x)\cdot\nabla + V(x) = i\sum_{j=1}^3 a_j(x)\partial_j u + V(x) \tag{1.19}$$

and $\sum_{j\in\mathbb{Z}} \varphi_j(\lambda) = 1$, with $\varphi_j(\lambda) = \varphi(\frac{\lambda}{2^j})$, $\varphi \in C_0^\infty(\mathbb{R})$, $\operatorname{supp}\varphi \subset [\frac{1}{2}, 2]$.

Remark 1.4. We can use the perturbed homogeneous Sobolev space in (1.17) because, the assumptions (1.5) and (1.6) imply that $\sigma_{sing}(-\Delta + V(x,D)) = \emptyset$ so the wave operators exist and are complete [24, 25, 31].

The key point in this work is the use of appropriate estimates of the resolvent $R_V(\lambda^2 \pm i0)$ defined as follows:

$$R_V(\lambda^2 \pm i0) f = \lim_{\varepsilon\to 0^+} R_V(\lambda^2 \pm i\varepsilon) f, \tag{1.20}$$

where

$$R_V(\lambda^2 \pm i\varepsilon) = [(\lambda^2 \pm i\varepsilon) + \Delta_V)]^{-1}, \tag{1.21}$$

with the notation $D = i^{-1}\nabla$. The operator in (1.18) has to be understood in the sense of the classical Friedrich's extension defined by the quadratic form

$$\begin{aligned}(-\Delta_V f, f) &= \int_{\mathbb{R}^3} |\nabla f(x)|^2\,dx + \int_{\mathbb{R}^3} V(x)\,|f(x)|^2\,dx \\ &\quad + \sum_{j=1}^3 \int_{\mathbb{R}^3} i a_j(x) f(x) \overline{\partial_j f(x)}\,dx, f \in C_0^\infty(\mathbb{R}^3),\end{aligned}$$

and the limit in (1.20) is taken in a suitable L^2 weighted sense.

More precisely, given any real a and $\delta > 0$, we define the spaces $L^2_{a,\delta}$ as the completion of $C_0^\infty(\mathbb{R}^3)$ respect to the following norms:

$$\|f\|^2_{L^2_{a,\delta}} := \int_{\mathbb{R}^3} |f|^2 |x|^{2a} W_\delta^2(|x|)dx, \quad \text{if } a > 0$$

and

$$\|f\|^2_{L^2_{a,\delta}} := \int_{\mathbb{R}^3} |f|^2 |x|^{2a} W_\delta^{-2}(|x|)dx, \quad \text{if } a < 0,$$

where the weights $W_\delta(|x|)$ are defined in (1.7).

The existence of the limit in (1.20) (known as limiting absorption principle [1, 4, 21, 25] can be established in the uniform operator norm

$$B(L^2_{1/2,\delta}, L^2_{-1/2,\delta}) \quad \forall \delta > 0.$$

To verify the limiting absorption principle we use the following resolvent identities:

$$\begin{aligned} R_V(\lambda^2 \pm i\varepsilon) &= R_0(\lambda^2 \pm i\varepsilon) + iR_0(\lambda^2 \pm i\varepsilon)a \cdot \nabla R_V(\lambda^2 \pm i\varepsilon) \\ &\quad + R_0(\lambda^2 \pm i\varepsilon)VR_V(\lambda^2 \pm i\varepsilon), \\ R_V(\lambda^2 \pm i\varepsilon) &= R_0(\lambda^2 \pm i\varepsilon) + iR_V(\lambda^2 \pm i\varepsilon)a \cdot \nabla R_0(\lambda^2 \pm i\varepsilon) \\ &\quad + R_V(\lambda^2 \pm i\varepsilon)VR_0(\lambda^2 \pm i\varepsilon). \end{aligned}$$

The previous identities combined with the classical limiting absorption principle for the free resolvent imply

$$\begin{aligned} R_V(\lambda^2 \pm i0) &= R_0(\lambda^2 \pm i0) + iR_0(\lambda^2 \pm i0)a \cdot \nabla R_V(\lambda^2 \pm i0) \\ &\quad + R_0(\lambda^2 \pm i0)VR_V(\lambda^2 \pm i0), \end{aligned} \tag{1.22}$$

and

$$\begin{aligned} R_V(\lambda^2 \pm i0) &= R_0(\lambda^2 \pm i0) + iR_V(\lambda^2 \pm i0)a \cdot \nabla R_0(\lambda^2 \pm i0) \\ &\quad + R_V(\lambda^2 \pm i0)VR_0(\lambda^2 \pm i0). \end{aligned} \tag{1.23}$$

Several works have treated the potential type perturbation of the free wave equations. The case of purely potential perturbation $V(x)$ is considered in [6] under the following decay assumption:

$$|V(x)| \le \frac{C}{|x|^{4+\delta_0}}, \quad |x| \ge 1,$$

for some $C, \delta_0 > 0$. In [10] the previous assumption is weaken and the decay required at infinity is the following one: $|V(x)| \le \frac{C}{|x|^{3+\delta_0}}$. The family of radial potentials $V(x) = \frac{c}{|x|^2}$, where $c \in \mathbb{R}^+$, are treated in the papers [27] and [9]. More precisely, the first paper treats the case of radial initial data, while in the second work general initial data are considered. In these papers dispersive estimates for the corresponding perturbed wave equations are established. In [14] the assumption (1.6) means that at infinity the potential is bounded from above by $C|x|^{-2-\varepsilon_0}$, while its behavior near $x = 0$ is dominated by constant times $|x|^{-2+\varepsilon_0}$. In this paper Strichartz type estimates for the corresponding perturbed wave equation are established. In this work we introduce a "short range" perturbation with symbol of order one and (1.5) means that at infinity our potential is bounded from above by $C|x|^{-1-\varepsilon_0}$, while its behavior near $x = 0$ is dominated by constant times $|x|^{-1+\varepsilon_0}$. It is clear that the assumption (1.5), (1.6) are quite general and allow one to consider non radially symmetric potentials.

The work is organized as follows. In the section 2 we prove some estimates for the operators $R_0(\lambda^2 \pm i0)$. In section 3 we give some estimates for the perturbed resolvent $R_V(\lambda^2 \pm i0)$. In section 4 we prove theorems 1.1, 1.2, and 1.3.

2. Free Resolvent Eestimates

This section is devoted to prove of some estimates satisfied by the free resolvent operator $R_0(\lambda^2 \pm i0)$.

Lemma 2.1. *The family of operators $R_0(\lambda^2 \pm i0)$ satisfies the following estimates:*

(i) *For any $\delta, \delta' > 0$ there exists a real constant $C = C(\delta, \delta') > 0$ such that for any $\lambda > 0$:*
$$\||x|^{-\frac{1}{2}} W_\delta^{-1} R_0(\lambda^2 \pm i0) f\|_{L^2} \leq \frac{C}{\lambda} \||x|^{\frac{1}{2}} W_{\delta'} f\|_{L^2} \tag{2.1}$$

(ii) *For any $\delta, \delta', \epsilon > 0$ that satisfy $0 < \epsilon < 2\delta'$, there exists $C = C(\delta, \delta', \epsilon) > 0$ such that for any $\lambda > 0$:*
$$\||x|^{-\frac{1}{2}} W_\delta^{-1} R_0(\lambda^2 \pm i0) f\|_{L^2} \leq C \||x|^{\frac{3+\epsilon}{2}} W_{\delta'} f\|_{L^2} \tag{2.2}$$

(iii) *For any $\delta, \delta' > 0$ there exists a real constant $C = C(\delta, \delta') > 0$ such that for any $\lambda > 0$:*
$$\||x|^{-\frac{1}{2}} W_\delta^{-1} R_0(\lambda^2 \pm i0) f\|_{L^2} \leq \frac{C}{\lambda^{\frac{\delta'}{2+\delta'}}} \||x|^{\frac{3}{2}} W_{\delta'} f\|_{L^2} \tag{2.3}$$

(iv) *For any $\delta, \delta' > 0$ and for $s \in [1/2, 3/2]$, there exists a real constant $C = C(\delta, \delta') > 0$ such that for any $\lambda \in \mathbb{R}$:*
$$\||x|^{-s} W_\delta^{-1} R_0(\lambda^2 \pm i0) f\|_{L^2} \leq C \||x|^{2-s} W_{\delta'} f\|_{L^2} \tag{2.4}$$

(v) *For any $\delta, \delta' > 0$ there exists a real constant $C = C(\delta, \delta') > 0$ such that for any $\lambda > 0$:*
$$\||x|^{-\frac{3}{2}} W_\delta^{-1} R_0(\lambda^2 \pm i0) f\|_{L^2} \leq \frac{C}{\lambda^{\frac{\delta'}{2+\delta'}}} \||x|^{\frac{1}{2}} W_{\delta'} f\|_{L^2} \tag{2.5}$$

(vi) *For any $\delta > 0$ there exists a real constant $C = C(\delta) > 0$ such that for any $\lambda \geq 0$:*
$$\||x|^{-\frac{1}{2}} W_\delta^{-1} R_0(\lambda^2 \pm i0) f\|_{L^2} \leq C \|f\|_{L^1} \tag{2.6}$$

(vii) *For any $\delta, \delta' > 0$ and for $s \in [1/2, 3/2]$, there exists a real constant $C = C(\delta, \delta') > 0$ such that for any $\lambda > 0$:*
$$\||x|^{-s} W_\delta^{-1} \nabla R_0(\lambda^2 \pm i0) f\|_{L^2} \leq C \ \||x|^{s} W_\delta f\|_{L^2}. \tag{2.7}$$

Proof. In the sequel we will use the following representation formula for the operator $R_0(\lambda^2 \pm i0)$:
$$R_0(\lambda^2 \pm i0) f(x) = \frac{1}{4\pi} \int \frac{e^{\pm i\lambda|x-y|}}{|x-y|} f(y) dy. \tag{2.8}$$
The proof of (2.1) can be found in [1] and [5]. The proof of (2.2), (2.3), (2.4), (2.5), and (2.6) can be found in [14]. The proof of (2.7) can be found in [31, 37, 21], where slightly different spaces have been used. $\square$

Lemma 2.2. *Assume that the perturbation $V(x,D)$ satisfies Assumption* (1.5), (1.6). *Then the following estimates are satisfied: For any $\delta, \delta' > 0$ there exists a real constant $C := C(\delta,\delta') > 0$ such that for any $\lambda \geq 0$,*

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} R_0(\lambda^2 \pm i0) V(x,D) f\|_{L^2} \leq C\||x|^{-\frac{1}{2}} W_{\delta'}^{-1} f\|_{L^2}, \tag{2.9}$$

$$\||x|^{\frac{1}{2}} W_\delta V(x,D) R_0(\lambda^2 \pm i0) f\|_{L^2} \leq C\||x|^{\frac{1}{2}} W_{\delta'} f\|_{L^2}. \tag{2.10}$$

Proof. We split the proof of (2.9) into two step.
Step 1. Estimate of

$$iR_0(\lambda^2 \pm i0) a \cdot \nabla f. \tag{2.11}$$

We have the formula

$$iR_0(\lambda^2 \pm i0) a \cdot \nabla f = iR_0(\lambda^2 \pm i0) \nabla (a \cdot f) - iR_0(\lambda^2 \pm i0)(\nabla a) \cdot f \tag{2.12}$$

From the functional calculus we have $[\nabla, R_0(\lambda^2 \pm i0)] = 0$, so we rewrite (2.12) as

$$iR_0(\lambda^2 \pm i0) a \cdot \nabla f := i\nabla R_0(\lambda^2 \pm i0)(a \cdot f) - iR_0(\lambda^2 \pm i0)(\nabla a) \cdot f. \tag{2.13}$$

We have

$$\begin{aligned} &\||x|^{-\frac{1}{2}} W_\delta^{-1} iR_0(\lambda^2 \pm i0) a \cdot \nabla f\|_{L^2} \\ &\leq C\||x|^{-\frac{1}{2}} W_\delta^{-1} i\nabla R_0(\lambda^2 \pm i0) a f\|_{L^2} + C\||x|^{-\frac{1}{2}} W_\delta^{-1} iR_0(\lambda^2 \pm i0)(\nabla a) f\|_{L^2}. \end{aligned} \tag{2.14}$$

We can estimate now the first term in the right-hand side of (2.14). Using (2.7), we obtain

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} i\nabla R_0(\lambda^2 \pm i0) a f\|_{L^2} \leq C\|\||x|^{-\frac{1}{2}} W_{\delta''}^{-1} a f\|_{L^2}. \tag{2.15}$$

By assumption (1.5) and choosing $0 < \delta'' < \epsilon_0$, $\delta_a \leq \epsilon_0 - \delta''$ we have

$$\begin{aligned} \||x|^{-\frac{1}{2}} W_\delta^{-1} iR_0(\lambda^2 \pm i0) a \cdot \nabla f\|_{L^2} &\leq C\||x|^{\frac{1}{2}} W_{\delta''}^{-1} a f\|_{L^2} \\ &\leq C\||x|^{-\frac{1}{2}} W_{\epsilon_0 - \delta''}^{-1} f\|_{L^2} \\ &\leq C\||x|^{-\frac{1}{2}} W_{\delta_a}^{-1} f\|_{L^2}. \end{aligned} \tag{2.16}$$

For the second term in the right-hand side of (2.14), we use the estimates (2.4) and obtain

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} iR_0(\lambda^2 \pm i0)(\nabla a) f\|_{L^2} \leq C\||x|^{\frac{3}{2}} W_{\delta''} \nabla a f\|_{L^2}. \tag{2.17}$$

By (1.5), choosing $0 < \delta'' < \epsilon_0$, $\delta_b \leq \epsilon_0 - \delta''$, we have

$$\begin{aligned} \||x|^{-\frac{1}{2}} W_\delta^{-1} iR_0(\lambda^2 \pm i0) a \cdot \nabla f\|_{L^2} &\leq C\||x|^{\frac{3}{2}} W_{\delta''} (\nabla a) f\|_{L^2} \\ &\leq C\||x|^{-\frac{1}{2}} W_{\epsilon_0 - \delta''} f\|_{L^2} \\ &\leq C\||x|^{-\frac{1}{2}} W_{\delta_b}^{-1} f\|_{L^2}. \end{aligned} \tag{2.18}$$

From the fact that $\delta_b < \delta_a$ we put $\delta' \leq \delta_b$. Then (2.14) becomes

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} iR_0(\lambda^2 \pm i0) a \cdot \nabla f\|_{L^2} \leq C\||x|^{-\frac{1}{2}} W_{\delta'}^{-1} f\|_{L^2}. \tag{2.19}$$

Step 2. Estimate of

$$R_0(\lambda^2 \pm i0) V f. \tag{2.20}$$

From assumption (1.5), we see that $|\nabla a_j(x)| \leq \frac{C_0 \delta}{|x|^2 W_{\epsilon_0}(x)}$. Then we proceed as in Step 1 to obtain

$$\||x|^{-\frac{1}{2}} W_\delta^{-1} R_0(\lambda^2 \pm i0) V f\|_{L^2} \leq C\||x|^{-\frac{1}{2}} W_{\delta'}^{-1} f\|_{L^2}. \tag{2.21}$$

Taking into account estimates (2.19) and (2.21), we arrive at

$$\begin{aligned}
&\||x|^{-\frac{1}{2}}W_\delta^{-1}R_0(\lambda^2 \pm i0)V(x,D)f\|_{L^2} \\
&\le \||x|^{-\frac{1}{2}}W_\delta^{-1}iR_0(\lambda^2 \pm i0)a\cdot\nabla f\|_{L^2} + \||x|^{-\frac{1}{2}}W_\delta^{-1}R_0(\lambda^2 \pm i0)Vf\|_{L^2} \\
&\le C\||x|^{-\frac{1}{2}}W_{\delta'}^{-1}f\|_{L^2}\,.
\end{aligned}$$

Thus (2.9) is established.

The Proof of (2.10) is the dual to the estimate (2.9) and it is omitted. □

3. Perturbed Resolvent Estimates

In this section we prove some estimates for the perturbed resolvent $R_V(\lambda^2 \pm i0)$.

Theorem 3.1. *Assume that the perturbation $V(x,D)$ satisfies the assumptions* (1.5) *and* (1.6)*. Then for any $0 < \delta < \epsilon_0/2$ there exists a family of operators $A_\lambda^\pm \in \mathcal{B}(L^2_{-\frac{1}{2},\delta}, L^2_{-\frac{1}{2},\delta})$ such that,*

$$A_\lambda^\pm \circ [I - R_0(\lambda^2 \pm i0)V(x,D)] = I = [I - R_0(\lambda^2 \pm i0)V(x,D)] \circ A_\lambda^\pm.$$

Moreover, there exists a constant $C = C(\delta) > 0$ such that,

$$\|A_\lambda^\pm f\|_{L^2_{-\frac{1}{2},\delta}} \le C\|f\|_{L^2_{-\frac{1}{2},\delta}}, \quad \forall\lambda \in \mathbb{R}.$$

Theorem 3.2. *Assume that the perturbation $V(x,D)$ satisfies the assumptions* (1.5) *and* (1.6)*. Then for any $0 < \delta < \epsilon_0/2$ there exists a family of operators $B_\lambda^\pm \in \mathcal{B}(L^2_{\frac{1}{2},\delta}, L^2_{\frac{1}{2},\delta})$ such that,*

$$B_\lambda^\pm \circ [I - V(x,D)R_0(\lambda^2 \pm i0)] = I = [I - V(x,D)R_0(\lambda^2 \pm i0)] \circ B_\lambda^\pm.$$

Moreover, there exists a constant $C = C(\delta) > 0$ such that

$$\|B_\lambda^\pm f\|_{L^2_{\frac{1}{2},\delta}} \le C\|f\|_{L^2_{\frac{1}{2},\delta}}, \quad \forall\lambda \in \mathbb{R}.$$

We have

$$R_0(\lambda^2 \pm i0)V(x,D) = iR_0(\lambda^2 \pm i0)a\cdot\nabla + R_0(\lambda^2 \pm i0)V. \tag{3.1}$$

Now we need the following lemmas.

Lemma 3.3. *Assume that the potential V satisfies assumptions* (1.6)*. Then*

(1) *The operators $R_0(\lambda^2 \pm i0)V$ are compact in the space $B(L^2_{-\frac{1}{2},\delta}, L^2_{-\frac{1}{2},\delta'})$, provided that δ, δ' are small. Moreover the following estimate is satisfied:*

$$\|R_0(\lambda^2 \pm i0)V\|_{B(L^2_{-\frac{1}{2},\delta}, L^2_{-\frac{1}{2},\delta'})} \to 0,$$

as $\lambda \to \infty$.

(2) *The operators $VR_0(\lambda^2 \pm i0)$ are compact in the space $B(L^2_{\frac{1}{2},\delta}, L^2_{\frac{1}{2},\delta'})$, provided that δ, δ' are small. Moreover the following estimate is satisfied:*

$$\|VR_0(\lambda^2 \pm i0)\|_{B(L^2_{\frac{1}{2},\delta}, L^2_{\frac{1}{2},\delta})} \to 0,$$

as $\lambda \to \infty$.

Proof. The proof of (1) can be found in [14, Theorem III.1 and Lemma III.1]. The proof of (2) is the dual of (1) where

$$VR_0(\lambda^2 \pm i0) = (R_0(\lambda^2 \mp i0)V)^*$$

and is omitted. $\square$

Lemma 3.4. *Assume that the potential $ia\cdot\nabla$ satisfies assumptions* (1.5). *Then*

1. *The operators $iR_0(\lambda^2 \pm i0)a\cdot\nabla$ are compact in the space $B(L^2_{-\frac12,\delta}, L^2_{-\frac12,\delta'})$, provided that δ,δ' are small.*
2. *The operators $ia\cdot\nabla R_0(\lambda^2 \pm i0)$ are compact in the space $B(L^2_{\frac12,\delta}, L^2_{\frac12,\delta'})$, provided that δ,δ' are small.*

Proof. For part (1), we follow the proof in [14]. Let $\{f_n\}$ be a sequence bounded in $L^2_{-\frac12,\delta}$ and let $g_n := iR_0(\lambda^2 \pm i0)a\cdot\nabla\ f_n$. We split the proof in two cases:
Case 1. Compactness in $B_{2r}\setminus B_{\frac{1}{2r}}$, for $0<r<\infty$. The estimate (2.9) implies that if δ,δ' are small, then

$$iR_0(\lambda^2 \pm i0)a\cdot\nabla\ \in B(L^2_{-\frac12,\delta}, L^2_{-\frac12,\delta'}). \tag{3.2}$$

In the proceeding of the proof, we use the representation (2.13) for the operator (2.11) acting on $L^2_{-\frac12,\delta}$. The estimate (3.2) implies that $\|g_n\|_{L^2(B_{2r}\setminus B_{\frac{1}{2r}})} \le C(r)$. Let now

$$\begin{aligned}
&\|\nabla g_n\|_{L^2(B_{2r}\setminus B_{\frac{1}{2r}})}\\
&\le C\|i(\Delta+\lambda^2)R_0(\lambda^2\pm i0)af\|_{L^2(B_{2r}\setminus B_{\frac{1}{2r}})}\\
&\quad + C\lambda^2\||x|^{-\frac12}W_\delta^{-1}iR_0(\lambda^2\pm i0)af\|_{L^2(B_{2r}\setminus B_{\frac{1}{2r}})}\\
&\quad + C\||x|^{-\frac12}W_\delta^{-1}i\nabla R_0(\lambda^2\pm i0)(\nabla a)f\|_{L^2(B_{2r}\setminus B_{\frac{1}{2r}})}\\
&\le C\|af\|_{L^2(B_{2r}\setminus B_{\frac{1}{2r}})} + C\lambda^2\||x|^{-\frac12}W_\delta^{-1}iR_0(\lambda^2\pm i0)af\|_{L^2(B_{2r}\setminus B_{\frac{1}{2r}})}\\
&\quad + C\||x|^{-\frac32}W_\delta^{-1}i\nabla R_0(\lambda^2\pm i0)(\nabla a)f\|_{L^2(B_{2r}\setminus B_{\frac{1}{2r}})}
\end{aligned} \tag{3.3}$$

With estimates (2.1), (2.7) and the assumption (1.5), we obtain

$$\|\nabla g_n\|_{L^2(B_{2r}\setminus B_{\frac{1}{2r}})} \le C(r,\lambda)\||x|^{-\frac12}W_\delta^{-1}f_n\|_{L^2(B_{2r}\setminus B_{\frac{1}{2r}})}$$

and from the boundness of $\{f_n\}$, $\|\nabla g_n\|_{L^2(B_{2r}\setminus B_{\frac{1}{2r}})} \le C(r,\lambda)$. So we have

$$\|\nabla g_n\|_{H^1(B_{2r}\setminus B_{\frac{1}{2r}})} \le C(r,\lambda).$$

The compactness of the Sobolev embedding due to Rellich-Kondrachov theorem implies that $\{g_n\}$ is compact $L^2(B_r\setminus B_{\frac1r})$ for any $1<r<\infty$.
Case 2. Compactness in $(\mathbb{R}^3\setminus B_r)\cup B_{\frac1r}$. To study compactness in this space, we use the inequality

$$\begin{aligned}
&\int_{(\mathbb{R}^3\setminus B_r)\cup B_{\frac1r}} g_n^2(|x|)W_\delta^{-2}(|x|)|x|^{-1}dx\\
&\le \Big(\sup_{\{(\mathbb{R}^3\setminus B_r)\cup B_{\frac1r}\}} W_\delta^{-1}(|x|)\Big)\int_{\mathbb{R}^3} g_n^2(|x|)W_\delta^{-1}(|x|)|x|^{-1}dx.
\end{aligned} \tag{3.4}$$

The definition of the weights $W_\delta(|x|)$ guarantees that for $\delta > 0$ there exist real constants $c_1(\delta), c_2(\delta)$ such that $c_1(\delta)W_\delta \le W_{\frac{\delta}{2}}^2 \le c_2(\delta)W_\delta$. This property combined with (3.4), where we chose $\delta' = \frac{\delta}{2}$, implies

$$\begin{aligned}
&\int_{(\mathbb{R}^3\setminus B_r)\cup B_{\frac{1}{r}}} g_n^2(|x|)W_\delta^{-2}(|x|)|x|^{-1}dx\\
&\le C\Big(\sup_{\{(\mathbb{R}^3\setminus B_r)\cup B_{\frac{1}{r}}\}} W_\delta^{-1}(|x|)\Big)\int_{\mathbb{R}^3} g_n^2(|x|)W_{\frac{\delta}{2}}^{-2}(|x|)|x|^{-1}dx\\
&\le C'\Big(\sup_{\{(\mathbb{R}^3\setminus B_r)\cup B_{\frac{1}{r}}\}} W_\delta^{-1}(|x|)\Big)\|f|_{L^2_{-\frac{1}{2},\delta}}.
\end{aligned}$$

Moreover $(\sup_{\{(\mathbb{R}^3\setminus B_r)\cup B_{\frac{1}{r}}\}} W_\delta^{-1}(|x|)) \to 0$ if $r \to \infty$ and it implies with an easy diagonal argument the compactness of the sequence $\{g_n\}$ in the space $L^2_{-\frac{1}{2},\delta}$.
Proof of (2) This is the dual to part (1) of this theorem. We can also proceed independently following [1], [21, Chapter XIV, Scattering Theory. lemma 14.5.1] or [40]. □

Proof of Theorem 3.1. Lemmas 3.3, 3.4 and the choice of δ (small perturbation) in the coefficients of the perturbing term (1.5) imply that the operators $[\mathrm{Id} - R_0(\lambda^2 \pm i0)V(x,D)]$ are injective in $B(L^2_{-\frac{1}{2},\delta})$ and are compact perturbation of the invertible operator Id. We can apply the Fredholm Alternative Theorem to obtain the existence of the operators $A^\pm_\lambda$. To prove the uniform bound $\|A^\pm_\lambda\|_{B(L^2_{-\frac{1}{2},\delta})} \le C$ we consider two cases.
Case 1: λ large. As a consequence of lemma 3.3, 3.4 there exists $\bar\lambda > 0$ such that if $\lambda > \bar\lambda$ then $\|R_0(\lambda^2 \pm i0)V(x,D)\|_{B(L^2_{-\frac{1}{2},\delta})} \le \frac{1}{2}$ and this implies that $\|[\mathrm{Id} - R_0(\lambda^2 \pm i0)V(x,D)]\|_{B(L^2_{-\frac{1}{2},\delta})} \ge \frac{1}{2}$ provided that $\lambda > \bar\lambda$. This uniform bound from below for the operators implies an uniform bound from above for their corresponding inverse operators $A^\pm_\lambda$.

Case 2: λ small. The boundedness of $\|A^\pm_\lambda\|_{B(L^2_{-\frac{1}{2},\delta})}$ for $\lambda < \bar\lambda$ is a consequence of the continuity of the family of operators $A^\pm_\lambda$ in the space $B(L^2_{-\frac{1}{2},\delta})$ with respect to the parameter $\lambda \in [0,\infty)$ and of the compactness of the interval $[0, \bar\lambda_p]$. □

The proof of Theorem 3.2 is analogous to the proof of theorem 3.1; therefore, we omit it.

Remark 3.5. The notion of resonances of an operator was introduced in quantum mechanics for Schrödinger operator. The resonances of an operator can be connected with poles of the associated resolvent function taken in some generalized sense. The problem of resonances arise in mathematical physics and in other field such as geometry. In our case this problem arises when we have perturbation of operator acting in some Banach spaces. Several works have treated the theory of resonances, we refer the reader to [2, 20, 28, 33, 38]. The remark suggest that resonances may exist in the case of electromagnetic perturbation of type $V(x,D) = ia(x)\cdot\nabla + V(x)$. To assure that resonances cannot exist we impose a smallness assumption (1.5) on a.

Theorem 3.6. *Assume that the perturbation $V(x,D)$ satisfies* (1.5) *and* (1.6)*. For each $0<\delta<\epsilon_0/2$ we have*

(i) *There exists a real constant $C=C(\delta)>0$ such that for any $\lambda\in\mathbb{R}$:*
$$\||x|^{-\frac{1}{2}}W_\delta^{-1}R_V(\lambda^2\pm i0)f\|_{L^2}\le\frac{C}{\lambda}\||x|^{\frac{1}{2}}W_\delta f\|_{L^2} \tag{3.5}$$

(ii) *For any $\epsilon>0$ that satisfy $0<\epsilon<2\delta$, there exists $C=C(\delta,\epsilon)>0$ such that for any $\lambda\in\mathbb{R}$:*
$$\||x|^{-\frac{1}{2}}W_\delta^{-1}R_V(\lambda^2\pm i0)f\|_{L^2}\le C\||x|^{\frac{3+\epsilon}{2}}W_{\delta'}f\|_{L^2} \tag{3.6}$$

(iii) *There exists a real constant $C=C(\delta)>0$ such that for any $\lambda\in\mathbb{R}$:*
$$\||x|^{-\frac{1}{2}}W_\delta^{-1}R_V(\lambda^2\pm i0)f\|_{L^2}\le\frac{C}{\lambda^{\frac{\delta'}{2+\delta'}}}\||x|^{\frac{3}{2}}W_{\delta'}\|_{L^2} \tag{3.7}$$

(iv) *For any $\delta,\delta'>0$ and for $s\in[1/2,3/2]$, there exists a real constant $C=C(\delta,\delta')>0$ such that for any $\lambda\in\mathbb{R}$:*
$$\||x|^{-s}W_\delta^{-1}R_V(\lambda^2\pm i0)f\|_{L^2}\le C\||x|^{2-s}W_{\delta'}f\|_{L^2} \tag{3.8}$$

(v) *There exists a real constant $C=C(\delta)>0$ such that for any $\lambda\in\mathbb{R}$:*
$$\||x|^{-\frac{3}{2}}W_{\delta'}^{-1}R_V(\lambda^2\pm i0)f\|_{L^2}\le\frac{C}{\lambda^{\frac{\delta'}{2+\delta'}}}\||x|^{\frac{1}{2}}W_\delta f\|_{L^2} \tag{3.9}$$

(vi) *For any $\delta>0$ there exists a real constant $C=C(\delta)>0$ such that for any $\lambda\in\mathbb{R}$:*
$$\||x|^{-\frac{1}{2}}W_\delta^{-1}R_V(\lambda^2\pm i0)f\|_{L^2}\le C\|f\|_{L^1}. \tag{3.10}$$

(vi) *For any $\delta,\delta'>0$ and for $s\in[1/2,3/2]$, there exists a real constant $C=C(\delta,\delta')>0$ such that for any $\lambda>0$:*
$$\||x|^{-s}W_\delta^{-1}\nabla R_V(\lambda^2\pm i0)f\|_{L^2}\le C\ \||x|^{s}W_\delta f\|_{L^2}. \tag{3.11}$$

Theorem 3.1 implies that the identity (1.22) can be written as:
$$[I-R_0(\lambda^2\pm i0)V(x,D)]R_V(\lambda^2\pm i0)=R_0(\lambda^2\pm i0),$$
and the following identity,
$$R_V(\lambda^2\pm i0)=A_\lambda^\pm R_0(\lambda^2\pm i0). \tag{3.12}$$

Theorem 3.2 implies that the identity (1.23) can be written now as:
$$R_V(\lambda^2\pm i0)[I-V(x,D)R_0(\lambda^2\pm i0)]=R_0(\lambda^2\pm i0),$$
and the following identity,
$$R_V(\lambda^2\pm i0)=R_0(\lambda^2\pm i0)B_\lambda^\pm. \tag{3.13}$$

Proof Theorem 3.6. Estimate (3.5) can be proved combining the identity (3.12) with the theorem 3.1 and estimate (2.1) in the following way:
$$\begin{aligned}\||x|^{-\frac{1}{2}}W_\delta^{-1}R_V(\lambda^2\pm i0)f\|_{L^2}&\le\||x|^{-\frac{1}{2}}W_\delta^{-1}A_\lambda^\pm R_0(\lambda^2\pm i0)f\|_{L^2}\\&\le C\||x|^{-\frac{1}{2}}W_\delta^{-1}R_0(\lambda^2\pm i0)f\|_{L^2}\\&\le C\||x|^{\frac{1}{2}}W_\delta f\|_{L^2}.\end{aligned}$$
Estimate (3.6) can be proved combining the identity (3.12) with the theorem 3.1 and estimate (2.2) as before. Estimate (3.7) can be proved combining the identity (3.12) with the theorem 3.1 and estimate (2.3) as before. Estimate (3.8) can be proved

combining the identity (3.12) with the theorem 3.1 and estimate (2.4) as before. Estimate (3.9) can be proved combining the identity (3.12) with the theorem 3.1 and estimate (2.5) as before. Estimate (3.10) can be proved combining the identity (3.12) with the theorem 3.1 and estimate (2.6) as before. Estimate (3.11) can be proved combining the identity (3.13) with the theorem 3.2 and estimate (3.11) . □

Theorem 3.7. *Assume that the perturbation $V(x,D)$ satisfies* (1.5), (1.6). *For each $0<\delta<\epsilon_0/2$ we have for any $\lambda\in\mathbb{R}$*

$$\||x|^{\frac12}W_\delta V(x,D)R_V(\lambda^2\pm i0)f\|_{L^2}\le C\||x|^{\frac12}W_{\delta'}f\|_{L^2}. \tag{3.14}$$

Proof. The resolvent identity implies

$$V(x,D)R_V(\lambda^2\pm i0)=V(x,D)R_0(\lambda^2\pm i0)+V(x,D)R_0(\lambda^2\pm i0)V(x,D)R_V(\lambda^2\pm i0).$$

From this we have

$$[I-V(x,D)R_0(\lambda^2\pm i0)]R_V(\lambda^2\pm i0)=V(x,D)R_0(\lambda^2\pm i0). \tag{3.15}$$

Following theorem 3.2 part (2), we have

$$V(x,D)R_V(\lambda^2\pm i0)=B_\lambda^\pm V(x,D)R_0(\lambda^2\pm i0)\,.$$

Combining this with estimate (2.10) obtain

$$\begin{aligned}\|V(x,D)R_V(\lambda^2\pm i0)f\|_{L^2_{\frac12,\delta}}&\le C\|B_\lambda^\pm V(x,D)R_0(\lambda^2\pm i0)f\|_{L^2_{\frac12,\delta}}\\&\le C\|V(x,D)R_0(\lambda^2\pm i0)f\|_{L^2_{\frac12,\delta}}\\&\le C\|f\|_{L^2_{\frac12,\delta}}.\end{aligned}$$

□

4. Proof of Main Estimates

In this section we prove the main theorems 1.1, 1.2, 1.3. We use the techniques of [23] and [35].

Proof of Theorem 1.1. Case 1. Wave equation. To prove (1.9), we take Fourier Transform in time variable in (1.1) to get

$$(\lambda^2+\Delta_V)\hat u(\lambda,x)=-\hat F(\lambda,x). \tag{4.1}$$

Using (1.20) and the limit absorption principle, we get

$$\hat u(\lambda,x)=-R_V(\lambda^2\pm i0)\hat F(\lambda,x). \tag{4.2}$$

and consequently

$$\nabla\hat u(\lambda,x)=-\nabla R_V(\lambda^2\pm i0)\hat F(\lambda,x). \tag{4.3}$$

Now we can use (3.5) and obtain

$$\||x|^{-\frac12}W_\delta^{-1}\nabla\hat u(\lambda,x)\|_{L^2}^2\le C\||x|^{\frac12}W_\delta\hat F(\lambda,x)\|_{L^2}^2. \tag{4.4}$$

Integrating over λ and using the Plancherel identity in time variable, we have

$$\||x|^{-\frac12}W_\delta^{-1}\nabla u(x,t)\|_{L^2_tL^2_x}\le C\||x|^{\frac12}W_\delta F(x,t)\|_{L^2_tL^2_x}. \tag{4.5}$$

To prove (1.10), we use, after the Fourier transform, the identity (4.2), the Theorem 3.1 and the perturbed resolvent estimate (3.10).

To prove (1.11), we apply the Fourier Transform to obtain

$$V(x,D)\hat u(\lambda,x)=V(x,D)R_V(\lambda^2\pm i0)\hat F(\lambda,x)\,. \tag{4.6}$$

Then using the estimate (3.14) we have

$$\||x|^{\frac{1}{2}} W_\delta V(x,D)\hat{u}(\lambda,x)\|_{L^2} \le C\||x|^{\frac{1}{2}} W_\delta \hat{F}(\lambda,x)\|_{L^2}\,. \tag{4.7}$$

Consequently,

$$\||x|^{\frac{1}{2}} W_\delta V(x,D)u(x,t)\|_{L^2_tL^2_x} \le C\||x|^{\frac{1}{2}} W_\delta F(x,t)\|_{L^2_tL^2_x}. \tag{4.8}$$

Remark 4.1. The constants in (1.9), (1.10), (1.11) are all independent of λ.

Case 2. Dirac equation. The Dirac equation can be treated as the wave equation. In fact we write the solution of (1.3) as the following integral equation:

$$u = \int_0^t U(t-s)F(u(s),V(x,D))ds, \tag{4.9}$$

where $F(u(s),V(x,D)) = a\cdot\nabla u + F(t,x)$ and $U(t)$ denote the propagator of the free Dirac equation given by

$$U(t) = \cos(t\sqrt{-\Delta}) - \gamma_0(\gamma^j\partial_j)\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}. \tag{4.10}$$

A reduction to the wave equation can be done by applying the operator $\Box$ to the solution (4.9) and using the relation

$$\partial_\mu\partial^\mu u = 0. \tag{4.11}$$

So the estimates (1.9), (1.10) and (1.11) remain valid. $\square$

Proof of Theorem 1.2. The proof of non-homogeneous case (1.2) is the analogous of the perturbed wave equation (1.1). However we have to replace λ^2 by $\lambda > 0$ in the definitions (1.20), (1.21), (2.8) and in the estimates for free and perturbed resolvent in the section 2 and 3. $\square$

Proof of Theorem 1.3. For the homogeneous case, the TT^* argument [17, 22] combined with the estimates (1.9) imply (1.16). $\square$

Remark 4.2. By the definition of the perturbed Besov space we have $\dot{H}^s_V := \dot{B}^s_{V,2,2}$, for any $s\in\mathbb{R}$, so we can replace $\dot{H}^{1/2}_V$ by $\dot{B}^{1/2}_{V,2,2}$ in the (1.16).

Remark 4.3. One can also consider the following Cauchy problems for the perturbed wave equation and the Dirac equation:

$$\begin{gathered} \Box u + ia(x)\cdot\nabla u + V(x)u = 0,\\ u(0) = f, \quad \partial_t u(0) = g\,. \end{gathered} \tag{4.12}$$

and

$$\begin{gathered} i\gamma_\mu\partial_\mu u + ia(x)\cdot\nabla u + V(x)u = 0, \quad t\in\mathbb{R},\ x\in\mathbb{R}^3,\\ u(0,x) = f, \end{gathered} \tag{4.13}$$

As in the case of Schödinger equation, the TT^* argument combined with the estimates (1.9) applied to the problem (4.12), for any $\delta,\delta' > 0$, yields

$$\||x|^{-\frac{1}{2}} W_\delta^{-1}\nabla u(x,t)\|_{L^2_tL^2_x} \le C(\|f\|_{\dot{H}^1_V} + \|g\|_{L^2}).$$

For problem (4.13), with any $\delta,\delta' > 0$, the following holds:

$$\||x|^{-\frac{1}{2}} W_\delta^{-1}\nabla u(x,t)\|_{L^2_tL^2_x} \le C\|f\|_{\dot{H}^1_V},$$

where, in the previous estimates, we used the $L^2 - L^2$ boundness of the operator $\frac{\nabla}{\sqrt{-\Delta_V}}$ given by the following lemma.

Lemma 4.4. *The operator* $\frac{\nabla}{\sqrt{-\Delta_V}}$, *where* ∇ *is the gradient on* $\mathbb{R}^3$ *and* $-\Delta_V$ *is defined by the* (1.18) *satisfies the estimate*

$$\Big\| \frac{\nabla}{\sqrt{-\Delta_V}} f \Big\|_{L^2} \le C\|f\|_{L^2}, \quad f \in L^2. \tag{4.14}$$

Proof. One can rewrite the left-hand side of (4.14) as

$$\Big(\frac{\nabla}{\sqrt{-\Delta_V}} f, \frac{\nabla}{\sqrt{-\Delta_V}} f \Big). \tag{4.15}$$

Setting in the (4.15) $g = \frac{1}{\sqrt{-\Delta_V}} f$, we obtain

$$\begin{aligned} (\nabla g, \nabla g) &\le C(-\Delta_V f, f) \\ &\le C_1(-\Delta f, f) + i(a \cdot \nabla f, f) + \int V|f|^2, \end{aligned} \tag{4.16}$$

where as in the previous estimate we used the smallness assumption (1.5). So (4.14) is established. □

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