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# BOUNDARY MONOTONICITY FORMULAE AND APPLICATIONS TO FREE BOUNDARY PROBLEMS I: THE ELLIPTIC CASE

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Abstract. We derive a monotonicity formula at boundary points for a class of nonlinear elliptic partial differential equations, including the obstacle problem case, quenching, a free boundary problem with Bernoulli-type free boundary condition as well as the blow-up case. As application model we prove – for Dirichlet boundary data satisfying certain assumptions – the global existence of a classical solution of the free boundary problem with Bernoulli-type free boundary condition in two and three dimensions.

## 1. Introduction

Monotonicity formulae have proved useful in: Deriving growth estimates, analyzing asymptotic behavior, proving regularity, and investigating behavior that is neither of a microscopic nor of a very large order (global analysis). Unfortunately most of the known monotonicity formulae are valid only in an interior setting or in special boundary cases, for example the convex domain case (cf. [9]). Only recently progress has been made by B. White who succeeded in deriving a boundary monotonicity formula for the Plateau problem (see [6],[17]). In parabolic problems, boundary monotonicity formulae are desirable for the investigation of interior points, too, as the existing monotonicity identities are formulae for the Cauchy problem; cut-off or perturbation arguments have been only partly successful (see for example [14] and [11]).

In this first elliptic paper we derive boundary monotonicity identities for the following class of semilinear elliptic equations containing free boundary problems as well as the blow-up case:

$$\Delta u = \frac{\lambda_+}{2} p \max(u,0)^{p-1} - \frac{\lambda_-}{2} p \max(-u,0)^{p-1} \quad \text{for } p \in (0,+\infty) - \{2\}$$

and for $p = 0$,

$$\Delta u = 0 \quad \text{in } \{u > 0\} \cup \{u < 0\},$$
$$|\nabla \max(u,0)|^2 - |\nabla \max(-u,0)|^2 = \lambda_+ - \lambda_- \,.$$

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In the interior case, the identities coincide with those derived by the author in [16].

As model application we prove a global regularity result for the free boundary problem with Bernoulli-type free boundary condition,

$$\Delta u = 0 \text{ in } \Omega \cap \{u > 0\}, |\nabla u| = 1 \text{ on } \Omega \cap \partial\{u > 0\}, u = u_D \text{ on } \partial\Omega. \tag{1.1}$$

The applications of (1.1) are as various as the modelling of jets and cavities [3], electro-chemical machining [13] and optimal heat conductors [8]. As singular limit problem of a reaction-diffusion equation it has also been used as a model for the propagation of equidiffusional premixed flames with high activation energy [4]. For the mathematical background of (1.1) see [1]. For a boundary regularity result in a two-dimensional special setting see [2].

The natural regularity for a solution $u$ of this problem is Lipschitz regularity [1]. On the other hand, the harmonic function with Lipschitz boundary values – a special solution of problem (1.1) – is not necessarily Lipschitz continuous on $\bar{\Omega}$. We construct therefore another solution of the problem, the *minimal solution*, which we prove to be Lipschitz continuous under a growth assumption on $u_D$.

Our main result here (Theorem 4.2) is that, assuming further conditions on $u_D$, there exists in two and three dimensions a classical solution of (1.1) on $\bar{\Omega}$, i.e. $\partial\{u > 0\}$ is locally a $C^{1,\beta}$-surface on $\bar{\Omega}$ and $u$ satisfies the condition $|\nabla u| = 1$ on $\partial\{u > 0\} \cap \bar{\Omega}$.

Applications of the monotonicity formulae in other areas like regularity at corners or cusps, and behavior of solutions in irregular domains seem feasible. Concerning the application to the above Bernoulli-type free boundary problem, there is a result by Karakhanyan, Kenig and Shahgholian for the smooth separation case – i.e. tangential touch of the fixed and the free boundary – which uses different methods [12].

## 2. Notation

We denote by $\chi_A$ the characteristic function of the set $A$, by $x \cdot y$ the Euclidean inner product on $\mathbb{R}^n$, by $|x|$ the Euclidean norm in $\mathbb{R}^n$, by $B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$ the ball of center $x_0$ and radius $r$, and by $e_i$ the $i$-th unit vector in $\mathbb{R}^n$. We shall often use abbreviations for inverse images like $\{u > 0\} := \{x \in \Omega : u(x) > 0\}, \{x_n > 0\} := \{x \in \mathbb{R}^n : x_n > 0\}$ etc. and occasionally we shall employ the decomposition $x = (x', x_n)$ of a vector $x \in \mathbb{R}^n$. By $\nu$ we will always refer to the outer normal on a given surface, by $\nabla u \cdot \nu$ to the normal derivative of the function $u$ and by $\nabla_\theta u = \nabla u - (\nabla u \cdot \nu)\nu$ to the tangential gradient of $u$ on the surface. Finally $\mathcal{H}^s$ shall denote the $s$-dimensional Hausdorff measure.

## 3. The Elliptic Monotonicity Formula

In this section we derive the monotonicity formula in the elliptic case. By a translation we take the point at which we derive the monotonicity formula to be the origin.

3.1. **Assumptions.**

(1) $\Omega$ is an open set in $\mathbb{R}^n$ and $\partial\Omega - \{0\}$ is of class $C^1$, i.e. for each $x \in \partial\Omega - \{0\}$, $\Omega$ is in some neighborhood of $x$ the subgraph of a $C^1$-function.

(2) $p \in [0, +\infty) - \{2\}, r_0 > 0, u \in H^{1,2}(B_{r_0}(0) \cap \Omega)$ and $\nabla u$ has an $L^2$-trace on $B_{r_0}(0) \cap \partial\Omega$.

(3) $\alpha = \frac{2}{2-p}$ and $\nabla u(x) \cdot x - \alpha u(x) = 0 \mathcal{H}^{n-1}$-a.e. on $\{x \in B_{r_0}(0) \cap \partial\Omega : \nu(x) \cdot x = 0\}$, i.e. $u$ is homogeneous on the "cone-part" of the boundary.

(4) $u$ is a *variational solution* in the sense of Definition 3.1 in [16]: a) for $p \in (0, +\infty) - \{2\}$, of the equation

$$\Delta u = \frac{\lambda_+}{2} p \max(u, 0)^{p-1} - \frac{\lambda_-}{2} p \max(-u, 0)^{p-1} . \tag{3.1}$$

b) for $p = 0$, of the problem

$$\begin{gathered} \Delta u = 0 \quad \text{ in } \{u > 0\} \cup \{u < 0\}, \\ |\nabla \max(u, 0)|^2 - |\nabla \max(-u, 0)|^2 = \lambda_+ - \lambda_- \quad \text{on } \partial\{u > 0\} \cup \partial\{u < 0\}. \end{gathered} \tag{3.2}$$

For the sake of completeness let us recall the definition of a variational solution [16, Definition 3.1]. We define $u \in H^{1,2}_{\mathrm{loc}}(\Omega)$ to be a *variational solution of (3.1), (3.2)*, if $u \in C^0(\Omega) \cap C^2(\Omega \cap (\{u > 0\} \cup \{u < 0\})), (\chi_{\{u>0\}} u^{p-1} + \chi_{\{u<0\}} (-u)^{p-1}) \in L^1_{\mathrm{loc}}(\Omega)$ for $p \in (0, 1)$ and the first variation with respect to domain variations of the functional

$$F(v) := \int_\Omega \left(|\nabla v|^2 + \lambda_+ \chi_{\{v>0\}} v^p + \lambda_- \chi_{\{v<0\}} (-v)^p\right)$$

vanishes at $v = u$, i.e., for any $\phi \in C^1_0(\Omega; \mathbb{R}^n)$,

$$\begin{aligned} 0 &= -\frac{d}{d\epsilon} F(u(x + \epsilon\phi(x)))|_{\epsilon=0} \\ &= \int_\Omega \Big(|\nabla u|^2 \div \phi - 2\nabla u D\phi \nabla u + \lambda_+ \chi_{\{u>0\}} u^p \div \phi + \lambda_- \chi_{\{u<0\}} (-u)^p \div \phi\Big) . \end{aligned} \tag{3.3}$$

**Remark 3.1.** It follows that minimizers of the energy are variational solutions, provided that they are continuous and satisfy the $L^1$-bound.

**Extension of the data:** we cover $\{x \in \partial\Omega : \nu(x) \cdot x \neq 0\}$ up to a set of vanishing $n-1$-dimensional Hausdorff measure by countably many disjoint balls $B_{\rho_j}(x_j)$ such that $x_j \in \partial\Omega$ and $B_{\rho_j}(x_j) \cap \partial\Omega \subset \{x \in \partial\Omega : \nu(x) \cdot x \neq 0\}$ is a $C^1$-graph of a function on $x^\perp$ for each $x \in B_{\rho_j}(x_j)$.

Now we define $D_j := \{\theta x : x \in B_{\rho_j}(x_j) \cap \partial\Omega, \theta \geq 1\}$ and we define *homogeneous extensions* $g_j : D_j \to \mathbb{R}$ and $h_j : D_j \to \mathbb{R}^n$ as follows: for $\theta \geq 1$ and $x \in B_{\rho_j}(x_j) \cap \partial\Omega$, let $g_j(\theta x) := \theta^\alpha u(x)$ and let $h_j(\theta x) := \theta^{\alpha-1} \nabla u(x)$. Moreover we set $\sigma_j := \operatorname{sgn} \nu(x_j) \cdot x_j$.

**Remark 3.2.** Observe that the union of the graphs of $g_j$ and $u$ is in general not a graph in $\mathbb{R}^{n+1}$.

**Theorem 3.3** (Elliptic Monotonicity Formula)**.** *Suppose that*

$$\int_{B_{r_0}(0)\cap\Omega} |u|^p + \sum_{j=1}^\infty \int_{B_{r_0}(0)\cap D_j} \Big(|h_j|^2 + |h_j \cdot \nabla g_j| + g_j^2 + |g_j|^p\Big) < +\infty.$$

*Let us define functions*

$$I(r) = r^{-n-2(\alpha-1)} \int_{B_r(0)\cap\Omega} \Big(|\nabla u|^2 + \lambda_+ \chi_{\{u>0\}} u^p + \lambda_- \chi_{\{u<0\}} (-u)^p\Big) ,$$

$$I^B(r) = \alpha r^{1-n-2\alpha} \int_{\partial B_r(0)\cap\Omega} u^2 d\mathcal{H}^{n-1} ,$$

$$E_j(r) = r^{-n-2(\alpha-1)} \int_{B_r(0)\cap D_j} \Big( -|h_j|^2 + 2h_j \cdot \nabla g_j + \lambda_+ \chi_{\{g_j>0\}} g_j^p + \lambda_- \chi_{\{g_j<0\}} (-g_j)^p \Big) ,$$

$$E_j^B(r) = \alpha r^{1-n-2\alpha} \int_{\partial B_r(0)\cap D_j} g_j^2 d\mathcal{H}^{n-1} .$$

*Then* $\Phi_0(r) := I(r) - I^B(r) + \sum_{j=1}^\infty \sigma_j (E_j(r) - E_j^B(r))$ *satisfies for a.e.* $0 < \rho < \sigma < r_0$ *the monotonicity identity*

$$\Phi_0(\sigma) - \Phi_0(\rho) = \int_\rho^\sigma r^{-n-2(\alpha-1)} \int_{\partial B_r(x_0)\cap\Omega} 2 \Big( \nabla u \cdot \nu - \alpha \frac{u}{r} \Big)^2 d\mathcal{H}^{n-1} dr \ \geq 0 \ .$$

We defer the proof of this theorem to the Appendix and go into the applications first.

**Corollary 3.4.** *Suppose in addition that the following suprema are finite:*

$$\sup_{r\in(0,r_0)} r^{-n-2(\alpha-1)} \int_{B_r(0)\cap\Omega} \Big( |\min(\lambda_+,0)| \chi_{\{u>0\}} u^p + |\min(\lambda_-,0)| \chi_{\{u<0\}} (-u)^p \Big) ,$$

$$\sup_{r\in(0,r_0)} r^{1-n-2\alpha} \int_{\partial B_r(0)\cap\Omega} u^2 d\mathcal{H}^{n-1} ,$$

$$\sup_{r\in(0,r_0)} \sum_{j=1}^\infty \int_{B_r(0)\cap D_j} \Big( |h_j|^2 + |h_j \cdot \nabla g_j| + |\min(\sigma_j\lambda_+,0)| \chi_{\{g_j>0\}} g_j^p + |\min(\sigma_j\lambda_-,0)| \chi_{\{g_j<0\}} (-g_j)^p \Big) ,$$

$$\sup_{r\in(0,r_0)} \sum_{j=1}^\infty \max(\sigma_j,0) r^{1-n-2\alpha} \int_{\partial B_r(0)\cap D_j} g_j^2 d\mathcal{H}^{n-1} .$$

*Then* $\Phi_0(r) \searrow \Phi_0(0)$ *as* $r \searrow 0$, *and for any open* $D \subset\subset \mathbb{R}^n$ *and* $k \geq k(D)$ *the sequence* $u_k(x) = \frac{u(\rho_k x)}{{\rho_k}^\alpha}$ *is bounded in* $H^{1,2}(D \cap \Omega_k)$, *where* $\Omega_k = \{{\rho_k}^{-1} y : y \in \Omega\}$. *Moreover,*

$$\chi_{\Omega_k} (\nabla u_k(x) \cdot x - \alpha u_k(x)) \to 0 \text{ strongly in } L^2_{\rm loc}(\mathbb{R}^n) \text{ as } k \to \infty.$$

*Proof.* Defining $D_{jk} = \{{\rho_k}^{-1} y : y \in D_j\}$ and calculating, for $0 < R < \infty$,

$$I(R\rho_k) = R^{-n-2(\alpha-1)} \int_{B_R(0)\cap\Omega_k} \Big( |\nabla u_k|^2 + \lambda_+ \chi_{\{u_k>0\}} {u_k}^p + \lambda_- \chi_{\{u_k<0\}} (-u_k)^p \Big),$$

$$I^B(R\rho_k) = \alpha R^{1-n-2\alpha} \int_{\partial B_R(0)\cap\Omega_k} {u_k}^2 d\mathcal{H}^{n-1} ,$$

$$E(R\rho_k) = R^{-n-2(\alpha-1)} \int_{B_R(0)\cap D_{jk}} \Big( -|h_j|^2 + 2h_j \cdot \nabla g_j + \lambda_+ \chi_{\{g_j>0\}} g_j^p + \lambda_- \chi_{\{g_j<0\}} (-g_j)^p \Big),$$

$$E^B(R\rho_k) = \alpha R^{1-n-2\alpha} \int_{\partial B_R(0)\cap D_{jk}} g_j^2 d\mathcal{H}^{n-1} ,$$

we infer from the monotonicity formula Theorem 3.3 and from the assumed growth estimate that $u_k$ is bounded in $H^{1,2}(D \cap \Omega_k)$ for $k \geq k(D)$. Since $\Phi_0$ is non-decreasing and bounded in $(0, r_0)$, we know that $\Phi_0$ has a right limit at 0. Consequently, as $k \to \infty$,

$$\begin{aligned}
0 &\leftarrow \Phi_0(\rho_k S) - \Phi_0(\rho_k R) \\
&= \int_R^S r^{-n-2(\alpha-1)} \int_{\partial B_r(0)\cap\Omega_k} 2\left(\nabla u_k \cdot \nu - \alpha\frac{u_k}{r}\right)^2 d\mathcal{H}^{n-1} dr \\
&= \int_{(B_S(0)-B_R(0))\cap\Omega_k} 2|x|^{-n-2\alpha}\left(\nabla u_k(x)\cdot x - \alpha u_k(x)\right)^2 .
\end{aligned}$$

□

## 4. Application Example: Boundary Regularity for a Free Boundary Problem with a Bernoulli-type Condition on the Free Boundary

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ whose boundary is of class $C^{2,\gamma}$ for some $\gamma \in (0,1)$ and let $u_D : \partial\Omega \to [0, +\infty)$ satisfy $u_D \in C^{2,\gamma}(\overline{\{u_D > 0\}})$ as well as the non-degeneracy condition $\nabla_\theta u_D \neq 0$ on the boundary of $\{u_D > 0\}$ relative to $\partial\Omega$; here $\nabla_\theta$ denotes the tangential gradient. Let us extend $u_D$ to a smooth function on $\bar{\Omega}$. Moreover, let $u$ be the *minimal solution* of the free boundary problem $\Delta u = 0$ in $\Omega \cap \{u > 0\}, |\nabla u| = 1$ on $\Omega \cap \partial\{u > 0\}$ and $u = u_D$ on $\partial\Omega$. A minimal solution is a function with $u_D$-boundary data on $\partial\Omega$, satisfying for each open $\Omega' \subset \Omega$ the following conditions:

(1) $u$ is a global minimizer of the energy
$$E_{\Omega'}(w) = \int_{\Omega'} (|\nabla w|^2 + \chi_{\{w>0\}})$$
on the affine subspace $\{w \in H^{1,2}(\Omega') : w - u \in H_0^{1,2}(\Omega')\}$.

(2) For each global minimizer $v$ of the energy $E_{\Omega'}$ on the affine subspace $\{w \in H^{1,2}(\Omega') : w - v \in H_0^{1,2}(\Omega')\}$, satisfying $v \geq u$ on $\partial\Omega'$ (that is, $\max(u - v, 0) \in H_0^{1,2}(\Omega')$), we have $v \geq u$ in $\Omega'$.

For the following reason there exists a minimal solution and this minimal solution is unique: let $v_1, v_2$ be a global minimizer of the energy $E_{\Omega'}$ on the affine subspace $\{w \in H^{1,2}(\Omega') : w - v_1 \in H_0^{1,2}(\Omega')\}, \{w \in H^{1,2}(\Omega') : w - v_2 \in H_0^{1,2}(\Omega')\}$, respectively, and assume that $v_1$ and $v_2$ satisfy $v_1 \leq v_2$ on $\partial\Omega'$. Then

$$E_{\Omega'}(v_1) + E_{\Omega'}(v_2) = E_{\Omega'}(\min(v_1, v_2)) + E_{\Omega'}(\max(v_1, v_2)),$$

$E_{\Omega'}(\min(v_1, v_2)) \geq E_{\Omega'}(v_1)$ and $E_{\Omega'}(\max(v_1, v_2)) \geq E_{\Omega'}(v_2)$.

Consequently $E_{\Omega'}(\min(v_1, v_2)) = E_{\Omega'}(v_1)$ and $E_{\Omega'}(\max(v_1, v_2)) = E_{\Omega'}(v_2)$, implying that $\min(v_1, v_2)$ is a global minimizer on $\{w \in H^{1,2}(\Omega') : w - v_1 \in H_0^{1,2}(\Omega')\}$ and $\max(v_1, v_2)$ is a global minimizer on $\{w \in H^{1,2}(\Omega') : w - v_2 \in H_0^{1,2}(\Omega')\}$. Therefore, $u$ defined by $u(x) := \inf\{v(x) : v$ is a global minimizer of $E_\Omega$ with respect to $u_D$-boundary data $\}$ is the unique function with properties 1) and 2): first, by [1, 1.3] there exists a global minimizer, so the set in the above infimum is non-empty. Next, as the family of global minimizers with respect to $u_D$-boundary data is locally in $\Omega$ equicontinuous, we may reduce the family in the definition of $u$ to a countable family $(w_m)_{m\in\mathbb{N}}$ of global minimizers with respect to $u_D$-boundary data. Since $\min(w_1, \dots, w_m)$ is for each $m \in \mathbb{N}$ a global minimizer with respect to $u_D$-boundary data, the limit $u$, too, must by weak lower-semicontinuity of the energy be a global minimizer with respect to $u_D$-boundary data. Consider now $v$ as in property 2): extending the function $\min(v,u)$ by $u$ outside $\Omega'$, we see that the extended function is contained in the set of the above infimum. Consequently $\min(v,u) \geq u$ in $\Omega'$. Last, taking two functions $u_1$ and $u_2$ with properties 1) and 2), we may set $\Omega' := \Omega$ and test property 2) for $u_1$ with $u_2$ and vice versa to obtain the uniqueness.

Incidentally – going back to $v_1$ and $v_2$ defined above – the strong maximum principle tells us that $0 < v_1(x_0) = v_2(x_0)$ for some $x_0 \in \Omega'$ implies that $v_1 = v_2$ in the connected component of $x_0$. In particular, $v_1 < v_2$ on $\partial\Omega'$ implies $v_1 \leq v_2$ in $\Omega'$.

Although the harmonic function with $u_D$ boundary data – corresponding to the "maximal solution" in our concept – can except in the case $u_D \equiv 0$ never be Lipschitz continuous on $\bar\Omega$ (in the two-dimensional case this is a consequence of Corollary 3.4), Lipschitz continuity of the *minimal solution* can be ensured by an assumption on $u_D$. More precisely:

**Proposition 4.1** (Lipschitz continuity)**.** *Let $\Omega$ and $u_D$ be as above and let $u$ be the minimal solution with respect to $\Omega$ and $u_D$. If $(\Omega, u_D)$ satisfies in addition for*

$$R_0 := \begin{cases} \left((\frac{2}{n})^{-\frac{2}{n-2}} - 1\right)^{-1/2} \operatorname{diam}(\Omega), & \text{if } n \geq 3 \\ (e-1)^{-1/2} \operatorname{diam}(\Omega), & \text{if } n = 2, \end{cases}$$

$$\phi_{x_0,R_0}(x) := \begin{cases} \frac{R_0}{n-2}\left(1 - (\frac{|x-x_0|}{R_0})^{2-n}\right), & \text{if } n \geq 3 \\ -R_0 \log \frac{R_0}{|x-x_0|}, & \text{if } n = 2 \end{cases}$$

$$K^-_\delta(y_0) := \{x \in \mathbb{R}^n : \nabla_\theta u_D(y_0) \cdot (x - y_0) \leq -\delta |\nabla_\theta u_D(y_0)|\}$$

*the condition*

$$\begin{aligned} &u_D \leq \max(\phi_{x_0,R_0}, 0) \text{ on } \partial\Omega \text{ for some } \delta > 0, \text{each } y_0 \in \partial\Omega \cap \{u_D = 0\} \\ &\text{and some } x_0 \in \partial B_{R_0}(y_0) \cap \{x \in \mathbb{R}^n : \nu(y_0) \cdot (x - y_0) > 0\} \cap K^-_\delta(y_0), \end{aligned} \tag{4.1}$$

*then $u$ is Lipschitz continuous on $\bar\Omega$ and $\partial\{u > 0\}$ cannot approach $\partial\Omega$ tangentially. More precisely: $u = 0$ in an open neighborhood of the interior of $\{u_D = 0\}$ relative to $\partial\Omega$, and there is $\kappa \in (0,1)$ such that for each $y_0$ in the boundary of $\{u_D = 0\}$ relative to $\partial\Omega$, the free boundary is in an open neighborhood of $y_0$ contained in $\{x \in \mathbb{R}^n : -\kappa|x - y_0| \leq \mu(y_0) \cdot (x - y_0) \leq \kappa|x - y_0|\}$. Here $\mu(y_0)$ denotes the outward pointing unit co-normal on the relative boundary of $\{u_D = 0\}$.*

*Proof.* Let $y_0$ and $x_0$ as in the Proposition. Since $u$ is the minimal solution, we can compare it to $\max(\phi_{x_0,R_0}, 0)$ in $\Omega$ to obtain the growth estimate

$$u(y_0 + x) \leq |x| \tag{4.2}$$

as well as

$$u = 0 \text{ in } B_{R_0}(x_0) \cap \Omega\,;$$

note that by the choice of $R_0$ and by [1, 2.6], $\max(\phi_{x_0,R_0}, 0)$ is a global minimizer in $\Omega$. On the other hand, as $u$ is subharmonic in $\Omega$, we may at a point $y_0$ on the relative boundary of $\{u_D = 0\}$ define $K^+_{\tilde\delta} := \{x \in \mathbb{R}^n : \nabla_\theta u_D(y_0) \cdot (x - y_0) \geq \tilde\delta |\nabla_\theta u_D(y_0)|\}$, and make use of the non-degeneracy of $u_D$ and compare $u$ to the global minimizer $\max(-\phi_{x_1,R_1}, 0)$ (cf. [1, 2.6]) where $x_1 \in \partial B_{R_1}(y_0) \cap \{x \in \mathbb{R}^n : \nu(y_0) \cdot (x - y_0) > 0\} \cap K^+_{\tilde\delta}$ such that $R_1 \geq c, x_1 \notin \Omega$ and $u_D \geq \max(-\phi_{x_1,R_1}, 0)$ on $\partial\Omega \cap B_{R_1}(x_1)$;

such constants $c > 0$ and $\tilde\delta > 0$ depending only on $n, \Omega, u_D$ exist because of the non-degeneracy of $u_D$ as well as the smoothness of $\partial\Omega$ and $u_D|_{\overline{\{u_D>0\}}}$. We obtain $u > 0$ in $\Omega \cap B_{R_1}(x_1)$. Both balls $B_{R_1}(x_1)$ and $B_{R_0}(x_0)$ are touching the point $y_0$, implying the non-tangential touch of the free boundary.

To prove the Lipschitz continuity, let us first derive a bound for the normal derivative of $u$ on $\partial\Omega \cap \{u > 0\}$. Let $z_0 \in \partial\Omega \cap \{u > 0\}$ be a point close to $\{u_D = 0\}$ and let $R_2 := \operatorname{dist}(z_0, \{u_D = 0\})$. By the already proven part (the non-tangential touch) we know that $u > 0$ in $\Omega \cap B_{\tilde c R_2}(z_0)$ where $\tilde c \in (0, 1)$ is a constant depending on $n, \Omega, u_D$ and the chosen relative neighborhood of $\{u_D = 0\}$. Scaling $v(x) = \frac{u(z_0+\tilde c R_2 x)}{\tilde c R_2}$ and $\tilde\Omega = \frac{1}{\tilde c R_2}(\Omega - z_0)$ we infer from the above growth estimate that $v$ is a harmonic function in $B_1(0) \cap \tilde\Omega$ satisfying $|v| \le \frac{2}{\tilde c}$ in $B_1(0)$. Since we assumed $z_0$ to be close to $\{u_D = 0\}$, $\tilde\Omega$ is near $B_1(0)$ a domain close to a half-space and its smoothness is uniform in $z_0$. Thus we may apply local boundary regularity for harmonic functions (see for example [10, Corollary 6.7]) and obtain a bound for $\|\nabla v\|_{L^\infty(\tilde\Omega\cap B_{\frac12}(0))}$ depending only on $n, \Omega, u_D$ and the chosen relative neighborhood of $\{u_D = 0\}$. Scaling back, we obtain a uniform bound for $|\nabla u|$ at points of $\partial\Omega$ that are close to $\{u_D = 0\}$.
At points of $\partial\Omega$ that are relatively far from $\{u_D = 0\}$, the distance to the set $\{u = 0\}$ is estimated from below by a positive constant, and we combine the bound for $u$, i.e. $u \le \sup_{\partial\Omega} u_D$, with local boundary regularity for harmonic functions to derive a bound for $|\nabla u|$.

Thus $\nabla u$ is bounded on $\{u > 0\} \cap \partial\Omega$, and the fact that

$$\limsup_{x\to x_0\in\Omega\cap\partial\{u>0\}} |\nabla u(x)| \le 1$$

(see [1, Remark 6.4]) together with the maximum principle yields the Lipschitz continuity of $u$ on $\bar\Omega$. □

**Theorem 4.2.** *Let $n = 2, 3$, let $\Omega$ be a bounded domain of class $C^{2,\gamma}$ and let $u_D : \partial\Omega \to [0, +\infty)$ satisfy $u_D \in C^{2,\gamma}(\overline{\{u_D > 0\}})$, the non-degeneracy condition $\nabla_\theta u_D \ne 0$ on the boundary of $\{u_D > 0\}$ relative to $\partial\Omega$ as well as condition (4.1). Furthermore let us assume that the boundary data satisfy*

$$|\nabla_\theta u_D(x)\cdot(x - x_0) - \alpha u_D(x)| \le C|\nu(x)\cdot(x - x_0)|$$

*on $\partial\Omega \cap B_{r_0}(x_0)$ for some $C < \infty$ and every $x_0 \in \partial\Omega \cap \{u_D = 0\}$. Last, let $u$ be the minimal solution of the free boundary problem $\Delta u = 0$ in $\Omega \cap \{u > 0\}, |\nabla u| = 1$ on $\Omega \cap \partial\{u > 0\}$ and $u = u_D$ on $\partial\Omega$. Then $\partial\{u > 0\}$ is locally a $C^{1,\beta}$-surface on $\bar\Omega$ for some $\beta \in (0, 1)$, and $u$ satisfies the condition $|\nabla u| = 1$ on $\partial\{u > 0\} \cap \bar\Omega$.*

The following lemma stating uniqueness of the blow-up limit will be crucial in the proof of the theorem.

**Lemma 4.3.** *Let the assumptions of Theorem 4.2 be satisfied. Then, at each point $x_0$ of the boundary of $\{u_D = 0\}$ relative to $\partial\Omega$, $u(x_0 + rx)/r$ converges on each compact subset of $\{\nu(x_0)\cdot(x - x_0) < 0\}$ to exactly one of the two half-plane solutions $h_1$ and $h_2$ as $r \to 0$. Here*

$$h_1(x) = \max(x\cdot\nabla_\theta u_D(x_0) + x_n\sqrt{1 - |\nabla_\theta u_D(x_0)|^2}, 0)\,,$$
$$h_2(x) = \max(x\cdot\nabla_\theta u_D(x_0) - x_n\sqrt{1 - |\nabla_\theta u_D(x_0)|^2}, 0)\,.$$

*Proof.* Observe that as

$$\nabla g_j(x)\cdot\nu(x) = \frac{\alpha u_D(x) - \nabla_\theta u_D(x)\cdot(x-x_0)}{(x-x_0)\cdot\nu(x)} \quad \text{on } \partial D_j,$$

the assumption $|\nabla_\theta u_D(x)\cdot(x-x_0) - \alpha u_D(x)| \le C|\nu(x)\cdot(x-x_0)|$ ensures that the $g_j$-integrals in the monotonicity formula stay bounded as $r\to 0$. By a rotation we may take $\nu(x_0)$ to be $e_n$ where $e_n$ is the $n$-th unit vector in $\mathbb{R}^n$. By the assumptions and by (4.2), $u_r(x) = u(x_0+rx)/r$ converges on $\{x_n = 0\}$ to $\max(x\cdot w, 0)$ for some $w \in \mathbb{R}^{n-1}\cap\overline{B_1(0)} - \{0\}$. Each limit $u_0$ of $(u_r)_{r\in(0,1)}$ with respect to a sequence $r_k\to 0$ as $k\to\infty$ is harmonic in the open set $\{x_n > 0\}\cap\{u_0 > 0\}$, homogeneous of degree 1 (cf. Corollary 3.4) and a global minimizer of the energy $E_{B_1(0)}$ with respect to $u_0$-boundary values (cf. [1, Lemma 5.4]). In the two-dimensional case, it follows from the homogeneity and from the fact that $u_0$ is harmonic in the open set $\{u_0 > 0\}\cap\{x_n > 0\}$ that $u_0$ is linear in each connected component of $\{u_0 > 0\}\cap\{x_n > 0\}$ and must therefore be a half-plane solution $\max(x\cdot v, 0)$ satisfying $v\in\partial B_1(0)$ (cf. [15, Corollary 3.3]).
In the three-dimensional case we proceed as follows: according to Proposition 4.1, the free boundary $\partial\{u_0 > 0\}$ does not touch $\{x_n = 0\}\cap\{\nabla_\theta u_D(x_0)\cdot x < 0\}$. Moreover, by [15, Corollary 2.9], the set $\{u_0 = 0\}$ is the finite union of convex cones with vertex at the origin. But then the connected component of $\{u_0 = 0\}^0$ touching $\{x_n = 0\}$ must be the restriction of some half-space $\{x\cdot v < 0\}$ (where $v\in\partial B_1(0)$) to the set $\{x_n > 0\}$. Since $u_0$ satisfies on $\{x_n > 0\}\cap\partial\{x\cdot v > 0\}$ the boundary conditions $u_0 = 0$ and $\partial_v u_0 = -1$, we obtain from the unique solvability of the Cauchy problem that $u_0(x) = \max(x\cdot v, 0)$.

Furthermore, in the two- *and* three-dimensional case, $|v'|^2 + v_n^2 = 1$. Note that $v' = \nabla_\theta u_D(x_0)$. Thus $v_n = \sqrt{1-|\nabla_\theta u_D(x_0)|^2}$ or $v_n = -\sqrt{1-|\nabla_\theta u_D(x_0)|^2}$. Therefore $h_1$ and $h_2$ make up the whole $\omega$-limit set with respect to $r\to 0$, and we obtain uniqueness of the blow-up limit. □

*Proof of the Theorem.* Let us consider a point $x_1$ in the boundary of $\{u_D = 0\}$ relative to $\partial\Omega$. For small $\delta_1 > 0$ and each $x_0\in\Omega\cap B_{\delta_1}(x_1)\cap\partial\{u > 0\}$, we obtain from Lemma 4.3 that $u$ is in $B_r(x_0)\cap\Omega$ close to some half-plane solution $\max(x\cdot v(x_1), 0)$ (in particular, $u = 0$ in $B_r(x_0)\cap\{(x-x_0)\cdot v(x_1) < -\delta_2\}$ and $u > 0$ in $B_r(x_0)\cap\{(x-x_0)\cdot v(x_1) > \delta_2\}$). Thus interior regularity theory (see [1, Theorem 8.1]) implies that $\partial\{u > 0\}$ is in $B_{\frac{r}{4}}(x_0)$ the graph of a $C^{1,\beta}$-function in the direction of $-v(x_1)$ with a uniformly bounded $C^{1,\beta}$-norm. We obtain that for each point $x_2\in\overline{\partial\{u > 0\}}\cap\partial\Omega$ (note that this set coincides by Proposition 4.1 with the boundary of $\{u_D = 0\}$ relative to $\partial\Omega$), $\partial\{u > 0\}\cap\Omega$ is in an open neighborhood of $x_2$ the graph of a $C^{1,\beta}$-function in the direction of $-v(x_2)$.

Combining this with interior regularity results (we refer to [1, Theorem 8.3] for the two-dimensional case and to [5] for the three-dimensional case), this yields the statement of our theorem. □

## 5. Appendix

*Proof of the monotonicity formula:* For small positive $\kappa$,

$$\eta_\kappa(x) := \max(0, \min(1, \frac{1}{\kappa}(r-|x|))), \quad \xi_\kappa(x) := \min(1, \frac{1}{\kappa}\mathrm{dist}(x, \Omega^c))\,,$$

we take after approximation $\phi_\kappa(x) := \eta_\kappa(x)\xi_\kappa(x)x$ as test function in (3.3). We obtain

$$0 = I_1^\kappa + I_2^\kappa + I_3^\kappa\,, \tag{5.1}$$

where

$$I_1^\kappa = \int \eta_\kappa\xi_\kappa\Big(n|\nabla u|^2 - 2|\nabla u|^2 + n\lambda_+\chi_{\{u>0\}}u^p + n\lambda_-\chi_{\{u<0\}}(-u)^p\Big),$$

$$\begin{aligned} I_2^\kappa = \int \xi_\kappa\Big(&|\nabla u|^2\nabla\eta_\kappa\cdot x - 2\nabla u\cdot x\nabla u\cdot\nabla\eta_\kappa + \lambda_+\chi_{\{u>0\}}u^p\nabla\eta_\kappa\cdot x \\ &+ \lambda_-\chi_{\{u<0\}}(-u)^p\nabla\eta_\kappa\cdot x\Big), \end{aligned}$$

$$\begin{aligned} I_3^\kappa = \int \eta_\kappa\Big(&|\nabla u|^2\nabla\xi_\kappa\cdot x - 2\nabla u\cdot x\nabla u\cdot\nabla\xi_\kappa + \lambda_+\chi_{\{u>0\}}u^p\nabla\xi_\kappa\cdot x \\ &+ \lambda_-\chi_{\{u<0\}}(-u)^p\nabla\xi_\kappa\cdot x\Big)\,. \end{aligned}$$

The first integral

$$I_1^\kappa \to \int_{B_r(0)\cap\Omega}\Big[n\Big(|\nabla u|^2 + \lambda_+\chi_{\{u>0\}}u^p + \lambda_-\chi_{\{u<0\}}(-u)^p\Big) - 2|\nabla u|^2\Big]$$

as $\kappa\to 0$. The second integral $I_2^\kappa$ approaches

$$-\int_{\partial B_r(0)\cap\Omega}\Big[r\Big(|\nabla u|^2 + \lambda_+\chi_{\{u>0\}}u^p + \lambda_-\chi_{\{u<0\}}(-u)^p\Big) - 2r(\nabla u\cdot\nu)^2\Big]\,d\mathcal{H}^{n-1}$$

for a.e. $r\in(0,r_0)$ as $\kappa\to 0$. The third integral $I_3^\kappa$ approaches

$$-\int_{B_r(0)\cap\partial\Omega}\Big[\nu\cdot x\Big(|\nabla u|^2 + \lambda_+\chi_{\{u>0\}}u^p + \lambda_-\chi_{\{u<0\}}(-u)^p\Big) - 2\nabla u\cdot x\nabla u\cdot\nu\Big]\,d\mathcal{H}^{n-1}$$

as $\kappa\to 0$. Furthermore the fact that $\max(u,\theta)$ and $-\min(u,-\theta)$ are for small positive $\theta$ subsolutions satisfying

$$\begin{gathered} \Delta\max(u,\theta) - \frac{\lambda_+}{2}p\chi_{\{u>\theta\}}u^{p-1} \ge 0, \\ \Delta(-\min(u,-\theta)) - \frac{\lambda_-}{2}p\chi_{\{u<-\theta\}}(-u)^{p-1} \ge 0 \end{gathered}$$

implies that the distributions

$$\begin{gathered} \Delta\max(u,\theta) - \frac{\lambda_+}{2}p\chi_{\{u>\theta\}}u^{p-1}, \\ \Delta(-\min(u,-\theta)) - \frac{\lambda_-}{2}p\chi_{\{u<-\theta\}}(-u)^{p-1} \end{gathered}$$

are non-negative finite $\sigma$-additive measures with support in $\partial\{u>\theta\}$ and $\partial\{u<-\theta\}$, respectively. Since $\frac{\lambda_+}{2}p\chi_{\{u>\theta\}}u^{p-1} \to \frac{\lambda_+}{2}p\chi_{\{u>0\}}u^{p-1}$ in $L^1_{\rm loc}(B_r(0)\cap\Omega)$ as $\theta\to 0+$, we obtain that $\Delta\max(u,\theta) \rightharpoonup \Delta\max(u,0)$ weakly-* in $(C^0_0(B_r(0)\cap\Omega))^*$ as $\theta\to 0+$ and that

$$\operatorname{supp}(\Delta\max(u,0) - \frac{\lambda_+}{2}p\chi_{\{u>0\}}u^{p-1}) \subset \partial\{u>0\}\,. \tag{5.2}$$

Approximating $\max(u,0)$ by mollified functions we now see that

$$\int\nabla\max(u,0)\cdot\nabla\zeta = -\int\zeta\Delta\max(u,0)$$

for any $\zeta \in C^0(\overline{B_r(0)\cap\Omega}) \cap H_0^{1,2}(B_r(0)\cap\Omega)$. An analogous formula holds for $-\min(u,0)$. Using this and (5.2) one can now easily derive the formula

$$\begin{aligned}\int_{B_r(0)\cap\Omega} |\nabla u|^2 = \int_{\partial B_r(0)\cap\Omega} u\nabla u\cdot\nu d\mathcal{H}^{n-1} + \int_{B_r(0)\cap\partial\Omega} u\nabla u\cdot\nu d\mathcal{H}^{n-1}\\ - \frac{p}{2}\int_{B_r(0)\cap\Omega} \big(\lambda_+\chi_{\{u>0\}}u^p + \lambda_-\chi_{\{u<0\}}(-u)^p\big)\end{aligned} \tag{5.3}$$

for a.e. $r\in(0,r_0)$. Next, multiplying the limit identity of (5.1) by $-r^{-n-2(\alpha-1)-1}$ we get for a.e. $r\in(0,r_0)$

$$0 = \mathrm{Int}_1 + \mathrm{Bou}_1 + \mathrm{Int}_2 + \mathrm{Int}_3 + \mathrm{Bou}_2 + \mathrm{Bou}_1^\Omega\,, \tag{5.4}$$

where

$$\begin{aligned}\mathrm{Int}_1 &= -(n+2(\alpha-1))r^{-n-2(\alpha-1)-1}\\ &\quad\times\int_{B_r(0)\cap\Omega}\Big(|\nabla u|^2 + \lambda_+\chi_{\{u>0\}}u^p + \lambda_-\chi_{\{u<0\}}(-u)^p\Big)\,,\\ \mathrm{Int}_2 &= 2(\alpha-1)r^{-n-2(\alpha-1)-1}\int_{B_r(0)\cap\Omega}\Big(\lambda_+\chi_{\{u>0\}}u^p + \lambda_-\chi_{\{u<0\}}(-u)^p\Big)\,,\\ \mathrm{Int}_3 &= (2(\alpha-1)+2)r^{-n-2(\alpha-1)-1}\int_{B_r(0)\cap\Omega}|\nabla u|^2\,,\\ \mathrm{Bou}_1 &= r^{-n-2(\alpha-1)}\int_{\partial B_r(0)\cap\Omega}\Big(|\nabla u|^2 + \lambda_+\chi_{\{u>0\}}u^p + \lambda_-\chi_{\{u<0\}}(-u)^p\Big)d\mathcal{H}^{n-1}\,,\\ \mathrm{Bou}_2 &= -2r^{-n-2(\alpha-1)}\int_{\partial B_r(0)\cap\Omega}(\nabla u\cdot\nu)^2 d\mathcal{H}^{n-1}\,,\\ \mathrm{Bou}_1^\Omega &= r^{-n-2(\alpha-1)-1}\int_{B_r(0)\cap\partial\Omega}\Big[\nu\cdot x\Big(|\nabla u|^2 + \lambda_+\chi_{\{u>0\}}u^p\\ &\quad + \lambda_-\chi_{\{u<0\}}(-u)^p\Big) - 2\nabla u\cdot x\nabla u\cdot\nu\Big]\,d\mathcal{H}^{n-1}\,.\end{aligned}$$

By (5.3), identity (5.4) becomes

$$0 = \mathrm{Int}_1 + \mathrm{Bou}_1 + \mathrm{Int}_4 + \mathrm{Bou}_3 + \mathrm{Bou}_4 + \mathrm{Bou}_5 + \mathrm{Bou}_2^\Omega\,, \tag{5.5}$$

where

$$\begin{aligned}\mathrm{Int}_4 &= (2(\alpha-1) - \frac{p}{2}(2(\alpha-1)+2))r^{-n-2(\alpha-1)-1}\\ &\quad\times\int_{B_r(0)\cap\Omega}\Big(\lambda_+\chi_{\{u>0\}}u^p + \lambda_-\chi_{\{u<0\}}(-u)^p\Big) = 0\end{aligned}$$

by the definition of the value $\alpha$,

$$\begin{aligned}\mathrm{Bou}_3 &= -2r^{-n-2(\alpha-1)}\int_{\partial B_r(0)\cap\Omega}\Big(\nabla u\cdot\nu - \alpha\frac{u}{r}\Big)^2 d\mathcal{H}^{n-1},\\ \mathrm{Bou}_4 &= -2\alpha r^{-n-2(\alpha-1)-1}\int_{\partial B_r(0)\cap\Omega}u\nabla u\cdot\nu d\mathcal{H}^{n-1}\,,\\ \mathrm{Bou}_5 &= 2\alpha^2 r^{-n-2(\alpha-1)-2}\int_{\partial B_r(0)\cap\Omega}u^2 d\mathcal{H}^{n-1}\,,\\ \mathrm{Bou}_2^\Omega &= r^{-n-2(\alpha-1)-1}\int_{B_r(0)\cap\partial\Omega}\Big[\nu\cdot x\Big(|\nabla u|^2 + \lambda_+\chi_{\{u>0\}}u^p + \lambda_-\chi_{\{u<0\}}(-u)^p\Big)\end{aligned}$$

$$- 2(\nabla u\cdot x-\alpha u)\nabla u\cdot\nu\Big]\,d\mathcal{H}^{n-1}\,.$$

Note that $\mathrm{Bou}_3$ is the integrand on the right-hand side of monotonicity identity Theorem 3.3. Moreover,

$$\begin{aligned}
&\mathrm{Int}_1+\mathrm{Bou}_1+\mathrm{Bou}_4+\mathrm{Bou}_5\\
&=-\frac{\partial}{\partial r}\sum_{j=1}^{\infty}\sigma_j\alpha r^{1-n-2\alpha}\int_{\partial B_r(0)\cap D_j} g_j^2 d\mathcal{H}^{n-1}\\
&\quad+\frac{\partial}{\partial r}\big(r^{-n-2(\alpha-1)}\int_{B_r(0)\cap\Omega}\big(|\nabla u|^2+\lambda_+\chi_{\{u>0\}}u^p+\lambda_-\chi_{\{u<0\}}(-u)^p\big)\\
&\quad-\alpha r^{-n-2(\alpha-1)-1}\int_{\partial B_r(0)\cap\Omega}u^2 d\mathcal{H}^{n-1}\Big)
\end{aligned}$$

for a.e. $r\in(0,r_0)$. Consequently, for a.e. $r\in(0,r_0)$,

$$\begin{aligned}
&\Phi_0'(r)\\
&=-\mathrm{Bou}_3+r^{-n-2(\alpha-1)-1}\sum_{j=1}^{\infty}\sigma_j\int_{B_r(0)\cap\partial D_j}\Big[\nu\cdot x\Big(|\nabla u|^2-|\nabla g_j|^2+\lambda_+(\chi_{\{u>0\}}u^p\\
&\quad-\chi_{\{g_j>0\}}g_j^p)+\lambda_-(\chi_{\{u<0\}}(-u)^p-\chi_{\{g_j<0\}}(-g_j)^p)+|h_j-\nabla g_j|^2\Big)\\
&\quad-2(\nabla u\cdot x-\alpha u)\nabla u\cdot\nu\Big]\,d\mathcal{H}^{n-1}\\
&=-\mathrm{Bou}_3+r^{-n-2(\alpha-1)-1}\sum_{j=1}^{\infty}\sigma_j\int_{B_r(0)\cap\partial D_j}\Big[-2(\nabla u\cdot x-\alpha u)\nabla u\cdot\nu+\nu\cdot x|\nabla u|^2\\
&\quad+2(\nabla g_j\cdot x-\alpha g_j)\nabla g_j\cdot\nu-\nu\cdot x|\nabla g_j|^2+\nu\cdot x|h_j-\nabla g_j|^2\Big]\,d\mathcal{H}^{n-1}\,.
\end{aligned}$$

Since $\nabla u=\nabla g_j+\nabla(u-g_j)\cdot\nu\nu=:\nabla g_j+z_j\nu$ on $\partial D_j$, we obtain

$$\begin{aligned}
&\Phi_0'(r)\\
&=-\mathrm{Bou}_3+r^{-n-2(\alpha-1)-1}\sum_{j=1}^{\infty}\sigma_j\int_{B_r(0)\cap\partial D_j}\Big[2(\nabla g_j\cdot x-\alpha g_j)\nabla g_j\cdot\nu\\
&\quad-\nu\cdot x|\nabla g_j|^2-2(\nabla g_j\cdot x-\alpha g_j+z_j x\cdot\nu)(\nabla g_j\cdot\nu+z_j)+\nu\cdot x|\nabla g_j|^2\\
&\quad+z_j^2 x\cdot\nu+2z_j\nabla g_j\cdot\nu x\cdot\nu+\nu\cdot x|h_j-\nabla g_j|^2\Big]\,d\mathcal{H}^{n-1}\\
&=-\mathrm{Bou}_3+r^{-n-2(\alpha-1)-1}\sum_{j=1}^{\infty}\sigma_j\int_{B_r(0)\cap\partial D_j}\Big[-z_j^2 x\cdot\nu+\nu\cdot x|h_j-\nabla g_j|^2\Big]\,d\mathcal{H}^{n-1}\\
&=-\mathrm{Bou}_3
\end{aligned}$$

for a.e. $r\in(0,r_0)$.

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