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A LIMIT SET TRICHOTOMY FOR ORDER-PRESERVING SYSTEMS ON TIME SCALES

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Abstract. In this paper we derive a limit set trichotomy for abstract order-preserving 2-parameter semiflows in normal cones of strongly ordered Banach spaces. Additionally, to provide an example, Müller's theorem is generalized to dynamic equations on arbitrary time scales and applied to a model from population dynamics.

1. Introduction

In certain relevant situations it happens that a dynamical system preserves a (partial) order relation on its state space. These systems are called *order-preserving* or *monotone* and the ground for their qualitative theory was laid by Krasnoselskii in his two books [19, 20]. Meanwhile many others made further important contributions for different types of such dynamical systems like (semi-)flows of ordinary differential equation [15, 16, 17, 30], functional differential equations [31, 1], semilinear parabolic equations [18, 34], ordinary difference equations [17, 33], [21, 22], [26, 13], random dynamical systems [2] or general skew-product flows [7]; compare also the monographs [32] and [8] for numerous examples and applications.

The essential property of order-preserving dynamical systems is that they possess a surprisingly simple asymptotic behavior. In fact Krause et al. [21, 22] proved a so-called *limit set trichotomy* (cf. also [25] for nonautonomous difference equations or [3] for random dynamical systems), describing the only three asymptotic scenarios of such systems under a certain kind of concavity.

In this paper we prove such a limit set trichotomy for a general model of non-expansive dynamical processes, namely 2-*parameter semiflows* in normal cones on time scales. They include the solution operators of dynamic equations on time scales (cf. [14, 6]) and in particular of nonautonomous difference and differential equations. Beyond the unification aspect, dynamic equations on time scales are predestinated to describe the interaction of biological species with hibernation periods. The crucial point is that we provide sufficient criteria for the nonexpansiveness of

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such solution operators in terms of concavity and cooperativity conditions on the right-hand sides of the corresponding equations.

On this occasion we generalize the classical theorem of Müller (cf., e.g., [24]) to dynamic equations in real Banach spaces. Thereby we closely follow the arguments of [36], who considers finite-dimensional ordinary differential equations and orderings with respect to arbitrary cones. However, although our state spaces are allowed to be infinite-dimensional, we have to make the assumption that cones have nonempty interior. The use of arbitrary order cones instead of $\mathbb{R}_+^d$ even in finite dimensions has the advantage that certain equations are cooperative (see Definition 5.6) with respect to an ordering different from the component-wise.

2. Preliminaries

Let $\mathbb{T}$ be an arbitrary *time scale*, i.e., a canonically ordered closed subset of the real axis $\mathbb{R}$. Since we are interested in the asymptotic behavior of systems on such sets $\mathbb{T}$, it is reasonable to assume that $\mathbb{T}$ is unbounded above in the whole paper. Moreover, $\mathbb{T}$ is called *homogeneous*, if $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = h\mathbb{Z}$, $h > 0$. A $\mathbb{T}$-*interval* is the intersection of a real interval with the set $\mathbb{T}$, for $a, b \in \mathbb{R}$ we write $[a, b]_{\mathbb{T}} := [a, b] \cap \mathbb{T}$ and (half-)open intervals are defined analogously. (X, d) denotes a metric space from now on.

Definition 2.1. A mapping $\varphi : \{(t, \tau) \in \mathbb{T}^2 : \tau \le t\} \times X \to X$ is denoted as a 2-*parameter semiflow on* X, if the mappings $\varphi(t, \tau, \cdot) = \varphi(t, \tau) : X \to X$, $\tau \le t$, satisfy the following properties:

(i) $\varphi(\tau, \tau)x = x$ for all $\tau \in \mathbb{T}$, $x \in X$,
(ii) $\varphi(t, s)\varphi(s, \tau) = \varphi(t, \tau)$ for all $\tau, s, t \in \mathbb{T}$, $\tau \le s \le t$,
(iii) $\varphi(\,\cdot, \cdot)x : \{(t, \tau) \in \mathbb{T}^2 : \tau \le t\} \to X$ is continuous for all $x \in X$.

Remark 2.2. *(1)* Sometimes 2-parameter semiflows are also called *(evolutionary) processes* (cf., e.g., [12, p. 100, Definition 1.1]).

(2) To provide some concepts from classical (1-parameter) semiflows, we denote a point $x_0 \in X$ as an *equilibrium* of φ, if $\varphi(t, \tau)x_0 = x_0$ for all $\tau \le t$ holds. Moreover, for $\tau \in \mathbb{T}$ and $x \in X$, the *orbit emanating from* (τ, x) is

$$\gamma_\tau^+(x) := \{\varphi(t, \tau)x \in X : \tau \le t\}$$

and the ω-*limit set of* (τ, x) is given by

$$\omega_\tau^+(x) := \bigcap_{\tau \le t} \operatorname{closure}\{\varphi(s, \tau)x \in X : t \le s\}.$$

Equivalently, $\omega_\tau^+(x)$ consists of all the points $x^* \in X$ such that there exists a sequence $t_n \to \infty$ in $\mathbb{T}$ with $x^* = \lim_{n\to\infty} \varphi(t_n, \tau)x$. A subset $U \subset X$ is denoted as *forward invariant*, if $\varphi(t, \tau)U \subset U$ holds for $\tau \le t$.

Example 2.3. *(1)* For homogeneous time scales $\mathbb{T}$, any strongly continuous discrete ($\mathbb{T} := h\mathbb{Z}$, $h > 0$) or continuous ($\mathbb{T} := \mathbb{R}$) 1-parameter semiflow $\{\phi_t\}_{t\ge 0}$ evidently generates a 2-parameter semiflow φ via $\varphi(t, \tau) := \phi_{t-\tau}$.

(2) Let X be some Banach space V and $f : \mathbb{T} \times V \to V$. Then the standard examples for 2-parameter-semiflows are the solution operators $\varphi(t, \tau, \cdot) : V \to V$, $\tau \le t$, of nonautonomous difference equations $v(t + 1) = f(t, v(t))$, $t \in \mathbb{T} := \mathbb{Z}$, or of nonautonomous ordinary differential equations $\dot v(t) = f(t, v(t))$, $t \in \mathbb{T} := \mathbb{R}$ in V, provided that in the ODE case, f is e.g. measurable in t, (locally) Lipschitzian

in v, and satisfies a certain growth condition to exclude finite escape times. The general situation, where $\mathbb{T}$ is an arbitrary closed subset of the reals, occurs in the context of dynamic equations $v^\Delta = f(t,v)$ on time scales (see Section 5).

(3) Let $r \geq 0$ be real, $X := C([-r,0],\mathbb{R}^d)$ the space of continuous functions endowed with the sup-norm, and $f : \mathbb{R} \times X \to \mathbb{R}^d$ be continuous and (locally) Lipschitzian in the second argument. Then, if no finite escape times appear, the solution $v(\cdot,\tau,v^0) : [\tau,\infty) \to \mathbb{R}^d$ of the retarded functional differential equation

$$\dot v(t) = f(t,v_t),$$
$$v_t(\theta) := v(t+\theta) \quad \text{for all } \theta \in [-r,0]$$

satisfying the initial condition $v_\tau = v^0$ for $\tau \in \mathbb{R}$, $v^0 \in X$, defines a 2-parameter semiflow on X with $\mathbb{T} = \mathbb{R}$ via $\varphi(t,\tau)v^0 := v_t(\cdot,\tau,v^0)$ (cf. [12]).

(4) Criteria for more abstract nonautonomous evolutionary equations to generate a 2-parameter semiflow can be found in [4] and the references therein.

We need some further terminology. A self-mapping $\Phi : X \to X$ will be called *nonexpansive* (on (X,d)), if $d(\Phi x, \Phi \bar x) \leq d(x,\bar x)$ for all $x,\bar x \in X$, and Φ will be called *contractive*, if $d(\Phi x,\Phi \bar x) < d(x,\bar x)$ for all $x,\bar x \in X$, $x \neq \bar x$. If P is a nonempty set, then a family of parameter-dependent self-mappings $\Phi(p) : X \to X$, $p \in P$, is called *uniformly contractive*, if there exists a continuous function $c : X \times X \to \mathbb{R}_+$, such that the following two conditions are fulfilled (cf. [25]):

(i) $c(x,\bar x) < d(x,\bar x)$ for all $x,\bar x \in X$, $x \neq \bar x$,
(ii) $d(\Phi(p)x,\Phi(p)\bar x) \leq c(x,\bar x)$ for all $p \in P$, $x,\bar x \in X$.

Assume from now on that the metric space X is a cone V_+ in a real Banach space $(V,\|\cdot\|)$. Recall that a *cone* is a nonempty closed convex set $V_+ \subset V$ such that $\alpha V_+ \subset V_+$ for $\alpha \geq 0$ and $V_+ \cap (-V_+) = \{0\}$. Moreover, define $V_+^* := V_+ \setminus \{0\}$. Any cone defines a partial order relation on V via $u \leq v$, if $v-u \in V_+$, which is preserved under addition and scalar multiplication with nonnegative reals. Furthermore, we write $u < v$ when $u \leq v$ and $u \neq v$. If V_+ has nonempty interior $\operatorname{int} V_+$, we say that V is *strongly ordered* and write $u \ll v$, if $v - u \in \operatorname{int} V_+$. A cone V_+ is called *normal*, if there exists a real number $M \geq 0$ such that $\|u\| \leq M\|v\|$ for all $u,v \in V_+$ with $u \leq v$. In fact, without loss of generality, one can assume the norm $\|\cdot\|$ to be monotone, i.e., $\|u\| \leq \|v\|$, if $u \leq v$; otherwise an equivalent norm on V can be found for which $M = 1$ (cf. [29]). Finally we define the *order interval* $[u,v] := \{w \in V : u \leq w \leq v\}$ for $u,v \in V$, $u \leq v$. Explicit examples of normal cones and strongly ordered Banach spaces can be found in, e.g., [10, pp. 219ff].

Although forthcoming results on the boundedness of orbits are stated in the norm topology on V_+, our contractivity condition for 2-parameter semiflows will be formulated in a different metric topology:

Definition 2.4. (i) The equivalence classes under the equivalence relation defined by $u \sim v$, if there exists $\alpha > 0$ such that $\alpha^{-1}u \leq v \leq \alpha u$ on the cone V_+ are called the *parts* of V_+.
(ii) Let C be a part of V_+. Then $p : C \times C \to \mathbb{R}_+$,

$$p(u,v) := \inf\{\log\alpha : \alpha^{-1}u \leq v \leq \alpha u\} \quad \text{for all } u,v \in C,$$

defines a metric on C called the *part metric* of C.

Remark 2.5. *(1)* u and v lie in the same part, if and only if $p(u,v) < \infty$.

(2) Clearly $\operatorname{int} V_+$ is a part and the closure of every part is also a convex cone in the Banach space V. For a proof of the fact that p is a metric on C and for other properties of the part metric we refer to [5] or [8, pp. 83–86].

(3) If the cone V_+ is normal, then $\operatorname{int} V_+$ is a complete metric space with respect to the part metric p (cf. [35]).

Norm distance and the part metric are related by the following inequality:

Lemma 2.6. (a) *If V_+ is normal with monotone norm, then*
$$\|v - \bar{v}\| \leq \big(2e^{p(v,\bar{v})} - e^{-p(v,\bar{v})} - 1\big) \min\{\|v\|, \|\bar{v}\|\} \quad \textit{for all } v, \bar{v} \in V_+^*,$$
(b) $p|_{\operatorname{int} V_+ \times \operatorname{int} V_+}$ *is continuous in the norm topology on* $\operatorname{int} V_+ \times \operatorname{int} V_+$.

Proof. See [21, Lemma 2.3] for (a), while assertion (b) can be found in [25, Proof of Theorem 2]. □

3. A Limit Set Trichotomy

The following theorem is a clear manifestation of the general experience that contractivity drastically simplifies the possible long-term behavior of a dynamical system. It is the main result in the abstract part of this paper.

In the autonomous discrete time case a limit set trichotomy was discovered (and so named) by Krause and Ranft [22] and generalized in [21] to infinite-dimensional autonomous difference equations; in addition, [25] considers such nonautonomous systems.

Theorem 3.1 (Limit Set Trichotomy). *Let $V_+ \subset V$ be a normal cone,* $\operatorname{int} V_+ \neq \emptyset$ *and assume that φ is a 2-parameter semiflow on V_+ with the following properties:*

(i) *There exists a real $T > 0$ such that for all $t, \tau \in \mathbb{T}$ satisfying $T \leq t - \tau$, one has $\varphi(t,\tau)V_+^* \subset \operatorname{int} V_+$ and that the mapping $\varphi(t,\tau)|_{\operatorname{int} V_+}$ is nonexpansive,*
(ii) *for all $(\tau, v) \in \mathbb{T} \times V_+$ every bounded orbit $\gamma_\tau^+(v)$ is relatively compact in the norm topology.*

Then for every $\tau \in \mathbb{T}$ the following trichotomy holds, i.e., precisely one of the following three cases applies:

(a) *For all $v \in V_+^*$ the orbits $\gamma_\tau^+(v)$ are unbounded in norm,*
(b) *for all $v \in V_+$ the orbits $\gamma_\tau^+(v)$ are bounded in norm and for all $v \in V_+^*$ we have $\lim_{t\to\infty} \|\varphi(t,\tau)v\| = 0$,*
(c) *for all $v \in V_+$ the orbits $\gamma_\tau^+(v)$ are bounded in norm, the ω-limit sets $\omega_\tau^+(v)$ are nonempty and for all $v \in V_+^*$ they have a nontrivial accumulation point.*

If, moreover, $\omega_\tau^+(v) \subset \operatorname{int} V_+ \cup \{0\}$ for all $v \in V_+^*$ and the mappings $\varphi(t,\tau)|_{\operatorname{int} V_+}$ are uniformly contractive for all $t, \tau \in \mathbb{T}$ with $T \leq t - \tau$, then in case (c) we have
$$\lim_{t\to\infty} \big[\varphi(t,\tau)v_1 - \varphi(t,\tau)v_2\big] = 0 \quad \textit{for all } v_1, v_2 \in V_+^*. \tag{3.1}$$

Remark 3.2. *(1)* Condition (3.1) implies that all ω-limit sets $\omega_\tau^+(v)$, $v \in V_+^*$, are identical, and it excludes the existence of two different equilibria of φ. In fact, if φ possesses an equilibrium $v_0 \in V_+^*$, then (3.1) guarantees $\omega_\tau^+(v) = \{v_0\}$ for all $v \in V_+^*$. In the "autonomous" situation of a homogeneous time scale $\mathbb{T}$ and a 2-parameter semiflow induced by a 1-parameter semiflow (cf. Example 2.3(1)), the assumption $\omega_\tau^+(v) \subset \operatorname{int} V_+ \cup \{0\}$ becomes superfluous. This yields by the invariance of $\omega_\tau^+(v)$ and hypothesis (ii), i.e., $\omega_\tau^+(v) = \varphi(t,\tau)\omega_\tau^+(v) \subset \operatorname{int} V_+$ for $T \leq t - \tau$.

(2) One can show a stronger limit set trichotomy, if φ is induced by a discrete 1-parameter semiflow (cf. [21, Theorem 3.1]). More results in the finite-dimensional situation $V_+ = \mathbb{R}^d_+$ can be found in [22, Theorems 1, 2] and related topics are contained in [33, Theorem 1.1] or [26, Theorem 4.1]. Furthermore, [25, Lemma 4] provides sufficient conditions for the right-hand side of nonautonomous difference equations to generate a uniformly contractive 2-parameter semigroup. On general time scales, stronger limit set trichotomies can be found in [28] under the assumption that φ is uniformly ascending.

(3) We also briefly comment the situation when φ comes from an ordinary differential equation ($\mathbb{T} = \mathbb{R}$). For autonomous cooperative systems in $\mathbb{R}^2$, a prototype result has been given by [15, Theorem 2.3]. If φ comes from a time-periodic equation, [30, Theorem 3.1] proved a "limit set dichotomy" under certain assumptions on the Floquet multipliers. Similar results are given by [22, Theorems 3, 4]; [18, Theorem 6.8] considers general continuous 1-parameter semiflows, and [7, Theorem 3.1] proves a limit set trichotomy for order-preserving skew-product flows.

(4) Finally, the case of random dynamical systems is considered in [3, Theorem 4.2] and [8, pp. 123–124, Theorem 4.4.1].

Proof. Let $\tau \in \mathbb{T}$ be arbitrary, but fixed. If (a) holds, then obviously (b) and (c) cannot hold. If (a) does not hold, then there exists a $v_1 \in V_+^*$ such that the orbit $\gamma_\tau^+(v_1)$ is bounded, i.e., $\|\varphi(t,\tau)v_1\| \le M$ for some $M \ge 0$ and all $t \ge \tau$. Now we show that in this case every orbit $\gamma_\tau^+(v)$, $v \in V_+$, is bounded in norm. Let the vector $v_2 \in V_+$ be arbitrary. Then either (i) $\gamma_\tau^+(v_2)$ is bounded or (ii) there exists a $t' \in \mathbb{T}$ with $T \le t' - \tau$ such that $\varphi(t',\tau)v_2 \ne 0$. In case (ii) it follows from assumption (i) that $\varphi(t,\tau)v_1, \varphi(t,\tau)v_2 \in \operatorname{int} V_+$ for $t \ge t'$. The Remarks 2.5 (1) and (2) imply $K := p(\varphi(t',\tau)v_1, \varphi(t',\tau)v_2) < \infty$. Using the 2-parameter semiflow property of φ (cf. Definition 2.1(ii)) together with the fact that the mappings $\varphi(t,t')$, $t \ge t' + T$, are nonexpansive, we obtain

$$p(\varphi(t,\tau)v_1, \varphi(t,\tau)v_2) \le K \quad \text{for } t \ge t' + T\,.$$

Consequently, Lemma 2.6(a) provides the estimate

$$\|\varphi(t,\tau)v_2\| \le \|\varphi(t,\tau)v_2 - \varphi(t,\tau)v_1\| + \|\varphi(t,\tau)v_1\| \le 2e^K M$$

for all $t \ge t' + T$, proving that $\gamma_\tau^+(v_2)$ is bounded.

Now we show that either (b) or (c) holds. By assumption (ii) the orbits $\gamma_\tau^+(v)$, $v \in V_+$, are relatively compact and therefore $\omega_\tau^+(v) \ne \emptyset$, moreover, the relation $\omega_\tau^+(v) = \{0\}$ is equivalent to $\lim_{t\to\infty} \varphi(t,\tau)v = 0$. We show that

$$\omega_\tau^+(v_1) = \{0\} \text{ for a single } v_1 \in V_+^* \quad \Longrightarrow \quad \omega_\tau^+(v) = \{0\} \text{ for any } v \in V_+^*\,.$$

To this end, we assume that there exist $v_1, v_2, v_2^* \in V_+^*$ with $\omega_\tau^+(v_1) = \{0\}$ and $v_2^* \in \omega_\tau^+(v_2) \setminus \{0\}$. Then there exists a sequence $t_n \to \infty$ in $\mathbb{T}$ with

$$\lim_{n\to\infty} \varphi(t_n,\tau)v_1 = 0 \quad \text{and} \quad \lim_{n\to\infty} \varphi(t_n,\tau)v_2 = v_2^*\,,$$

where we assume without lost of generaliry that $t_0 = \tau$ and $t_{n+1} - t_n \ge T$, which implies by assumption (i) that $\varphi(t_n,\tau)v_i \in \operatorname{int} V_+$ for $i = 1,2$ and $n \in \mathbb{N}$. Using the 2-parameter semiflow property and the fact that the mappings $\varphi(t_{n+1},t_n)$, $n \in \mathbb{N}$, are nonexpansive, we get

$$p(\varphi(t_n,\tau)v_1, \varphi(t_n,\tau)v_2) \le \dots \le p(\varphi(t_1,\tau)v_1, \varphi(t_1,\tau)v_2) \le p(v_1,v_2)$$

for $n \in \mathbb{N}$. Choosing $N \in \mathbb{N}$ such that $\|\varphi(t_n, \tau)v_1\| \le \|\varphi(t_n, \tau)v_2\|$ for $n \ge N$, Lemma 2.6(a) implies the contradiction

$$\begin{aligned}\|\varphi(t_n,\tau)v_2\| &\le \|\varphi(t_n,\tau)v_2 - \varphi(t_n,\tau)v_1\| + \|\varphi(t_n,\tau)v_1\| \\ &\le 3e^{p(v_1,v_2)}\|\varphi(t_n,\tau)v_1\| \to 0 \quad \text{for } n \to \infty ,\end{aligned}$$

proving that either (b) or (c) is true.

It remains to show (3.1) under the additional assumptions that the mappings $\varphi(t, s)$, $T \le t - s$, are uniformly contractive, and that $\omega_\tau^+(v) \in \operatorname{int} V_+ \cup \{0\}$ for all $v \in V_+^*$. Assume that (3.1) does not hold. Then there exists $v_1, v_2 \in V_+^*$, an $\varepsilon > 0$ and a sequence $t_n \to \infty$ in $\mathbb{T}$ with

$$\|\varphi(t_n,\tau)v_1 - \varphi(t_n,\tau)v_2\| \ge \varepsilon \quad \text{for all } n \in \mathbb{N}, \tag{3.2}$$

where we assume without lost of generaliry $t_1 \ge \tau + T$ and $t_{n+1} - t_n \ge T$, which implies that $\varphi(t_n, \tau)v_i \ne 0$ for $i = 1, 2$ and $n \in \mathbb{N}$. Since the orbits $\gamma_\tau^+(v_1)$ and $\gamma_\tau^+(v_2)$ are relatively compact there exists a subsequence of $(t_n)_{n\in\mathbb{N}}$, which we denote by $(t_n)_{n\in\mathbb{N}}$ again, such that the limits

$$v_1^* := \lim_{n\to\infty} \varphi(t_n,\tau)v_1 \quad \text{and} \quad v_2^* := \lim_{n\to\infty} \varphi(t_n,\tau)v_2$$

exist. By assumption $v_1^*, v_2^* \in \operatorname{int} V_+ \cup \{0\}$ and by (3.2) $v_1^* \ne v_2^*$. We can also rule out that $v_1^* = 0$ and $v_2^* \in \operatorname{int} V_+$, since in this case, choosing $N \in \mathbb{N}$ such that $\|\varphi(t_n, \tau)v_1\| \le \|\varphi(t_n, \tau)v_2\|$ for $n \ge N$, Lemma 2.6(a) would imply

$$\begin{aligned}\|\varphi(t_n,\tau)v_2\| &\le \|\varphi(t_n,\tau)v_2 - \varphi(t_n,\tau)v_1\| + \|\varphi(t_n,\tau)v_1\| \\ &\le 3e^{p(\varphi(t_1,\tau)v_1,\varphi(t_1,\tau)v_2)}\|\varphi(t_n,\tau)v_1\| \to 0 \quad \text{for } n \to \infty ,\end{aligned}$$

contradicting $v_1^* \ne v_2^*$. Hence we have $v_1^*, v_2^* \in \operatorname{int} V_+$. The 2-parameter semiflow property and the fact that the mappings $\varphi(t_{n+1}, t_n)$, $n \in \mathbb{N}$, are uniformly contractive, imply the estimates

$$\begin{aligned}p(\varphi(t_n,\tau)v_1,\varphi(t_n,\tau)v_2) &\le c(\varphi(t_n,\tau)v_1,\varphi(t_n,\tau)v_2) < \dots \\ &< p(\varphi(t_1,\tau)v_1,\varphi(t_1,\tau)v_2) \le c(\varphi(t_1,\tau)v_1,\varphi(t_1,\tau)v_2) .\end{aligned}$$

Since monotone and bounded sequences converge and the mappings p and c are continuous in (v_1^*, v_2^*) by Lemma 2.6(b) and assumption, respectively, we get

$$p(v_1^*, v_2^*) = c(v_1^*, v_2^*)$$

in contradiction to $c(v_1^*, v_2^*) < p(v_1^*, v_2^*)$, thus proving that (3.1) holds. □

4. Subhomogeneous Order-Preserving Semiflows

The results of Section 3 are only helpful, if one has verifiable conditions which guarantee that a 2-parameter semiflow is nonexpansive with respect to the part-metric. Sufficient conditions will be given by the following notions.

Definition 4.1. Let $U \subset V_+$. A 2-parameter semiflow φ on V_+ is said to be

(i) *order-preserving on* U if

$$u, v \in U,\ u \le v \quad \Longrightarrow \quad \varphi(t,\tau)u \le \varphi(t,\tau)v \quad \text{for all } \tau \le t;$$

(ii) *strictly order-preserving on* U, if it is order-preserving on U and

$$u, v \in U,\ u < v \quad \Longrightarrow \quad \varphi(t,\tau)u < \varphi(t,\tau)v \quad \text{for all } \tau \le t;$$

(iii) *strongly order-preserving on* U, if $\operatorname{int} V_+ \neq \emptyset$, if φ is order-preserving on U and

$$u, v \in U,\ u \ll v \quad \Longrightarrow \quad \varphi(t,\tau)u \ll \varphi(t,\tau)v \quad \text{for all } \tau \leq t.$$

We now introduce a class of order-preserving 2-parameter semiflows which possess a certain concavity property we call *subhomogeneity* (sometimes also named sublinearity). Subhomogeneity means concavity for the particular case in which one of the reference points is 0, hence asks less and is thus more general than classical concavity. The autonomous version of this property plays an important role in many studies and applications, see [19, 20], [21, 22], [30, 33] and the references therein.

Definition 4.2. A 2-parameter semiflow φ, which is order-preserving on V_+, is said to be

(i) *subhomogeneous*, if for any $v \in V_+$ and for any $\alpha \in (0,1)$ we have

$$\alpha\varphi(t,\tau)v \leq \varphi(t,\tau)\alpha v \quad \text{for all } \tau < t; \tag{4.1}$$

(ii) *strictly subhomogeneous*, if we have in addition for any $v \in \operatorname{int} V_+$ the strict inequality

$$\alpha\varphi(t,\tau)v < \varphi(t,\tau)\alpha v \quad \text{for all } \tau < t. \tag{4.2}$$

Remark 4.3. Inequality (4.1) holds automatically for $t = \tau$ and for $\alpha \in \{0,1\}$; it can be equivalently rewritten as follows: For any $v \in V_+$ and for any $\alpha > 1$ we have

$$\varphi(t,\tau)\alpha v \leq \alpha\varphi(t,\tau)v \quad \text{for all } \tau < t \tag{4.3}$$

and $\varphi(t,\tau)\alpha v < \alpha\varphi(t,\tau)v$ instead of (4.2), respectively.

Lemma 4.4. *Let φ be a subhomogeneous 2-parameter semiflow on V_+, which is order-preserving on V_+. Then*

(a) *φ preserves the equivalence relation from Definition 2.4(i) and is nonexpansive under the part metric on every part C of V_+.*
(b) *If, moreover, φ is strictly subhomogeneous on a part C of V_+, it is contractive under the part metric on C.*

Remark 4.5. It is easy to see that a contractive 2-parameter semiflow possesses at most one equilibrium in C.

Proof. (a) It follows from (4.1) and (4.3) that, if for $v, \bar{v} \in C$ and some $\alpha \geq 1$ the estimate $\alpha^{-1}v \leq \bar{v} \leq \alpha\bar{v}$ implies

$$\alpha^{-1}\varphi(t,\tau)v \leq \varphi(t,\tau)\bar{v} \leq \alpha\varphi(t,\tau)v \quad \text{for all } \tau \leq t$$

and hence by the definition of the part metric

$$p(\varphi(t,\tau)v, \varphi(t,\tau)\bar{v}) \leq p(v,\bar{v}) \quad \text{for all } \tau \leq t.$$

(b) Analogously, under the assumption (4.2), it follows

$$\alpha^{-1}\varphi(t,\tau)v < \varphi(t,\tau)\bar{v} < \alpha\varphi(t,\tau)v \quad \text{for all } t > \tau$$

and this leads to the contractivity of φ. $\square$

5. Order-Preserving Dynamic Equations

Supplementing our explanations from Section 2, we need some further terminology. I_V denotes the identity map on V. The *dual cone* V'_+ of V_+ is the set of all linear and continuous mappings $v' : V \to \mathbb{R}$ such that $\langle v, v' \rangle \geq 0$ for all $v \in V_+$, where $\langle v, v' \rangle := v'(v)$ is the duality mapping. If V is a Hilbert space, then V'_+ can be identified with a subset of V through the Riesz representation theorem (cf. [23, p. 104, Theorem 2.1]). The elements of $V^\star_+ := V'_+ \setminus \{0\}$ are called *supporting forms* and we define $\mathcal{L}(V_+) := \{T \in \mathcal{L}(V) : T(V_+) \subset V_+\}$. Any such operator $T \in \mathcal{L}(V_+)$ is called *positive*, and *strictly positive*, if $Tv = 0$ implies $v = 0$ for any $v \in V_+$.

First of all, we can characterize the (interior) points of a cone in terms of linear functionals.

Lemma 5.1. *For any $v \in V$ the following holds:*

(a) $v \in V_+ \Leftrightarrow \langle v, v' \rangle \geq 0$ *for all* $v' \in V^\star_+$,
(b) $v \in \operatorname{int} V_+ \Leftrightarrow \langle v, v' \rangle > 0$ *for all* $v' \in V^\star_+$.

Proof. See [10, p. 221, Proposition 19.3]. □

At this point we introduce some further notation concerning the calculus on time scales (see also [6]). Remember that $\mathbb{T}$ is a closed subset of $\mathbb{R}$ which is assumed to be unbounded above. $\sigma(t) := \inf\{s \in \mathbb{T} : t < s\}$ defines the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ and $\mu(t) := \sigma(t) - t$ the *graininess* of $\mathbb{T}$. A point $t \in \mathbb{T}$ is called *right-dense*, if $\sigma(t) = t$ and otherwise *right-scattered*. Analogously, in case $\sup\{s \in \mathbb{T} : s < t\} = t$, the point $t \in \mathbb{T}$ is said to be *left-dense*. It is worth to mention that all the results of this paper remain true with obvious modifications, if the time scale $\mathbb{T}$ is replaced by a general measure chain (cf. [14]).

Now we will discuss dynamic equations in real Banach spaces of the form

$$v^\Delta = f(t, v), \tag{5.1}$$

where the *right-hand side* $f : \mathbb{T} \times V \to V$ satisfies the following assumptions:

(H0) V is a strongly ordered Banach space with cone V_+, i.e., $\operatorname{int} V_+ \neq \emptyset$,
(H1) $f : \mathbb{T} \times V \to V$ is rd-continuously differentiable with respect to the second variable, i.e., the partial derivative $D_2 f : \mathbb{T} \times V \to V$ is assumed to exist and, furthermore, is rd-continuous,
(H2) for any $\tau \in \mathbb{T}$ and $v \in V$ the solution $t \mapsto \varphi(t, \tau, v)$ of (5.1) starting at time τ in v exists for all $\tau \leq t$.

Under the condition (H1) the solutions of (5.1) exist and are unique locally in forward time (cf. [6, p. 324, Theorem 8.20]), with continuous partial derivatives $D_3\varphi(t, \tau, v) \in \mathcal{L}(V)$, $\tau \leq t$, $v \in V$ (cf. [27, pp. 47–48, Satz 1.2.22]) and by the absence of finite escape times, obviously φ defines a 2-parameter semiflow on V (cf. [27, p. 42, Korollar 1.2.19]), where all the results from the previous sections apply to φ under certain assumptions on f.

Lemma 5.2 (Characterization of Forward Invariance)**.** *The following three statements are equivalent.*

(a) *The cone V_+ is forward invariant for* (5.1).
(b) *For every right-dense $t_0 \in \mathbb{T}$, any $v \in \partial V_+$, $v' \in V^\star_+$ such that $\langle v, v' \rangle = 0$ satisfy $\langle f(t_0, v), v' \rangle \geq 0$, and, for every right-scattered $t_0 \in \mathbb{T}$ any $v \in V_+$ satisfies $v + \mu(t_0) f(t_0, v) \in V_+$.*

(c) *For every* $t_0 \in \mathbb{T}$ *it holds*

$$\lim_{h \searrow \mu(t_0)} \frac{\operatorname{dist}(v + hf(t_0, v), V_+)}{h} = 0, \tag{5.2}$$

if $v \in \partial V_+$ *and* t_0 *is right-dense, or,* $v \in V_+$ *and* t_0 *is right-scattered.*

Remark 5.3. The condition (5.2) provides a descriptive geometric interpretation: In a right-scattered point $t_0 \in \mathbb{T}$ it simply means that $v + \mu(t)f(t, v) \in V_+$, if $v \in V_+$. In a right-dense point $t_0 \in \mathbb{T}$, and at a boundary point $\nu(t_0) \in \partial V_+$ with a tangent, $f(t_0, \nu(t_0))$ and hence the vector $\nu^\Delta(t_0)$ have to be directed to the interior of V_+, i.e., this vector does not point into the outer half space. Both conditions force the solutions to remain in V_+.

Proof. Let $t_0 \in \mathbb{T}$ be arbitrary. We proceed in four steps:

(I) In case of a right-dense t_0 the equivalence of (b) and (c) is shown in [9, p. 51, Example 4.1]. In a right-scattered t_0 the relation (5.2) obviously holds, if and only if $v + \mu(t_0)f(t_0, v) \in V_+$, since V_+ is closed.

(II) We show that the forward invariance of V_+ implies (b). Thereto let $t_0 \in \mathbb{T}$, $v \in V_+$ be arbitrary and let ν be the solution of (5.1) with $\nu(t_0) = v$. In a right-scattered t_0 the invariance of V_+ implies

$$v + \mu(t_0)f(t_0, v) = \nu(t_0) + \mu(t_0)f(t_0, \nu(t_0)) = \nu(\sigma(t_0)) \in V_+\,.$$

On the other hand, if t_0 is right-dense and $v \in \partial V_+$, choose $v' \in V_+^\star$ such that $\langle v, v'\rangle = 0$ (cf. Lemma 5.1). Then the assumption $\langle f(t_0, v), v'\rangle < 0$ would imply the existence of a right-sided $\mathbb{T}$-neighborhood N of t_0 with $\langle \nu(t), v'\rangle < 0$ for $t \in N$ and hence the contradiction $\nu(t) \notin V_+$ (cf. Lemma 5.1(a)).

(III) In the remaining two steps we show the forward invariance of V_+ under the condition (b). For the present, we strengthen (b) to the hypothesis that for any right-dense point t_0, every $v \in \partial V_+$ and every $v' \in V_+^\star$ such that $\langle v, v'\rangle = 0$, one has $\langle f(t_0, v), v'\rangle > 0$. Thus, let ν denote a solution of (5.1) starting at $\tau \in \mathbb{T}$ in $\nu(\tau) \in \operatorname{int} V_+$. If the claim were false, then there exists a finite $t^* \in \mathbb{T}$ given by

$$t^* := \sup T, \quad T := \{t \geq \tau : \nu(s) \in V_+ \text{ for all } s \in [\tau, t]_{\mathbb{T}}\}.$$

Since the cone V_+ is closed we have $\nu(t^*) \in V_+$. The point t^* is right-dense, because otherwise $\nu(\sigma(t^*)) = \nu(t^*) + \mu(t^*)f(t^*, \nu(t^*)) \in V_+$ would yield the contradiction $t^* < \sigma(t^*) \in T$. Moreover, $\nu(t^*) \in \partial V_+$, because the assumption $\nu(t^*) \in \operatorname{int} V_+$ would imply the existence of a neighborhood $U \subset V_+$ of $\nu(t^*)$ and a $\mathbb{T}$-neighborhood N of t^* with $\nu(t) \in U$ for $t \in N$, since ν is continuous as the solution of (5.1). This again contradicts the definition of t^*. Now by Lemma 5.1(b) there exists a supporting form $v' \in V_+^\star$ such that $\langle \nu(t^*), v'\rangle = 0$ and by definition t^* is the time when the solution ν leaves V_+, which by Lemma 5.1(a) gives us $\langle \nu(t), v'\rangle \leq 0$ for t from a right-sided neighborhood of t^*. One finds $\langle \frac{\nu(t)-\nu(t^*)}{t-t^*}, v'\rangle \leq 0$ and in the limit $t \searrow t^*$, we therefore obtain the contradiction

$$0 \geq \langle \nu^\Delta(t^*), v'\rangle = \langle f(t^*, \nu(t^*)), v'\rangle > 0$$

with a view to the above (strengthened) hypothesis.

(IV) The verification of the assumption under the general hypothesis yields as follows: For arbitrary reals $\varepsilon > 0$ and some fixed $e \in \operatorname{int} V_+$ one can apply the above step (III) to the solution ν_ε of the dynamic equation

$$v^\Delta = f(t, v) + \varepsilon e$$

and consequently, for any $\tau \in \mathbb{T}$ and $v_0 \in V_+$ one obtains $\nu_\varepsilon(t) \in V_+$ for $\tau \le t$, provided that $\nu_\varepsilon(\tau) \in V_+$. Now let ν denote the solution of (5.1) satisfying $\nu(\tau) = \nu_\varepsilon(\tau)$. One should bear in mind that $\nu_\varepsilon(t)$ is continuous in (ε, t) (cf. [27, p. 39, Satz 1.2.17]) and consequently uniformly continuous on the set $K \times [0, \varepsilon_0]$, where $K \subset [\tau, \infty)_\mathbb{T}$ is a compact $\mathbb{T}$-interval and $\varepsilon_0 > 0$ arbitrary. By a standard argument, the solutions ν_ε converge to ν uniformly on K as $\varepsilon \searrow 0$. $\square$

Corollary 5.4. *Let $V_+ \subset V$ be a normal cone and assume that*

(H3) *in any left-dense $t_0 \in \mathbb{T}$ there exists a left-sided $\mathbb{T}$-neighborhood $N_0(t_0)$ of t_0 such that $f(s, 0) \in V_+$ for all $s \in N_0(t_0)$.*

Then V_+^* is forward invariant for (5.1)*, if and only if every right-dense $t_0 \in \mathbb{T}$, any $v \in \partial V_+$, $v' \in V_+^\star$ such that $\langle v, v'\rangle = 0$ satisfy $\langle f(t_0, v), v'\rangle \ge 0$, and, for every right-scattered $t_0 \in \mathbb{T}$ any $v \in V_+^*$ satisfies $v + \mu(t_0) f(t_0, v) \in V_+^*$.*

Proof. We have to show two directions:

($\Rightarrow$) If V_+^* is a forward invariant set, then the assertion can be shown analogously to step (II) in the proof of Lemma 5.2.

($\Leftarrow$) Using the induction principle (cf. [6, p. 4, Theorem 1.7]) we deduce the statement

$$\mathcal{A}(t) : v \neq 0 \quad \Longrightarrow \quad \varphi(t, \tau)v \neq 0 \quad \text{for all } \tau \le t.$$

Above all, choose $v \in V_+^*$ arbitrarily.

- $\mathcal{A}(\tau)$ obviously holds since $\varphi(\tau, \tau)v = v$.
- Let $t \ge \tau$ be right-scattered and $\mathcal{A}(t)$ be true. Then by the 2-parameter semiflow property and the assumption one immediately gets
$$\varphi(\sigma(t), \tau)v = \varphi(t, \tau)v + \mu(t) f(t, \varphi(t, \tau)v) \neq 0$$
i.e., $\mathcal{A}(\sigma(t))$ holds.
- Let $t \ge \tau$ be right-dense and assume that $\mathcal{A}(t)$ is valid. Then $\varphi(t, \tau)v \neq 0$ implies that $\varphi(s, \tau)v \neq 0$ in a right-sided $\mathbb{T}$-neighborhood N of t. Hence $\mathcal{A}(t)$ yields $\varphi(s, \tau)v \neq 0$ for $s \in N$.
- Let $t \ge \tau$ be left-dense and $\mathcal{A}(s)$ be true for $s < t$. We want to show $\mathcal{A}(t)$ and proceed indirectly, i.e., assume that we have $\varphi(t, \tau)v_0 = 0$ for some $v_0 \in V_+^*$. Since (H3) holds, we get from Lemma 5.2 that
$$\begin{aligned} 0 \le \varphi(s, \tau)v_0 &= -\int_s^t f(\rho, \varphi(\rho, \tau)v_0)\, \Delta\rho \\ &\le \int_s^t \big[f(\rho, 0) - f(\rho, \varphi(\rho, \tau)v_0)\big]\, \Delta\rho \quad \text{for all } s \in N_0(t) \end{aligned}$$
and Hypothesis (H1) implies that $C(\rho) := \sup_{h \in [0,1]} \|D_2 f(\rho, h\varphi(\rho, \tau)v_0)\|$ exists as an rd-continuous function in $\rho \in N_0(t)$. By assumption the cone V_+ is normal and therefore
$$\begin{aligned} \|\varphi(s, \tau)v_0\| &\le \int_s^t \|f(\rho, 0) - f(\rho, \varphi(\rho, \tau)v_0)\|\, \Delta\rho \\ &\le -\int_t^s C(\rho)\|\varphi(\rho, \tau)v_0\|\, \Delta\rho \quad \text{for all } s \in N_0(t). \end{aligned}$$
Due to the limit relation $\lim_{\rho \nearrow t} \mu(t) = 0$ one can choose a left-sided $\mathbb{T}$-neighborhood $N \subset N_0(t) \cap [\tau, t]_\mathbb{T}$ such that we have $C(\rho)\mu(\rho) < 1$ for $\rho \in N \setminus \{t\}$. Thus $-C(\rho)$ is positively regressive on $N \setminus \{t\}$ and from the

Gronwall lemma (cf. [6, p. 256, Theorem 6.4]) we obtain $\varphi(s,\tau)v_0 = 0$ for $s \in N$. This contradicts $\mathcal{A}(s)$.

Hence the proof of Corollary 5.4 is complete. □

Before stating the next result we refer to [27, p. 54, Definition 1.3.5] for the definition of the *transition operator* $\Phi_A(t,\tau) \in \mathcal{L}(V)$ of a linear dynamic equation

$$v^\Delta = A(t)v \tag{5.3}$$

in the nonregressive case. Now the forward invariance of V_+ with respect to (5.3) is a necessary and sufficient condition for the positivity of $\Phi_A(t,\tau)$.

Corollary 5.5. *Let $A : \mathbb{T} \to \mathcal{L}(V)$ be rd-continuous and $t,\tau \in \mathbb{T}$. Then the following statements are equivalent:*

(a) $\Phi_A(t,\tau) \in \mathcal{L}(V_+)$ *for* $\tau \le t$.

(b) *For every right-dense $t \ge \tau$, $v \in \partial V_+$ and $v' \in V_+^\star$ satisfying $\langle v, v'\rangle = 0$, the inequality $\langle A(t)v, v'\rangle \ge 0$ holds, and, moreover, for every right-scattered $t \ge \tau$, $v \in V_+$ the inclusion $v + \mu(t)A(t)v \in V_+$ holds.*

(c) *For every $t \ge \tau$ it holds*

$$\lim_{h \searrow \mu(t)} \frac{\operatorname{dist}(v + hA(t)v, V_+)}{h} = 0,$$

if $v \in \partial V_+$ and t is right-dense, or, $v \in V_+$ and t is right-scattered.

Proof. Evidently Lemma 5.2 applies to (5.3) and therefore V_+ is forward invariant with respect to (5.3), which, in turn, yields $\Phi_A(t,\tau)V_+ \subset V_+$ for $\tau \le t$. □

Adopting terminology introduced in [15], we denote a nonvoid subset $U \subset V$ as V_+*-convex*, if for any $u, v \in U$ such that $u \le v$, the whole line segment between u and v is contained in U, i.e., $u + h(v - u) \in U$ for $h \in [0,1]$. Evidently the cone V_+ itself is V_+-convex. With all the above preliminaries at hand, we can proceed to an appropriate definition of cooperativity.

Definition 5.6. Let $U \subset V$ be V_+-convex. A dynamic equation of the form (5.1) is called

(i) V_+*-cooperative on* U, if for all right-dense $t \in \mathbb{T}$, $u \in U$, $v \in \partial V_+$ and $v' \in V_+^\star$ such that $\langle v, v'\rangle = 0$, the inequality $\langle v', D_2 f(t,u)v\rangle \ge 0$ holds and, moreover, if for every right-scattered $t \in \mathbb{T}$, $u \in U$, $v \in V_+$ the inclusion $v + \mu(t)D_2 f(t,u)v \in V_+$ holds,

(ii) *strictly* V_+*-cooperative on* U, if (5.1) is V_+-cooperative on U, satisfies (H3), and if for every right-scattered $t \in \mathbb{T}$, $u \in U$, $v \in V_+$ the implication $v + \mu(t)D_2 f(t,u)v = 0 \Rightarrow v = 0$ holds.

Remark 5.7. Fix $\tau \in \mathbb{T}$ and $u \in U$ arbitrarily.

(1) Since the partial derivative $D_3\varphi(\cdot,\tau,u) : [\tau,\infty)_\mathbb{T} \to \mathcal{L}(V)$ solves the variational equation

$$X^\Delta = D_2 f(t, \varphi(t,\tau,u))X \tag{5.4}$$

to the initial condition $X(\tau) = I_V$ on $[\tau,\infty)_\mathbb{T}$ (cf. [27, pp. 47–48, Satz 1.2.22]), by Corollary 5.5 the dynamic equation (5.1) is V_+-cooperative on the set U, if and only if $D_3\varphi(t,\tau,u) \in \mathcal{L}(V_+)$ holds for $\tau \le t$ and $u \in U$.

(2) Assume that V_+ is normal and that (5.1) satisfies (H3). By using Corollary 5.4 instead of Lemma 5.2 in the proof of Corollary 5.5, it is not difficult to

see that (5.1) is strictly V_+-cooperative on U, if and only if $D_3\varphi(t,\tau,u)$ is strictly positive for $\tau \le t$ and $u \in U$.

Example 5.8. Let $V_+ = \mathbb{R}^d_+$ be the nonnegative orthant in the Banach space $V = \mathbb{R}^d$. Then $\mathcal{L}(V_+)$ is (isomorphic to) the set $\mathbb{R}^{d\times d}_+$ of nonnegative matrices. A mapping $f : \mathbb{T} \times \mathbb{R}^d \to \mathbb{R}^d$ is $\mathbb{R}^d_+$-cooperative, if in each right-dense point t the off-diagonal elements of $D_2 f(t,u) \in \mathbb{R}^{d\times d}$ are nonnegative and, if in each right-scattered point t the matrix $I_{\mathbb{R}^d} + \mu(t)D_2 f(t,u)$ is nonnegative for every $u \in U$. In the case of ordinary differential equations, where $\mathbb{T} = \mathbb{R}$ consists of right-dense points, this definition coincides with the one from [15].

Remark 5.9 (Euler discretization of cooperative ODEs)**.** Consider an $\mathbb{R}^d_+$-cooperative ordinary differential equation $\dot v = f(t,v)$. Then according to Example 5.8 its Euler discretization $\nu(t_{n+1}) = \nu(t_n) + (t_{n+1} - t_n) f(t_n, \nu(t_n))$ on a discrete time scale $\mathbb{T} = \{t_n\}_{n\in\mathbb{N}_0}$ with $t_{n+1} > t_n$, is also $\mathbb{R}^d_+$-cooperative if the matrix $I_{\mathbb{R}^d} + (t_{n+1} - t_n) D_2 f(t_n, u)$ is nonnegative. Since the off-diagonal elements of $D_2 f(t,u)$ are nonnegative, this is true, if the diagonal entries $a_{ii}(t_n,u)$, $i = 1,\dots,d$, of the matrix $D_2 f(t_n,u)$ satisfy the condition $a_{ii}(t_n,u) \ge \frac{-1}{t_{n+1}-t_n}$ with the stepsize $\mu(t_n) = t_{n+1} - t_n$ of the Euler discretization.

Theorem 5.10 (Müller's Theorem)**.** *Let $U \subset V$ be V_+-convex.*

(a) *If* (5.1) *is V_+-cooperative on U, then φ is order-preserving on U.*
(b) *Conversely, if φ is order-preserving on U, then the dynamic equation* (5.1) *is V_+-cooperative on U.*

Proof. (a) Choose $u, v \in U$ with $u \le v$ and because of the V_+-convexity of U one has $u + h(v-u) \in U$ for $h \in [0,1]$. Then the mean value theorem (cf. [23, p. 341, Theorem 4.2]) yields

$$\varphi(t,\tau,v) - \varphi(t,\tau,u) = \int_0^1 D_3\varphi(t,\tau,u + h(v-u))(v-u)\,dh \quad \text{for all } \tau \le t$$

and since $D_3\varphi(t,\tau,w) \in \mathcal{L}(V_+)$, $w \in U$, we obtain $D_3\varphi(t,\tau,w)(v-u) \in V_+$. Now convexity of the integral implies the claim $\varphi(t,\tau,v) \le \varphi(t,\tau,u)$.

(b) Using the fact that $D_3\varphi(t,\tau,v)$ solves the dynamic equation (5.4) in $\mathcal{L}(V_+)$, we get the assertion (b) with a view to Corollary 5.5. $\square$

The next part of this section is dedicated to sufficient conditions for strictly and strongly order-preserving mappings. Since we have not assumed regressivity of f (cf. [6, pp. 321–322, Definition 8.14(ii)]) the mapping $\varphi(t,\tau) : V \to V$, $\tau \le t$, needs not to be a homeomorphism. Hence the arguments of [32, pp. 32–33, Proof of Proposition 1.1] do not apply directly.

Corollary 5.11. *Let $U \subset V$ be V_+-convex. If for a V_+-cooperative system* (5.1) *on U one of the following conditions*

(i) *$I_V + \mu(t) f(t,\cdot) : V \to V$ in one-to-one on U for any $t \in \mathbb{T}$,*
(ii) *$I_V + \mu(t) f(t,\cdot) : V \to V$ is strictly order-preserving on U for any $t \in \mathbb{T}$*

holds, then φ is strictly order-preserving on U.

Proof. We proceed in two steps:

(I) To show that (i) implies (ii), fix arbitrary $t \in \mathbb{T}$, $u \in U$ and abbreviate $F(u) := u + \mu(t) f(t,u)$. Observing the fact

$$\varphi(\sigma(t),t)u = u + \mu(t) f(t,u) = F(u), \tag{5.5}$$

it is evident that F is strictly order-preserving on U.

(II) We apply the induction principle (cf. [6, p. 4, Theorem 1.7]) to the statement

$$\mathcal{A}(t) : u < v \quad \Longrightarrow \quad \varphi(t,\tau)u < \varphi(t,\tau)v \quad \text{for all } \tau \leq t.$$

First of all, choose $u, v \in U$, $u < v$ arbitrarily.

- $\mathcal{A}(\tau)$ is clearly satisfied since $\varphi(\tau,\tau)u = u$.
- Let $t \geq \tau$ be right-scattered and $\mathcal{A}(t)$ be true. Then by 2-parameter semiflow property and (ii) one obtains
$$\varphi(\sigma(t),\tau)u \overset{(5.5)}{=} \varphi(t,\tau)u + \mu(t)f(t,\varphi(t,\tau)u) < \\ < \varphi(t,\tau)v + \mu(t)f(t,\varphi(t,\tau)v) \overset{(5.5)}{=} \varphi(\sigma(t),\tau)v\,,$$
i.e., $\mathcal{A}(\sigma(t))$ holds.
- Let $t \geq \tau$ be right-dense and assume that $\mathcal{A}(t)$ is valid. Then there exists a $\mathbb{T}$-neighborhood N of t such that $\varphi(s,t) : V \to V$ is a homeomorphism for $s \in N$ and in particular one-to-one. Hence $\mathcal{A}(t)$ yields $\varphi(s,\tau)u < \varphi(s,\tau)v$ for $s \in N$.
- Let $t \geq \tau$ be left-dense and $\mathcal{A}(s)$ be true for $s < t$. Similar to the above we get that $\varphi(t,s) : V \to V$ is a homeomorphism for s in some $\mathbb{T}$-neighborhood of t, which gives us $\varphi(t,\tau)u < \varphi(t,\tau)v$, i.e., $\mathcal{A}(t)$ is valid.

Thus the proof is complete. □

Corollary 5.12. *Let $U \subset V$ be V_+-convex. If for a V_+-cooperative system* (5.1) *on U one of the following conditions*

(i) *$I_V + \mu(t)D_2f(t,u) \in \mathcal{L}(V)$ is onto for any $u \in U$ and any $t \in \mathbb{T}$,*
(ii) *$I_V + \mu(t)f(t,\cdot) : V \to V$ is strongly order-preserving on U for any $t \in \mathbb{T}$*

holds, then φ is strongly order-preserving on U.

Remark 5.13. In the case of ordinary differential equations, where $\mathbb{T} = \mathbb{R}$ consists of right-dense points, we have $\mu(t) \equiv 0$ on $\mathbb{T}$, and both conditions (i) and (ii) in Corollary 5.11 and 5.12 are dispensable. Therefore, solutions of V_+-cooperative ODEs are always strictly and strongly order-preserving.

Proof. We proceed in two steps again:

(I) To show that (i) implies (ii) fix arbitrary $t \in \mathbb{T}$, $u, v \in U$ with $u \ll v$ and use the notation from the proof of Corollary 5.11. Then Theorem 5.10(a) yields that F maps the order-interval $[u,v]$ into the order-interval $[F(u),F(v)]$. Now we prove that the latter set has nonempty interior, which guarantees $F(u) \ll F(v)$. To do so, pick some $w \in \operatorname{int}[u,v]$ arbitrarily. Using the hypothesis (i) we see that F must be locally open in a neighborhood of w by the Surjective Mapping Theorem (cf. [23, p. 397, Theorem 3.5]). Consequently, we obtain the inclusion $F(w) \in \operatorname{int}[F(u),F(w)]$ and F is strongly order-preserving.

(II) We apply the induction principle (cf. [6, p. 4, Theorem 1.7]) to the statement

$$\mathcal{A}(t) : u \ll v \quad \Longrightarrow \quad \varphi(t,\tau)u \ll \varphi(t,\tau)v \quad \text{for all } \tau \leq t.$$

First of all, choose $u, v \in U$, $u \ll v$ arbitrarily.

- $\mathcal{A}(\tau)$ is clearly satisfied since $\varphi(\tau,\tau)u = u$.
- The implication $\mathcal{A}(t) \Rightarrow \mathcal{A}(\sigma(t))$ for right-scattered $t \geq \tau$ results as in the corresponding part of the proof of Corollary 5.11 with the relation $<$ replaced by $\ll$.

- Let $t \geq \tau$ be right-dense and assume that $\mathcal{A}(t)$ is valid. Then there exists a $\mathbb{T}$-neighborhood N of t such that $\varphi(s,t) : V \to V$ is a homeomorphism for $s \in N$. Since $[\varphi(t,\tau)u, \varphi(t,\tau)v]$ has nonempty interior by the induction hypothesis $\mathcal{A}(t)$, also $\operatorname{int}[\varphi(s,\tau)u, \varphi(s,\tau)v] \neq \emptyset$ holds for $s \in N$, which is equivalent to $\varphi(s,\tau)u \ll \varphi(s,\tau)v$.
- Let $t \geq \tau$ be left-dense and $\mathcal{A}(s)$ be true for $s < t$. Similar to the above we get that $\varphi(t,s) : V \to V$ is a homeomorphism for s in some $\mathbb{T}$-neighborhood of t, which, in turn, yields $\varphi(t,\tau)u \ll \varphi(t,\tau)v$, i.e., $\mathcal{A}(t)$ is valid.

Thus the proof is complete. □

So far, Theorem 5.10 provides a criterion that the solution operator φ of (5.1) is order-preserving. In order to apply Theorem 3.1, and in reference to Lemma 4.4, we need additional conditions for the subhomogeneity of φ.

Lemma 5.14. *Let* (5.1) *be V_+-cooperative on V_+. Then*

(a) *φ is subhomogeneous, if and only if*

$$D_3\varphi(t,\tau,v)v \leq \varphi(t,\tau,v) \quad \text{for all } \tau \leq t,\ v \in V_+; \tag{5.6}$$

(b) *φ is strictly subhomogeneous, if*

$$D_3\varphi(t,\tau,v)v < \varphi(t,\tau,v) \quad \text{for all } \tau < t,\ v \in V_+^*.$$

Proof. (a) Let $\tau \leq t$ be fixed in $\mathbb{T}$. Consider for $v' \in V_+^\star$ and $v \in V_+$ the function $\phi_{v',v} : (0,\infty) \to \mathbb{R}$, $\phi_{v',v}(\alpha) := \frac{1}{\alpha}\langle \varphi(t,\tau)\alpha v, v'\rangle$. We show that φ is subhomogeneous, if and only if $\phi_{v',v}$ is decreasing for all $v' \in V_+^\star$, $v \in V_+$:
($\Rightarrow$) If φ is subhomogeneous, then for arbitrary $0 < \alpha \leq \beta$ there holds the inequality $\frac{\alpha}{\beta}\varphi(t,\tau)\beta v \leq \varphi(t,\tau)\alpha v$, i.e., we have $\frac{1}{\beta}\varphi(t,\tau)\beta v \leq \frac{1}{\alpha}\varphi(t,\tau)\alpha v$. By Lemma 5.1(a) this implies that $\phi_{v',v}$ is decreasing.
($\Leftarrow$) Conversely, let $\phi_{v',v}$ be decreasing in $0 < \alpha < 1$. Then $\phi_{v',v}(1) \leq \phi_{v',v}(\alpha)$, and since $v' \in V_+^\star$ was arbitrary, we readily obtain $\varphi(t,\tau)v \leq \frac{1}{\alpha}\varphi(t,\tau)\alpha v$ from Lemma 5.1(a).
By assumption on f, the function $\phi_{v',v}$ is differentiable and the chain rule implies

$$\phi'_{v',v}(\alpha) = \frac{\langle \alpha D_3\varphi(t,\tau,\alpha v)v - \varphi(t,\tau,\alpha v), v'\rangle}{\alpha^2} \quad \text{for all } \alpha > 0.$$

Thus the subhomogeneity of the mapping φ is equivalent to the property that $\langle \alpha D_3\varphi(t,\tau,\alpha v)v - \varphi(t,\tau,\alpha v), v'\rangle \leq 0$, i.e., by Lemma 5.1(a) to the condition (5.6).

(b) Now let $\tau < t$ be arbitrary points in $\mathbb{T}$. Along the same lines as in (a), one shows that φ is strictly subhomogeneous, if and only if the mapping $\phi_{v',v}$ is strictly decreasing. This property, in turn, is necessary for $\phi'_{v',v}(\alpha) < 0$, $0 < \alpha$, and by Lemma 5.1(b) we obtain the assertion. □

Theorem 5.15. *Let* (5.1) *be V_+-cooperative on V_+. Then*

(a) *φ is subhomogeneous, if*

$$D_2f(t,v)v \leq f(t,v) \quad \text{for all } t \in \mathbb{T},\ v \in V_+; \tag{5.7}$$

(b) *φ is strictly subhomogeneous, if, moreover,* (5.1) *is strictly V_+-cooperative on V_+ and*

$$D_2f(t,v)v < f(t,v) \quad \text{for all } t \in \mathbb{T},\ v \in V_+^*.$$

Proof. Let $\tau \leq t$ and $u \in V_+$ be fixed.

(a) We are going to show that the mapping $\Lambda : [\tau, \infty)_{\mathbb{T}} \to V$, $\Lambda(t) := \varphi(t, \tau, u) - D_3\varphi(t, \tau, u)u$ has values in the cone V_+. Thereto consider

$$\begin{aligned}\Lambda^{\Delta}(t) &\overset{(5.1)}{=} f(t, \varphi(t, \tau, u)) - D_2 f(t, \varphi(t, \tau, u))D_3\varphi(t, \tau, u)u = \\ &= D_2 f(t, \varphi(t, \tau, u))\Lambda(t) + l(t)\end{aligned}$$

with $l(t) := f(t, \varphi(t, \tau, u)) - D_2 f(t, \varphi(t, \tau, u))\varphi(t, \tau, u)$. Since $l : [\tau, \infty)_{\mathbb{T}} \to V$ is rd-continuous and since $D_3\varphi(\cdot, \tau, u)$ solves (5.4) with respect to the initial condition $X(\tau) = I_V$, the variation of constants formula (cf. [27, p. 56, Satz 1.3.11]) yields

$$\Lambda(t) = \int_{\tau}^{t} \Psi_u(t, \sigma(s))l(s)\,\Delta s,$$

where $\Psi_u(t, \tau) \in \mathcal{L}(V)$ is the transition operator of (5.4). By assumption, (5.1) is V_+-cooperative on V_+ and similarly to Remark 5.7(1) one sees the inclusion $\Psi_u(t, \tau) \in \mathcal{L}(V_+)$ for $\tau \leq t$. Furthermore, (5.7) implies $l(t) \in V_+$ and by the convexity of the Cauchy-integral on $\mathbb{T}$ it follows $\Lambda(t) \in V_+$ for $\tau \leq t$. Now Lemma 5.14(a) leads to the assertion.

(b) Proceed like in the proof of (a). Here Remark 5.7(2) yields that $\Psi_u(t, \tau)$ is strictly positive and the assertion follows from Lemma 5.14(b). □

6. Application: Symbiotic Interaction

In the following last section we demonstrate the importance of the limit set trichotomy from Theorem 3.1 in an application from biology within the calculus on time scales. Thereto we restrict our considerations to time scales of the form

$$\mathbb{T} = \bigcup_{n \in \mathbb{N}_0} [\tau_n, t_n],$$

where $(\tau_n)_{n\in\mathbb{N}_0}$, $(t_n)_{n\in\mathbb{N}_0}$ are real sequences with $\lim_{n\to\infty} \tau_n = \lim_{n\to\infty} t_n = \infty$ and $\tau_n \leq t_n < \tau_{n+1}$ for all $n \in \mathbb{N}_0$. Hence we have a continuous ODE dynamical behavior of (5.1) on the intervals $[\tau_n, t_n]$, $n \in \mathbb{N}_0$, while the dynamic on the "gaps" (t_n, τ_{n+1}) is discrete, i.e., difference equation-like. For technical reasons we additionally assume that the differences $\tau_{n+1} - \tau_n$, $n \in \mathbb{N}_0$, are bounded above by some real $T \geq 0$.

Consider a symbiotic interaction between $d \geq 2$, e.g., insect populations, i.e., an interaction that results in a benefit between the populations. The life span of each population is given by the interval $[\tau_n, t_n]$, $n \in \mathbb{N}_0$, which can be interpreted as a summer period. Suppose that just before the populations die out, eggs are laid at time $t = t_n$ and hatch after the winter period (t_n, τ_{n+1}) at time $t = \tau_{n+1}$. During the winter, a certain amount of eggs dies, but to prevent each species from dying out, an exterior influence adds additional eggs. If $v_i(t_n) \geq 0$, $n \in \mathbb{N}_0$, denotes the biomass of the ith, $i = 1, \dots, d$, population at time $t = t_n$, we model this behavior over the winter periods with the equations

$$v_i(\tau_{n+1}) = q_i(t_n)v_i(t_n) + p_i(t_n) \quad \text{for all } i = 1, \dots, d \tag{6.1}$$

and $n \in \mathbb{N}_0$, where $q_i(t_n) \in [0, 1[$ describes the natural decay in the winter and $p_i(t_n) > 0$ the external "seed". The equation (6.1) guarantees that we have the inclusion $v(\tau_{n+1}) \in \operatorname{int}\mathbb{R}^d_+$ after each winter — independent of $v(t_n) \in \mathbb{R}^d_+$. For the continuous growth we lean on [22] and consider the ODEs

$$\dot{v}_i = v_i F_i(v,t) \quad \text{for all } i = 1,\dots,d, \tag{6.2}$$

on the intervals $[\tau_n, t_n]$, $n \in \mathbb{N}_0$, where the mappings $F_i : \mathbb{R}^d \times \mathbb{T} \to \mathbb{R}$ are continuously differentiable in each state space variable $v_1,\dots,v_d$. Obviously the boundary $\partial\mathbb{R}^d_+$ is forward invariant with respect to (6.2) and therefore any solution of (6.2) cannot leave the standard cone $\mathbb{R}^d_+$ for times $t \in [\tau_n, t_n]$, $n \in \mathbb{N}_0$. Combining both situations, we arrive at a dynamic equation (5.1) with right-hand side $f = (f_1,\dots,f_d)$ and

$$f_i(t,v) := \begin{cases} v_i F_i(v,t) & \text{for } t \in [\tau_n, t_n) \\ \frac{q_i(t)-1}{\mu(t)} v_i + \frac{p_i(t)}{\mu(t)} & \text{for } t = t_n\,. \end{cases}$$

If we assume that the ODE (6.2) has no finite escape times, then the mapping f satisfies the assumptions (H1)–(H2). In addition, the standard cone $\mathbb{R}^d_+$ is forward-invariant with respect to (5.1).

As a canonical state space for (5.1) we consider the cone $V_+ = \mathbb{R}^d_+$, which evidently satisfies the assumption (H0), and is $\mathbb{R}^d_+$-convex, since the nonnegative orthant is convex. Under the assumption

(C) $D_j F_i(u,t) \geq 0$ for all $u \in \mathbb{R}^d_+$, $i \neq j$, and $t \in \bigcup_{n\in\mathbb{N}_0}[\tau_n, t_n)$,

the system (5.1) is $\mathbb{R}^d_+$-cooperative on $\mathbb{R}^d_+$ and we obtain from Theorem 5.10(a) that its solution φ is order-preserving. On the other hand, if we suppose

(S) $\sum_{j=1}^d v_j D_j F_i(v,t) \leq 0$ for all $v \in \mathbb{R}^d_+$, $i = 1,\dots,d$ and $t \in \bigcup_{n\in\mathbb{N}_0}[\tau_n, t_n)$,

then using Theorem 5.15(a) one can show that φ is also subhomogeneous. So, due to Lemma 4.4(a), $\varphi(t,\tau)$, $\tau \leq t$, must be nonexpansive with respect to the part metric on $\mathbb{R}^d_+$. Finally, since each $\mathbb{T}$-interval of length greater or equal than T contains a right-scattered point, we have $\varphi(t,\tau,v) \in \operatorname{int}\mathbb{R}^d_+$ for $T \leq t - \tau$ and $v \in \mathbb{R}^d_+$. Therefore the assumptions of Theorem 3.1 are satisfied and our limit set trichotomy applies. In particular, if p_i is bounded away from zero, we can exclude case (b) of Theorem 3.1 and all solutions of the general nonautonomous dynamic equation (5.1) are either unbounded, or bounded with nonempty ω-limit sets.

Example 6.1 (Kolmogorov systems)**.** A particularly relevant special case of the symbiotic interaction discussed above, are so-called *Kolmogorov systems* which have the following biological interpretation (cf. [11]): Think of a hierarchy of species $v_1,\dots,v_d$, where $v_i(t)$ is the biomass of the ith species. In this hierarchy, v_i interacts only with v_{i-1} and v_{i+1}. Such a hierarchy may occur in steep mountain side or in a lake, where each population dominates a specific altitude or depth, respectively, but is obliged to cooperate with other populations in the (narrow) overlap of their zones of dominance. So we only modify the law for the continuous growth and consider the system of ODEs

$$\begin{gathered} \dot{v}_1 = v_1 F_1(v_1, v_2, t) \\ \dot{v}_i = v_i F_i(v_{i-1}, v_i, v_{i+1}, t) \quad \text{for all } i = 2,\dots,d-1 \\ \dot{v}_d = v_d F_d(v_{d-1}, v_d, t) \end{gathered} \tag{6.3}$$

to describe the behavior on the intervals $[\tau_n, t_n]$, where the mappings $F_1,\dots,F_d$ are continuously differentiable in their state space variables. Furthermore, the conditions (C) and (S) reduce to

- $D_2F_1(v_1, v_2, t), D_1F_i(v_1, v_2, v_3, t), D_3F_i(v_1, v_2, v_3, t), D_1F_d(v_1, v_2, t) \geq 0$ for all $v_1, v_2, v_3 \in \mathbb{R}_+$, $i = 2, \dots, d-1$, and $t \in \bigcup_{n \in \mathbb{N}_0} [\tau_n, t_n)$,
- $\sum_{j=1}^{2} v_j D_j F_1(v_1, v_2, t) \leq 0$ and $\sum_{j=1}^{3} v_j D_j F_i(v_1, v_2, v_3, t) \leq 0$, as well as $\sum_{j=1}^{2} v_j D_j F_d(v_1, v_2, t) \leq 0$ for all $v_1, v_2, v_3 \in \mathbb{R}_+$, $i = 2, \dots, d-1$, and $t \in \bigcup_{n \in \mathbb{N}_0} [\tau_n, t_n)$,

respectively. They guarantee that the right-hand side of (5.1) is $\mathbb{R}_+^d$-cooperative and generates a subhomogeneous 2-parameter semiflow. Consequently our limit set trichotomy from Theorem 3.1 applies. Explicit biological systems modelled by (6.3) can be found in [11].

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