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FINITE ORDER SOLUTIONS OF COMPLEX LINEAR DIFFERENTIAL EQUATIONS

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Abstract. We shall consider the growth of solutions of complex linear homogeneous differential equations

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0$$

with entire coefficients. If one of the intermediate coefficients in exponentially dominating in a sector and f is of finite order, then a derivative $f^{(j)}$ is asymptotically constant in a slightly smaller sector. We also find conditions on the coefficients to ensure that all transcendental solutions are of infinite order. This paper extends previous results due to Gundersen and to Belaïdi and Hamani.

1. Introduction

It is well known that all solutions of the linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = 0 \tag{1.1}$$

are entire functions, provided the coefficients $A_0(z) \not\equiv 0$, $A_1(z), \dots, A_{k-1}(z)$ are entire. A classical result, due to Wittich, tells that all solutions of (1.1) are of finite order of growth if and only if all coefficients $A_j(z)$, $j = 0, \dots, k-1$, are polynomials. For a complete analysis of possible orders in the polynomial case, see [6]. If some (or all) of the coefficients are transcendental, a natural question is to ask when and how many solutions of finite order may appear. Partial results have been available since a paper of Frei [2]. In its all generality, however, the problem remains open. Our starting point for this paper is a result due to Gundersen in [3]:

Theorem 1.1. *Let $A_0(z) \not\equiv 0, A_1(z)$ be entire functions such that for some real constants $\alpha > 0$, $\beta > 0$, $\theta_1 < \theta_2$ we have*

$$|A_1(z)| \geq \exp((1 + o(1))\alpha|z|^\beta),$$
$$|A_0(z)| \leq \exp(o(1)|z|^\beta)$$

as $z \to \infty$ in the sector $S(0) : \theta_1 \leq \arg z \leq \theta_2$. Given $\varepsilon > 0$ small enough, let $S(\varepsilon)$ denote the sector $\theta_1 + \varepsilon \leq \arg z \leq \theta_2 - \varepsilon$. If f is a nontrivial solution of

$$f'' + A_1(z)f' + A_0(z)f = 0 \tag{1.2}$$

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of finite order, then the following conditions hold:

(i) *There exists a constant $b \neq 0$ such that $f(z) \to b$ as $z \to \infty$ in $S(\varepsilon)$. Indeed,*
$$|f(z) - b| \leq \exp(-(1 + o(1))\alpha|z|^{\beta}).$$

(ii) *For each integer $k \geq 1$,*
$$|f^{(k)}(z)| \leq \exp(-(1 + o(1))\alpha|z|^{\beta})$$
as $z \to \infty$ in $S(\varepsilon)$.

This result has been recently generalized to the higher order case (1.1) by Belaïdi and Hamani, see [1], as follows:

Theorem 1.2. *Let $A_0(z) \not\equiv 0, A_1(z), \dots, A_{k-1}(z)$ be entire functions such that for some real constants $\alpha > 0, \beta > 0, \theta_1 < \theta_2$, we have*
$$|A_1(z)| \geq \exp((1 + o(1))\alpha|z|^{\beta}),$$
$$|A_j(z)| \leq \exp(o(1)|z|^{\beta}), \quad j = 0, 2, 3, \dots, k-1,$$
as $z \to \infty$ in $S(0)$. If f is a nontrivial solution of (1.1) of finite order, then the following conditions hold, provided $\varepsilon > 0$ is small enough:

(i) *There exists a constant $b \neq 0$ such that $f(z) \to b$ as $z \to \infty$ in $S(\varepsilon)$. Indeed,*
$$|f(z) - b| \leq \exp(-(1 + o(1))\alpha|z|^{\beta}).$$

(ii) *For each integer $k \geq 1$,*
$$|f^{(k)}(z)| \leq \exp(-(1 + o(1))\alpha|z|^{\beta})$$
as $z \to \infty$ in $S(\varepsilon)$.

A natural question is now to ask about a counterpart of Theorem 1.2 in the case that the coefficient in the same sense as in Theorem 1.2 is $A_s(z)$ instead of $A_1(z)$. We are going to present such a counterpart in this paper, see Theorem 2.1 below. As the proof now is more complicated than the corresponding proof of Theorem 1.2, see [1], we express the growth conditions for the coefficients more explicitly. In fact, making use of $o(1)$ only as in Theorem 1.2 might leave some doubts on the necessary uniformity in the course of the proof.

2. Notation and results

Given $\varepsilon > 0$, and $\theta_1, \theta_2 \in [0, 2\pi), \theta_1 < \theta_2$, we denote by $S(\theta_1, \theta_2, \varepsilon)$, resp. $S(R, \theta_1, \theta_2, \varepsilon)$, the sector $\{z | \theta_1 + \varepsilon \leq \arg z \leq \theta_2 - \varepsilon\}$, resp. the truncated sector $S(\theta_1, \theta_2, \varepsilon) \cap \{|z| \geq R\}$. If the sector boundaries are clear, and there is no possibility of confusion, we apply the shorter notations $S(\varepsilon)$, resp. $S(R, \varepsilon)$. In the proof of Theorem 1 below, we agree that whenever stating that $r \geq r_j$ to indicate that $|z| = r$ has to be large enough, we always assume that $r_j \geq r_{j-1}$. Hence, in such a situation, all corresponding previous conditions for $r \geq r_{j-1}$ remain valid, without saying this explicitly in what follows.

Theorem 2.1. *Let $\theta_1 < \theta_2$ be given to fix a sector $S(0)$, let $k \geq 2$ be a natural number, and let $\delta > 0$ be any real number such that $k\delta < 1$. Suppose that*

$A_0(z), \dots A_{k-1}(z)$ with $A_0(z) \not\equiv 0$ are entire functions such that for real constants $\alpha > 0$, $\beta > 0$, we have, for some $s = 1, \dots, k-1$,

$$|A_s(z)| \geq \exp((1+\delta)\alpha|z|^\beta), \tag{2.1}$$

$$|A_j(z)| \leq \exp(\delta\alpha|z|^\beta) \tag{2.2}$$

for all $j = 0, \dots, s-1, s+1, \dots, k-1$ whenever $|z| = r \geq r_\delta$ in the sector $S(0)$. Given $\varepsilon > 0$ small enough, if f is a transcendental solution of finite order $\rho < \infty$ of the linear differential equation

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \dots + A_1(z)f' + A_0(z)f = 0, \tag{2.3}$$

then the following conditions hold:

(i) *There exists $j \in \{0, \dots, s-1\}$ and a complex constant $b_j \neq 0$ such that $f^{(j)}(z) \to b_j$ as $z \to \infty$ in the sector $S(\varepsilon)$. More precisely,*

$$|f^{(j)}(z) - b_j| \leq \exp(-(1-k\delta)\alpha|z|^\beta) \tag{2.4}$$

in $S(\varepsilon)$, provided $|z|$ is large enough.

(ii) *For each integer $m \geq j+1$,*

$$|f^{(m)}(z)| \leq \exp(-(1-k\delta)\alpha|z|^\beta) \tag{2.5}$$

in $S(3\varepsilon)$ for all $|z|$ large enough.

Remark. In Theorem 2.1, it may happen that $j < s-1$. Indeed, $f(z) = e^z + 1$ satisfies

$$f''' + 2e^{-z}f'' - e^z f' + (-2 + e^z)f = 0. \tag{2.6}$$

Obviously, (2.6) fulfills the assumptions of Theorem 2.1 in the sector $\frac{2\pi}{3} < \theta < \frac{3\pi}{4}$. In this example, $A_2(z) = 2e^{-z}$ is the dominating coefficient, while we have $j = 0$.

Remark. The following two theorems are natural counterparts to [3, Theorem 5], and [1, Theorem 1.6], respectively to [1, Theorem 1.7].

Theorem 2.2. *Let $A_0(z) \not\equiv 0$, $A_1(z), \dots, A_{k-1}(z)$, $k \geq 2$, be entire functions, let $\alpha > 0$, $\beta > 0$ be given constants, let $\delta > 0$ be a real number such that $k\delta < 1$ and let s be an integer such that $1 \leq s \leq k-1$. Suppose that (i) $\rho(A_j) < \beta$ for $j \neq s$ and (ii) for any given $\varepsilon > 0$, there exists two finite collections of real numbers (ϕ_m) and (θ_m) that satisfy $\phi_1 < \theta_1 < \phi_2 < \theta_2 < \dots < \phi_n < \theta_n < \phi_{n+1} = \phi_1 + 2\pi$, such that*

$$\sum_{m=1}^{n} (\phi_{m+1} - \theta_m) < \varepsilon, \tag{2.7}$$

$$|A_s(z)| \geq \exp((1+\delta)\alpha|z|^\beta) \tag{2.8}$$

as $z \to \infty$ in $\phi_m \leq \arg z \leq \theta_m$, $m = 1, \dots, n$. Then every transcendental solution f of (2.3) is of infinite order.

Invoking the iterated order $\rho_p(f) := \limsup_{r\to\infty} (\log_p T(r,f))/\log r$ for entire functions, we add our final theorem which is a simple extension of [1, Theorem 1.7].

Theorem 2.3. *Let $A_0(z), \dots, A_{k-1}(z)$ be entire functions such that for some integer s, $1 \leq s \leq k-1$, we have $\rho_p(A_j) \leq \alpha < \beta = \rho_p(A_s) \leq +\infty$ for all $j \neq s$. Then every transcendental solution f of (2.3) satisfies $\rho_p(f) \geq \rho_p(A_s)$.*

3. Preparations for proofs

To prove Theorem 2.1, we need two preparatory lemmas. The first one is a simple extension of [3, Lemma 4]. See also [5, Lemma 3].

Lemma 3.1. *Let $f(z)$ be an entire function, and suppose that $|f^{(k)}(z)|$ is unbounded on a ray $\arg z = \theta$. Then there exists a sequence $z_n = r_n e^{i\theta}$ tending to infinity such that $f^{(k)}(z_n) \to \infty$ and that*

$$\left|\frac{f^{(j)}(z_n)}{f^{(k)}(z_n)}\right| \le \frac{1}{(k-j)!}(1+o(1))|z_n|^{k-j}, \tag{3.1}$$

provided $j < k$.

Proof. Let $M(r,\theta,f^{(k)})$ denote the maximum modulus of $f^{(k)}$ on the line segment $[0, re^{i\theta}]$. Clearly, we may construct a sequence of points $z_n = r_n e^{i\theta}$, $r_n \to \infty$, such that $M(r_n,\theta,f^{(k)}) = |f^{(k)}(r_n e^{i\theta})| \to \infty$ as $n \to \infty$. For each n, we obtain by $(k-j)$-fold iterated integration along the line segment $[0, z_n]$,

$$\begin{aligned} f^{(j)}(z_n) =& f^{(j)}(0) + f^{(j+1)}(0)z_n + \dots + \frac{1}{(k-j-1)!}f^{(k-1)}(0)z_n^{k-j-1} \\ &+ \int_0^{z_n} \dots \int_0^{z_n} f^{(k)}(t)dt \dots dt. \end{aligned} \tag{3.2}$$

Therefore, by an elementary triangle inequality estimate,

$$\begin{aligned} |f^{(j)}(z_n)| \le& |f^{(j)}(0)| + |f^{(j+1)}(0)|r_n + \dots \\ &+ \frac{1}{(k-j-1)!}|f^{(k-1)}(0)|r_n^{k-j-1} + \frac{1}{(k-j)!}|f^{(k)}(z_n)|r_n^{k-j}. \end{aligned} \tag{3.3}$$

The assertion immediately follows. □

Lemma 3.2. *Given $\alpha > 0$, $\beta > 0$, $K \ge \frac{1}{2}$ and $0 < \eta < \frac{1}{2}$, the integral $I(r) := \int_r^{+\infty} \exp(-K\alpha t^\beta)dt$ converges. More precisely, if $\beta > 1$, then $I(r) \le \exp(-K\alpha r^\beta)$, whenever $r^{\beta-1} \ge \frac{1}{K\alpha\beta}$ and if $\beta \le 1$, then $I(r) \le \exp(-(K-\eta)\alpha r^\beta)$ for any given $\eta \in (0,1/2)$, provided $\eta\alpha r^\beta \ge (1-\beta)\log r + \log\frac{2}{\alpha\beta}$.*

Remark. Observe that the lower bound obtained for r above is independent of K, in both cases. Moreover, if we take r large enough, say $r \ge r_0 \ge r_\delta$, then $I(r) \le \exp(-(K-\eta)\alpha r^\beta)$, in both cases again.

Proof. Clearly,

$$\begin{aligned} I(r) &= \int_r^{+\infty} \exp(-K\alpha t^\beta)dt = \frac{K\alpha\beta r^{\beta-1}}{K\alpha\beta r^{\beta-1}} \int_r^{+\infty} \exp(-K\alpha t^\beta)dt \\ &\le -\frac{1}{K\alpha\beta r^{\beta-1}} \int_r^{+\infty} -K\alpha\beta t^{\beta-1}\exp(-K\alpha t^\beta)dt \\ &= \frac{1}{K\alpha\beta r^{\beta-1}}\exp(-K\alpha r^\beta). \end{aligned} \tag{3.4}$$

If now $\beta > 1$, it suffices to have $K\alpha\beta r^{\beta-1} \ge 1$, as required. If $\beta \le 1$, we may write the above estimate for $I(r)$ as

$$I(r) \le \frac{2}{\alpha\beta} r^{1-\beta}\exp(-\eta\alpha r^\beta)\exp(-(K-\eta)\alpha r^\beta). \tag{3.5}$$

Provided $\frac{2}{\alpha\beta}r^{1-\beta}\exp(-\eta\alpha r^\beta) \le 1$, we again have the required estimate. □

4. Proofs of theorems

Proof of Theorem 2.1. **Boundedness of $f^{(s)}$ in $S(\varepsilon)$.** Recall first that by [4], Corollary 1, there exists a set $E \subset [0, 2\pi)$ of linear measure zero such that for all $k \geq s \geq 0$, all $j = s+1, \dots, k$, and all $r \geq r_1$,

$$\Big|\frac{f^{(j)}(z)}{f^{(s)}(z)}\Big| \leq |z|^{(j-s)(\rho-1+\varepsilon)} \leq |z|^{k\rho} \tag{4.1}$$

along any ray $\arg z = \psi$ such that $\psi \in [0, 2\pi) \setminus E$, provided $0 < \varepsilon < 1$.

Suppose now, for a while, that $|f^{(s)}(z)|$ is unbounded on some ray $\arg z = \phi \in S(0) \setminus E$. By Lemma 3.1, there exists a sequence of points $z_n = r_n e^{i\phi}$, $r_n \to \infty$ such that $f^{(s)}(z_n) \to \infty$ and so

$$\Big|\frac{f^{(j)}(z_n)}{f^{(s)}(z_n)}\Big| \leq \frac{1}{(s-j)!}(1+o(1))|z_n|^{s-j} \leq 2|z_n|^k \tag{4.2}$$

for all $j = 0, \dots, s-1$ and all n large enough, say $|z_n| \geq r_2$. From (2.3), we next conclude that

$$\begin{aligned}|A_s| \leq& \Big|\frac{f^{(k)}}{f^{(s)}}\Big| + |A_{k-1}|\Big|\frac{f^{(k-1)}}{f^{(s)}}\Big| + \dots + |A_{s+1}|\Big|\frac{f^{(s+1)}}{f^{(s)}}\Big| \\ &+ |A_{s-1}|\Big|\frac{f^{(s-1)}}{f^{(s)}}\Big| + \dots + |A_1|\Big|\frac{f'}{f^{(s)}}\Big| + |A_0|\Big|\frac{f}{f^{(s)}}\Big|.\end{aligned} \tag{4.3}$$

Combining now (2.2), (4.1) and (4.2) with the above estimate (4.3) for A_s, it is straightforward to see that $|A_s(z_n)| \leq \exp(3\delta\alpha r_n^{\beta})$ for all n large enough in the sequence z_n on the ray $\arg z = \phi$, contradicting (2.1). Therefore, $|f^{(s)}(z)|$ remains bounded on all rays $\arg z = \phi \in S(0) \setminus E$. By a standard application of the Phragmén–Lindelöf principle, we conclude that $f^{(s)}(z)$ is bounded, say $|f^{(s)}(z)| \leq M$, in the whole sector $S(\varepsilon)$.

Preliminary estimate for $|f^{(m)}(z)|$, $m \leq s$. We now proceed to show that $|f^{(m)}(z)| = O(|z|^{s-m})$ on any ray $\arg z = \phi \in S(0) \setminus E$, for all $m \leq s$. Of course, it suffices to consider $m < s$. By $(s-m)$-fold iterated integration along the ray under consideration, see (3.3),

$$\begin{aligned}|f^{(m)}(z)| \leq& |f^{(m)}(0)| + |f^{(m+1)}(0)||z| + \dots + \frac{1}{(s-m-1)!}|f^{(s-1)}(0)||z|^{s-m-1} \\ &+ M\int_0^{|z|}\dots\int_0^{|z|} dt\dots dt = O(|z|^{s-m}).\end{aligned} \tag{4.4}$$

Proof of the second assertion for $m = s$. Writing now (2.3) in the form

$$f^{(s)} = \frac{1}{A_s}(f^{(k)} + A_{k-1}f^{(k-1)} + \dots + A_{s+1}f^{(s+1)} + A_{s-1}f^{(s-1)} + \dots A_1 f' + A_0 f), \tag{4.5}$$

and recalling (4.1), (4.4), the boundedness of $f^{(s)}$ and the assumptions (2.1) and (2.2), we conclude that whenever $r \geq r_3$, we get

$$|f^{(s)}(z)| \leq \exp(-(1-\delta)\alpha|z|^{\beta}) \tag{4.6}$$

along any ray $\arg z = \phi \in S(\varepsilon) \setminus E$. By the Phragmén–Lindelöf principle again, (4.6) remains true in the sector $S(2\varepsilon)$, proving the second assertion in the case of $m = s$.

Proof of the second assertion for $m > s$. We may now restrict ourselves to the sector $S(3\varepsilon)$. We assume that $r \geq r_4$ is large enough to satisfy that for an arbitrary $z = re^{i\theta} \in S(3\varepsilon)$, the disk $\Gamma(z)$ of radius at most $\rho = \max_{s<m\leq k}((m-s)!)^{1/(m-s)}$, centered at z, is contained in $S(2\varepsilon)$, i.e. we must take $r_4 \geq \rho/\sin\varepsilon$. Given now $m > s$, we may use (4.6) in the Cauchy formula to see that

$$|f^{(m)}(z)| \leq \frac{(m-s)!}{2\pi} \int_{\Gamma(z)} \frac{|f^{(s)}(\zeta)|}{|z-\zeta|^{m-s+1}} d\zeta. \tag{4.7}$$

By the selection of ρ above, we may combine (4.6) and (4.7) to conclude that

$$|f^{(m)}(z)| \leq \exp(-(1-\delta)\alpha|z|^{\beta}). \tag{4.8}$$

Proof of the first assertion for $j = s-1$. Fix now $\theta \in S(2\varepsilon)$, and define

$$a_s := \int_0^{+\infty} f^{(s)}(te^{i\theta})e^{i\theta}dt = \lim_{R\to\infty} \int_0^R f^{(s)}(te^{i\theta})e^{i\theta}dt. \tag{4.9}$$

By (4.6), it is not difficult to see that $a_s \in \mathbb{C}$. Moreover, the definition of a_s is independent of θ. Indeed, integrating $f^{(s)}(\zeta)$ along the sector boundary $0 \to Re^{i\phi} \to Re^{i\theta} \to 0$, and using (4.6) to conclude that the integral of $f^{(s)}(\zeta)$ over the arc $[Re^{i\phi}, Re^{i\theta}]$ tends to zero as $R \to \infty$, the independence from θ immediately follows. Define now $b_{s-1} := f^{(s)}(0) + a_s$, and suppose that $b_{s-1} \neq 0$. Let $z = re^{i\phi}$ be an arbitrary point in $S(2\varepsilon)$ such that $r \geq r_4$. Then, since

$$f^{(s-1)}(z) - b_{s-1} = \int_0^z f^{(s)}(\zeta)d\zeta - \int_0^{+\infty} f^{(s)}(te^{i\phi})e^{i\phi}dt, \tag{4.10}$$

we may apply (4.6) and Lemma 3.2 to conclude that

$$\begin{aligned} |f^{(s-1)}(z) - b_{s-1}| &= \Big|\int_0^z f^{(s)}(\zeta)d\zeta - \int_0^\infty f^{(s)}(te^{i\phi})e^{i\phi}dt\Big| \\ &= \Big|\int_\infty^z f^{(s)}(te^{i\phi})e^{i\phi}dt\Big| \leq \int_{|z|}^\infty |f^{(s)}(te^{i\phi})|dt \\ &\leq \int_r^\infty \exp(-(1-\delta)\alpha t^{\beta})dt \\ &\leq \exp(-(1-2\delta)\alpha r^{\beta}), \end{aligned} \tag{4.11}$$

provided that $r \geq r_4$. Since we assumed that $b_{s-1} \neq 0$, we have completed the proof of the first assertion in this case.

Proof of the first assertion for $j < s-1$, the first part. We now have $b_{s-1} = 0$. To continue, we define a_{s-1} replacing $f^{(s)}$ by $f^{(s-1)}$ in (4.9), and $b_{s-2} := f^{(s-2)}(0) + a_{s-1}$. To estimate $f^{(s-2)}(z) - b_{s-2}$, we apply Lemma 3.2 and $|f^{(s-1)}(z)| \leq \exp(-(1-2\delta)\alpha r^{\beta})$ in place of (4.6) exactly as in (4.11) to obtain

$$|f^{(s-2)}(z) - b_{s-2}| \leq \exp(-(1-3\delta)\alpha r^{\beta}). \tag{4.12}$$

for $r \geq r_4$.

We may now continue inductively. If $b_j \neq 0$ for some $j = s-t$, $t = 2, \dots, s-1$, we obtain

$$|f^{(s-t)}(z) - b_{s-t}| \leq \exp(-(1-(t+1)\delta)\alpha r^{\beta}). \tag{4.13}$$

Otherwise, we have $b_{s-1} = b_{s-2} = \dots = b_1 = 0$, and we have the estimate

$$|f(z) - b_0| \leq \exp(-(1-(s+1)\delta)\alpha|z|^{\beta}). \tag{4.14}$$

If now $b_0 \neq 0$, we have proved the first assertion. It remains to show that the case $b_0 = 0$ is not possible.

Proof of the first assertion for $j < s-1$, the second part (impossibility of $b_{s-1} = \dots = b_0 = 0$). (a) First step. Writing now (2.3) in the form

$$\begin{aligned}-\frac{f^{(s)}}{f} =& \frac{A_0}{A_s} + \frac{A_1}{A_s}\frac{f'}{f} + \dots + \frac{A_{s-1}}{A_s}\frac{f^{(s-1)}}{f} \\ &+ \frac{A_{s+1}}{A_s}\frac{f^{(s+1)}}{f} + \dots + \frac{A_{k-1}}{A_s}\frac{f^{(k-1)}}{f} + \frac{1}{A_s}\frac{f^{(k)}}{f},\end{aligned} \tag{4.15}$$

we may use (2.1), (2.2) and (4.1) to conclude that

$$\Big|\frac{f^{(s)}(z)}{f(z)}\Big| \leq \exp(-(1-\delta)\alpha|z|^{\beta}) \tag{4.16}$$

in $S(2\varepsilon) \setminus E$. Therefore, by (4.16) and (4.14) with $b_0 = 0$, we infer that

$$|f^{(s)}(z)| \leq \exp(-(2-(s+2)\delta)\alpha r^{\beta}) \tag{4.17}$$

in $S(2\varepsilon) \setminus E$, hence in $S(2\varepsilon + \varepsilon/2)$ by the Phragmén–Lindelöf principle.
(b) Inductive step. Suppose now that we have been able to prove that the estimate

$$|f^{(s)}(z)| \leq \exp(-(T-((T-1)s+T)\delta)\alpha|z|^{\beta}) \tag{4.18}$$

holds good in the sector $S(2\varepsilon + \sum_{j=1}^{T-1}\frac{\varepsilon}{2^j})$. Combining now Lemma 3.2 with (4.18), we may repeat the reasoning in (4.11) to obtain

$$|f^{(s-1)}(z)| \leq \exp(-(T-((T-1)s+T)\delta - \delta)\alpha r^{\beta}). \tag{4.19}$$

Since $b_{s-1} = \dots = b_0 = 0$, we apply a parallel reasoning as in 4.6. of this proof above to get

$$|f(z)| \leq \exp(-(T-((T-1)s+T)\delta - s\delta)\alpha r^{\beta}), \tag{4.20}$$

valid in $S(r_4, 2\varepsilon + \sum_{j=1}^{T-1}\frac{\varepsilon}{2^j})$. Combining now (4.20) with (4.16), we obtain

$$|f^{(s)}(z)| \leq \exp(-(T+1-(Ts+T+1)\delta)\alpha|z|^{\beta})$$

in $S(2\varepsilon + \sum_{j=1}^{T-1}\frac{\varepsilon}{2^j}) \setminus E$, provided $r \geq r_4$. By the Phragmén–Lindelöf principle, this inequality remains valid in the whole sector $S(2\varepsilon + \sum_{j=1}^{\infty}\frac{\varepsilon}{2^j}) = S(3\varepsilon)$, completing the inductive step.
(c) Final conclusion. We have proved that, in this special case of $b_{s-1} = \dots = b_0 = 0$, the inequality (4.18) is valid in $S(3\varepsilon)$ for all $T \in \mathbb{N}$, provided $r \geq r_4$. Fix now a finite line segment in $S(r_4, 3\varepsilon)$. Since $k\delta < 1$, and $s+1 \leq k$, it follows that $T - ((T-1)s+T)\delta \to \infty$ as $T \to \infty$. Hence, $f^{(s)}$ vanishes identically on such a line segment. Therefore, by the standard uniqueness theorem of entire functions, f has to be a polynomial, a contradiction. □

Proof of Theorem 2.2. Suppose that f is a transcendental solution of (2.3) of finite order of growth. Given $\varepsilon > 0$, let (ϕ_m) and (θ_m) be as in the assumptions. From (2.8) and the supposition that $\rho(A_j) < \beta$ whenever $j \neq s$, we conclude by using Theorem 2.1(ii) that $|f^{(s)}(z)|$ is bounded in each of the sectors $\phi_m + 3\varepsilon \leq \arg z \leq \theta_m - 3\varepsilon, m = 1, \dots, n$. As ε is arbitrarily small, we infer from (2.7) and the Phragmén–Lindelöf principle that $|f^{(s)}(z)|$ must be bounded in the whole complex plane. By the Liouville theorem, f has to be a polynomial, a contradiction. □

Proof of Theorem 2.3. Suppose f is a transcendental solution of (2.3) such that $\rho_p(f) < \rho_p(A_s)$. Writing (2.3) in the form

$$A_s(z)f^{(s)} = -\sum_{j=0, j\neq s}^{k} A_j(z)f^{(j)}, \tag{4.21}$$

and making use of the elementary iterated order (in)equalities and the invariance of the iterated order under differentiation, we immediately observe that the left hand side of (4.21) is of iterated order $\rho_p(A_s)$, while the right hand side must be of iterated order $\leq \max_{j\neq s}(\rho_p(f), \rho_p(A_j)) < \rho_p(A_s)$, a contradiction. □

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