Electronic Journal of Differential Equations, Vol. 2004(2004), No. 62, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

MODIFIED WAVE OPERATORS FOR NONLINEAR SCHRÖDINGER EQUATIONS IN ONE AND TWO DIMENSIONS

NAKAO HAYASHI, PAVEL I. NAUMKIN,
AKIHIRO SHIMOMURA, & SATOSHI TONEGAWA

*Dedicated to Professor ShigeToshi Kuroda on his 70th birthday
and to Professor Masaru Yamaguchi on his 60th birthday*

Abstract. We study the asymptotic behavior of solutions, in particular the scattering theory, for the nonlinear Schrödinger equations with cubic and quadratic nonlinearities in one or two space dimensions. The nonlinearities are summation of gauge invariant term and non-gauge invariant terms. The scattering problem of these equations belongs to the long range case. We prove the existence of the modified wave operators to those equations for small final data. Our result is an improvement of the previous work [13].

1. Introduction

In this paper, we study the existence global solutions and scattering theory for the nonlinear Schrödinger equations

$$\mathcal{L}u = \mathcal{N}_n(u) + \mathcal{G}_n(u), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \tag{1.1}$$

in one or two space dimensions $n = 1$ and 2, where $\mathcal{L} = i\partial_t + \frac{1}{2}\Delta$ and

$$\mathcal{N}_1(u) = \lambda_1 u^3 + \lambda_2 \overline{u}^2 u + \lambda_3 \overline{u}^3,$$
$$\mathcal{N}_2(u) = \lambda_1 u^2 + \lambda_2 \overline{u}^2,$$
$$\mathcal{G}_n(u) = \lambda_0 |u|^{\frac{2}{n}} u$$

with $\lambda_0 \in \mathbb{R}$ and $\lambda_j \in \mathbb{C}$, $j = 1, 2, 3$. We construct a modified wave operator in L^2 to equation (1.1) for small final data $\phi \in H^{0,2} \cap \dot{H}^{-\delta}$ with $\frac{n}{2} < \delta < 2$, where the weighted Sobolev space is defined by

$$H^{m,s} = \{u \in \mathcal{S}'; \|u\|_{H^{m,s}} = \| \langle i\nabla\rangle^m \langle x\rangle^s u\|_{L^2} < \infty\},$$

where $\langle x\rangle = \sqrt{1 + |x|^2}$ and the homogeneous Sobolev space is

$$\dot{H}^m = \left\{u \in \mathcal{S}'; \|u\|_{\dot{H}^m} = \|(-\Delta)^{\frac{m}{2}} u\|_{L^2} < \infty\right\}.$$

We intend to weaken the assumption $\phi \in \dot{H}^{-4}$ from the previous work [13].

2000 *Mathematics Subject Classification.* 35Q55, 35B40, 35B38.
Key words and phrases. Modified wave operators, nonlinear Schrödinger equations.

Submitted March 10, 2004. Published April 21, 2004.

Many works have been devoted to the global existence and asymptotic behavior of solutions for the nonlinear Schrödinger equations. We remind the definition of the wave operators in the scattering theory for the linear Schrödinger equation. Assume that for a solution $u_f(t,x)$ of the free Schrödinger equation $\mathcal{L}u_f = 0$ with given initial data $u_f(0,x) = \phi(x)$, there exists a unique global in time solution $u(t,x)$ of the perturbed Schrödinger equation such that $u(t,x)$ behaves like free solution $u_f(t,x)$ as $t \to \infty$ (this case is called by the short range case, otherwise it is called by the long range case). Then we define a wave operator $\mathcal{W}_+$ by the mapping from ϕ to $u|_{t=0}$. In the long range case, ordinary wave operators do not exist and we have to construct modified wave operators including a suitable phase correction in their definition. Analogously we can define the wave operators and introduce the modified wave operators for the nonlinear Schrödinger equation.

We first recall several known results concerning the scattering problem for the nonlinear Schrödinger equation

$$\mathcal{L}u = \lambda |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n \tag{1.2}$$

with $\lambda \in \mathbb{R}$ and $p > 1$. We consider the existence of wave operators $W_\pm$ for equation (1.2). The wave operator W_+ is defined for equation (1.2) as follows. Let Σ be L^2 or a dense subset of it. Let $\phi \in \Sigma$, and define the free solution

$$u_f(t) = \mathcal{U}(t)\phi,$$

where

$$\mathcal{U}(t) \equiv e^{\frac{it}{2}\Delta}.$$

Note that u_f is the solution to the Cauchy problem of the free Schrödinger equation

$$\mathcal{L}u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n,$$
$$u(0,x) = \phi(x), \quad x \in \mathbb{R}^n.$$

If there exists a unique global solution u of equation (1.2) such that

$$\|u(t) - u_f(t)\|_{L^2} \to 0,$$

as $t \to +\infty$, then a mapping

$$\mathcal{W}_+ : \phi \mapsto u(0)$$

is well-defined on Σ. We call the mapping W_+ by the wave operator. The function ϕ is called by a final state, final data, a scattered state or scattered data. It is known that, when $p > 1 + \frac{2}{n}$ and $n \le 3$, there exist the wave operators $\mathcal{W}_\pm$ on a suitable weighted Sobolev space (see [3]). In the case of $n \ge 4$, the existence of wave operators is proved if $p > \frac{1}{4}(\sqrt{n^2+4n+36} - n + 2)$ in [3] and if $p = \frac{1}{4}(\sqrt{n^2+4n+36} - n + 2)$ in [11]. (Note that $1 + \frac{2}{n} < \frac{1}{4}(\sqrt{n^2+4n+36} - n + 2)$ if $n \ge 4$, so for the case $n \ge 4$ and $1 + \frac{2}{n} < p < \frac{1}{4}(\sqrt{n^2+4n+36} - n + 2)$ the problem is open). On the other hand, when $1 \le p \le 1 + \frac{2}{n}$, non-trivial solutions of equation (1.2) does not have a free profile in L^2, that is, we cannot define the wave operators on L^2 (see, e.g., [1]). Intuitive meaning of these facts is as follows. Recalling the well-known time decay estimates $\|u_f(t)\|_{L^2} = \|\phi\|_{L^2} = O(1)$, and $\|u_f(t)\|_{L^\infty} = O(t^{-\frac{n}{2}})$, we see that $\||u_f(t)|^p\|_{L^2} = O(t^{-\frac{n}{2}(p-1)})$. Roughly speaking, according to the linear scattering theory (the Cook-Kuroda method), wave operators exist if and only if $\||u_f(t)|^p\|_{L^2}$ is integrable with respect to t over the interval $[1,\infty)$, that is, $p > 1 + \frac{2}{n}$.

There are several results concerning the long range scattering for equation (1.2) in the critical case $p = 1 + \frac{2}{n}$. In the long range case, as we already mentioned, the usual wave operators do not exist, so we introduce the modified wave operators $\widetilde{\mathcal{W}}_+$ as follows. We construct a suitable modified free profile $A_+(t)$, and consider a unique solution $u(t)$ of equation (1.2) which approaches $A_+(t)$ in L^2 as $t \to \infty$:

$$\|u(t) - A_+(t)\|_{L^2} \to 0, \quad t \to \infty.$$

Then the mapping

$$\widetilde{\mathcal{W}}_+ : A_+(0) \mapsto u(0)$$

is called the modified wave operator. Ozawa [12] and Ginibre and Ozawa [2] proved the existence of modified wave operators for small final data in one space dimension and in two and three space dimensions, respectively, by the phase correction method. More precisely, they put a modified free profile of the form $A_+(t) = \mathcal{U}(t)e^{-iS(t,-i\nabla)}\phi$, where ϕ is a final state, and chose the phase function S such that $\|\mathcal{L}A_+(t) - |A(t)|^{\frac{2}{n}}A(t)\|_{L^2}$ decays faster than $\||\mathcal{U}(t)\phi|^{\frac{2}{n}}\mathcal{U}(t)\phi\|_{L^2} = O(t^{-1})$. Recently, Ginibre and Velo [4] have partially extended above results removing the size restrictions of the final data in the case of the nonlinearity $a(t)|u|^2u$. where $a(t)$ has a suitable growth rate with respect to t. The large time asymptotic behavior of solutions to the initial value problem for equation (1.2) with $1 \le n \le 3$ was studied and the asymptotic completeness of the wave operator was partially shown in [6]. The phase correction method is applicable only for the gauge invariant nonlinearities, like $\lambda|u|^{p-1}u$, where $\lambda \in \mathbb{R}$, because we can regard $|u|^{p-1}$ as a time dependent long range potential. We cannot apply the phase correction method to non-gauge invariant nonlinearities of the form u^p or $|u|^{p-1}u + u^p$, because we should consider the non-gauge invariant nonlinearity as a time dependent external force.

There are some results on the scattering theory for equation (1.1) in one or two space dimensions. In [10] it was shown the existence of the wave operator for equation (1.1) with $\mathcal{G}_n(u) = 0$ by using the method by Hörmander [8], where he studied the life span of solutions of nonlinear Klein-Gordon equations and in [13] it was constructed the modified wave operator for equation (1.1) by combining the methods in [8] and [12]. More precisely, the following two propositions were obtained in [13]:

Proposition 1.1. *Let $n = 1$, $\phi \in H^{0,3} \cap \dot{H}^{-4}$ and $\|\phi\|_{H^{0,3}} + \|\phi\|_{\dot{H}^{-4}}$ be sufficiently small. Then there exists a unique global solution u of (1.1) such that $u \in C(\mathbb{R}^+; L^2)$,*

$$\sup_{t\ge1} t^b\|u(t) - u_p(t)\|_{L^2} + \sup_{t\ge1} t^b\Big(\int_t^\infty \|u(\tau) - u_p(\tau)\|_{L^\infty}^4\, d\tau\Big)^{1/4} < \infty,$$

where $\frac{1}{2} < b < 1$, and

$$u_p(t) = \frac{1}{(it)^{\frac{n}{2}}} e^{\frac{ix^2}{2t}} \widehat{\phi}(\frac{x}{t}) \exp\big(-i\lambda_0|\widehat{\phi}(\frac{x}{t})|^{\frac{2}{n}} \log t\big).$$

Proposition 1.2. *Let $n = 2$, $\phi \in H^{0,4} \cap \dot{H}^{-4}$, $x\phi \in \dot{H}^{-2}$ and $\|\phi\|_{H^{0,4}} + \|\phi\|_{\dot{H}^{-4}} + \|x\phi\|_{\dot{H}^{-2}}$ be sufficiently small. Then there exists a unique global solution u of equation (1.1) such that $u \in C(\mathbb{R}^+; L^2)$,*

$$\sup_{t\ge1} t^b\|u(t) - u_p(t)\|_{L^2} + \sup_{t\ge1} t^b\Big(\int_t^\infty \|u(\tau) - u_p(\tau)\|_{L^4}^4\, d\tau\Big)^{1/4} < \infty,$$

where $\frac{1}{2} < b < 1$.

Throughout this paper, we denote the norm of a Banach space $\mathbf{Z}$ by $\|\cdot\|_{\mathbf{Z}}$. Our purpose in this paper is to improve the condition on a final data $\phi \in \dot{H}^{-4}$. In order to explain the reason why the previous proof by [10] and [8] requires such a condition, we give briefly the idea of paper [13] on the example of the Cauchy problem

$$\mathcal{L}u = u^2, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2. \tag{1.3}$$

If a solution u of (1.3) behaves like a free solution $\mathcal{U}(t)\phi$ as $t \longrightarrow \infty$ for a given ϕ, then. $u_0(t,x) = \frac{1}{(it)^{\frac{n}{2}}} e^{\frac{ix^2}{2t}} \widehat{\phi}(\frac{x}{t})$ can be considered as an approximate solution of (1.3) since

$$\mathcal{U}(t)\phi = \frac{1}{it} e^{\frac{ix^2}{2t}} \widehat{\phi}\big(\frac{x}{t}\big) + O\big(t^{-1-\alpha} \big\||x|^{2\alpha}\phi\big\|_{L^1}\big).$$

By a direct calculation we find that $\mathcal{L}(u-u_0) = u^2 - \frac{1}{2it^3} e^{\frac{ix^2}{2t}} \widehat{|\cdot|^2\phi}(\eta)$. with $\eta = \frac{x}{t}$. The last term of the right-hand side of the above equation is a remainder term which we denote by R. Hence the problem becomes

$$\mathcal{L}(u-u_0) = u^2 - u_0^2 + u_0^2 + R. \tag{1.4}$$

We find a solution in the neighborhood of u_0. however u_0^2 can not be considered as a remainder term since $\|u_0^2\|_{L^2} = t^{-1}\|\widehat{\phi}^2\|_{L^2}$. In order to cancel u_0^2 we try to find u_r such that $\mathcal{L}u_r - u_0^2$ is a remainder term. We put $u_r = t^{-b}P(\frac{x}{t})e^{\frac{iax^2}{2t}}$ to get $\mathcal{L}u_r = t^{-b}\frac{a(1-a)}{2}\frac{x^2}{t^2}P\big(\frac{x}{t}\big)e^{\frac{iax^2}{2t}} + R_1$ which implies that we should take $P(\eta) = \frac{2}{a(a-1)}\frac{1}{\eta^2}\widehat{\phi}(\eta)^2$ and $a = b = 2$ to cancel u_0^2 in the right-hand side of (1.4) and we note that R_1 contains a term like $t^{-4}e^{\frac{ix^2}{t}}\frac{1}{\eta^4}\widehat{\phi}(\eta)^2$. Thus we get

$$\mathcal{L}(u-u_0-u_r) = u^2 - u_0^2 + R + R_1.$$

This is the reason why we require a vanishing condition of $\widehat{\phi}(\eta)$ at the origin.

Our main result in the present paper is the following.

Theorem 1.3. *Let $\phi \in H^{0,2} \cap \dot{H}^{-\delta}$ and $\|\phi\|_{H^{0,2}} + \|\phi\|_{\dot{H}^{-\delta}}$ be sufficiently small, where $\frac{n}{2} < \delta < 2$. Then there exists a unique global solution u of (1.1) such that $u \in C(\mathbb{R}^+; L^2)$,*

$$\sup_{t\geq 1} t^{\frac{\delta}{2}}\|u(t)-u_p(t)\|_{L^2} + \sup_{t\geq 1} t^{\frac{\delta}{2}}\Big(\int_t^\infty \|u(\tau)-u_p(\tau)\|_{X_n}^4\,d\tau\Big)^{1/4} < \infty$$

where $X_1 = L^\infty, X_2 = L^4$,

$$u_p(t) = \frac{1}{(it)^{\frac{n}{2}}} e^{\frac{ix^2}{2t}} \widehat{\phi}\big(\frac{x}{t}\big)\exp\big(-i\lambda_0\big|\widehat{\phi}\big(\frac{x}{t}\big)\big|^{\frac{2}{n}}\log t\big).$$

Furthermore the modified wave operator

$$\widetilde{\mathcal{W}}_+ : \phi \mapsto u(0)$$

is well-defined.

Similar result holds for the negative time.

Remark 1.4. If we consider the asymptotic behavior of solutions to the Cauchy problem for equation (1.1) with initial data $u(0,x) = \phi_0(x)$, $x \in \mathbb{R}^n$, then we see from Theorem 1.3 that for any initial data ϕ_0 belonging to the range of the modified wave operator $\widetilde{\mathcal{W}}_+$, there exists a unique global solution $u \in C(\mathbb{R}^+; L^2)$ of the Cauchy problem for equation (1.1) which has a modified free profile u_p. More precisely, u satisfies the asymptotic formula of Theorem 1.3. However it is not clear how to describe the initial data beloging to the range of the operator $\widetilde{\mathcal{W}}_+$.

Remark 1.5. If $\phi \in H^{0,2}$ and $\widehat{\phi}(0) = 0$, then $\phi \in H^{0,2} \cap \dot{H}^{-\alpha}$ for $0 \le \alpha < 1 + \frac{n}{2}$ with $n = 1, 2$. This follows from the fact that $\dot{H}^0 = L^2 \supset H^{0,2}$ and the following inequalities:

(a) $\| |\cdot|^{-\alpha} f\|_{L^2} \le C \| |\cdot|^{-\alpha+1} \nabla f\|_{L^2}$ for $\alpha > \frac{n+1}{2}$, provided that $f(0) = 0$,
(b) $\| |\cdot|^{-\alpha+1} f\|_{L^2} \le C\|f\|_{H^{1,0}}$ for $1 < \alpha < 1 + \frac{n}{2}$ with $n = 1, 2$.

Note that this implies that $\int \phi(x)dx = 0$ and $\phi \in H^{0,2}$, then $\phi \in H^{0,2} \cap \dot{H}^{-\alpha}$.
Proof of (a): From the equality

$$f(\xi) = f(\xi) - f(0) = \int_0^1 \frac{d}{dt} f(t\xi)dt = \int_0^1 \xi \cdot \nabla f(t\xi)dt.$$

and Schwarz' inequality, it follows that

$$|f(\xi)|^2 \le |\xi|^2 \int_0^1 |\nabla f(t\xi)|^2 dt.$$

Therefore, we have

$$\begin{aligned} \| |\cdot|^{-\alpha} f\|_{L^2}^2 &= \int \frac{1}{|\xi|^{2\alpha}} |f(\xi)|^2 d\xi \le \int \frac{1}{|\xi|^{2\alpha-2}} \int_0^1 |\nabla f(t\xi)|^2 dt d\xi \\ &= \int_0^1 \int \frac{1}{|\xi|^{2\alpha-2}} |\nabla f(t\xi)|^2 d\xi dt = \int_0^1 \int \frac{t^{2\alpha-2}}{|\eta|^{2\alpha-2}} |\nabla f(\eta)|^2 \frac{d\eta}{t^n} dt \\ &= \frac{1}{2\alpha - 1 - n} \| |\cdot|^{-\alpha+1} \nabla f\|_{L^2}^2 \end{aligned}$$

for $\alpha > \frac{n+1}{2}$.
Proof of (b) : We split the norm on the left hand side as follows:

$$\| |\cdot|^{-\alpha+1} f\|_{L^2} \le \| |\cdot|^{-\alpha+1} f\|_{L^2(|\cdot| \ge 1)} + \| |\cdot|^{-\alpha+1} f\|_{L^2(|\cdot|<1)} = I_1 + I_2.$$

Since $\alpha \ge 1$, it is easily seen that $I_1 \le \|f\|_{L^2}$. By the Hölder inequality, we have

$$I_2 \le \| |\cdot|^{-\alpha+1}\|_{L^p(|\cdot|<1)} \|f\|_{L^q(|\cdot|<1)},$$

where $2 \le p, q \le \infty$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Here, we put $(p, q) = (2, \infty)$ for $n = 1$ and $(p, q) = \left(\frac{\alpha}{\alpha-1}, \frac{2\alpha}{2-\alpha}\right)$ for $n = 2$ so that we have $\| |\cdot|^{-\alpha+1}\|_{L^p(|\cdot|<1)} < \infty$ and $\|f\|_{L^q(|\cdot|<1)} \le \|f\|_{L^q} \le C\|f\|_{H^{1,0}}$ by the Sobolev embedding.

Remark 1.6. In the previous paper [7], we considered the Cauchy problem for the cubic nonlinear Schrödinger equation

$$\begin{gathered} iu_t + \frac{1}{2} u_{xx} = \mathcal{N}(u), \quad x \in \mathbb{R},\ t > 1 \\ u(1, x) = u_1(x), \quad x \in \mathbb{R}, \end{gathered}$$

where $\mathcal{N}(u) = \lambda_1 u^3 + \lambda_2 \overline{u}^2 u + \lambda_3 \overline{u}^3$. $\lambda_j \in \mathbb{C}$. $j = 1, 2, 3$. It was shown that there exists a global small solution $u \in C([1, \infty), L^\infty)$, if the initial data u_1 belong to some analytic function space and are sufficiently small. For the coefficients λ_j it was assumed that there exists $\theta_0 > 0$ such that

$$\operatorname{Re}\big(\frac{\lambda_1}{\sqrt{3}}e^{2ir} - i\lambda_2 e^{-2ir} + \frac{\lambda_3}{\sqrt{3}}e^{-4ir}\big) \geq C > 0,$$
$$\operatorname{Im}\big(\frac{\lambda_1}{\sqrt{3}}e^{2ir} - i\lambda_2 e^{-2ir} + \frac{\lambda_3}{\sqrt{3}}e^{-4ir}\big)r \geq Cr^2,$$

for all $|r| < \theta_0$. and also it was assumed that the initial data $u_1(x)$ are such that

$$\big|\arg e^{-\frac{i}{2}\xi^2}\widehat{u_1}(\xi)\big| < \theta_0, \quad \inf_{|\xi|\leq 1}|\widehat{u_1}(\xi)| \geq C\varepsilon,$$

where ε is a small positive constant depending on the size of the initial data in a suitable norm. Moreover it was shown that there exist unique final states $\mathcal{W}_+, r_+ \in L^\infty$ and $0 < \gamma < 1/20$ such that the asymptotic statement

$$u(t,x) = \frac{(it)^{-\frac{1}{2}}W_+(\frac{x}{t})e^{\frac{ix^2}{2t}}}{\sqrt{1+\chi\big(\frac{x}{t}\big)|W_+\big(\frac{x}{t}\big)|^2\log\frac{t^2}{t+x^2}}} + O\Big(t^{-\frac{1}{2}}\big(1+\log\frac{t^2}{t+x^2}\big)^{-\frac{1}{2}-\gamma}\Big)$$

is valid for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$, where $\gamma > 0$ and $\chi(\xi)$ is given by

$$\chi(\xi) = \operatorname{Re}\Big(\frac{\lambda_1}{\sqrt{3}}\exp(2ir_+(\xi)) - i\lambda_2\exp(-2ir_+(\xi)) + \frac{\lambda_3}{\sqrt{3}}\exp(-4ir_+(\xi))\Big).$$

This asymptotic formula shows that, in the short range region $|x| < \sqrt{t}$. the solution has an additional logarithmic time decay comparing with the corresponding linear case. Thus we can see that the vanishing condition at the origin on the Fourier transform of the final data seems to be essential for our result in the present paper.

For the convenience of the reader we now state the strategy of the proof. We consider the linearized version of equation (1.1)

$$\mathcal{L}u = \mathcal{N}_n(v) + \mathcal{G}_n(v), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^n.$$

We take

$$u_0(t,x) = \frac{1}{(it)^{\frac{n}{2}}}e^{\frac{ix^2}{2t}}\widehat{\phi}\big(\frac{x}{t}\big)\exp\big(-i\lambda_0|\widehat{\phi}\big(\frac{x}{t}\big)|^{\frac{2}{n}}\log t\big)$$

as the first approximation for solutions to (1.1). By a direct calculation we get

$$\mathcal{L}u_0 = \mathcal{G}_n(u_0) + R_1(t),$$

where $R_1(t)$ is a remainder term. Hence

$$\mathcal{L}(u-u_0) = \mathcal{N}_n(v) + \mathcal{G}_n(v) - \mathcal{G}_n(u_0) + R_1.$$

We define the second approximation u_1 for solutions of (1.1) as

$$u_1(t) = -i\int_\infty^t \mathcal{U}(t-\tau)\mathcal{N}_n(u_0)\,d\tau$$

which implies that

$$\mathcal{L}u_1 = \mathcal{N}_n(u_0)$$

and

$$\begin{aligned} u(t) - u_0(t) &= -i\int_\infty^t \mathcal{U}(t-\tau)(\mathcal{N}_n(v) - \mathcal{N}_n(u_0) + \mathcal{G}_n(v) - \mathcal{G}_n(u_0))\,d\tau \\ &\quad - i\int_\infty^t \mathcal{U}(t-\tau)R_1(\tau)\,d\tau + u_1(t). \end{aligned}$$

We define the function space

$$X = \{f \in C([T,\infty); \mathbf{L}^2); \|f\|_X < \infty\}$$

$$\|f\|_X = \sup_{t\in[T,\infty)} t^b \|f(t) - u_0(t)\|_{L^2} + \sup_{t\in[T,\infty)} t^b (\int_t^\infty \|f(t) - u_0(t)\|^4_{X_n} \,dt)^{1/4},$$

where

$$X_1 = L^\infty,\ X_2 = L^4,\ b > \frac{n}{4}.$$

In order to get the result we need to prove the following estimate for $u_1(t)$,

$$\|u_1(t)\| + (\int_t^\infty \|u_1(\tau)\|^4_{X_n}\,d\tau)^{1/4} \le C(\||\cdot|^{-\widetilde{\delta}}\widehat{\phi}\| + \|\phi\|_{H^{0,2}})^{1+\frac{2}{n}} t^{-\widetilde{\delta}/2},$$

for $n/2 < \widetilde{\delta} < 2$. which is the main estimate of the present paper. Note that the choice of u_1 differs from that used in the previous papers.

2. Preliminaries

Lemma 2.1. *We have for $\omega \ne 1$. $f, g \in L^1 \cap L^2$ and $h \in C^2$,*

$$\begin{aligned}
&\int_\infty^t h(i\tau)\mathcal{U}(t-\tau)\Delta(e^{\frac{i\omega x^2}{2\tau}} e^{ig(\frac{x}{\tau})\log\tau} f(\frac{x}{\tau}))\,d\tau \\
&= -\frac{2i\omega}{1-\omega} h(it) e^{\frac{i\omega x^2}{2t}} e^{ig\left(\frac{x}{t}\right)\log t} f\left(\frac{x}{t}\right) \\
&\quad - \frac{2\omega}{(1-\omega)^2}\int_\infty^t \Big(\sum_{(F,k)} F(i\tau) e^{\frac{i\omega x^2}{2\tau}} e^{ig(\frac{x}{\tau})\log\tau} k(\frac{x}{\tau}) \\
&\quad - i\omega\mathcal{U}(t-\tau)\int_\infty^\tau \sum_{(F,k)} F'(is) e^{\frac{i\omega x^2}{2s}} e^{ig(\frac{x}{s})\log s} k(\frac{x}{s})\,ds \\
&- i\omega\mathcal{U}(t-\tau)\int_\infty^\tau \sum_{(F,k)} F(is) e^{\frac{i\omega x^2}{2s}} e^{ig(\frac{x}{s})\log s} \frac{1}{s} k(g - \frac{in}{2})(\frac{x}{s})\,ds\Big)\,d\tau + R(t),
\end{aligned}$$

where the summation is taken over $(F,k) = (h', f), (h\tau^{-1}, f(g - in/2))$,

$$\begin{aligned}
R(t) = &-\frac{i\omega}{(1-\omega)^2}\int_\infty^t \mathcal{U}(t-\tau)\int_\infty^\tau \sum_{(F,k)} F(is) R_{0,k}(s)\,ds\,d\tau \\
&+ \frac{1}{1-\omega}\int_\infty^t h(i\tau)\mathcal{U}(t-\tau) R_{0,f}(\tau)\,d\tau,
\end{aligned}$$

and

$$\begin{aligned}
R_{0,k}(t) = &\, e^{\frac{i\omega x^2}{2t}} k(\frac{x}{t}) \Delta e^{ig\left(\frac{x}{t}\right)\log t} + 2i\frac{1}{t^2}\sum \partial_j g(\frac{x}{t}) \partial_j k(\frac{x}{t}) e^{\frac{i\omega x^2}{2t}} e^{ig(\frac{x}{t})\log t}\log t \\
&+ \frac{1}{t^2}(\Delta k)(\frac{x}{t}) e^{\frac{i\omega x^2}{2t}} e^{ig\left(\frac{x}{t}\right)\log t}.
\end{aligned}$$

Proof. By a direct computation we find that

$$(2i\omega\partial_t + \Delta) e^{\frac{i\omega x^2}{2t}} e^{ig\left(\frac{x}{t}\right)\log t} f\left(\frac{x}{t}\right) = -2\omega\frac{1}{t} f(g - \frac{id}{2})(\frac{x}{t}) e^{\frac{i\omega x^2}{2t}} e^{ig\left(\frac{x}{t}\right)\log t} + R_{0,f}(t),$$

where

$$\begin{aligned}R_{0,f}(t) &= e^{\frac{i\omega x^2}{2t}} f(\frac{x}{t})\Delta e^{ig\left(\frac{x}{t}\right)\log t} + 2i\frac{1}{t^2}\sum(\partial_j g\cdot\partial_j f)(\frac{x}{t})e^{\frac{i\omega x^2}{2t}} e^{ig(\frac{x}{t})\log t}\log t\\ &\quad + \frac{1}{t^2}(\Delta f)(\frac{x}{t})e^{\frac{i\omega x^2}{2t}} e^{ig\left(\frac{x}{t}\right)\log t}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathcal{U}(-t)\Delta(e^{\frac{i\omega x^2}{2t}} e^{ig\left(\frac{x}{t}\right)\log t} f(\frac{x}{t}))\\ &= -\partial_t(\mathcal{U}(-t)2i\omega(e^{\frac{i\omega x^2}{2t}} e^{ig\left(\frac{x}{t}\right)\log t} f(\frac{x}{t}))) + \omega\mathcal{U}(-t)\Delta(e^{\frac{i\omega x^2}{2t}} e^{ig\left(\frac{x}{t}\right)\log t} f(\frac{x}{t}))\\ &\quad + \mathcal{U}(-t)(-2\omega\frac{1}{t}f(g-\frac{in}{2})(\frac{x}{t})e^{\frac{i\omega x^2}{2t}} e^{ig\left(\frac{x}{t}\right)\log t} + R_{0,f}(t))\end{aligned}$$

from which it follows that

$$\begin{aligned}&\mathcal{U}(-t)\Delta(e^{\frac{i\omega x^2}{2t}} e^{ig\left(\frac{x}{t}\right)\log t} f(\frac{x}{t}))\\ &= -\frac{2i\omega}{1-\omega}\partial_t(\mathcal{U}(-t)e^{\frac{i\omega x^2}{2t}} e^{ig\left(\frac{x}{t}\right)\log t} f(\frac{x}{t}))\\ &\quad - \frac{2\omega}{1-\omega}\mathcal{U}(-t)\Big(\frac{1}{t}f(g-\frac{in}{2})(\frac{x}{t})e^{\frac{i\omega x^2}{2t}} e^{ig\left(\frac{x}{t}\right)\log t}\Big)\\ &\quad + \frac{1}{1-\omega}\mathcal{U}(-t)R_{0,f}(t).\end{aligned} \tag{2.1}$$

Hence

$$\begin{aligned}&\int_\infty^t h(i\tau)\mathcal{U}(t-\tau)\Delta(e^{\frac{i\omega x^2}{2\tau}} e^{ig(\frac{x}{\tau})\log\tau} f(\frac{x}{\tau}))\,d\tau\\ &= -\frac{2i\omega}{1-\omega}\mathcal{U}(t)\int_\infty^t h(i\tau)\partial_\tau(\mathcal{U}(-\tau)e^{\frac{i\omega x^2}{2\tau}} e^{ig(\frac{x}{\tau})\log\tau} f(\frac{x}{\tau}))\,d\tau\\ &\quad - \frac{2\omega}{1-\omega}\int_\infty^t h(i\tau)\mathcal{U}(t-\tau)\frac{1}{\tau}f(g-\frac{in}{2})(\frac{x}{\tau})e^{\frac{i\omega x^2}{2\tau}} e^{ig(\frac{x}{\tau})\log\tau}\,d\tau\\ &\quad + R_{1,f}(t)\\ &= -\frac{2i\omega}{1-\omega}h(it)e^{\frac{i\omega x^2}{2t}} e^{ig\left(\frac{x}{t}\right)\log t} f(\frac{x}{t})\\ &\quad - \frac{2\omega}{1-\omega}\int_\infty^t h'(i\tau)\mathcal{U}(t-\tau)e^{\frac{i\omega x^2}{2\tau}} e^{ig(\frac{x}{\tau})\log\tau} f(\frac{x}{\tau})\,d\tau\\ &\quad - \frac{2\omega}{1-\omega}\int_\infty^t h(i\tau)\mathcal{U}(t-\tau)\frac{1}{\tau}f(g-\frac{in}{2})(\frac{x}{\tau})e^{\frac{i\omega x^2}{2\tau}} e^{ig(\frac{x}{\tau})\log\tau}\,d\tau\\ &\quad + R_{1,f}(t),\end{aligned} \tag{2.2}$$

where

$$R_{1,f}(t) = \frac{1}{1-\omega}\int_\infty^t h(i\tau)\mathcal{U}(t-\tau)R_{0,f}(\tau)\,d\tau.$$

We write

$$\begin{aligned}
&F(i\tau)\mathcal{U}(-\tau)e^{\frac{i\omega x^2}{2\tau}}e^{ig(\frac{x}{\tau})\log\tau}k(\frac{x}{\tau})\\
&=\partial_\tau(\mathcal{U}(-\tau)\int_\infty^\tau F(is)e^{\frac{i\omega x^2}{2s}}e^{ig(\frac{x}{s})\log s}k(\frac{x}{s})\,ds)\\
&\quad+\frac{i}{2}\mathcal{U}(-\tau)\int_\infty^\tau F(is)\Delta(e^{\frac{i\omega x^2}{2s}}e^{ig(\frac{x}{s})\log s}k(\frac{x}{s}))\,ds\\
&=\partial_\tau(\mathcal{U}(-\tau)\int_\infty^\tau F(is)e^{\frac{i\omega x^2}{2s}}e^{ig(\frac{x}{s})\log s}k(\frac{x}{s})\,ds)\\
&\quad+\omega F(i\tau)\mathcal{U}(-\tau)e^{\frac{i\omega x^2}{2\tau}}e^{ig(\frac{x}{\tau})\log\tau}k(\frac{x}{\tau})\\
&\quad-\omega\mathcal{U}(-\tau)\int_\infty^\tau iF'(is)e^{\frac{i\omega x^2}{2s}}e^{ig(\frac{x}{s})\log s}k(\frac{x}{s})\,ds\\
&\quad-i\omega\mathcal{U}(-\tau)\int_\infty^\tau F(is)e^{\frac{i\omega x^2}{2s}}e^{ig(\frac{x}{s})\log s}\frac{1}{s}k(g-\frac{in}{2})(\frac{x}{s})\,ds\\
&\quad+\frac{i}{2}\mathcal{U}(-\tau)\int_\infty^\tau F(is)R_{0,k}(s)\,ds
\end{aligned}$$

hence

$$\begin{aligned}
&(1-\omega)F(i\tau)\mathcal{U}(-\tau)e^{\frac{i\omega x^2}{2\tau}}e^{ig(\frac{x}{\tau})\log\tau}k(\frac{x}{\tau})\\
&=\partial_\tau(\mathcal{U}(-\tau)\int_\infty^\tau F(is)e^{\frac{i\omega x^2}{2s}}e^{ig(\frac{x}{s})\log s}k(\frac{x}{s})\,ds)\\
&\quad-\omega\mathcal{U}(-\tau)\int_\infty^\tau iF'(is)e^{\frac{i\omega x^2}{2s}}e^{ig(\frac{x}{s})\log s}k(\frac{x}{s})\,ds\\
&\quad-i\omega\mathcal{U}(-\tau)\int_\infty^\tau F(is)e^{\frac{i\omega x^2}{2s}}e^{ig(\frac{x}{s})\log s}\frac{1}{s}k(g-\frac{in}{2})(\frac{x}{s})\,ds\\
&\quad+\frac{i}{2}\mathcal{U}(-\tau)\int_\infty^\tau F(is)R_{0,k}(s)\,ds.
\end{aligned}\tag{2.3}$$

We apply (2.3) with $(F,k)=(h',f)$ or $(F,k)=(h\tau^{-1},f(g-in/2))$ to the right-hand side of (2.1) to get the desired result. □

In the next lemma we state the Strichartz estimate for $\int_s^t\mathcal{U}(t-\tau)f(\tau)\,d\tau$ obtained by Yajima [14].

Lemma 2.2. *For any pairs (q,r) and (q',r') such that $0\le\frac{2}{q}=\frac{n}{2}-\frac{n}{r}<1$ and $0\le\frac{2}{q'}=\frac{n}{2}-\frac{n}{r'}<1$. for any (possibly unbounded) interval I and for any $s\in\overline{I}$ the Strichartz estimate*

$$(\int_I\Big\|\int_s^t\mathcal{U}(t-\tau)f(\tau)\,d\tau\Big\|_{L^r}^q\,dt)^{\frac{1}{q}}\le C(\int_I\|f(t)\|_{L^{\overline{r}'}}^{\overline{q}'}\,dt)^{\frac{1}{\overline{q}'}},$$

is true with a constant C independent of I and s, where $\frac{1}{r}+\frac{1}{\overline{r}}=1$ and $\frac{1}{q}+\frac{1}{\overline{q}}=1$.

Denote

$$\begin{gathered}
\widetilde{R}_1(t)=\int_\infty^t\mathcal{U}(t-\tau)\int_\infty^\tau F(is)R_{0,k}(s)\,ds\,d\tau\\
\widetilde{R}_2(t)=\int_\infty^t\mathcal{U}(t-\tau)h(i\tau)R_{0,k}(\tau)\,d\tau,
\end{gathered}$$

where

$$\begin{aligned}R_{0,k}(t) &= e^{\frac{i\omega x^2}{2t}} k(\frac{x}{t}) \Delta e^{ig\left(\frac{x}{t}\right)\log t} + 2i\frac{1}{t^2}\sum \partial_j g(\frac{x}{t})\partial_j k(\frac{x}{t}) e^{\frac{i\omega x^2}{2t}} e^{ig(\frac{x}{t})\log t}\log t \\ &\quad + \frac{1}{t^2}(\Delta k)(\frac{x}{t}) e^{\frac{i\omega x^2}{2t}} e^{ig\left(\frac{x}{t}\right)\log t}.\end{aligned}$$

Lemma 2.3. *Let*

$$|F(it)| \le C|t|^{-2-\frac{n}{2}}, \quad |h(it)| \le C|t|^{-1-\frac{n}{2}}.$$

Then

$$\begin{aligned}&\|\widetilde{R}_j(t)\|_{L^2} + (\int_t^\infty \|\widetilde{R}_j(t)\|_{X_n}^4 \, dt)^{1/4} \\ &\le Ct^{-2}(\|\Delta k\|_{L^2} + \|\nabla k\cdot\nabla g\|_{L^2}\log t + \|k\Delta g\|_{L^2}\log t + \|k\nabla g\cdot\nabla g\|_{L^2}(\log t)^2),\end{aligned}$$

where $X_1 = L^\infty, X_2 = L^4$.

Proof. We have by the Strichartz estimate (see Lemma 2.2)

$$\begin{aligned}&\|\widetilde{R}_j(t)\|_{L^2} + \Big(\int_t^\infty \Big\|\widetilde{R}_j(t)\Big\|_{X_n}^4 \, dt\Big)^{1/4} \\ &\le C\int_t^\infty \Big(\int_\tau^\infty |s|^{-2-\frac{2}{n}} \left\|R_{0,k}(s)\right\|_{L^2} \, ds + |\tau|^{-1-\frac{2}{n}} \left\|R_{0,k}(\tau)\right\|_{L^2}\Big) \, d\tau.\end{aligned}$$

It is easy to see that

$$\begin{aligned}&\left\|R_{0,k}(t)\right\|_{L^2} \\ &\le Ct^{-2+\frac{2}{n}}(\left\|\Delta k\right\|_{L^2} + \left\|\nabla k\cdot\nabla g\right\|_{L^2}\log t + \left\|k\Delta g\right\|_{L^2}\log t + \left\|k\nabla g\cdot\nabla g\right\|_{L^2}(\log t)^2).\end{aligned}$$

Therefore, we have the result of the lemma. □

Lemma 2.4. *Assume that* $|G(it)| + |t||G'(it)| \le C|t|^{-q-\frac{n}{2}}$, *then*

$$\begin{aligned}&\Big\|\int_\infty^t G(i\tau) e^{\frac{i\omega x^2}{2\tau}} e^{ig(\frac{x}{\tau})\log s} k(\frac{x}{\tau}) \, d\tau\Big\|_{L^p} \\ &\le \begin{cases} Ct^{-\frac{\delta}{2}-q+1-\frac{n}{2}(1-\frac{2}{p})} \||\cdot|^{-\delta}k\|_{L^p} \\ +Ct^{-\frac{\widetilde{\delta}}{2}-q+1-\frac{n}{2}(1-\frac{2}{p})}(\||\cdot|^{1-\widetilde{\delta}}\nabla k\|_{L^p} + \||\cdot|^{1-\widetilde{\delta}}k\nabla g\|_{L^p}\log t), \\ \qquad \text{for } 0<\delta,\widetilde{\delta}<2,\ 1\le p<\infty, \\ Ct^{-\frac{\delta}{2}-q+1-\frac{n}{2}(1-\frac{1}{p})} \||\cdot|^{-\delta}k\|_{L^\infty} \\ +Ct^{-\frac{\widetilde{\delta}}{2}-q+1-\frac{n}{2}(1-\frac{1}{p})}(\||\cdot|^{1-\widetilde{\delta}}\nabla k\|_{L^\infty} + \||\cdot|^{1-\widetilde{\delta}}k\nabla g\|_{L^\infty}\log t), \\ \qquad \text{for } 0<\delta,\widetilde{\delta}<2-\frac{n}{p},\ 1\le p<\infty. \end{cases}\end{aligned}$$

Proof. Using the identity

$$\frac{1}{1-\frac{i\omega x^2}{2\tau}}\partial_t \tau e^{\frac{i\omega x^2}{2\tau}} = e^{\frac{i\omega x^2}{2\tau}}$$

we have

$$\begin{aligned}
&\int_\infty^t G(i\tau)e^{\frac{i\omega x^2}{2\tau}}e^{ig(\frac{x}{\tau})\log\tau}k(\frac{x}{\tau})\,d\tau\\
&=\int_\infty^t G(i\tau)e^{ig(\frac{x}{\tau})\log\tau}k(\frac{x}{\tau})\Big(\frac{1}{1-\frac{i\omega x^2}{2\tau}}\partial_\tau\tau e^{\frac{i\omega x^2}{2\tau}}\Big)\,d\tau\\
&=G(it)k(\frac{x}{t})e^{ig(\frac{x}{t})\log t}\Big(\frac{1}{1-\frac{i\omega x^2}{2t}}te^{\frac{i\omega x^2}{2t}}\Big)\\
&-\int_\infty^t \tau e^{\frac{i\omega x^2}{2\tau}}\partial_\tau\Big(G(i\tau)k(\frac{x}{\tau})\frac{1}{1-\frac{i\omega x^2}{2\tau}}e^{ig(\frac{x}{\tau})\log\tau}\Big)\,d\tau.
\end{aligned}$$

We also obtain

$$\begin{aligned}
&\Big\|G(it)k(\frac{x}{t})e^{ig(\frac{x}{t})\log t}\big(\frac{1}{1-\frac{i\omega x^2}{2t}}te^{\frac{i\omega x^2}{2t}}\big)\Big\|_{L^p}\\
&\le Ct^{-\frac{\delta}{2}-q+1-\frac{n}{2}}\Big(\int\big(\frac{\big|\frac{x}{t^{1/2}}\big|^\delta}{1+\big|\frac{x}{t^{1/2}}\big|^2}\Big|\frac{x}{t}\Big|^{-\delta}k(\frac{x}{t})\big)^pdx\Big)^{1/p}\\
&\le\begin{cases} Ct^{-\frac{\delta}{2}-q+1-\frac{n}{2}(1-\frac{2}{p})}\||\cdot|^{-\delta}k\|_{L^p}, & 0<\delta<2, 1\le p<\infty\\
Ct^{-\frac{\delta}{2}-q+1-\frac{n}{2}(1-\frac{1}{p})}\||\cdot|^{-\delta}k\|_{L^\infty}, & 0<\delta<2-\frac{n}{p}, 1\le p<\infty\end{cases}
\end{aligned}$$

and in the same way we get

$$\begin{aligned}
&\Big\|te^{\frac{i\omega x^2}{2t}}\partial_t\Big(G(it)k(\frac{x}{t})\frac{1}{1-\frac{i\omega x^2}{2t}}e^{ig\left(\frac{x}{t}\right)\log t}\Big)\Big\|_{L^p}\\
&\le\begin{cases} Ct^{-\frac{\delta}{2}-q-\frac{n}{2}(1-\frac{2}{p})}\Big\||\cdot|^{-\delta}k\Big\|_{L^p}\\
+Ct^{-\frac{\tilde\delta}{2}-q-\frac{n}{2}(1-\frac{2}{p})}(\Big\||\cdot|^{1-\tilde\delta}\nabla k\Big\|_{L^p}+\Big\||\cdot|^{1-\tilde\delta}k\nabla g\Big\|_{L^p}\log t),\\
\quad\text{for } 0<\delta,\tilde\delta<2,\ 1\le p<\infty,\\
Ct^{-\frac{\delta}{2}-q-\frac{n}{2}(1-\frac{1}{p})}\Big\||\cdot|^{-\delta}k\Big\|_{L^\infty}\\
+Ct^{-\frac{\tilde\delta}{2}-q-\frac{n}{2}(1-\frac{1}{p})}(\Big\||\cdot|^{1-\tilde\delta}\nabla k\Big\|_{L^\infty}+\Big\||\cdot|^{1-\tilde\delta}k\nabla g\Big\|_{L^\infty}\log t),\\
\quad\text{for } 0<\delta,\tilde\delta<2-\frac{n}{2},\ 1\le p<\infty.\end{cases}
\end{aligned}$$

Hence we have the result of the lemma. □

3. Proof of Theorem 1.3

We consider the linearized version of equation (1.1)

$$\mathcal{L}u=\mathcal{N}_n(v)+\mathcal{G}_n(v),\quad (t,x)\in\mathbb{R}\times\mathbb{R}^n. \tag{3.1}$$

We take

$$u_0(t,x)=\frac{1}{(it)^{\frac{n}{2}}}e^{\frac{ix^2}{2t}}\widehat{\phi}(\frac{x}{t})\exp\Big(-i\lambda_0|\widehat{\phi}(\frac{x}{t})|^{\frac{2}{n}}\log t\Big)$$

as the first approximation for solutions of (3.1). By a direct calculation we get

$$\mathcal{L}u_0=\mathcal{G}_n(u_0)+R_1,$$

where

$$\begin{aligned} R_1(t) &= \frac{1}{(it)^{\frac{n}{2}}} e^{\frac{ix^2}{2t}} \widehat{\phi}\big(\frac{x}{t}\big) \frac{1}{2} \Delta \exp(-i\lambda_0 |\widehat{\phi}\big(\frac{x}{t}\big)|^{\frac{2}{n}} \log t) \\ &- \frac{2}{n}\lambda_0 \frac{1}{t^2} \frac{1}{(it)^{\frac{n}{2}}} e^{\frac{ix^2}{2t}} \nabla\widehat{\phi}\big(\frac{x}{t}\big) \exp(-i\lambda_0 |\widehat{\phi}\big(\frac{x}{t}\big)|^{\frac{2}{n}} \log t) \\ &\times 2 \operatorname{Re} \nabla\widehat{\phi}\big(\frac{x}{t}\big) \overline{\widehat{\phi}\big(\frac{x}{t}\big)} |\widehat{\phi}\big(\frac{x}{t}\big)|^{\frac{2}{n}-2} \log t \\ &+ \frac{1}{2} \frac{1}{(it)^{\frac{n}{2}}} e^{\frac{ix^2}{2t}} t^{-2} \Delta\widehat{\phi}\big(\frac{x}{t}\big) \exp(-i\lambda_0 |\widehat{\phi}\big(\frac{x}{t}\big)|^{\frac{2}{n}} \log t). \end{aligned}$$

Hence

$$\mathcal{L}(u - u_0) = \mathcal{N}_n(v) + \mathcal{G}_n(v) - \mathcal{G}_n(u_0) + R_1.$$

By Lemma 2.2 we obtain

$$\begin{aligned} &\Big\| \int_t^\infty \mathcal{U}(t-\tau) R_1(\tau)\, d\tau \Big\|_{L^2} + \Big(\int_t^\infty \Big\| \int_t^\infty \mathcal{U}(t-\tau) R_1(\tau)\, d\tau \Big\|_{X_n}^4 dt \Big)^{1/4} \\ &\le C \int_t^\infty \|R_1(\tau)\|_{L^2}\, d\tau \le C t^{-1} (\log t)^2 \|\phi\|_{H^{0,2}}^{1+\frac{2}{n}} \end{aligned} \tag{3.2}$$

since by the Hölder inequality we have

$$\begin{aligned} &\|R_1(t)\|_{L^2} \\ &\le C t^{-2} \|\Delta\widehat{\phi}\|_{L^2} + C t^{-2} (\log t)^2 \|\widehat{\phi}\|_{L^\infty}^{\frac{2}{n}-1} \|\nabla\widehat{\phi}\|_{L^4}^2 + C t^{-2} (\log t) \|\widehat{\phi}\|_{L^\infty}^{\frac{2}{n}} \|\Delta\widehat{\phi}\|_{L^2} \\ &\le C t^{-2} (\log t)^2 \|\phi\|_{H^{0,2}}^{1+\frac{2}{n}}. \end{aligned}$$

We now define u_1 as

$$u_1(t) = -i \int_\infty^t \mathcal{U}(t-\tau) \mathcal{N}_n(u_0)\, d\tau$$

which implies $\mathcal{L} u_1 = \mathcal{N}_n(u_0)$ and

$$\begin{aligned} u(t) - u_0(t) &= -i \int_\infty^t \mathcal{U}(t-\tau) (\mathcal{N}_n(v) - \mathcal{N}_n(u_0) + \mathcal{G}_n(v) - \mathcal{G}_n(u_0))\, d\tau \\ &- i \int_\infty^t \mathcal{U}(t-\tau) R_1(\tau)\, d\tau + u_1(t). \end{aligned} \tag{3.3}$$

Note that

$$i\partial_t u_1(t) = \mathcal{N}_n(u_0) + \frac{i}{2} \int_\infty^t \mathcal{U}(t-\tau) \Delta \mathcal{N}_n(u_0)\, d\tau. \tag{3.4}$$

Now, we define the function space

$$X = \big\{ f \in C([T,\infty); L^2); \|f\|_X < \infty \big\}, \text{ where}$$

$$\|f\|_X = \sup_{t \in [T,\infty)} t^b \|f(t) - u_0(t)\|_{L^2} + \sup_{t \in [T,\infty)} t^b \Big(\int_t^\infty \|f(t) - u_0(t)\|_{X_n}^4 dt \Big)^{1/4},$$

and

$$X_1 = L^\infty, \quad X_2 = L^4, \quad b > \frac{n}{4}.$$

Let X_ρ be a closed ball in X with a radius ρ and a center u_0. Let $v \in X_\rho$. From (3.4) and Lemma 2.1 it follows that

$$\begin{aligned} i\partial_t u_1(t) = \mathcal{N}_n(u_0) + \frac{i}{2} \sum_{(\omega,h,g,f)} \Big(&- \frac{2i\omega}{1-\omega} h(it) e^{\frac{i\omega x^2}{2t}} e^{ig(\frac{x}{t})\log t} f(\frac{x}{t}) \\ &- \frac{2\omega}{(1-\omega)^2} \int_\infty^t \Big(\sum_{(F,k)} F(i\tau) e^{\frac{i\omega x^2}{2\tau}} e^{ig(\frac{x}{\tau})\log \tau} k(\frac{x}{\tau}) \\ &- i\omega \mathcal{U}(t-\tau) \int_\infty^\tau \sum_{(F,k)} F'(is) e^{\frac{i\omega x^2}{2s}} e^{(\frac{x}{s})\log s} k(\frac{x}{s})\, ds \\ &- i\omega \mathcal{U}(t-\tau) \int_\infty^\tau \sum_{(F,k)} F(is) e^{\frac{i\omega x^2}{2s}} e^{ig(\frac{x}{s})\log s} \frac{1}{s} k(g - \frac{in}{2})(\frac{x}{s})\, ds \Big)\, d\tau + R(t), \end{aligned}$$

where the summation with respect to (ω, h, g, f) is taken over

$$\begin{aligned} (\omega, h, g, f) = &\Big(3, (it)^{-3/2}, \lambda_0 |\hat{\phi}(\frac{x}{t})|^2, \lambda_1 \hat{\phi}(\frac{x}{t})^3\Big), \\ &\Big(-1, (-i)^{-1/2} t^{-3/2}, \lambda_0 |\hat{\phi}(\frac{x}{t})|^2, \lambda_2 \hat{\phi}(\frac{x}{t}) \overline{\hat{\phi}(\frac{x}{t})}^2\Big), \\ &\Big(-3, (-it)^{-3/2}, \lambda_0 |\hat{\phi}(\frac{x}{t})|^2, \lambda_3 \overline{\hat{\phi}(\frac{x}{t})}^3\Big), \end{aligned}$$

when $n = 1$, and

$$(\omega, h, g, f) = \Big(2, (it)^{-1}, \lambda_0 |\hat{\phi}(\frac{x}{t})|, \lambda_1 \hat{\phi}(\frac{x}{t})^2\Big), \Big(-2, (-it)^{-1}, \lambda_0 |\hat{\phi}(\frac{x}{t})|, \lambda_2 \overline{\hat{\phi}(\frac{x}{t})}^2\Big),$$

when $n = 2$, and the summation with respect to (F, k) is taken over $(F, k) = (h', f), (h\tau^{-1}, f(g - in/2))$. We have

$$\begin{aligned} &\mathcal{G}_n(v) - \mathcal{G}_n(u_0) \\ &= \lambda_0 |v|^{\frac{2}{n}} v - \lambda_0 |u_0|^{\frac{2}{n}} u_0 \\ &= \lambda_0 (|v|^{\frac{2}{n}} - |u_0|^{\frac{2}{n}})(v - u_0) + \lambda_0 (|v|^{\frac{2}{n}} - |u_0|^{\frac{2}{n}}) u_0 + \lambda_0 |u_0|^{\frac{2}{n}} (v - u_0)\,. \end{aligned}$$

Therefore, by the Strichartz estimate we get

$$\begin{aligned} &\Big\| \int_t^\infty \mathcal{U}(t-\tau)(\mathcal{G}_n(v) - \mathcal{G}_n(u_0))\, d\tau \Big\|_{L^2} \\ &+ \Big(\int_t^\infty \Big\| \int_t^\infty \mathcal{U}(t-\tau)(\mathcal{G}_n(v) - \mathcal{G}_n(u_0))\, d\tau \Big\|_{L^4}^4 dt \Big)^{1/4} \\ &\le C \Big(\int_t^\infty \|v(\tau) - u_0(\tau)\|_{L^2}^2\, d\tau \Big)^{\frac{1}{2}} \Big(\int_t^\infty \|v(\tau) - u_0(\tau)\|_{L^4}^4\, d\tau \Big)^{1/4} \\ &\quad + C \int_t^\infty \|v(\tau) - u_0(\tau)\|_{L^2} \|u_0(\tau)\|_{L^\infty}\, d\tau \\ &\le C\rho^2 t^{-2b+\frac{1}{2}} + C t^{-b} \rho \|\phi\|_{L^1}, \end{aligned} \tag{3.5}$$

for $n = 2$. Also

$$
\begin{aligned}
&\Big\| \int_t^\infty \mathcal{U}(t-\tau)(\mathcal{G}_n(v) - \mathcal{G}_n(u_0))\, d\tau \Big\|_{L^2} \\
&\quad + \Big(\int_t^\infty \Big\| \int_t^\infty \mathcal{U}(t-\tau)(\mathcal{G}_n(v) - \mathcal{G}_n(u_0))\, d\tau \Big\|_{X_1}^4 dt \Big)^{1/4} \\
&\le C \Big(\int_t^\infty \| |v(\tau) - u_0(\tau)|^3 \|_{L^1}^{\frac{4}{3}}\, d\tau \Big)^{3/4} \\
&\quad + C \int_t^\infty \| |v(\tau) - u_0(\tau)| |u_0(\tau)|^2 \|_{L^2}\, d\tau \\
&\le C \Big(\int_t^\infty \|v(\tau) - u_0(\tau)\|_{L^\infty}^{\frac{4}{3}} \|v(\tau) - u_0(\tau)\|_{L^2}^{\frac{8}{3}} d\tau \Big)^{3/4} \\
&\quad + C \int_t^\infty \|v(\tau) - u_0(\tau)\|_{L^2} \|u_0(\tau)\|_{L^\infty}^2\, d\tau \\
&\le C \Big(\int_t^\infty \|v(\tau) - u_0(\tau)\|_{L^\infty}^4\, d\tau \Big)^{\frac{1}{4}} \Big(\int_t^\infty \|v(\tau) - u_0(\tau)\|_{L^2}^4 d\tau \Big)^{1/2} \\
&\quad + C \int_t^\infty \|v(\tau) - u_0(\tau)\|_{L^2} \|u_0(\tau)\|_{L^\infty}^2\, d\tau \\
&\le C\rho t^{-b} \Big(\int_t^\infty \rho^4 \tau^{-4b} d\tau \Big)^{1/2} + C\rho \|\phi\|_{L^1}^2 \int_t^\infty \tau^{-b-1} d\tau \\
&\le C\rho^3 t^{-3b+\frac{1}{2}} + Ct^{-b}\rho \|\phi\|_{L^1}^2,
\end{aligned}
\tag{3.6}
$$

for $n = 1$, where we have used the facts that $b > n/4$ and

$$
|\mathcal{G}_n(v) - \mathcal{G}_n(u_0)| \le C(|v - u_0|^{\frac{2}{n}} + |u_0|^{\frac{2}{n}})|v - u_0|.
$$

Similarly, we see that the above estimate holds valid with $\mathcal{G}_n$ replaced by $\mathcal{N}_n$. Thus by (3.2), (3.3), (3.5) and (3.6)

$$
\begin{aligned}
&\|u(t) - u_0(t)\|_{L^2} + \Big(\int_t^\infty \|u(\tau) - u_0(\tau)\|_{X_n}^4\, d\tau \Big)^{1/4} \\
&\le C\rho^{1+\frac{2}{n}} t^{-(1+\frac{2}{n})b+\frac{1}{2}} + Ct^{-b}\rho \|\phi\|_{L^1}^{\frac{2}{n}} + Ct^{-1}(\log t)^2 \|\phi\|_{H^{0,2}}^{1+\frac{2}{n}} \\
&\quad + \|u_1(t)\|_{L^2} + \Big(\int_t^\infty \|u_1(\tau)\|_{X_n}^4\, d\tau \Big)^{1/4}.
\end{aligned}
\tag{3.7}
$$

To get the result we now estimate $u_1(t)$. By Lemma 2.1, Lemma 2.3 and Lemma 2.4 we get

$$
\|u_1(t)\|_{L^2} + \Big(\int_t^\infty \|u_1(\tau)\|_{X_n}^4 d\tau \Big)^{1/4} \le C(\| |\cdot|^{-\tilde{\delta}} \widehat{\phi} \|_{L^2} + \|\phi\|_{H^{0,2}})^{1+\frac{2}{n}} t^{-\frac{\tilde{\delta}}{2}}, \tag{3.8}
$$

for $\frac{n}{2} < \widetilde{\delta} < 2$. where we have used the fact that

$$\begin{aligned}
&\Big\| \int_t^\infty \int_s^\infty \mathcal{U}(s-\tau) f(\tau)\, d\tau\, ds \Big\|_{X_n} \\
&\le C \int_t^\infty s^{-\alpha} s^{\alpha} \Big\| \int_s^\infty \mathcal{U}(s-\tau) f(\tau)\, d\tau \Big\|_{X_n} ds \\
&\le C \Big(\int_t^\infty s^{-\frac{4}{3}\alpha}\, ds \Big)^{3/4} \Big(\int_t^\infty s^{4\alpha} \Big\| \int_s^\infty \mathcal{U}(s-\tau) f(\tau)\, d\tau \Big\|_{X_n}^4 ds \Big)^{1/4} \\
&\le C t^{-\alpha+\frac{3}{4}} \Big(\int_t^\infty s^{4\alpha} \Big\| \int_s^\infty \mathcal{U}(s-\tau) f(\tau)\, d\tau \Big\|_{X_n}^4 ds \Big)^{1/4}
\end{aligned}$$

with $\alpha \ge 1$. from which it follows that

$$\begin{aligned}
&\Big(\int_{\widetilde{t}}^\infty \Big\| \int_t^\infty \int_s^\infty \mathcal{U}(s-\tau) f(\tau)\, d\tau\, ds \Big\|_{X_n}^4 dt \Big)^{1/4} \\
&\le C \Big(\int_{\widetilde{t}}^\infty t^{-4\alpha+3} \Big(\int_t^\infty \Big\| \int_s^\infty \mathcal{U}(s-\tau) \tau^\alpha f(\tau)\, d\tau \Big\|_{X_n}^4 ds \Big) dt \Big)^{1/4} \\
&\le C \Big(\int_{\widetilde{t}}^\infty t^{-4\alpha+3} \Big(\int_t^\infty \|\tau^\alpha f(\tau)\|_{L^2}\, d\tau \Big)^4 dt \Big)^{1/4} \\
&\le C t^{-\alpha+1-\beta} \sup_t t^\beta \int_t^\infty \|\tau^\alpha f(\tau)\|_{L^2}\, d\tau \\
&\le C t^{-\beta} \sup_t t^\beta \int_t^\infty \|\tau^\alpha f(\tau)\|_{L^2}\, d\tau.
\end{aligned}$$

By virtue of (3.7) and (3.8), taking $\frac{n}{2} < \widetilde{\delta} < 2, b = \frac{\widetilde{\delta}}{2}$. we get

$$\begin{aligned}
&\|u(t) - u_0(t)\|_{L^2} + \Big(\int_t^\infty \|u(\tau) - u_0(\tau)\|_{X_n}^4 d\tau \Big)^{1/4} \\
&\le C(\||\cdot|^{-\widetilde{\delta}} \widehat{\phi}\| + \|\phi\|_{H^{0,2}})^{1+\frac{2}{n}} t^{-b}.
\end{aligned} \tag{3.9}$$

Since the norm of the final state $\|\phi\|_{H^{0,2}} + \|\phi\|_{\dot{H}^{-\delta}}$ is sufficiently small, estimate (3.9) implies that there exists a sufficiently small radius $\rho > 0$ such that the mapping $\mathcal{M}v = u$. defined by equation (3.1), transforms the set X_ρ into itself. In the same way as in the proof of estimate (3.9) we find that $\mathcal{M}$ is a contraction mapping in X_ρ. This completes the proof of the theorem.

Acknowledgment. The first author would like to thank Professor Jean-Marc Delort for useful comments and discussions on this subject. The third and the fourth authors are grateful to Mr. Yuichiro Kawahara for communicating his recent work [9]. Finally, the authors would like to thank Professor Tohru Ozawa for letting us know the paper [11].

References

[1] J.E. Barab, *Nonexistence of asymptotically free solutions for nonlinear Schrödinger equations*, J. Math. Phys. 25 (1984), 3270–3273.
[2] J. Ginibre and T. Ozawa, *Long range scattering for nonlinear Schrödinger and Hartree equations in space dimension $n \ge 2$* , Comm. Math. Phys. 151 (1993), 619–645.
[3] J. Ginibre, T. Ozawa and G. Velo, *On existence of the wave operators for a class of nonlinear Schrödinger equations*, Ann. Inst. H. Poincaré, Physique Théorique 60 (1994), 211–239.

[4] J. Ginibre and G. Velo, *Long range scattering and modified wave operators for some Hartree type equations III, Gevrey spaces and low dimensions*, J. Differential Equations 175 (2001), 415–501.
[5] N. Hayashi, T. Mizumachi and P.I. Naumkin, *Time decay of small solutions to quadratic nonlinear Schrödinger equations in 3D*, Differential Integral Equations 16 (2003), 159–179.
[6] N. Hayashi and P.I. Naumkin, *Asymptotics for large time of solutions to the nonlinear Schrödinger and Hartree equations*, Amer. J. Math. 120 (1998), 369–389.
[7] N. Hayashi and P.I. Naumkin, *Large time behavior for the cubic nonlinear Schrödinger equation*, Canad. J. Math. 54 (2002), 1065–1085.
[8] L. Hörmander, *Lectures on Nonlinear Hyperbolic Differential Equations*, Mathématiques & Applications 26, Springer.
[9] Y. Kawahara, *Asymptotic behavior of solutions for nonlinear Schrödinger equations with quadratic nonlinearities*, Master Thesis, Osaka University, 2004, (in Japanese).
[10] K. Moriyama, S. Tonegawa and Y. Tsutsumi, *Wave operators for the nonlinear Schrödinger equation with a nonlinearity of low degree in one or two dimensions*, Commun. Contemp. Math. 5 (2003), 983–996.
[11] K. Nakanishi and T. Ozawa, *Remarks on scattering for nonlinear Schrödinger equations*, NoDEA. 9 (2002), 45-68.
[12] T. Ozawa, *Long range scattering for nonlinear Schrödinger equations in one space dimension*, Comm. Math. Phys. 139 (1991), 479–493.
[13] A. Shimomura and S. Tonegawa, *Long range scattering for nonlinear Schrödinger equations in one and two space dimensions*, Differential Integral Equations 17 (2004), 127-150.
[14] K. Yajima, *Existence of solutions for Schrödinger evolution equations*, Comm. Math. Phys. 110 (1987), 415–426.

Nakao Hayashi
Department of Mathematics, Graduate School of Science, Osaka University, Osaka, Toyonaka, 560-0043, Japan
E-mail address: nhayashi@math.wani.osaka-u.ac.jp

Pavel I. Naumkin
Instituto de Matemáticas, UNAM Campus Morelia, AP 61-3 (Xangari), Morelia CP 58089, Michoacán, Mexico
E-mail address: pavelni@matmor.unam.mx

Akihiro Shimomura
Department of Mathematics, Gakushuin University, 1-5-1 Mejiro, Toshima-ku, Tokyo 171-8588, Japan
E-mail address: simomura@math.gakushuin.ac.jp

Satoshi Tonegawa
College of Science and Technology, Nihon University, 1-8-14 Kanda-Surugadai, Chiyoda-ku, Tokyo 101-8308, Japan
E-mail address: tonegawa@math.cst.nihon-u.ac.jp