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BLOW-UP OF SOLUTIONS TO A NONLINEAR WAVE EQUATION

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Abstract. We study the solutions to the the radial 2-dimensional wave equation

$$\chi_{tt} - \frac{1}{r}\chi_r - \chi_{rr} + \frac{\sinh 2\chi}{2r^2} = g,$$
$$\chi(1,r) = \chi_\circ \in \dot{H}^\gamma_{\rm rad}, \quad \chi_t(1,r) = \chi_1 \in \dot{H}^{\gamma-1}_{\rm rad},$$

where $r = |x|$ and x in $\mathbb{R}^2$. We show that this Cauchy problem, with values into a hyperbolic space, is ill posed in subcritical Sobolev spaces. In particular, we construct a function $g(t,r)$ in the space $L^p([0,1]L^q_{\rm rad})$, with $\frac{1}{p} + \frac{2}{q} = 3-\gamma$, $0 < \gamma < 1$, $p \geq 1$, and $1 < q \leq 2$, for which the solution satisfies $\lim_{t\to 0} \|\bar{\chi}\|_{\dot{H}^\gamma_{\rm rad}} = \infty$. In doing so, we provide a counterexample to estimates in [1].

1. Introduction

In this paper, we study the properties of the solutions to the Cauchy problem

$$\chi_{tt} - \frac{1}{r}\chi_r - \chi_{rr} + \frac{\sinh(2\chi)}{2r^2} = g(t,r), \tag{1.1}$$
$$\chi(1,r) = \chi_\circ \in \dot{H}^\gamma_{\rm rad}, \quad \chi_t(1,r) = \chi_1 \in \dot{H}^{\gamma-1}_{\rm rad}, \tag{1.2}$$

where $g \in L^p([0,1]L^q_{\rm rad})$, $\frac{1}{p} + \frac{2}{q} = 3-\gamma$, $0 < \gamma < 1$, $p \geq 1$, $1 < q \leq 2$, $r = |x|$, $x \in \mathbb{R}^2$, and $t \in [0,1]$.

The homogeneous problem, i.e. (1.1) with $g \equiv 0$, has been investigated by several authors; see for example [1, p. 89-141], [10], [11], and references therein. For the non-homogeneous problem, we construct function g in $L^p([0,1]L^q_{\rm rad})$, and a solution $\bar{\chi}$ to (1.1)–(1.2) such that $\lim_{t\to 0} \|\bar{\chi}\|_{\dot{H}^\gamma_{\rm rad}} = \infty$. This implies that (1.1)–(1.2) is an ill posed problem, and provides a counter example to [1, pages 3–4, Eq. (9)].

Equation (1.1) is obtained from the wave map equation with a source term when $x \in \mathbb{R}^2$ and the target is the hyperboloid $\mathcal{H}^2 : u_1^2 + u_2^2 - u_3^2 = -1$, $\mathcal{H}^2 \hookrightarrow \mathbb{R}^3$. Let us consider the equation

$$u_{tt} - \triangle u - (|u_t|^2 - |\nabla_x u|^2)u = q(t,x),$$

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when $x \in \mathbb{R}^2$ and the target is the hyperboloid $\mathcal{H}^2 : u_1^2 + u_2^2 - u_3^2 = -1$, $\mathcal{H}^2 \hookrightarrow \mathbb{R}^3$. Equation (1) is a wave map equation with a source term, where

$$\begin{gathered}
|u_t|^2 = u_{1t}^2 + u_{2t}^2 - u_{3t}^2, \\
|\nabla_x u|^2 = |\nabla_{x_1} u|^2 + |\nabla_{x_2} u|^2, \\
|\nabla_{x_i} u|^2 = u_{1x_i}^2 + u_{2x_i}^2 - u_{3x_i}^2, \quad i = 1, 2.
\end{gathered}$$

Let $q = (q_1, q_2, q_3)$, with

$$\begin{gathered}
q_1 = \cosh\chi\cos\phi_1(\chi_{tt} - \Delta\chi) + \frac{1}{r^2}\sinh\chi\cos\phi_1 + \frac{\sinh^3\chi\cos\phi_1}{r^2}, \\
q_2 = \cosh\chi\sin\phi_1(\chi_{tt} - \Delta\chi) + \frac{1}{r^2}\sinh\chi\sin\phi_1 + \frac{\sinh^3\chi\sin\phi_1}{r^2}, \\
q_3 = \sinh\chi g.
\end{gathered}$$

For the hyperboloid $\mathcal{H}^2$, we have the parametric representation

$$u = (u_1, u_2, u_3), \tag{1.3}$$
$$u_1 = \sinh\chi\cos\phi_1, \tag{1.4}$$
$$u_2 = \sinh\chi\sin\phi_1, \tag{1.5}$$
$$u_3 = \cosh\chi, \quad \chi \geq 0, \quad \phi_1 \in [0, 2\pi]. \tag{1.6}$$

Let $x_1 = r\cos\phi_1$, $x_2 = r\sin\phi_1$. Then from (1), we get that χ satisfies (1.1).

Our main result is as follows.

Theorem 1.1. *Let $\frac{1}{p} + \frac{2}{q} = 3 - \gamma$, $p \geq 1$, $1 < q \leq 2$, $0 < \gamma < 1$. Then there exist function $g \in L^p([0,1]L^q_{\rm rad})$ and solution $\bar{\chi}$ to (1.1)–(1.2) such that*

$$\lim_{t \to 0} \|\bar{\chi}\|_{\dot{H}^\gamma_{\rm rad}} = \infty.$$

Note that the case $p = 1$, $q = 2$ which is the energy case can not be reached by the setting in this theorem.

2. Preliminary results

Let f be a real-valued function satisfying

(H1) $f \in \mathcal{C}^2[0, \infty)$,
(H2) $f(0) = 0$, $f(1) = f'(1) = 0$.

As an example of a function satisfying (H1)-(H2), we have

$$f(x) = (1 - x)^2 x. \tag{2.1}$$

Certainly $f \in \mathcal{C}^2[0, \infty)$; therefore, (H1) holds. Note that $f(0) = 0$ and $f'(x) = -2(1 - x)x + (1 - x)^2$ thus $f(1) = f'(1) = 0$; therefore, (H2) holds.

In addition, assume that:

(H3) $\frac{1}{p} + \frac{2}{q} = 3 - \gamma$, $0 < \gamma < 1$, $p \geq 1$, $\quad q > 1$
(H4) $0 < \alpha \leq 2 - q$, $\beta > 0$
(H5) Either $\beta > \alpha$ with $\frac{\beta}{\alpha} < \frac{q}{2p(q-1)}$ or $\beta < \alpha$ with $\frac{\beta}{\alpha} > \frac{q(2p-1)}{2p}$.

Note that when (H3) holds, $q \leq 2$; because if $q > 2$ then $3 - \gamma = \frac{1}{p} + \frac{2}{q} < 1 + 1 = 2$, from where $\gamma > 1$ which contradicts $0 < \gamma < 1$.

Let $z^{1/\alpha} = \frac{r}{t^{\beta/\alpha}}$. Then $z^{2/\alpha} = \frac{r^2}{t^{2\beta/\alpha}}$,

$$\frac{\partial z^{2/\alpha}}{\partial t} = -\frac{2\beta}{\alpha}\frac{1}{t} z^{2/\alpha}, \quad \frac{\partial^2 z^{2/\alpha}}{\partial t^2} = \frac{2\beta(\alpha+2\beta)}{\alpha^2}\frac{1}{t^2} z^{2/\alpha},$$
$$\frac{\partial z^{2/\alpha}}{\partial r} = \frac{2}{t^{\frac{\beta}{\alpha}}} z^{\frac{1}{\alpha}}, \quad \frac{\partial^2 z^{2/\alpha}}{\partial r^2} = \frac{2}{t^{\frac{2\beta}{\alpha}}}.$$

Let f be a function satisfying (H1)–(H2) and let

$$\chi_\circ = \begin{cases} f(r^2) & \text{for } r \le 1, \\ 0 & \text{for } r \ge 1, \end{cases} \tag{2.2}$$

$$\chi_1 = \begin{cases} -\frac{2\beta}{\alpha} r^2 f'(r^2) & \text{for } r \le 1, \\ 0 & \text{for } r \ge 1, \end{cases} \tag{2.3}$$

$$B_1 = \frac{4\beta^2}{\alpha^2} z^{4/\alpha} f''\big(z^{2/\alpha}\big) + \frac{2\beta(\alpha+2\beta)}{\alpha^2} z^{2/\alpha} f'\big(z^{2/\alpha}\big), \tag{2.4}$$

$$B_2 = z^{2/\alpha} f''(z^{2/\alpha}) + f'(z^{2/\alpha}), \tag{2.5}$$

$$g = \begin{cases} \frac{B_1}{t^2} - \frac{4}{t^{2\beta/\alpha}} B_2 + \frac{\sinh(2\bar\chi)}{2r^2} & \text{for } r \le t^{\beta/\alpha}, \\ 0 & \text{for } r \ge t^{\beta/\alpha}, \end{cases} \tag{2.6}$$

and let

$$\bar\chi = \begin{cases} f(z^{2/\alpha}) & \text{for } r \le t^{\beta/\alpha} \\ 0 & \text{for } r \ge t^{\beta/\alpha}. \end{cases} \tag{2.7}$$

Note that $\bar\chi$ is a solution of (1.1)–(1.2). Indeed, for $z \le 1$ we have

$$\bar\chi_t = -\frac{2\beta}{t\alpha} z^{2/\alpha} f'(z^{2/\alpha}),$$
$$\bar\chi_{tt} = \frac{4\beta^2}{\alpha^2}\frac{1}{t^2} z^{\frac{4}{\alpha}} f''(z^{2/\alpha}) + \frac{2\beta(\alpha+2\beta)}{\alpha^2}\frac{z^{2/\alpha}}{t^2} f'(z^{2/\alpha}),$$
$$\bar\chi_r = \frac{2}{t^{\beta/\alpha}} z^{\frac{1}{\alpha}} f'(z^{2/\alpha}),$$
$$\bar\chi_{rr} = \frac{4}{t^{\frac{2\beta}{\alpha}}} z^{2/\alpha} f''(z^{2/\alpha}) + \frac{2}{t^{\frac{2\beta}{\alpha}}} f'(z^{2/\alpha}).$$

Then, for $z \le 1$, we have

$$\bar\chi_{tt} - \frac{1}{r}\bar\chi_r - \bar\chi_{rr} + \frac{\sinh 2\bar\chi}{2r^2} = \frac{B_1}{t^2} - \frac{4}{t^{\frac{2\beta}{\alpha}}} B_2 + \frac{\sinh(2\bar\chi)}{2r^2} \equiv g,$$

Lemma 2.1. *Under Assumptions (H1)-(H5), the function g defined by* (2.6) *is in the space $L^p([0,1]L^q_{\rm rad})$.*

Proof. Note that

$$\|g\|^q_{L^q_{\rm rad}} = \int_0^{t^{\beta/\alpha}} |g|^q r\,dr = \int_0^{t^{\beta/\alpha}} \Big|\frac{B_1}{t^2} - \frac{4}{t^{\frac{2\beta}{\alpha}}} B_2 + \frac{\sinh(2\bar\chi)}{2r^2}\Big|^q r\,dr.$$

By making the change of variable $r = z^{\frac{1}{\alpha}} t^{\beta/\alpha}$,

$$dr = t^{\beta/\alpha}\frac{1}{\alpha} z^{\frac{1}{\alpha}-1} dz, \quad r\,dr = t^{\frac{2\beta}{\alpha}}\frac{1}{\alpha} z^{\frac{2}{\alpha}-1} dz$$

(with t fixed) and

$$\begin{aligned}\|g\|^q_{L^q_{\rm rad}} &= \frac{t^{\frac{2\beta}{\alpha}}}{\alpha}\int_0^1 \Big|B_1\frac{1}{t^2} - \frac{4}{t^{\frac{2\beta}{\alpha}}}B_2 + \frac{sinh(2\bar\chi)}{2t^{\frac{2\beta}{\alpha}}z^{2/\alpha}}\Big|^q z^{\frac{2}{\alpha}-1}dz\\
&\le c_1\frac{t^{\frac{2\beta}{\alpha}-2q}}{\alpha}\int_0^1 |B_1|^q z^{\frac{2}{\alpha}-1}dz + c_2\frac{t^{\frac{2\beta}{\alpha}(1-q)}}{\alpha}\int_0^1 |B_2|^q z^{\frac{2}{\alpha}-1}dz\\
&\quad + c_3\frac{t^{\frac{2\beta}{\alpha}(1-q)}}{\alpha}\int_0^1\Big|\frac{\sinh(2\bar\chi)}{2z^{2/\alpha}}\Big|^q z^{\frac{2}{\alpha}-1}dz.\end{aligned}$$

Let

$$I_1 = \int_0^1 |B_1|^q z^{\frac{2}{\alpha}-1}dz,\quad I_2 = \int_0^1 |B_2|^q z^{\frac{2}{\alpha}-1}dz,\quad I_3 = \int_0^1\Big|\frac{\sinh(2\bar\chi)}{2z^{2/\alpha}}\Big|^q z^{\frac{2}{\alpha}-1}dz.$$

From the definition of B_1 and B_2 and since (H1) and (H4) hold (we note that $\alpha\in(0,1)$ because $1<q\le 2$), we have $|B_1|^q z^{\frac{2}{\alpha}-1}\in\mathcal{C}[0,1]$, $|B_2|^q z^{\frac{2}{\alpha}-1}\in\mathcal{C}[0,1]$. Then $I_1<\infty$, $I_2<\infty$. Now we consider I_3. Let B be constant for which

$$B - 2\cosh(2\bar\chi)\bar\chi' \ge 0.$$

Such a constant B exists because $\bar\chi(z^{2/\alpha})$ and $\bar\chi'(z^{2/\alpha})$ are bounded functions for $z\in[0,1]$ and $\operatorname{supp}\bar\chi\subset[0,1]$, $\operatorname{supp}\bar\chi'\subset[0,1]$. Let

$$p(z) = Bz^{2/\alpha} - \sinh(2\bar\chi).$$

Then

$$p'(z) = \frac{2}{\alpha}z^{\frac{2}{\alpha}-1}(B - 2\cosh(2\bar\chi)\bar\chi') \ge 0\quad \forall z\in[0,1].$$

Consequently, $p(z)$ is an increasing function for $z\in[0,1]$. Therefore, $p(z)\ge p(0)=0$ for all $z\in[0,1]$ or $Bz^{2/\alpha}\ge sinh(2\bar\chi)$ for $z\in[0,1]$. Then

$$I_3 \le \int_0^1\Big|\frac{Bz^{2/\alpha}}{2z^{2/\alpha}}\Big|^q z^{\frac{2}{\alpha}-1}dz \equiv \text{const} < \infty.$$

Consequently,

$$\|g\|^q_{L^q_{\rm rad}} \le \bar c_1 t^{\frac{2\beta}{\alpha}-2q} + \bar c_2 t^{\frac{2\beta}{\alpha}(1-q)}$$

and from here

$$\|g\|_{L^q_{\rm rad}} \le \tilde c_1 t^{\frac{2\beta}{\alpha q}-2} + \tilde c_2 t^{\frac{2\beta}{\alpha q}-\frac{2\beta}{\alpha}},$$

$$\|g\|^p_{L^p([0,1]L^q_{\rm rad})} \le \overline{\overline{c_1}}\int_0^1 t^{\frac{2\beta p}{\alpha q}-2p}dt + \overline{\overline{c_2}}\int_0^1 t^{\frac{2\beta p}{\alpha q}-\frac{2\beta p}{\alpha}}dt < \infty$$

because (H5) holds. Here c_1, c_2, c_3, $\bar c_1$, $\bar c_2$, $\tilde c_1$, $\tilde c_2$, $\overline{\overline{c_1}}$, $\overline{\overline{c_2}}$ are positive constants. $\square$

As special notation we have

$$\dot H^s(\mathbb{R}^n) \equiv \dot F^s_{2,2}(\mathbb{R}^n),\quad 0<s<\infty$$

where (see [1, p. 94, def. 2])

$$\dot F^s_{2,2}(\mathbb{R}^n) = \dot B^s_{2,2}(\mathbb{R}^n),\quad -\infty<s<\infty.$$

As in [5, p. 30-31], when $f(r)$ is a function with compact support in $[0,1]$, we have

$$\|f\|_{\dot H^\gamma_{\rm rad}} := \Big(\int_0^1 h^{-1-2\gamma}\|\Delta_h f\|^2_{L^2_{\rm rad}}dh\Big)^{1/2},$$

where $\Delta_h f = f(r+h) - f(r)$, $0<\gamma<1$.

Lemma 2.2. *Under assumptions (H1)–(H5), the function* $\chi_\circ$ *defined by* (2.2) *is in the space* $\dot{H}^{\gamma}_{\rm rad}$.

Proof. By definition of the norm,

$$\|\chi_\circ\|^2_{\dot{H}^{\gamma}_{\rm rad}} = \int_0^1 h^{-(1+2\gamma)} \int_0^1 |f((r+h)^2) - f(r^2)|^2 r\,dr\,dh$$

Using the Mean Value Theorem,

$$\begin{aligned}
\|\chi_\circ\|^2_{\dot{H}^{\gamma}_{\rm rad}} &= \int_0^1 h^{-(1+2\gamma)} \int_0^1 |f'(\xi)|^2[(r+h)^2 - r^2]^2 r\,dr\,dh \\
&= \int_0^1 h^{-(1+2\gamma)} \int_0^1 |f'(\xi)|^2(h^2+2rh)^2 r\,dr\,dh \\
&\le Q_1 \int_0^1 h^{-(1+2\gamma)} \int_0^1 (h^4 r + 4r^2h^3 + 4r^3h^2)\,dr\,dh \\
&= Q_1 \int_0^1 h^{-(1+2\gamma)} \big(\frac{h^4}{2} + \frac{4}{3}h^3 + h^2\big)dh \\
&= Q_1 \frac{1}{2(4-2\gamma)} + \frac{4}{3}Q_1\frac{1}{3-2\gamma} + Q_1\frac{1}{2-2\gamma},
\end{aligned}$$

where Q_1 is positive constant. Therefore, $\|\chi_\circ\|_{\dot{H}^{\gamma}_{\rm rad}} \le Q_2$ for some constant Q_2 and $\chi_\circ \in \dot{H}^{\gamma}_{\rm rad}$. □

Lemma 2.3. *Under Assumptions (H1)-(H5), the function* χ_1 *defined by* (2.3) *is in the space* $\dot{H}^{\gamma-1}_{\rm rad}$.

Proof. We have that $L^2_{\rm rad} \hookrightarrow \dot{H}^{\gamma-1}_{\rm rad}$. On the other hand

$$\|\chi_1\|^2_{L^2_{\rm rad}} = \frac{4\beta^2}{\alpha^2}\int_0^1 r^5|f'(r^2)|^2 dr = \frac{2\beta^2}{\alpha^2}\int_0^1 r^4|f'(r^2)|^2 dr^2 < \infty$$

because $f'(r^2) \in \mathcal{C}[0,1]$. Consequently, $\chi_1 \in \dot{H}^{\gamma-1}_{\rm rad}$. □

Proof of main result

Let (H1)-(H5) hold. By (2.1), $f(z^{2/\alpha}) = (1-z^{2/\alpha})^2 z^{2/\alpha}$. Let g be defined by (2.6) and $\bar{\chi}$ by (2.7). From Lemma 2.1, $g \in L^p([0,1]L^q_{\rm rad})$. Also, we have $\bar{\chi}(1,r) = \chi_\circ$, $\bar{\chi}_t(1,r) = \chi_1$. From Lemma 2.2, $\bar{\chi}(1,r) \in \dot{H}^{\gamma}_{\rm rad}$, and from Lemma 2.3, $\bar{\chi}_t(1,r) \in \dot{H}^{\gamma-1}_{\rm rad}$. Therefore, $\bar{\chi}$ is solution of (1.1)–(1.2). Let $\theta = t^{\beta/\alpha}$ which is in $(0,1]$. Then

$$\|\bar{\chi}\|^2_{\dot{H}^{\gamma}_{\rm rad}} = \int_0^1 h^{-(1+2\gamma)} \int_0^\theta \Big|f\Big(\Big(\frac{r+h}{\theta}\Big)^2\Big) - f\Big(\frac{r^2}{\theta^2}\Big)\Big|^2 r\,dr\,dh\Big) =: I. \tag{2.8}$$

Then

$$I \ge \int_{(\sqrt{2}-\frac{1}{2})\theta}^1 h^{-(1+2\gamma)} \int_{\frac{\theta}{2}}^{\frac{\theta}{\sqrt{3}}} \Big|f\Big(\Big(\frac{r+h}{\theta}\Big)^2\Big) - f\Big(\frac{r^2}{\theta^2}\Big)\Big|^2 r\,dr\,dh.$$

and

$$\begin{aligned}
f\Big(\Big(\frac{r+h}{\theta}\Big)^2\Big) - f\Big(\frac{r^2}{\theta^2}\Big) &= \Big[1 - \frac{(r+h)^2}{\theta^2}\Big]^2 \frac{(r+h)^2}{\theta^2} - \Big(1-\frac{r^2}{\theta^2}\Big)^2\frac{r^2}{\theta^2} \\
&= \frac{[\theta^2 - (r+h)^2]^2(r+h)^2 - (\theta^2 - r^2)^2 r^2}{\theta^6}.
\end{aligned}$$

Note that the numerator of the above expression is

$$\begin{aligned}L &:= [\theta^2-(r+h)^2]^2(r+h)^2-(\theta^2-r^2)^2r^2\\ &= (\theta^2-r^2)(2rh+h^2)(\theta^2-3r^2)+(3r^2-2\theta^2)(2rh+h^2)^2+(2rh+h^2)^3.\end{aligned}$$

For $r\in(\frac{\theta}{2},\frac{\theta}{\sqrt{3}})$ we have

$$\begin{aligned}L &\ge (3r^2-2\theta^2)(2rh+h^2)^2+(2rh+h^2)^3\\ &= (2rh+h^2)^2(3r^2-2\theta^2+2rh+h^2)\\ &= (2rh+h^2)^2[2r^2+(r+h)^2-2\theta^2].\end{aligned}$$

For $h\in((\sqrt{2}-\frac12)\theta,1)$ and $r\in(\frac{\theta}{2},\frac{\theta}{\sqrt{3}})$,

$$\begin{gathered}r+h\ge\frac{\theta}{2}+\sqrt{2}\theta-\frac{1}{2}\theta=\sqrt{2}\theta,\\ (r+h)^2\ge 2\theta^2.\end{gathered}$$

Therefore, for $h\in((\sqrt{2}-\frac12)\theta,1)$ and $r\in(\frac{\theta}{2},\frac{\theta}{\sqrt{3}})$,

$$\begin{gathered}L\ge(2rh+h^2)^2 2r^2\ge h^4r^2,\\ \Big|f\big((\frac{r+h}{\theta})^2\big)-f\big(\frac{r^2}{\theta^2}\big)\Big|^2\ge\frac{h^8r^4}{\theta^{12}}.\end{gathered}$$

Consequently

$$\begin{aligned}I &\ge \int_{(\sqrt{2}-\frac12)\theta}^{1} h^{-(1+2\gamma)}\int_{\frac{\theta}{2}}^{\frac{\theta}{\sqrt{3}}}\frac{h^8r^5}{\theta^{12}}\,dr\,dh\\ &= \frac{1}{6\theta^{12}}\int_{(\sqrt{2}-\frac12)\theta}^{1} h^{7-2\gamma}\Big(\frac{\theta^6}{27}-\frac{\theta^6}{64}\Big)dh\\ &= \frac{37}{6\times27\times64\theta^6.(8-2\gamma)}\Big[1-(\sqrt{2}-\frac12)^{8-2\gamma}\theta^{8-2\gamma}\Big]\\ &= \frac{37}{6\times27\times64\theta^6.(8-2\gamma)}-\frac{37}{6\times27\times64(8-2\gamma)(\sqrt{2}-\frac12)^{8-2\gamma}}\theta^{2-2\gamma}\end{aligned}$$

which approcahes zero as $\theta\to0$. From this statement and (2.8), we obtain

$$\lim_{t\to0}\|\bar\chi\|_{\dot H^\gamma_{\rm rad}}=\infty.$$

2.1. **Comments.** Let (H1)-(H5) hold. By (2.1), $f(z^{2/\alpha})=(1-z^{2/\alpha})^2z^{2/\alpha}$. Then the function $\bar\chi$ defined by (2.7) is a solution to the Cauchy problem

$$\chi_{tt}-\frac{1}{r}\chi_r-\chi_{rr}=g_1(t,r), \tag{2.9}$$

$$\chi(1,r)=\chi_\circ\in\dot H^\gamma_{\rm rad},\quad \chi_t(1,r)=\chi_1\in\dot H^{\gamma-1}_{\rm rad}, \tag{2.10}$$

where $\chi_\circ$ and χ_1 are the functions defined with (2.2) and (2.3),

$$g_1=\begin{cases}\frac{B_1}{t^2}-\frac{4}{t^{2\beta/\alpha}}B_2 & \text{for } r\le t^{\beta/\alpha}\\ 0 & \text{for } r\ge t^{\beta/\alpha}.\end{cases}$$

Then because of (H5),

$$\|g_1\|^p_{L^p([0,1]L^q_{\rm rad})}\le l_1\int_0^1 t^{\frac{2\beta p}{\alpha q}-2p}dt+l_2\int_0^1 t^{\frac{2\beta p}{\alpha q}-\frac{2\beta p}{\alpha}}dt<\infty,$$

where l_1, l_2 are positive constants. In [2, Corollary 1.3] it is proved that

$$\|\bar{\chi}\|_{\mathcal{C}([0,1]\dot{H}^{\gamma}_{\mathrm{rad}})} \leq \|\chi_\circ\|_{\dot{H}^{\gamma}_{\mathrm{rad}}} + \|\chi_1\|_{\dot{H}^{\gamma-1}_{\mathrm{rad}}} + \|g_1\|_{L^p([0,1]L^q_{\mathrm{rad}})}.$$

Therefore, $\bar{\chi}$ is in $\mathcal{C}([0,1]\dot{H}^{\gamma}_{\mathrm{rad}})$.

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