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SOLUTION CURVES OF $2m$-TH ORDER BOUNDARY-VALUE PROBLEMS

BRYAN P. RYNNE

Abstract. We consider a boundary-value problem of the form $Lu = \lambda f(u)$, where L is a $2m$-th order disconjugate ordinary differential operator ($m \ge 2$ is an integer), $\lambda \in [0, \infty)$, and the function $f : \mathbb{R} \to \mathbb{R}$ is C^2 and satisfies $f(\xi) > 0$, $\xi \in \mathbb{R}$. Under various convexity or concavity type assumptions on f we show that this problem has a smooth curve, $\mathcal{S}_0$, of solutions (λ, u), emanating from $(\lambda, u) = (0, 0)$, and we describe the shape and asymptotes of $\mathcal{S}_0$. All the solutions on $\mathcal{S}_0$ are positive and all solutions for which u is stable lie on $\mathcal{S}_0$.

1. Introduction

For any integer $m \ge 2$, we consider the $2m$-th order boundary-value problem

$$(-1)^m u^{(2m)}(x) + \sum_{i=0}^{m-1} (-1)^i p_i u^{(2i)}(x) = \lambda f(u(x)), \quad x \in (-1, 1), \tag{1.1}$$

$$u^{(i)}(-1) = u^{(i)}(1) = 0, \quad i = 0, \dots, m-1, \tag{1.2}$$

where $p_i \ge 0$, $i = 0, \dots, m-1$, are constants and $u^{(i)}$ is the ith derivative of $u \in C^{2m}[-1, 1]$, the number $\lambda \in \mathbb{R}_+ := [0, \infty)$, and the function $f : \mathbb{R} \to \mathbb{R}$ is C^2 and satisfies

$$f(\xi) > 0, \quad \xi \in \mathbb{R} \tag{1.3}$$

(we are only interested here in positive solutions so the behaviour of $f(\xi)$ when $\xi < 0$ is irrelevant). We assume that (1.3) holds throughout the paper and, under various additional assumptions on f, we show that (1.1)–(1.2) has a curve of solutions (λ, u) in $\mathbb{R}_+ \times C^{2m}[-1, 1]$, emanating from $(\lambda, u) = (0, 0)$, and we describe the shape of this curve. All the solutions on this curve are positive (that is, u is positive on $(-1, 1)$), and any solutions for which u is stable lie on this curve.

In the second order case ($m = 1$) this problem has been considered in many papers, for example [3], [4], [6], [11], [12], [13], [14], [17]. Detailed results for this case are obtained in [3] and [14] by using quadrature to derive explicit formulae for $\lambda = \lambda(\rho)$, $u = u(\rho) \in C^2[-1, 1]$ as functions of a parameter $\rho \ge 0$, with $\rho = |u(\rho)|_0$, such that for each $\rho \ge 0$, the pair $(\lambda(\rho), u(\rho))$ is a solution. The results on the shape of the curve of solutions are then obtained by investigating the function $\rho \to \lambda(\rho)$.

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Such a formula for the solutions is not available in the higher order case. Similar results, for the second order case, are also obtained in Section 4 of [4], and in [11], [12], [13], where the strategy is to use the implicit function theorem to construct a solution curve in $\mathbb{R} \times C^2[-1,1]$, and then investigate the structure of this curve directly. The approach we adopt is similar to this, for the higher order case, and we obtain most of the results obtained in the above papers for the second order case. However, many of the standard tools for second order differential equations used in these papers, such as the the maximum principle, the Sturm comparison theorem and simplicity of the zeros of solutions of linear equations, are not available in the higher order case. This leads to considerable complication in some of the proofs here and forces us to use the sophisticated theory of 'disconjugate' differential operators described in [8].

Higher order problems ($m > 1$) have also been investigated recently. For applications to elasticity see [2], [15] and [18], and the references therein. For general nth-order problems see, for example, [9] and [10], and the references therein (these papers allow the order n to be odd, and the boundary conditions are more general than here; the boundary conditions considered here are of the type considered in [1] and [10], with $n = 2m$ and $k = p = m$).

2. Preliminary results

For any integer $r \geq 0$, let $C^r[-1,1]$ denote the standard Banach space of real valued, r-times continuously differentiable functions defined on $[-1,1]$, with the norm $|u|_r = \sum_{i=0}^{r} |u^{(i)}|_0$, where $|\cdot|_0$ denotes the usual sup-norm on $C^0[-1,1]$. For any u, $v \in C^0[-1,1]$, let $\langle u, v\rangle = \int_{-1}^{1} uv$. Let $H^{2m}(-1,1)$ denote the standard Sobolev space of order $2m$ on $(-1,1)$, with the standard norm, which will be denoted by $\|\cdot\|_{2m}$.

For any $u \in C^{2m}[-1,1]$, let $S(u)$ denote the number of changes of sign of u in $(-1,1)$, and let $Z(u)$ denote the number of zeros of u in $(-1,1)$ (in the applications below, u will be a non-trivial solution of a differential equation so these numbers will be finite). If all the zeros of u in $(-1,1)$ are simple then $S(u) = Z(u)$.

Let

$$X = \{u \in C^{2m}[-1,1] : u \text{ satisfies } (1.2)\}, \quad Y = C^0[-1,1].$$

We define the operator $L : X \to Y$ by

$$Lu := (-1)^m u^{(2m)} + \sum_{i=0}^{m-1} (-1)^i p_i u^{(2i)}, \quad u \in X.$$

It can be verified that $\langle Lu, v\rangle = \langle u, Lv\rangle$, for all u, $v \in X$ and

$$\langle Lu, u\rangle > 0, \quad 0 \neq u \in X,$$

that is, L is formally self-adjoint and positive definite on X with respect to the inner product $\langle \cdot\,, \cdot\rangle$. It is shown in Remark 2.1 of [16] that this positive definiteness of L implies that the disconjugacy condition imposed on L in [8] holds, and hence all the results of [8] hold for the above L (in [8], L need not be formally self-adjoint, and the order of L is denoted by n, and may be odd). In particular, L has the following factorisation: there exists functions $\rho_i \in C^{2m-i}[-1,1]$, with $\rho_i > 0$ on $[-1,1]$, $i = 0, \dots, 2m$, such that, if we define

$$L_0 w := \rho_0 w, \quad L_i w := \rho_i (L_{i-1} w)', \quad i = 1, \dots, 2m,$$

for any $w \in C^{2m}[-1,1]$, then L has the form

$$Lu = (-1)^m L_{2m} u, \quad u \in X.$$

We note that the term $(-1)^m$ is not included in the definition of L in [8]. This sign factor is convenient here (in particular, for the spectral properties of L), but must be borne in mind when results from [8] are quoted. The functions $L_i u$, $i = 0, \dots, 2m-1$, will be called *quasi-derivatives* of u. For any $w \in C^{(2m)}[-1,1]$ we let $\nu(\pm 1, w)$ denote the total number of quasi-derivatives $L_i w(\pm 1)$, $i = 0, \dots, 2m-1$, which are zero. If $u \in X$ then the boundary conditions (1.2) imply that $L_i u(\pm 1) = 0$, $i = 0, \dots, m-1$, so that $\nu(\pm 1, u) \geq m$. Furthermore, Corollary 3 of [8] shows that the operator $L : X \to Y$ is non-singular.

We also need some results from [8] regarding the eigenvalue problem

$$Lu = \mu p u, \tag{2.1}$$

for functions $p \in C^0[-1,1]$ with $Z(p) = 0$. For convenience we state these results in the following lemma.

Lemma 2.1. *There exists a strictly increasing sequence of eigenvalues of* (2.1)*, denoted by $\mu_k(p) > 0$, $k = 1, 2, \dots$. Each eigenvalue μ_k has multiplicity one* (*both geometric and algebraic*)*, and any corresponding eigenfunction ϕ_k has only simple zeros in* $(-1,1)$*, and $Z(\phi_k) = k-1$. Also, $\phi_k^{(m)}(\pm 1) \neq 0$.*

For any $u \in X$, define $f(u) \in Y$ by $f(u)(x) = f(u(x))$, $x \in [-1,1]$. Then (1.1)–(1.2) can be rewritten as

$$Lu = \lambda f(u). \tag{2.2}$$

Let $\mathcal{S}$ denote the set of solutions (λ, u) of (2.2) in $\mathbb{R}_+ \times X$. Since L is non-singular, there are positive constants b_1, b_2 such that

$$b_1 |u|_{2m} \leq |Lu|_0 = \lambda |f(u)|_0 \leq b_2 |u|_{2m}, \quad (\lambda, u) \in \mathcal{S}. \tag{2.3}$$

Also, the only solution of (2.2) with $\lambda = 0$ is $(0,0)$. Let $\mathcal{S}_0$ denote the connected component of $\mathcal{S}$ which contains $(0,0)$. We will be primarily interested in the structure of $\mathcal{S}_0$, and the stability of the solutions on $\mathcal{S}_0$.

We say that a function $u \in X$ is *positive* if $u(x) > 0$ for $x \in (-1,1)$; we say that a solution $(\lambda, u) \in \mathcal{S}$ is *positive* if u is positive.

Lemma 2.2. *Every solution $(\lambda, u) \in \mathcal{S} \setminus \{(0,0)\}$ is positive and*

$$u^{(m)}(\pm 1) \neq 0. \tag{2.4}$$

Proof. If $u \equiv 0$ then since $f(0) > 0$ it follows from (2.2) that $\lambda = 0$. Now suppose that $u \not\equiv 0$ and $\lambda > 0$. Corollary 1 of [8] shows that $Z(u) \leq S(Lu) = S(f(u)) = 0$ (since $f(u)$ is positive), that is, u has no zeros in $(-1,1)$. If $u < 0$ on $(-1,1)$ then $0 < \langle Lu, u \rangle = \lambda \langle f(u), u \rangle \leq 0$, which is impossible. Finally, since u and Lu are positive we have $S(u) = S(Lu) = 0$, so setting $h = 2m$ in (6) in [8] yields,

$$0 \geq S(u) + \nu(-1,u) + \nu(1,u) - 2m - S(Lu) = \nu(-1,u) + \nu(1,u) - 2m$$

(it follows easily from its definition, which we will not repeat here, that the quantity $N_{2m}(u)$ occurring in [8] satisfies $N_{2m}(u) \geq \nu(-1,u) + \nu(1,u)$). This inequality, together with (1.2) and the definition of $\nu(\pm 1, u)$, shows that (2.4) must hold. $\square$

We now define a C^2 mapping $F : \mathbb{R} \times X \to Y$ by $F(\lambda, u) = Lu - \lambda f(u)$, $(\lambda, u) \in \mathbb{R} \times X$. Clearly, (2.2) is equivalent to the equation $F(\lambda, u) = 0$. At any $(\lambda, u) \in \mathbb{R} \times X$ the Fréchet derivative, $D_{(\lambda,u)}F(\lambda, u) : \mathbb{R} \times X \to Y$, is given by

$$D_{(\lambda,u)}F(\lambda, u)(\mu, v) = (L - \lambda f'(u))v - \mu f(u), \quad (\mu, v) \in \mathbb{R} \times X,$$

and this operator is Fredholm with index 1. We now show that if this derivative satisfies a suitable condition everywhere on $\mathcal{S}_0$ then $\mathcal{S}_0$ is a C^2 curve with a global C^2 parametrisation (this condition will be verified in the following sections under various hypotheses on f).

Lemma 2.3. *Suppose that at every $(\lambda, u) \in \mathcal{S}_0$ the operator $D_{(\lambda,u)}F(\lambda, u)$ is surjective. Then $\mathcal{S}_0$ has a C^2 parametrisation $s \to (\lambda(s), u(s)) : \mathbb{R}_+ \to \mathbb{R}_+ \times X$, such that $(\lambda(0), u(0)) = (0, 0)$, $\lambda_s(0) > 0$,*

$$\lim_{s\to\infty} \lambda(s)|f(u(s))|_0 = \infty, \tag{2.5}$$

and, for any $s \geq 0$, the s-derivative $(\lambda_s(s), u_s(s))$ satisfies

$$\lambda_s^2(s) + \sum_{i=0}^{2m} \langle u_s^{(i)}(s), u_s^{(i)}(s)\rangle = 1, \tag{2.6}$$

$$\big(L - \lambda(s)f'(u(s))\big)u_s(s) = \lambda_s(s)f(u(s)). \tag{2.7}$$

Proof. Since L is non-singular we can construct (using the implicit function theorem) a $\delta_0 > 0$ and a parametrisation $s \to (\lambda(s), u(s)) : [0, \delta_0) \to \mathbb{R}_+ \times X$ such that $(\lambda(0), u(0)) = (0, 0)$, $\lambda_s(0) > 0$, and (2.6) holds. Furthermore, the surjectivity hypothesis in the lemma implies that the set $\mathcal{S}_0 \setminus \{(0, 0)\}$ is a C^2 connected curve in $\mathbb{R}_+ \times X$ (see Sections 4.15 and 4.18 in [19] for details) and a local parametrization with the property (2.6) can be constructed for this curve near any $(\lambda, u) \in \mathcal{S}_0 \setminus \{(0, 0)\}$ (by the implicit function theorem). Thus the above local parametrisation near $(0, 0)$ can be extended to a maximal interval $[0, s_{\max})$. Suppose that $s_{\max} < \infty$, and for each $n = 1, 2, \dots$, let $s_n = s_{\max} - 1/n$, and let $\lambda_n = \lambda(s_n)$, $u_n = u(s_n)$. Then, by (2.6), $|\lambda_n| + \|u_n\|_{2m} \leq C$, for some $C > 0$, and so, by compactness of the embedding of $H^{2m}(-1, 1)$ in $C^{2m-1}[-1, 1]$, we may suppose that $\lambda_n \to \lambda_\infty$ and $u_n \to u_\infty$ in $C^{2m-1}[-1, 1]$, and hence, by (2.2), $u_n \to u_\infty$ in $C^{2m}[-1, 1]$, and $(\lambda_\infty, u_\infty) \in \mathcal{S}_0$. But now, by the above local result, the parametrisation can be extended to the right of $s_{\max}$, which contradicts the maximality of the interval $[0, s_{\max})$ and shows that the parametrisation extends to $[0, \infty)$.

A similar argument shows that

$$\lim_{s\to\infty} \big(|\lambda(s)| + \|u(s)\|_{2m}\big) = \infty. \tag{2.8}$$

Now suppose that there exists a sequence (s_n) in $\mathbb{R}_+$ such that $s_n \to \infty$ and the sequence $(|u_n|_{2m})$ is bounded. Then $\lambda_n \to \infty$ and $u_n \to u_\infty$ in $C^{2m-1}[-1, 1]$ (after taking a subsequence if necessary), and hence $|f(u_n)|_0 \to |f(u_\infty)|_0 > 0$ (by (1.3)). But these results contradict (2.3), which proves that $\lim_{s\to\infty} |u(s)|_{2m} = \infty$; (2.5) then follows from (2.3).

Finally, differentiating the equation $F(\lambda(s), u(s)) \equiv 0$ with respect to s, at any $s \geq 0$, yields (2.7). $\square$

Derivatives with respect to s will always be denoted by a subscript s, to avoid confusion with derivatives with respect to x.

Remark 2.4. The condition (2.6) says that the chosen parametrisation of the curve C_0 is a 'unit speed' parametrisation in the space $\mathbb{R} \times H^{2m}(-1,1)$.

Remark 2.5. Equation (2.7) says that the derivative $(\lambda_s(s), u_s(s))$ lies in the null space of $D_{(\lambda,u)}F(\lambda(s), u(s))$, and the surjectivity condition in Lemma 2.3 implies that this null-space is 1-dimensional, so (2.7) determines $(\lambda_s(s), u_s(s))$ uniquely, up to a scale factor, whose magnitude is determined by the unit speed condition (2.6).

We will also consider the stability of the solutions on $\mathcal{S}_0$ and relate this to the shape of $\mathcal{S}_0$. A solution $(\lambda, u) \in \mathcal{S}$ is said to be *stable* if all the eigenvalues of the operator $D_uF(\lambda, u) = L - \lambda f'(u)$ are strictly positive. Suppose that Lemma 2.3 holds and, for $s \geq 0$, let $\sigma(s)$ denote the principal (that is, the least) eigenvalue of the operator $L - \lambda(s)f'(u(s))$. By definition $(\lambda(s), u(s)) \in \mathcal{S}_0$ is stable if and only if $\sigma(s) > 0$. By standard continuous dependence results the function $\sigma(\cdot)$ is continuous on $\mathbb{R}_+$. Also, letting σ_0 denote the principal eigenvalue of L, we have $\sigma(0) = \sigma_0$, and our assumptions ensure that L is positive definite, that is $\sigma_0 > 0$. Thus, if s is sufficiently small, $(\lambda(s), u(s))$ is stable.

3. Increasing f

Throughout this section we suppose that

$$f'(\xi) > 0, \quad \xi > 0. \tag{3.1}$$

Lemma 3.1. *If $(\lambda, u) \in \mathcal{S}$ and*

$$(L - \lambda f'(u))w = 0, \quad 0 \neq w \in X, \tag{3.2}$$

then $Z(w) \neq 1$.

Proof. Differentiating (1.1) with respect to x and letting $v = u' := u^{(1)}$ yields

$$\widetilde{L}v = \lambda f'(u)v, \tag{3.3}$$

$$v^{(i)}(-1) = v^{(i)}(1) = 0, \quad i = 0, \dots, m-2, \tag{3.4}$$

where $\widetilde{L}$ is defined in the same way as L, except that we apply it to functions in $C^{2m}[-1,1]$, not just in X. From (1.2) and (3.2)–(3.4) we obtain, by integration by parts,

$$\lambda\langle f'(u)v, w\rangle = (-1)^m[v^{(m-1)}w^{(m)}]_{-1}^{1} + \lambda\langle v, f'(u)w\rangle$$

and hence,

$$u^{(m)}(-1)w^{(m)}(-1) = u^{(m)}(1)w^{(m)}(1). \tag{3.5}$$

Now, by (3.1), $f'(u) > 0$ on $(-1,1)$, so the results in Lemma 2.1 hold for the eigenvalue problem $Lw = \lambda f'(u)w$ and show that if $Z(w) = 1$ then $S(w) = 1$, and so (after multiplying w by -1 if necessary) $w^{(m)}(-1) > 0$, $(-1)^m w^{(m)}(1) < 0$. Also, since u is positive and satisfies (1.2) and (2.4), we must have $u^{(m)}(-1) > 0$, $(-1)^m u^{(m)}(1) > 0$. However, combining these results contradicts (3.5), which completes the proof. $\square$

Lemma 3.2. *If $(\lambda, u) \in \mathcal{S}_0$ then $\lambda < \mu_2(f'(u))$. Hence, if (3.2) holds then $\lambda = \mu_1(f'(u))$ and $Z(w) = 0$.*

Proof. It follows from Lemmas 2.1 and 3.1 that if $(\lambda, u) \in \mathcal{S}_0$ then $\lambda \neq \mu_2(f'(u))$. Since $0 < \mu_2(0))$, the first result follows from the continuity of the eigenvalues with respect to (λ, u) on $\mathcal{S}_0$. The second result now follows from this and Lemma 2.1, since the hypothesis implies that $\lambda = \mu_k(f'(u))$, for some $k \geq 1$, and w is a corresponding eigenfunction. □

Theorem 3.3. *$\mathcal{S}_0$ has the global parametrisation described in Lemma 2.3.*

Proof. We must show that if $(\lambda, u) \in \mathcal{S}_0$ then $D_{(\lambda,u)}F(\lambda, u)$ is surjective. Firstly, this is clearly true if the operator $L - \lambda f'(u)$ is non-singular. On the other hand, if $L - \lambda f'(u)$ is singular then, by Lemma 3.2, $\lambda = \mu_1(f'(u))$ and $(L - \lambda f'(u))w = 0$ for some positive $w \in X$. So, by Lemma 2.1,

$$1 = \dim N(L - \lambda f'(u)) = \operatorname{codim} R(L - \lambda f'(u))$$

(here, N and R denote null space and range respectively). Thus $D_{(\lambda,u)}F(\lambda, u)$ is surjective if $f(u) \notin R(L - \lambda f'(u))$, which by standard spectral theory is equivalent to $\langle f(u), w\rangle \neq 0$. However, this is clearly true since w and $f(u)$ are positive (by (1.3)), which proves that $D_{(\lambda,u)}F(\lambda, u)$ is surjective. □

The stability of solutions on the curve $\mathcal{S}_0$ is related to the shape of the curve as described in the following theorem. We note that (3.1), together with the results of [8], ensure that if $\sigma(s) \geq 0$ then $\sigma(s)$ is a simple eigenvalue, with a corresponding normalised, positive eigenfunction $\psi(s)$, satisfying the equation

$$(L - \lambda(s)f'(u(s)))\psi(s) = \sigma(s)\psi(s). \tag{3.6}$$

Theorem 3.4. *For $s \geq 0$, we have $\lambda_s(s) = 0$ if and only if $\sigma(s) = 0$. If $\lambda_s(s) = 0$ then $Z(u_s(s)) = 0$. If $\lambda_s(s) \neq 0$ then $\operatorname{sgn} \lambda_s(s) = \operatorname{sgn} \sigma(s)$. If $\lambda_s(s) > 0$ then $u_s(s)$ is positive.*

Proof. If $\lambda_s(s) = 0$ then, by (2.6), $u_s(s) \neq 0$ and, by (2.7),

$$(L - \lambda(s)f'(u(s)))u_s(s) = 0, \tag{3.7}$$

so by Lemma 3.2, $Z(u_s(s)) = 0$, and hence $\sigma(s) = 0$. Conversely, if $\sigma(s) = 0$ then comparing (3.6) with (2.7) shows that $\lambda_s(s) = 0$ (see Remark 2.5). Thus the set of zeros of λ_s and σ coincide; we denote the complement of this set in $\mathbb{R}_+$ by M. This proves the first part of the theorem.

To prove the second part we require the following result, see Theorem 3.6 in [4]: if $\lambda_s(s_0) = 0$ then there exists $\delta(s_0) > 0$ such that if $|s - s_0| < \delta(s_0)$ and $s \in M$ then the quantities $\sigma(s)\langle u_s(s), u_s(s)\rangle$ and $\lambda_s(s)\langle u_s(s), f(u(s))\rangle$ are both nonzero and have the same sign.

Define t_1 (respectively t_2) to be the supremum of the set of $t \in M$ (respectively $t \in \mathbb{R}_+$) such that $\operatorname{sgn} \lambda_s(s) = \operatorname{sgn} \sigma(s)$ for all $s \in [0, t] \cap M$. Suppose that $t_2 < \infty$. Since $\lambda_s(0) > 0$, $\sigma(0) > 0$, we have $0 < t_1 \leq t_2$. By continuity, $t_2 \notin M$, but by definition there exists $s \in M$ arbitrarily close to t_2 with $s > t_2$, and $\operatorname{sgn} \lambda_s(s) = -\operatorname{sgn} \sigma(s)$. Thus, by Theorem 3.6 of [4] (quoted above), the function $u_s(t_2)$ is negative. By a similar argument, $u_s(t_1)$ is positive. Thus, $t_1 < t_2$, and $[t_1, t_2] \cap M = \emptyset$. However, by the first part of the proof, for all $s \in [t_1, t_2]$ we have $Z(u_s(s)) = 0$, so by continuity the sign of $u_s(s)$ is constant for $s \in [t_1, t_2]$, which is a contradiction. Thus we deduce that $t_2 = \infty$, which proves the desired result.

Now suppose that $\lambda_s(s) > 0$ on some interval (t_3, t_4), with either $t_3 = 0$ or $\lambda_s(t_3) = 0$, and either $t_4 = \infty$ or $\lambda_s(t_4) = 0$. The above results show that $u_s(t_3)$ is positive. Let t_5 be the supremum of the set of $t \in [t_3, t_4]$ such that $u_s(s)$ is positive for all $s \in [t_3, t]$. Suppose that $t_5 < t_4$. Then, by (2.7) and continuity, there exists $\delta > 0$ such that if $|s - t_5| < \delta$ then $S(Lu_s(s)) = 0$ and so, by Corollary 1 of [8], $Z(u_s(s)) = 0$. However, this contradicts the definition of t_5, and so proves that $t_5 = t_4$, which completes the proof of the theorem. □

The following theorem shows that for any $s > 0$, $u(s)$ is even, with a single local maximum at $x = 0$ (this is easy to prove in the second order case).

Theorem 3.5. *For all $s > 0$, $u(s)$ is even and $u'(s)$ has exactly one zero in $(-1, 1)$ (at $x = 0$, since $u'(s)$ must be odd), so that $|u(s)|_0 = u(s)(0)$.*

Proof. Clearly, $u(0) = 0$ is even and the set

$$\Sigma := \{s \in \mathbb{R}_+ : u(s) \text{ is even}\}$$

is closed. Now suppose that $s_0 \in \Sigma$ and there exists a sequence (δ_n) such that $\delta_n \to 0$ and $0 \le s_0 + \delta_n \notin \Sigma$. For any $s \ge 0$, define $\widetilde{u}(s)$ by $\widetilde{u}(s)(x) := u(s)(-x)$, $x \in [-1, 1]$. Then the curve $s \to (\lambda(s), \widetilde{u}(s))$, $s \ge 0$, is a curve of solutions of (2.2) satisfying the properties in Lemma 2.3, with $(\lambda(s_0), \widetilde{u}(s_0)) = (\lambda(s_0), u(s_0))$, but which is distinct from the curve $s \to (\lambda(s), u(s))$ near $(\lambda(s_0), u(s_0))$. However, this contradicts the implicit function theorem construction of the local curve in the proof of Lemma 2.3. This shows that such a sequence cannot exist for any s_0, and hence the set Σ is also open in $\mathbb{R}_+$, which implies that $\Sigma = \mathbb{R}_+$.

Next, for any $s > 0$, let $v(s) := u'(s)$. Then $v(s)$ is odd and satisfies (3.3)–(3.4) (with $\lambda = \lambda(s)$, $u = u(s)$) and, by (2.4),

$$v^{(m-1)}(s)(\pm 1) \neq 0. \tag{3.8}$$

We will show that $S(v(s)) = 1$ for all $s > 0$.

Firstly, when s is small, $(\lambda(s), u(s)) \simeq s\gamma(1, f(0)\eta)$, in $\mathbb{R}_+ \times X$, where γ is a suitable scaling factor and $\eta \in X$ satisfies $L\eta = 1$ (η is even since, if not, the function $\widetilde{\phi}$ defined by $\widetilde{\phi}(x) = \eta(-x)$, $x \in [-1, 1]$, would provide a second solution of this equation, contradicting the non-singularity of L). Thus, for small s, we have $S(v(s)) = S(\zeta)$, where $\zeta := \eta'$, and since ζ is non-trivial and odd we must have $S(\zeta) \ge 1$. Now, differentiating the equation $L\eta = 1$ yields $\widetilde{L}\zeta = 0$, and hence $L_{2m-1}\zeta = c_1$, for some constant c_1. If $c_1 \neq 0$ then (6) in [8] (with $h = 2m - 1$) shows that

$$0 \ge S(\zeta) + N_{2m-1}(\zeta) - (2m - 1) - S(L_{2m-1}\zeta) \ge S(\zeta) - 1$$

(again, we omit the definition of $N_{2m-1}(\zeta)$ given in [8], but we note that (3.4) implies that $N_{2m-1}(\zeta) \ge 2m - 2$), and hence $S(\zeta) = 1$ in this case. Now suppose that $c_1 = 0$, and so $L_{2m-2}\zeta = c_2$, for some constant c_2. By Corollary 3 in [8] it follows from this, together with (3.4), that $c_2 = 0$ implies that $\zeta = 0$, so we must have $c_2 \neq 0$. On the other hand, repeating the above argument (with $h = 2m - 2$) yields $0 \ge S(\zeta)$, which contradicts $S(\zeta) \ge 1$, so we conclude that $c_1 \neq 0$, and hence $S(\zeta) = 1$.

Now suppose that there exists $s_0 > 0$ such that $v(s_0)$ has a double zero in $(-1, 1)$. We consider the quantity $N(v(s_0))$ defined in [8] (the definition is somewhat complicated so we will not repeat it here; essentially, $N(v(s_0))$ is a count of the multiple zeros of $v(s_0)$). Since v satisfies (3.3) it follows from Lemma 1 in [8] and the definition of $N(v(s_0))$ that

$$\nu(-1, v(s_0)) + \nu(1, v(s_0)) + 2 \le N(v(s_0)) \le 2m \tag{3.9}$$

(the 2 here is a lower bound on the contribution from the assumed double zero of $v(s_0)$). However, (3.4) implies that $\nu(\pm 1, v(s_0)) \ge m - 1$, so we must have $N(v(s_0)) = 2m$. Lemma 1 in [8] now shows that $\nu(\pm 1, v(s_0))$ are even (odd) if m is even (odd), which implies that we must actually have $\nu(\pm 1, v(s_0)) \ge m$ (this parity argument relies on the positivity of f', that is, on (3.1)). But then (3.9) is contradictory, so we conclude that $v(s)$ cannot have a double zero in $(-1, 1)$ for any $s > 0$.

It follows from this that $Z(v(s)) = S(v(s))$ and that $S(v(s)) = S(\zeta) = 1$, for all $s > 0$, since if $S(v(s))$ changes at some $s = s_0$ then, by (3.8), $v(s_0)$ would have a double zero in $(-1, 1)$. □

Remark 3.6. In certain second order problems $\mathcal{S}_0$ can be parametrised by $|u|_0$, that is, the function $s \to |u(s)|_0$ is strictly increasing on $\mathbb{R}_+$, see for instance, [6]. We cannot show this here for the whole of $\mathcal{S}_0$, but Theorem 3.4 shows that this holds on the stable portions of $\mathcal{S}_0$, that is, $|u(\cdot)|_0$ is strictly increasing on any interval on which $\lambda_s(\cdot) > 0$.

We now examine the behaviour of $\mathcal{S}_0$ as $s \to \infty$.

Theorem 3.7. *We have*

$$\lim_{s \to \infty} |u(s)|_0 = \infty. \tag{3.10}$$

If, in addition, the limit $\gamma_\infty := \lim_{\xi \to \infty} f(\xi)/\xi$ exists then $\lambda_\infty := \lim_{s \to \infty} \lambda(s)$ exists, and $\lambda_\infty = \sigma_0/\gamma_\infty$ (we allow $\gamma_\infty = \infty$ (or $\gamma_\infty = 0$) here, in which case $\lambda_\infty = 0$ (or $\lambda_\infty = \infty$).

Proof. We follow the proof of Lemma 4 in [8] to a certain extent, but for brevity we merely describe the necessary changes. As in [8], for any $u \in X$ let

$$r(u, x) = \Big(\sum_{i=0}^{2m-1} (L_i u)^2(x) \Big)^{1/2}, \quad x \in [-1, 1].$$

Now let (s_n) be an arbitrary sequence in $\mathbb{R}_+$ with $s_n \to \infty$, and for each n let $\lambda_n = \lambda(s_n)$, $u_n = u(s_n)$. Following the proof in [8], without the normalization (11) used there, (16) in [8] becomes

$$\lambda_n \Big| \int_{-1/2}^{1/2} \frac{f(u_n(x))}{\rho_{2m}(x)}\, dx \Big| \le B r(u_n, -1/2), \tag{3.11}$$

while (17) becomes

$$|(L_q u_n)(x)| \le C r(u_n, -1/2), \quad -1/2 \le x \le 1/2, \quad 0 \le q \le 2m - 1, \tag{3.12}$$

for positive constants B, C (replacing α, δ, y_λ, n in [8] with $-1/2$, $1/2$, u_n, $2m$, respectively, here). Now, if the sequence $(r(u_n, -1/2))$ were bounded then it would follow from (3.11) and (3.12) (together with Theorem 3.5) that the sequences (λ_n) and $(|u_n|_0)$ are bounded, which would contradict (2.5). Thus we may suppose that $r(u_n, -1/2) \to \infty$.

Let $v_n = u_n/r(u_n,-1/2)$, $n = 1,2,\dots$. The argument in [8] shows that $v_n \to v_\infty$ in $C^{2m-2}[-1/2,1/2]$ and $L_{2m-1}v_n \to L_{2m-1}v_\infty$ pointwisely in $(-1/2,1/2]$ (after choosing a subsequence if necessary), and so either $v_\infty \not\equiv 0$ or, by definition,

$$\frac{r(u_n,1/2)}{r(u_n,-1/2)} \to 0.$$

A similar argument using the functions $\widetilde{v}_n = u_n/r(u_n,1/2)$, $n = 1,2,\dots$, and the limit $\widetilde{v}_n \to \widetilde{v}_\infty$ on $[-1/2,1/2]$, shows that either $\widetilde{v}_\infty \not\equiv 0$ or,

$$\frac{r(u_n,-1/2)}{r(u_n,1/2)} \to 0.$$

From these alternatives we conclude that we must have $v_\infty \not\equiv 0$ or $\widetilde{v}_\infty \not\equiv 0$ (or both). The first result of the theorem now follows immediately from this. To prove the second result we now suppose that the limit γ_∞ exists, and that $v_\infty \not\equiv 0$ (the case $\widetilde{v}_\infty \not\equiv 0$ is similar), and hence $v_\infty \ge \epsilon$, for some $\epsilon > 0$, on some non-trivial interval $J \subset [-1/2,1/2]$.

If $\gamma_\infty = \infty$ then $f(u_n(x))/r(u_n,-1/2) \to \infty$, uniformly for $x \in J$, and so it follows from (3.11) that $\lambda_n \to 0$.

If $\gamma_\infty = 0$ then $|f(u_n)|_0/|u_n|_0 \to 0$ (using $|u_n|_0 \to \infty$), so by (2.3), $\lambda_n \to \infty$.

If $0 < \gamma_\infty < \infty$ then, for n sufficiently large, $f(u_n) \ge \epsilon\gamma_\infty/2r(u_n,-1/2)$ on J, so by (3.11) the sequence (λ_n) must be bounded and, after taking a subsequence if necessary, we have $\lambda_n \to \lambda_\infty$, for some λ_∞. Also, $|f(u_n)|_0/|u_n|_0 \to \gamma_\infty$, so by (2.3), $|u_n|_{2m}/|u_n|_0 \le c$ for some constant c. Now, defining the functions $w_n = u_n/|u_n|_{2m}$, $n = 1,2,\dots$, we may suppose that $w_n \to w_\infty \ne 0$ in $C^{2m-1}[-1,1]$, and $f(u_n)/|u_n|_{2m} \rightharpoonup \gamma_\infty w_\infty$ in $L^2(-1,1)$ (by the argument on p. 648 of [7]). Thus, taking the limit in (2.2), it follows readily that w_∞ is a non-trivial, weak solution of the equation $Lw_\infty = \lambda_\infty\gamma_\infty w_\infty$, and hence $\lambda_\infty\gamma_\infty = \sigma_0$ (since $w_\infty \ge 0$ on $[-1,1]$). This completes the proof. $\square$

4. Convex f

Throughout this section we suppose that (3.1) holds and, in addition,

$$f''(\xi) > 0, \quad \xi > 0 \tag{4.1}$$

(thus f is convex). Hence the results of Sections 2 and 3 hold. Also, by (3.1) and (4.1) the limits γ_∞ (see Theorem 3.7) and $f'_\infty := \lim_{\xi\to\infty} f'(\xi)$ exist (we allow $f'_\infty = \infty$), and it can be verified that $\gamma_\infty = f'_\infty > 0$. Thus, by Theorem 3.7, we have the following result.

Lemma 4.1. *The limit λ_∞ exists and $\lambda_\infty = \sigma_0/f'_\infty < \infty$.*

We now study the shape of $\mathcal{S}_0$ further.

Lemma 4.2. *If, for some $s > 0$, $\lambda_s(s) = 0$ then $\lambda_{ss}(s) < 0$.*

Proof. Differentiating (2.7) with respect to s and using $\lambda_s(s) = 0$ yields

$$(L - \lambda(s)f'(u(s)))u_{ss}(s) - \lambda(s)f''(u(s))(u_s(s))^2 = \lambda_{ss}(s)f(u(s)). \tag{4.2}$$

Taking the inner product of this with $u_s(s)$, integrating by parts, and using (2.7) yields

$$-\lambda(s)\langle f''(u(s))(u_s(s))^2, u_s(s)\rangle = \lambda_{ss}(s)\langle f(u(s)), u_s(s)\rangle,$$

and so it follows from Theorem 3.4, (1.3) and (4.1) that $\lambda_{ss}(s) < 0$. $\square$

Since $\lambda_s(0) > 0$, Lemma 4.2 shows that there is at most one 'turning point' $s_t > 0$ such that $\lambda_s(s_t) = 0$, so Theorem 3.4 and Lemma 4.1 give the following result on the shape of $\mathcal{S}_0$.

Theorem 4.3. *$\mathcal{S}_0$ must look like one of the curves (a)–(c) in Fig. 1. Case (c) occurs if and only if $f'_\infty = \infty$. Furthermore, all the solutions on $\mathcal{S}_0$ are stable in case (a), while in cases (b) and (c) only the solutions on the 'lower' portion of the curve before the turning point are stable.*

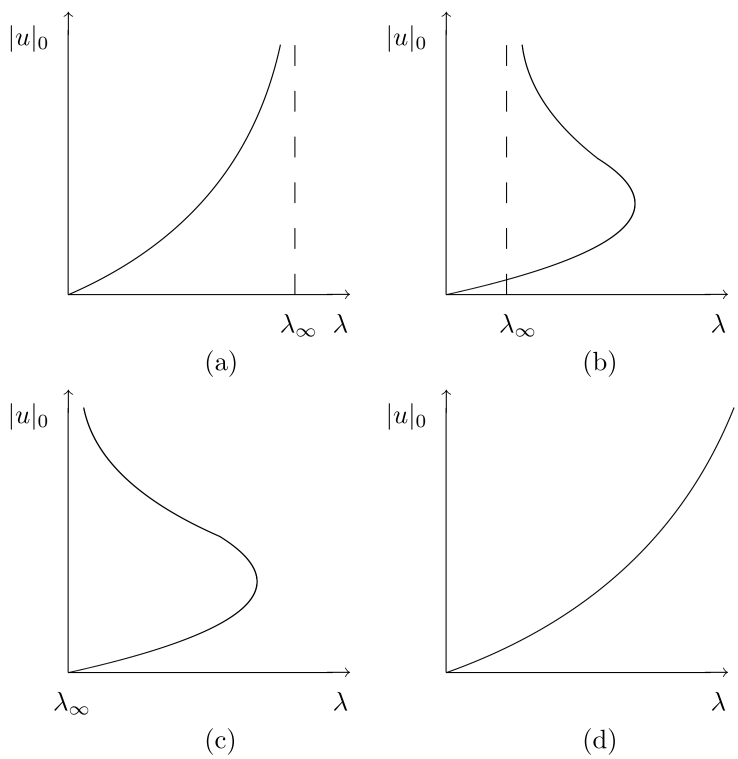

FIGURE 1. Possible forms of the solution curve $\mathcal{S}_0$

Note that, as mentioned in Remark 3.6, we have not shown that $\mathcal{S}_0$ can be parametrised by $|u|_0$, so we cannot preclude 'vertical oscillations' in the curves in Fig. 1, that is, multiple solutions for a given $|u|_0$.

If the turning point s_t exists we have the following simple estimate of the location of $\lambda(s_t)$ (analogous to Lemma 4.3 in [4]).

Lemma 4.4. *If s_t exists, then $\lambda(s_t) < \sigma_0/f'(0)$.*

Proof. By definition, the operator $L - \lambda(s_t)f'(u(s_t))$ is positive semi-definite on X, and by (4.1), $f'(u(s_t)) > f'(0)$ on $(-1, 1)$, so the operator $L - \lambda(s_t)f'(0)$ is positive definite, and hence $\lambda(s_t)f'(0) < \sigma_0$. □

We now give a necessary and sufficient condition for a turning point to exist, that is, to distinguish between cases (a) and (b). Defining $g(\xi) := f(\xi) - f'(\xi)\xi$, $\xi \geq 0$, it is clear that $g(0) > 0$ and, by (4.1), $g'(\xi) < 0$, $\xi > 0$.

Theorem 4.5. *Suppose that $f'_\infty < \infty$. Then $\mathcal{S}_0$ has a turning point if and only if $g(\xi_0) = 0$ for some $\xi_0 > 0$.*

Proof. Suppose that $\lambda_s(s) = 0$ for some $s > 0$. Taking the inner product of (2.7) with $u(s)$ yields

$$\lambda(s)\langle f'(u(s))u_s(s), u(s)\rangle = \langle Lu_s(s), u(s)\rangle = \langle u_s(s), Lu(s)\rangle = \lambda(s)\langle u_s(s), f(u(s))\rangle,$$

and hence $\langle u_s(s), g(u(s))\rangle = 0$. Now, by Theorem 3.4, $Z(u_s(s)) = 0$ so g must change sign, and hence we must have $g(\xi_0) = 0$ for some $\xi_0 > 0$.

Now suppose that $g(\xi_0) = 0$ for some $\xi_0 > 0$, and let $-\delta := g(\xi_0 + 1) < 0$, so that $g(\xi) \le -\delta$ for all $\xi \ge \xi_0 + 1$. We can choose a sequence (s_n) in $\mathbb{R}_+$ such that $u_n/|u_n|_{2m} \to w_\infty$ in $C^{2m-1}[-1, 1]$, with w_∞ positive (see the final part of the proof of Theorem 3.7), and hence $\lim_{n\to\infty} u_n(x) = \infty$ for $x \in (-1, 1)$. Letting

$$G_n = \{x \in [-1, 1] : u_n(x) > \xi_0 + 1\}, \quad B_n = \{x \in [-1, 1] : u_n(x) \le \xi_0 + 1\},$$

it follows that $\lim_{n\to\infty} |B_n| = 0$ (where $|B_n|$ denotes the Lebesgue measure of B_n). Hence, for sufficiently large n,

$$\begin{aligned}\langle f(u_n), u_n\rangle &\le \int_{B_n} f(u_n)u_n + \int_{G_n} \big(f'(u_n)(u_n)^2 - \delta\big) \\ &\le f(\xi_0 + 1)(\xi_0 + 1)|B_n| - \delta|G_n| + \langle f'(u_n)u_n, u_n\rangle \\ &< \langle f'(u_n)u_n, u_n\rangle,\end{aligned}$$

and so, by Theorem 4.1 and (4.1),

$$\begin{aligned}\lambda_\infty f'_\infty\langle u_n, u_n\rangle = \sigma_0\langle u_n, u_n\rangle &\le \langle Lu_n, u_n\rangle = \lambda_n\langle f(u_n), u_n\rangle \\ &< \lambda_n\langle f'(u_n)u_n, u_n\rangle < \lambda_n f'_\infty\langle u_n, u_n\rangle.\end{aligned}$$

Thus $\lambda_\infty < \lambda_n$ for sufficiently large n, so $\mathcal{S}_0$ must have a turning point. ☐

Remark 4.6. In the second order case it is shown in Theorem 3.2 of [14] that if g has a zero then $\mathcal{S}_0$ has a turning point; the discussion in [3] proves the reverse implication.

Remark 4.7. The above proof that the existence of a turning point implies that g has a zero did not use condition (4.1). Thus, if we merely suppose that (3.1) holds, and that g does not have a zero, then $\mathcal{S}_0$ must look like one of the curves (a) or (d) in Fig. 1, depending on whether $\gamma_\infty < \infty$ or $\gamma_\infty = \infty$ (positivity of g implies that the function $f(\xi)/\xi$ is increasing, so γ_∞ exists).

In the second order case it can be shown that there is only one curve of solutions, see Theorem 1 of [6] and the argument at the bottom of p. 1016 of [12]. Although we cannot prove here that the only solutions of (2.2) lie on $\mathcal{S}_0$, the following theorem shows that all the stable solutions do.

Theorem 4.8. *If $(\lambda, u) \in \mathcal{S}$ is stable then $(\lambda, u) \in \mathcal{S}_0$.*

Proof. Suppose, on the contrary, that there exists a stable solution $(\lambda_1, u_1) \in \mathcal{S}\backslash\mathcal{S}_0$. We may also suppose, without loss of generality, that $0 < f'_\infty < \infty$ (by suitably redefining f on the interval $[1 + |u_1|_0, \infty)$, if necessary, which does not affect the solution (λ_1, u_1)). Now, by stability, $\langle (L - \lambda_1 f'(u_1))v, v\rangle > 0$ for all non-zero $v \in X$, and hence, $\lambda_1 < \mu_1(f'(u_1))$. Thus the proof of Theorem 3.3 shows that there is a global, connected C^2 solution curve $\mathcal{S}_1 \subset \mathbb{R}_+ \times X$ with $(\lambda_1, u_1) \in \mathcal{S}_1$, and that we may apply the implicit function theorem at any point $(\lambda, u) \in \mathcal{S}_1$ (the proof relies on the point $(\lambda_1, u_1) \in \mathcal{S}_1$ satisfying $\lambda_1 < \mu_2(f'(u_1))$, which follows from the stability assumption). All the other results proved above for $\mathcal{S}_0$ also hold for $\mathcal{S}_1$. Since $\mathcal{S}_0$ and $\mathcal{S}_1$ are closed and connected, but not equal to each other, they must be disjoint. In particular, $\mathcal{S}_1$ must be bounded away from the point $(0,0)$. Also, Lemma 4.2 shows that $\mathcal{S}_1$ cannot be a 'loop' (homeomorphic to the unit circle), so it must be an 'open curve' (homeomorphic to $\mathbb{R}$), and there is a global unit speed parametrisation $t \to (\lambda^1(t), u^1(t)) : \mathbb{R} \to \mathbb{R}_+ \times X$, with $(\lambda^1(0), u^1(0)) = (\lambda_1, u_1)$, and

$$\lim_{t\to\pm\infty} |u^1(t)|_0 = \infty, \quad \lim_{t\to\pm\infty} \lambda^1(t) = \sigma_0/f'_\infty.$$

However, the results in [5] on bifurcation from infinity at a 'simple' eigenvalue show that this cannot happen (there cannot be two curves with the same 'limit at infinity'), and hence (λ_1, u_1) cannot exist. □

5. Concave f

Throughout this section we suppose that (3.1) holds and, in addition,

$$f''(\xi) < 0, \quad \xi > 0 \tag{5.1}$$

(thus f is concave). Again, the results of Sections 2 and 3 hold, and the limits γ_∞, f'_∞ exist, with $0 \le \gamma_\infty = f'_\infty < \infty$ here.

Theorem 5.1. *For all $s \ge 0$, $\lambda_s(s) > 0$ and $\sigma(s) > 0$, and $\lambda_\infty = \sigma_0/f'_\infty$ Theorem 4.8 also holds.*

Proof. Changing the sign of f'' in the proof of Lemma 4.2 shows that if $\lambda_s(s) = 0$ then $\lambda_{ss}(s) > 0$ (by (5.1)). However, since $\lambda_s(0) > 0$, this precludes the existence of a point $s > 0$ for which $\lambda_s(s) = 0$, so we must have $\lambda_s(s) > 0$ for all s. It now follows from Theorem 3.4 that $\sigma(s) > 0$ for all s. The value of λ_∞ follows from Theorem 3.7, and the proof of Theorem 4.8 also holds here. □

Corollary 5.2. *$\mathcal{S}_0$ must look like one of the curves (a) or (d) in Fig. 1, depending on whether $f'_\infty < \infty$ or $f'_\infty = 0$. All the solutions on $\mathcal{S}_0$ are stable, and all stable solutions lie on $\mathcal{S}_0$.*

6. Decreasing f

Throughout this section we suppose that

$$f'(\xi) \le 0, \quad \xi > 0. \tag{6.1}$$

Here, f'_∞ may not exist, but γ_∞ exists, with $\gamma_\infty = 0$. The results of Section 2 hold.

Theorem 6.1. *$\mathcal{S}_0$ has the global parametrisation described in Lemma 2.3. Also, $\lambda_s(s) > 0$ and $\sigma(s) > 0$ for all $s \ge 0$, and $\lambda_\infty = \infty$. Hence, for each $\lambda > 0$, equation (2.2) has a solution $(\lambda, u) \in \mathcal{S}_0$; for fixed λ this solution is unique (thus, $\mathcal{S} = \mathcal{S}_0$).*

Proof. If $(\lambda, u) \in \mathcal{S}_0$ then it follows from (6.1) and the positivity of L that the operator $L - \lambda f'(u)$ is positive definite, and hence non-singular, so the surjectivity hypothesis in Lemma 2.3 holds. Next, if we had $\lambda_s(s) = 0$, for some $s > 0$, then (3.7) would hold, with non-zero $u_s(s)$, which would contradict the non-singularity of $L - \lambda f'(u(s))$. Thus $\lambda_s(s) \ne 0$ for all $s \ge 0$, so by continuity we must have $\lambda_s(s) > 0$ (since $\lambda_s(0) > 0$). Similarly, $\sigma(s) > 0$ for all $s \ge 0$. Next, by (6.1), $|f(u(s))|_0 = f(0)$, so it follows from (2.5) that $\lambda_\infty = \infty$.

Finally, suppose that for some $\lambda > 0$ there exists $u_i \in X$ such that $Lu_i = \lambda f(u_i)$, $i = 1, 2$. If $u_1 - u_2 \neq 0$ then by (6.1)

$$0 < \langle L(u_1 - u_2), u_1 - u_2\rangle = \lambda\langle f(u_1) - f(u_2), u_1 - u_2\rangle \leq 0,$$

and this contradiction shows that we must have $u_1 - u_2 = 0$. $\square$

The following result shows that in this case the whole of $\mathcal{S}_0$ can be parametrised by $|u|_0$, see Remark 3.6.

Corollary 6.2. *The function $|u(\cdot)|_0$ is strictly increasing on $\mathbb{R}_+$.*

Proof. For any $s \geq 0$ the operator $L - \lambda f'(u(s))$ is positive definite so the results of [8] apply to this operator (see Section 2), and hence, by (2.7) and Corollary 1 of [8], $Z(u_s(s)) = 0$. Since $u_s(0)$ is positive, $u_s(s)$ must be positive for all $s \geq 0$, which proves the result. $\square$

Corollary 6.3. *$\mathcal{S}_0$ must look like the curve* (d) *in Fig. 1. All the solutions on $\mathcal{S}_0 = \mathcal{S}$ are stable.*

Remark 6.4. It is not clear if $\lim_{s\to\infty} |u(s)|_0 = \infty$, in general in this case, although this is true if $f(\xi) \geq \delta > 0$ for all $\xi \geq 0$. To see this, choose a positive function $\phi \in X$, and observe that

$$\langle u(s), L\phi\rangle = \langle Lu(s), \phi\rangle = \lambda(s)\langle f(u(s)), \phi\rangle \geq \delta\lambda(s)\langle 1, \phi\rangle \to \infty.$$

7. S-shaped $\mathcal{S}_0$

Throughout this section we suppose that (3.1) holds, so that the results of Sections 2 and 3 hold on $\mathcal{S}_0$, and we will give a sufficient condition for $\mathcal{S}_0$ to be 'S-shaped', that is, to have at least two turning points. We first give a sufficient condition for a turning point to occur in $\mathcal{S}_0$ which gives more explicit information on where the turning point occurs than the necessary and sufficient condition given in Theorem 4.5. This result is based on Theorem 1 in [13], and the proof here is an adaptation of the proof in [13] to deal with the higher order equation.

We will require the following notation. For $\xi \geq 0$, let $F(\xi) = \int_0^\xi f(t)\,dt$ and $G(\xi) = 2F(\xi) - \xi f(\xi)$. Note that $G'(\xi) = g(\xi)$ (where $g(\xi) := f(\xi) - f'(\xi)\xi$ was used in Theorem 4.5), and $g'(\xi) = -f''(\xi)\xi$. Clearly, $G(0) = 0$ and $G'(0) = g(0) > 0$.

Lemma 7.1. *Suppose that there exist numbers ξ_0, α such that $0 < \xi_0 < \alpha$ and*

$$g(\xi) > 0, \quad 0 < \xi < \xi_0, \quad \textit{and} \quad g(\xi) < 0, \quad \xi_0 < \xi < \alpha, \tag{7.1}$$

$$G(\alpha) \leq 0. \tag{7.2}$$

Then for any s_α satisfying $|u(s_\alpha)|_0 = \alpha$, we have $\lambda_s(s_\alpha) \leq 0$ (*by* (3.10)*, there exists at least one such s_α*).

Proof. Suppose, on the contrary, that $\lambda_s(s_\alpha) > 0$, and so, by Theorem 3.4, $\sigma(s_\alpha) > 0$. For any $\tau > 0$ let $\Phi_\tau := \tau\psi(s_\alpha) + u'(s_\alpha)$.

Lemma 7.2. *For any $\tau > 0$, $Z(\Phi_\tau) = 1$, and the zero of Φ_τ is simple.*

Proof. From (1.2), (2.4) and Theorem 3.5, Φ_τ satisfies

$$\Phi_\tau^{(i)}(\pm 1) = 0, \quad i = 0, \dots, m-2, \tag{7.3}$$

$$\Phi_\tau^{(m-1)}(\pm 1) = u^{(m)}(s_\alpha)(\pm 1) \neq 0, \tag{7.4}$$

$$\Phi_\tau(0) = \tau\psi(s_\alpha)(0) > 0. \tag{7.5}$$

It follows from (7.4) that there exists $\delta_\tau > 0$ such that

$$\pm\Phi_\tau(x) > 0, \quad 0 < |x \pm 1| < \delta_\tau. \tag{7.6}$$

Hence, by (7.5), Φ_τ changes sign in $(0,1)$ and $S(\Phi_\tau) \ge 1$.

Next, from (3.6) and (3.3), Φ_τ satisfies the differential equation

$$\widetilde{L}_\alpha \Phi_\tau := (\widetilde{L} - \lambda(s_\alpha) f'(u(s_\alpha)))\Phi_\tau = \tau\sigma(s_\alpha)\psi(s_\alpha).$$

Furthermore, since $\sigma(s_\alpha) > 0$, the operator $\widetilde{L}_\alpha$ is disconjugate, so the results of [8] apply to this equation (see Section 2 above). In particular, by (6) in [8], we have

$$S(\Phi_\tau) \le S(\widetilde{L}_\alpha \Phi_\tau) + 2m - N_{2m}(\Phi_\tau) = 2m - N_{2m}(\Phi_\tau) \le 2 \tag{7.7}$$

(since $\psi(s_\alpha)$ is positive, and the boundary conditions (7.3) imply that $N_{2m}(\Phi_\tau) \ge 2m-2$, see [8]). Now, if Φ_τ had a double zero in $(-1,1)$ then this would contribute a further 2 to $N_{2m}(\Phi_\tau)$, so that $S(\Phi_\tau) = 0$, but we already know that $S(\Phi_\tau) \ge 1$, so Φ_τ can only have simple zeros in $(-1,1)$. Thus, $Z(\Phi_\tau) = S(\Phi_\tau) \ge 1$ and, by (7.6), $Z(\Phi_\tau)$ must be odd, so by (7.7) we have $Z(\Phi_\tau) = 1$. □

By Theorem 3.5 we can now define $x_\alpha \in (0,1)$ by $u(s_\alpha)(x_\alpha) = \xi_0$, and let $\tau_\alpha := -u'(s_\alpha)(x_\alpha)/\psi(s_\alpha)(x_\alpha) > 0$. Then $\Phi_{\tau_\alpha}(x_\alpha) = 0$ and $g(u(s_\alpha)(x_\alpha)) = 0$, so from (7.1) and Lemma 7.2, we have

$$\begin{aligned} &g(u(s_\alpha)) < 0 \quad\text{and}\quad \tau_\alpha\psi(s_\alpha) + u'(s_\alpha) > 0, \quad \text{on } (0, x_\alpha),\\ &g(u(s_\alpha)) > 0 \quad\text{and}\quad \tau_\alpha\psi(s_\alpha) + u'(s_\alpha) < 0, \quad \text{on } (x_\alpha, 1).\end{aligned}$$

Hence,

$$\begin{aligned} \frac{\tau_\alpha}{2}\langle g(u(s_\alpha)), \psi(s_\alpha)\rangle &= \tau_\alpha \int_0^1 g(u(s_\alpha))\psi(s_\alpha)\,dx < \int_0^1 g(u(s_\alpha))(-u'(s_\alpha))\,dx \\ &= -\int_0^1 \frac{d}{dx} G(u(s_\alpha))\,dx = G(\alpha) \le 0 \end{aligned}$$

(using Theorem 3.5). On the other hand, taking the inner product of (3.6) with $u(s_\alpha)$ and the inner product of (2.2) with $\psi(s_\alpha)$, and subtracting, yields

$$\lambda(s_\alpha)\langle g(u(s_\alpha)), \psi(s_\alpha)\rangle = \sigma(s_\alpha)\langle u(s_\alpha), \psi(s_\alpha)\rangle \ge 0.$$

This contradiction proves that we must have $\lambda_s(s_\alpha) \le 0$. □

We now give conditions for $\mathcal{S}_0$ to be S-shaped, in the sense of the following theorem.

Theorem 7.3. *Suppose that* (7.2) *holds, for some* $\alpha > 0$*, and that*

$$f''(\xi) > 0, \quad 0 < \xi \le \alpha. \tag{7.8}$$

Suppose also that γ_∞ *exists, with* $0 \le \gamma_\infty \le f'(0)$*. Then there exists* t_1, t_2 *such that: (i)* $0 < t_1 < t_2$*; (ii)* $\lambda_s(t_1) = \lambda_s(t_2) = 0$*; (iii)* $0 < \lambda(t_2) < \lambda(t_1) < \sigma_0/f'(0)$*; (iv)* $\lambda_\infty = \sigma_0/\gamma_\infty > \lambda(t_1)$*.*

Proof. It follows from (7.2) and (7.8) that $g(\alpha) < 0$ and there exists $\xi_0 \in (0,\alpha)$ such that (7.1) holds, and hence the result of Lemma 7.1 holds. Thus we can define $t_1 = \inf\{s \ge 0 : \lambda_s(s) \le 0\}$, and we have $|u(s)|_0 \le \alpha$ for $s \in [0, t_1]$. It now follows from (7.8) that (4.1) holds for $0 < \xi \le \alpha$, and so the results of Section 4 hold for $s \in [0, t_1]$. Thus $\lambda_s(t_1) = 0$, and it follows from Lemma 4.4 that $\lambda(t_1) < \sigma_0/f'(0)$, and from Lemma 4.2 that $\lambda_s(s) < 0$ for $s - t_1 > 0$ sufficiently small. Also, Theorem 3.7 shows that $\lambda_\infty = \sigma_0/\gamma_\infty$. Hence, letting t_2 be the point at which $\lambda(\cdot)$ attains its minimum on the interval $[t_1, \infty)$, the results (i)–(iv) follow immediately. $\square$

Remark 7.4. Figs. 1 (a) and (b) on p. 1012 of [12] illustrate the above result when $\mathcal{S}_0$ is 'exactly S-shaped' (that is, when $\mathcal{S}_0$ has exactly two turning points). Fig. (a) corresponds to $\gamma_\infty = 0$, while Fig. (b) corresponds to $\gamma_\infty > 0$. Exact S-shapedness is also obtained in [17]. We have been unable to obtain exact S-shapedness of $\mathcal{S}_0$ here — the arguments in [12] and [17] rely on the problem being second order.

It follows from Theorem 7.3 that if $\lambda \in (\lambda(t_2), \lambda(t_1))$ then equation (2.2) has at least three solutions $(\lambda, u(s_i)) \in \mathcal{S}_0$, $i = 1, 2, 3$, with $s_1 < t_1 < s_2 < t_2 < s_3$ and, by Theorem 3.4 and the geometry of $\mathcal{S}_0$, $(\lambda, u(s_1))$ will be stable, $(\lambda, u(s_2))$ will not be stable, and $(\lambda, u(s_3))$ (on the 'upper' branch) will be stable if $\lambda_s(s_3) > 0$ (that is, the curve $\mathcal{S}_0$ is moving to the right at this point). We cannot rule out the possibility of more turning points, or 'vertical' points where $\lambda_s = 0$ (so that the corresponding solution is not stable). However, we have the following corollary.

Corollary 7.5. *If the hypotheses of Theorem 7.3 hold then, for almost all $\lambda \in (\lambda(t_2), \lambda(t_1))$, equation* (2.2) *has at least two stable solutions.*

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Bryan P. Rynne
Mathematics Department, Heriot-Watt University, Edinburgh EH14 4AS, Scotland
E-mail address: bryan@ma.hw.ac.uk