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DYNAMIC CONTACT WITH SIGNORINI'S CONDITION AND SLIP RATE DEPENDENT FRICTION

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Abstract. Existence of a weak solution for the problem of dynamic frictional contact between a viscoelastic body and a rigid foundation is established. Contact is modelled with the Signorini condition. Friction is described by a slip rate dependent friction coefficient and a nonlocal and regularized contact stress. The existence in the case of a friction coefficient that is a graph, which describes the jump from static to dynamic friction, is established, too. The proofs employ the theory of set-valued pseudomonotone operators applied to approximate problems and a priori estimates.

1. Introduction

This work considers frictional contact between a deformable body and a moving rigid foundation. The main interest lies in the dynamic process of friction at the contact area. We model contact with the Signorini condition and friction with a general nonlocal law in which the friction coefficient depends on the slip velocity between the surface and the foundation. We show that a weak solution to the problem exists either when the friction coefficient is a Lipschitz function of the slip rate, or when it is a graph with a jump from the static to the dynamic value at the onset of sliding.

Dynamic contact problems have received considerable attention recently in the mathematical literature. The existence of the unique weak solution of the problem for a viscoelastic material with normal compliance was established in Martins and Oden [23]. The existence of solutions for the frictional problem with normal compliance for a thermoviscoelastic material can be found in Figueiredo and Trabucho [9]; when the frictional heat generation is taken into account in Andrews *et al.* [2], and when the wear of the contacting surfaces is allowed in Andrews *et al.* [3]. A general normal compliance condition was dealt with in Kuttler [15] where the usual restrictions on the normal compliance exponent were removed. The dynamic frictionless problem with adhesion was investigated in Chau *et al.* [4] and in Fernández *et al.* [8]. An important one-dimensional problem with slip rate dependent friction coefficient was investigated in Ionescu and Paumier [11], and then in Paumier and Renard [28]. Problems with normal compliance and slip rate dependent coefficient

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of friction were considered in Kuttler and Shillor [17] and a problem with a discontinuous friction coefficient, which has a jump at the onset of sliding, in Kuttler and Shillor [19]. The problem of bilateral frictional contact with discontinuous friction coefficient can be found in Kuttler and Shillor [18]. A recent substantial regularity result for dynamic frictionless contact problems with normal compliance was obtained in Kuttler and Shillor [20], and a regularity result for the problem with adhesion can be found in Kuttler *et al.* [21]. For additional publications we refer to the references in these papers, and also to the recent monographs Han and Sofonea [10] and Shillor *et al.* [30].

Dynamic contact problems with a unilateral contact condition for the normal velocity were investigated in Jarusek [12], Eck and Jarusek [7] and the one with the Signorini contact condition in Cocu [5], (see also references therein). In [5] the existence of a weak solution for the problem for a viscoelastic material with regularized contact stress and constant friction coefficient has been established, using the normal compliance as regularization. After obtaining the necessary a priori estimates, a solution was obtained by passing to the regularization limit.

The normal compliance contact condition was introduced in [23] to represent real engineering surfaces with asperities that may deform elastically or plastically. However, very often in mathematical and engineering publications it is used as a regularization or approximation of the Signorini contact condition which is an idealization and describes a perfectly rigid surface. Since, physically speaking, there are no perfectly rigid bodies and so the Signorini condition is necessarily an approximation, admittedly a very popular one. The Signorini condition is easy to write and mathematically elegant, but seems not to describe well real contact. Indeed, there is a low regularity ceiling on the solutions to models which include it and, generally, there are no uniqueness results, unlike the situation with normal compliance. Moreover, it usually leads to numerical difficulties, and most numerical algorithms use normal compliance anyway. Although there are some cases in quasistatic or static contact problems where using it seems to be reasonable, in dynamic situations it seems to be a poor approximation of the behavior of the contacting surfaces. We believe that in dynamic processes the Signorini condition is an approximation of the normal compliance, and not a very good one. On the other hand, there is no rigorous derivation of the normal compliance condition either, so the choice of which condition to use is, to a large extent, up to the researcher.

This work extends the recent result in [5] in a threefold way. We remove the compatibility condition for the initial data, since it is an artifact of the mathematical method and is unnecessary in dynamic problems. We allow for the dependence of the friction coefficient on the sliding velocity, and we take into account a possible jump from a static value, when the surfaces are in stick state, to a dynamic value when they are in relative sliding. Such a jump is often assumed in engineering publications. For the sake of mathematical completeness we employ the Signorini condition together with a regularized non-local contact stress.

The rest of the paper is structured as follows. In Section 2 we describe the model, its variational formulation, and the regularization of the contact stress. In Section 3 we describe approximate problems, based on the normal compliance condition. The existence of solutions for these problems is obtained by using the theory of set-valued pseudomonotone maps developed in [17]. A priori estimates on the approximate solutions are derived in Section 4 and by passing to the limit we establish Theorem

2.1 . In Section 5 we approximate the discontinuous friction coefficient, assumed to be a graph at the origin, with a sequence of Lipschitz functions, obtain the necessary estimates and by passing to the limit prove Theorem 5.2. We conclude the paper in Section 6.

2. The model and variational formulation

First, we describe the classical model for the process and the assumptions on the problem data. We use the isothermal version of the problem that was considered in [2, 3] (see also [13, 23, 6, 27]) and refer the reader there for a detailed description of the model. A similar setting, with constant friction coefficient, can be found in [5].

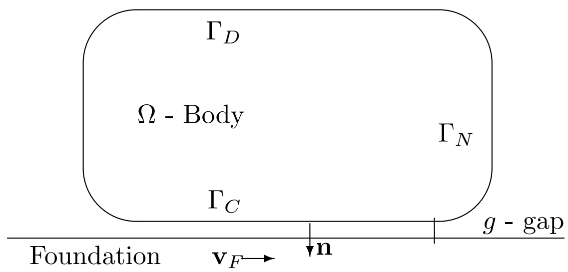

Figure 1. The physical setting; Γ_C is the contact surface.

We consider a viscoelastic body which occupies the reference configuration $\Omega \subset \mathbb{R}^N$ ($N = 2$ or 3 in applications) which may come in contact with a rigid foundation on the part Γ_C of its boundary $\Gamma = \partial\Omega$. We assume that Γ is Lipschitz, and is partitioned into three mutually disjoint parts Γ_D, Γ_N and Γ_C and has outward unit normal $\mathbf{n} = (n_1, \dots, n_N)$. The part Γ_C is the potential contact surface, Dirichlet boundary conditions are prescribed on Γ_D and Neumann's on Γ_N. We set $\Omega_T = \Omega \times (0, T)$ for $0 < T$ and denote by $\mathbf{u} = (u_1, \dots, u_N)$ the displacements vector, by $\mathbf{v} = (v_1, \dots, v_N)$ the velocity vector and the stress tensor by $\sigma = (\sigma_{ij})$, where here and below $i, j = 1, \dots, N$, a comma separates the components of a vector or tensor from partial derivatives, and "$'$" denotes partial time derivative, thus $\mathbf{v} = \mathbf{u}'$. The velocity of the foundation is $\mathbf{v}_F$, and the setting is depicted in Fig. 1.

The dynamic equations of motion, in dimensionless form, are

$$v_i' - \sigma_{ij,j}(\mathbf{u}, \mathbf{v}) = f_{Bi} \quad \text{in } \Omega_T, \tag{2.1}$$

where $\mathbf{f}_B$ represents the volume force acting on the body, all the variables are in dimensionless form, for the sake of simplicity we set the material density to be $\rho = 1$, and

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}(s)\,ds. \tag{2.2}$$

The initial conditions are

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}) \quad \text{in } \Omega, \tag{2.3}$$

where $\mathbf{u}_0$ is the initial displacement and $\mathbf{v}_0$ the velocity, both prescribed functions.

The body is held fixed on Γ_D and tractions $\mathbf{f}_N$ act on Γ_N. Thus,

$$\mathbf{u} = 0 \quad \text{on } \Gamma_D, \quad \sigma\mathbf{n} = \mathbf{f}_N \quad \text{on } \Gamma_N. \tag{2.4}$$

The foundation is assumed completely rigid, so we use the Signorini condition on the potential contact surface,

$$u_n - g \leq 0, \quad \sigma_n \leq 0, \quad (u_n - g)\sigma_n = 0 \quad \text{on } \Gamma_C. \tag{2.5}$$

Here, $u_n = \mathbf{u} \cdot \mathbf{n}$ is the normal component of $\mathbf{u}$ and $\sigma_n = \sigma_{ij} n_i n_j$ is the normal component of the stress vector or the contact pressure on Γ_C.

The material is assumed to be linearly viscoelastic with constitutive relation

$$\sigma(\mathbf{u}, \mathbf{v}) = A\varepsilon(\mathbf{u}) + B\varepsilon(\mathbf{v}), \tag{2.6}$$

where $\varepsilon(\mathbf{u})$ is the small strain tensor, A is the elasticity tensor, and B is the viscosity tensor, both symmetric and linear operators satisfying

$$(A\xi, \xi) \geq \delta^2 |\xi|^2, \quad (B\xi, \xi) \geq \delta^2 |\xi|^2,$$

for some δ and all symmetric second order tensors $\xi = \{\xi_{ij}\}$, i.e., both are coercive or elliptic. In components, the Kelvin-Voigt constitutive relation is

$$\sigma_{ij} = a_{ijkl} u_{k,l} + b_{ijkl} u'_{k,l},$$

where a_{ijkl} and b_{ijkl} are the elastic moduli and viscosity coefficients, respectively.

To describe the friction process we need additional notation. We denote the tangential components of the displacements by $\mathbf{u}_T = \mathbf{u} - (\mathbf{u} \cdot \mathbf{n})\mathbf{n}$ and the tangential tractions by $\sigma_{Ti} = \sigma_{ij} n_j - \sigma_n n_i$. The general law of dry friction, a version of which we employ, is

$$|\sigma_T| \leq \mu(|\mathbf{u}'_T - \mathbf{v}_F|)\, |\sigma_n|, \tag{2.7}$$

$$\sigma_T = -\mu(|\mathbf{u}'_T - \mathbf{v}_F|)\, |\sigma_n| \frac{\mathbf{u}'_T - \mathbf{v}_F}{|\mathbf{u}'_T - \mathbf{v}_F|} \quad \text{if} \quad \mathbf{u}'_T \neq \mathbf{v}_F. \tag{2.8}$$

This condition creates major mathematical difficulties in the weak formulation of the problem, since the stress σ does not have sufficient regularity for its boundary values to be well-defined. Nevertheless, some progress has been made in using this model in [7] and [12] (see also the references therein). However, there the contact condition was that of normal damped response, *i.e.*, the unilateral restriction was on the normal velocity rather than on the normal displacement, as in (2.5). Such a condition implies that once contact is lost, it is never regained, which in most applied situations is not the case, and, moreover, the mathematical difficulties in dealing with that model were considerable. These difficulties motivated [23] and many followers to model the normal contact between the body and the foundation by a *normal compliance* condition in which the normal stress is given as a function of surface resistance to interpenetration. This is usually justified by modeling the contact surface in terms of "surface asperities."

To overcome the difficulties in giving meaning to the trace of the stress on the contact surface we employ an averaged stress in the friction model. Thus, it is a locally averaged stress which controls the friction and the onset of sliding on the surface, however, we make no particular assumptions on the form of the averaging process. It may be of interest to investigate and deduce them from homogenization or experimental results (see [25] for a step in this direction). This procedure of averaging the stress has been employed earlier by the authors in [18] (see also references therein) and recently in [5] in a very interesting paper on the existence of weak solutions for a linear viscoelastic model with the Signorini boundary conditions (2.5) and an averaged Coulomb friction law. In this paper we consider a similar situation, but with a slip rate dependent coefficient of friction, and without the compatibility assumptions on the initial data which are assumed in [5]. The averaged form of the friction which we will employ involves replacing the normal stress σ_n with an averaged normal stress, $(\mathcal{R}\sigma)_n$, where $\mathcal{R}$ is an averaging operator to be described shortly. Thus, we employ the following law of dry friction,

$$|\sigma_T| \le \mu(|\mathbf{u}'_T - \mathbf{v}_F|)|(\mathcal{R}\sigma)_n|, \tag{2.9}$$

$$\sigma_T = -\mu(|\mathbf{u}'_T - \mathbf{v}_F|)|(\mathcal{R}\sigma)_n|\frac{\mathbf{u}'_T - \mathbf{v}_F}{|\mathbf{u}'_T - \mathbf{v}_F|} \quad \text{if} \quad \mathbf{u}'_T \ne \mathbf{v}_F. \tag{2.10}$$

Here, μ is the coefficient of friction, a positive bounded function assumed to depend on the relative slip $\mathbf{u}'_T - \mathbf{v}_F$ between the body and foundation. We could have let μ depend on $\mathbf{x} \in \Gamma_C$ as well, to model the local roughness of the contact surface, but we will not consider it here to simplify the presentation; however, all the results below hold when μ is a Lipschitz function of $\mathbf{x}$.

We assume that $\mathbf{v}_F \in L^\infty(0,T;(L^2(\Gamma_C))^N)$, and refer to [1, 14] for standard notation and concepts related to function spaces. The regularization $(\mathcal{R}\sigma)_n$ of the normal contact force in (2.9) and (2.10) is such that $\mathcal{R}$ is *linear* and

$$\mathbf{v}^k \to \mathbf{v} \text{ in } L^2(0,T;L^2(\Omega)^N) \text{ implies } (\mathcal{R}\sigma^k)_n \to (\mathcal{R}\sigma)_n \text{ in } L^2(0,T;L^2(\Gamma_c)) \tag{2.11}$$

There are a number of ways to construct such a regularization. For example, if $\mathbf{v} \in H^1(\Omega)$, one may extend it to $\mathbb{R}^N$ in such a way that the extended function $E\mathbf{v}$ satisfies $\|E\mathbf{v}\|_{1,\mathbb{R}^N} \le C\|\mathbf{v}\|_{1,\Omega}$, where C is a positive constant that is independent of $\mathbf{v}$, and define

$$\mathcal{R}\sigma \equiv A\varepsilon(E\mathbf{u} * \psi) + B\varepsilon(E\mathbf{v} * \psi), \tag{2.12}$$

where ψ is a smooth function with compact support, and "$*$" denotes the convolution operation. Thus, $\mathcal{R}\sigma \in C^\infty(\mathbb{R}^N)$ and $(\mathcal{R}\sigma)_n \equiv (\mathcal{R}\sigma)\mathbf{n}\cdot\mathbf{n}$ is well defined on Γ_C. In this way we average the displacements and the velocity and consider the stress determined by the averaged variables.

Another way to obtain an averaging operator satisfying (2.11) is as follows. Let $\psi : \Gamma_C \times \mathbb{R}^N \to \mathbb{R}$ be such that $\mathbf{y} \to \psi(\mathbf{x},\mathbf{y})$ is in $C_c^\infty(\Omega)$, ψ is uniformly bounded and, for $\mathbf{x} \in \Gamma_C$,

$$\mathcal{R}\sigma_n \equiv \mathcal{R}\sigma\mathbf{n}\cdot\mathbf{n} \quad \text{where} \quad \mathcal{R}\sigma(\mathbf{x}) \equiv \int_\Omega \sigma(\mathbf{y})\psi(\mathbf{x},\mathbf{y})dy.$$

Then, the operator is linear and it is routine to verify that (2.11) holds. Physically, this means that the normal component of stress, which controls the friction process, is averaged over a part of Ω, and to be meaningful, we assume that the support of $\psi(\mathbf{x},\cdot)$ is centered at $\mathbf{x} \in \Gamma_C$ and is small. Conditions (2.9) and (2.10) are the model for friction which we employ in this work. The tangential part of the traction is bounded by the so-called friction bound, $\mu(|\mathbf{u}'_T - \mathbf{v}_F|)|(\mathcal{R}\sigma)_n|$, which depends on the sliding velocity via the friction coefficient, and on the regularized contact stress. The surface point sticks to the foundation and no sliding takes place until $|\sigma_T|$ reaches the friction bound and then sliding commences and the tangential force has a direction opposite to the relative tangential velocity. The contact surface Γ_C is divided at each time instant into three parts: *separation* zone, *slip* zone and *stick* zone.

A new feature in the model is the dependence, which can be observed experimentally, of the friction coefficient on the magnitude of the slip rate $|\mathbf{u}'_T - \mathbf{v}_F|$.

We assume that the coefficient of friction is bounded, Lipschitz continuous and satisfies

$$|\mu(r_1) - \mu(r_2)| \leq \quad \mathrm{Lip}_\mu \, |r_1 - r_2| \, , \qquad \|\mu\|_\infty \leq K_\mu. \tag{2.13}$$

In Section 5 this assumption will be relaxed and we shall consider μ which is set-valued and models the jump from a static value to a dynamic value when sliding starts.

The classical formulation of the problem of *dynamic contact between a viscoelastic body and a rigid foundation* is:
Find the displacements $\mathbf{u}$ and the velocity $\mathbf{v} = \mathbf{u}'$, such that

$$\mathbf{v}' - \mathrm{Div}(\sigma(\mathbf{u}, \mathbf{v})) = \mathbf{f}_B \quad \text{in} \ \ \Omega_T, \tag{2.14}$$

$$\sigma(\mathbf{u}, \mathbf{v}) = A\varepsilon(\mathbf{u}) + B\varepsilon(\mathbf{v}), \tag{2.15}$$

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}(s) ds, \tag{2.16}$$

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{v}(\mathbf{x}, 0) = \mathbf{v}_0(\mathbf{x}) \quad \text{in } \Omega, \tag{2.17}$$

$$\mathbf{u} = 0 \ \text{ on } \Gamma_D, \quad \sigma\mathbf{n} = \mathbf{f}_N \quad \text{ on } \Gamma_N, \tag{2.18}$$

$$u_n - g \leq 0, \quad \sigma_n \leq 0, \quad (u_n - g)\sigma_n = 0 \quad \text{ on } \Gamma_C, \tag{2.19}$$

$$|\sigma_T| \leq \mu(|\mathbf{u}_T' - \mathbf{v}_F|) \, |(\mathcal{R}\sigma)_n| \, , \quad \text{ on } \Gamma_C, \tag{2.20}$$

$$\mathbf{u}_T' \neq \mathbf{v}_F \ \text{ implies } \quad \sigma_T = -\mu(|\mathbf{u}_T' - \mathbf{v}_F|)|(\mathcal{R}\sigma)_n| \frac{\mathbf{u}_T' - \mathbf{v}_F}{|\mathbf{u}_T' - \mathbf{v}_F|}. \tag{2.21}$$

We turn to the weak formulation of the problem, and to that end we need additional notation. V denotes a closed subspace of $(H^1(\Omega))^N$ containing the test functions $(C_c^\infty(\Omega))^N$, and γ is the trace map from $W^{1,p}(\Omega)$ into $L^p(\partial\Omega)$. We let $H = (L^2(\Omega))^N$ and identify H and H', thus,

$$V \subseteq H = H' \subseteq V'.$$

Also, we let $\mathcal{V} = L^2(0, T; V)$ and $\mathcal{H} = L^2(0, T; H)$.

Next, we choose the subspace V as follows. If the body is clamped over Γ_D, with $\mathrm{meas}\,\Gamma_D > 0$, then we set $V = \{\mathbf{u} \in H^1(\Omega))^N : \mathbf{u} = 0 \ \text{ on } \ \Gamma_D\}$. If the body is not held fixed, so that $\mathrm{meas}\,\Gamma_D = 0$, then it is free to move in space, and we set $V = (H^1(\Omega))^N$. We note that the latter leads to a noncoercive problem for the quasistatic model, the so-called punch problem, which in that context needs a separate treatment. We let U be a Banach space in which V is compactly embedded, V is also dense in U and the trace map from U to $(L^2(\partial\Omega))^N$ is continuous. We seek the solutions in the convex set

$$\mathcal{K} \equiv \left\{ \mathbf{w} \in \mathcal{V} : \mathbf{w}' \in \mathcal{V}', \ (w_n - g)_+ = 0 \text{ in } L^2(0, T; L^2(\Gamma_C)) \right\}. \tag{2.22}$$

Here, $(f)_+ = \max\{0, f\}$ is the positive part of f.

We shall need the following viscosity and elasticity operators, M and L, respectively, defined as: $M, L : V \to V'$,

$$\langle M\mathbf{u}, \mathbf{v} \rangle = \int_\Omega B\varepsilon(\mathbf{u})\varepsilon(\mathbf{v}) dx, \tag{2.23}$$

$$\langle L\mathbf{u}, \mathbf{v} \rangle = \int_\Omega A\varepsilon(\mathbf{u})\varepsilon(\mathbf{v}) dx. \tag{2.24}$$

It follows from the assumptions and Korn's inequality ([24, 26]) that M and L satisfy

$$\langle C\mathbf{u},\mathbf{u}\rangle \geq \delta^2\|\mathbf{u}\|_V^2 - \lambda|\mathbf{u}|_H^2, \quad \langle C\mathbf{u},\mathbf{u}\rangle \geq 0, \quad \langle C\mathbf{u},\mathbf{v}\rangle = \langle C\mathbf{v},\mathbf{u}\rangle, \tag{2.25}$$

where $C = M$ or L, for some $\delta > 0,\ \lambda \geq 0$. Next, we define $\mathbf{f} \in L^2(0,T;V')$ as

$$\langle \mathbf{f},\mathbf{z}\rangle_{\mathcal{V}',\mathcal{V}} = \int_0^T \int_\Omega \mathbf{f}_B\mathbf{z}\,dx\,dt + \int_0^T \int_{\Gamma_N} \mathbf{f}_N\mathbf{z}d\Gamma dt, \tag{2.26}$$

for $\mathbf{z} \in \mathcal{V}$. Here $\mathbf{f}_B \in L^2(0,T;H)$ is the body force and $\mathbf{f}_N \in L^2(0,T;L^2(\Gamma_N)^N)$ is the surface traction. Finally, we let

$$\gamma_T^* : L^2(0,T;L^2(\Gamma_C)^N) \to \mathcal{V}'$$

be defined as

$$\langle \gamma_T^*\xi,\mathbf{w}\rangle \equiv \int_0^T \int_{\Gamma_C} \xi\cdot\mathbf{w}_T d\Gamma dt. \tag{2.27}$$

The first of our two main results in this work is the existence of weak solutions to the problem, under the above assumptions.

Theorem 2.1. *Assume, in addition, that $u_0 \in V$, $u_{0n} - g \leq 0$ a.e. on Γ_C, $\mathbf{v}(0) = \mathbf{v}_0 \in H$, and let $\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}(s)ds$. Then, there exist $\mathbf{u} \in C([0,T];U) \cap L^\infty(0,T;V)$, $\mathbf{u} \in \mathcal{K}$, $\mathbf{v} \in L^2(0,T;V) \cap L^\infty(0,T;H)$, $\mathbf{v}' \in L^2(0,T;H^{-1}(\Omega)^N)$, and $\xi \in L^2(0,T;L^2(\Gamma_C)^N)$ such that*

$$\begin{aligned} &-(\mathbf{v}_0,\mathbf{u}_0-\mathbf{w}(0))_H + \int_0^T \langle M\mathbf{v},\mathbf{u}-\mathbf{w}\rangle dt + \int_0^T \langle L\mathbf{u},\mathbf{u}-\mathbf{w}\rangle dt \\ &- \int_0^T (\mathbf{v},\mathbf{v}-\mathbf{w}')_H dt + \int_0^T \langle \gamma_T^*\xi,\mathbf{u}-\mathbf{w}\rangle dt \\ &\leq \int_0^T \langle \mathbf{f},\mathbf{u}-\mathbf{w}\rangle dt, \end{aligned} \tag{2.28}$$

for all $\mathbf{w} \in \mathcal{K}_{\mathbf{u}}$, where

$$\mathcal{K}_{\mathbf{u}} \equiv \{\mathbf{w} \in \mathcal{V} : \mathbf{w}' \in \mathcal{V},\ (w_n - g)_+ = 0 \text{ in } L^2(0,T;L^2(\Gamma_C)),\ \ \mathbf{u}(T) = \mathbf{w}(T)\}. \tag{2.29}$$

Here, $\gamma_T^*\xi$ satisfies, for $\mathbf{w} \in \mathcal{K}_{\mathbf{u}}$,

$$\langle \gamma_T^*\xi,\mathbf{w}\rangle \leq \int_0^T \int_{\Gamma_C} \mu(|\mathbf{v}_T - \mathbf{v}_F|)\,|(\mathcal{R}\sigma)_n|\,(|\mathbf{v}_T - \mathbf{v}_F + \mathbf{w}_T| - |\mathbf{v}_T - \mathbf{v}_F|)\,d\Gamma dt. \tag{2.30}$$

The proof of this theorem will be given in Section 4. It is obtained by considering a sequence of approximate problems, based on the normal compliance condition studied in Section 3, where the relevant a priori estimates are derived.

3. Approximate Problems

The Signorini condition leads to considerable difficulties in the analysis of the problem. Therefore, we first consider approximate problems based on the normal compliance condition, which we believe is a more realistic model. We establish the unique solvability of these problems and obtain the necessary a priori estimates which will allow us to pass to the Signorini limit. These problems have merit in and of themselves.

We shall use the following two well known results, the first one can be found in Lions [22] and the other one in Simon [31] or Seidman [29] see also [16]).

Theorem 3.1 ([22])**.** *Let $p \geq 1$, $q > 1$, $W \subseteq U \subseteq Y$, with compact inclusion map $W \to U$ and continuous inclusion map $U \to Y$, and let*

$$S = \{\mathbf{u} \in L^p(0,T;W) : \mathbf{u}' \in L^q(0,T;Y), \|\mathbf{u}\|_{L^p(0,T;W)} + \|\mathbf{u}'\|_{L^q(0,T;Y)} < R\}.$$

Then S is precompact in $L^p(0,T;U)$.

Theorem 3.2 ([29, 31])**.** *Let W, U and Y be as above and for $q > 1$ let*

$$S_T = \{\mathbf{u} : \|\mathbf{u}(t)\|_W + \|\mathbf{u}'\|_{L^q(0,T;Y)} \leq R, \quad t \in [0,T]\}.$$

Then S_T is precompact in $C(0,T;U)$.

We turn to an abstract formulation of problem (2.1)–(2.10). We define the *normal compliance* operator $\mathbf{u} \to P(\mathbf{u})$, which maps $\mathcal{V}$ to $\mathcal{V}'$, by

$$\langle P(\mathbf{u}), \mathbf{w} \rangle = \int_0^T \int_{\Gamma_C} (u_n - g)_+ w_n \, d\Gamma dt. \tag{3.1}$$

It will be used to approximate the Signorini condition by penalizing it.

The abstract form of the approximate problem, with $0 < \varepsilon$, is as follows.
Problem $\mathcal{P}_\varepsilon$: Find $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ such that

$$\mathbf{v}' + M\mathbf{v} + L\mathbf{u} + \frac{1}{\varepsilon} P(\mathbf{u}) + \gamma_T^* \xi = \mathbf{f} \quad \text{in} \quad \mathcal{V}', \tag{3.2}$$

$$\mathbf{v}(0) = \mathbf{v}_0 \in H, \tag{3.3}$$

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}(s) ds, \quad \mathbf{u}_0 \in V \tag{3.4}$$

and for all $\mathbf{w} \in \mathcal{V}$,

$$\langle \gamma_T^* \xi, \mathbf{w} \rangle \leq \int_0^T \int_{\Gamma_C} \mu(|\mathbf{v}_T - \mathbf{v}_F|) \, |(\mathcal{R}\sigma)_n| \, (|\mathbf{v}_T - \mathbf{v}_F + \mathbf{w}_T| - |\mathbf{v}_T - \mathbf{v}_F|) d\Gamma dt. \tag{3.5}$$

The existence of solutions of the problem follows from a straightforward application of the existence theorem in [17], therefore,

Theorem 3.3. *There exists a solution to problem $\mathcal{P}_\varepsilon$.*

We assume, as in Theorem 2.1, that initially

$$u_{0n} - g \leq 0 \quad \text{on } \Gamma_C. \tag{3.6}$$

We turn to obtain estimates on the solutions of Problem $\mathcal{P}_\varepsilon$. In what follows C will denote a generic constant which depends on the data but is independent of $t \in [0,T]$ or ε, and whose value may change from line to line.

We multiply (3.2) by $\mathbf{v}\chi_{[0,t]}$, where

$$\chi_{[0,t]}(s) = \begin{cases} 1 & \text{if } s \in [0,t]\,, \\ 0 & \text{if } s \notin [0,t]\,, \end{cases}$$

and integrate over $(0,t)$. From (2.25) and (3.6) we obtain

$$\begin{aligned} &\frac{1}{2}\,|\mathbf{v}(t)|_H^2 - \frac{1}{2}\,|\mathbf{v}_0|_H^2 + \int_0^t (\delta^2\|\mathbf{v}\|_V^2 - \lambda|\mathbf{v}|_H^2)ds + \frac{1}{2}\langle L\mathbf{u}(t), \mathbf{u}(t)\rangle \\ &- \frac{1}{2}\langle L\mathbf{u}_0, \mathbf{u}_0\rangle + \frac{1}{2\varepsilon}\int_{\Gamma_C} ((u_n(t)-g)_+)^2\, d\Gamma + \int_0^t \int_{\Gamma_C} \xi\cdot\mathbf{v}_T\, d\Gamma ds \\ &\le \int_0^t \langle \mathbf{f}, \mathbf{v}\rangle ds. \end{aligned} \tag{3.7}$$

Since μ is assumed to be bounded, it follows from (3.5) that

$$\xi \in [-K_\mu\,|(\mathcal{R}\sigma)_n|\,,\, K_\mu\,|(\mathcal{R}\sigma)_n|\,]\,, \tag{3.8}$$

a.e. on Γ_C. Now, assumption (2.11) implies that there exists a constant C such that

$$\|(\mathcal{R}\sigma)_n\|_{L^2(0,t;L^2(\Gamma_c))} \le C\|\mathbf{v}\|_{L^2(0,t;H)}. \tag{3.9}$$

Therefore, we find from (3.8) that

$$\begin{aligned} \Big|\int_0^t \int_{\Gamma_C} \xi\cdot\mathbf{v}_T d\Gamma ds\Big| &\le \|\xi\|_{L^2(0,t;L^2(\Gamma_C))}\|\mathbf{v}_T\|_{L^2(0,t;(L^2(\Gamma_C))^N)} \\ &\le C\|\mathbf{v}\|_{L^2(0,t;H)}\|\mathbf{v}\|_{L^2(0,t;U)}, \end{aligned} \tag{3.10}$$

where U is the Banach space described above, such that $V \subset U$ compactly, and the trace into $(L^2(\partial\Omega)^N)$ is continuous. Now, the compactness of the embedding implies that for each $0 < \eta$

$$\|\mathbf{v}\|_{L^2(0,t;U)} \le \eta\|\mathbf{v}\|_{L^2(0,t;V)} + C_\eta\|\mathbf{v}\|_{L^2(0,t;H)}. \tag{3.11}$$

Therefore, (3.7) yields

$$\begin{aligned} &|\mathbf{v}(t)|_H^2 - |\mathbf{v}_0|_H^2 + \int_0^t (\delta^2\|\mathbf{v}\|_V^2 - \lambda|\mathbf{v}|_H^2)ds + (\delta^2\|\mathbf{u}(t)\|_V^2 - \lambda|\mathbf{u}(t)|_H^2) \\ &- \langle L\mathbf{u}_0, \mathbf{u}_0\rangle + \frac{1}{\varepsilon}\int_{\Gamma_C} ((u_n(t)-g)_+)^2 d\Gamma - C\|\mathbf{v}\|_{L^2(0,t;H)}\|\mathbf{v}\|_{L^2(0,t;U)} \\ &\le C_\eta \int_0^t \|\mathbf{f}\|_{\mathcal{V}'}^2 ds + \eta\int_0^t \|\mathbf{v}\|_V^2 ds. \end{aligned} \tag{3.12}$$

We obtain from (3.11) and (2.26),

$$\begin{aligned} &|\mathbf{v}(t)|_H^2 + \delta^2\int_0^t \|\mathbf{v}\|_V^2 ds + \delta^2\|\mathbf{u}(t)\|_V^2 + \frac{1}{\varepsilon}\int_{\Gamma_C} ((u_n(t)-g)_+)^2 d\Gamma \\ &\le C\int_0^t |\mathbf{v}(s)|_H^2\, ds + \eta\int_0^t \|\mathbf{v}\|_V^2 ds + \lambda|\mathbf{u}(t)|_H^2 + \langle L\mathbf{u}_0, \mathbf{u}_0\rangle \\ &+ |\mathbf{v}_0|_H^2 + C_\eta\int_0^t \|\mathbf{f}\|_{\mathcal{V}'}^2 ds + \eta\int_0^t \|\mathbf{v}\|_V^2 ds. \end{aligned} \tag{3.13}$$

Choosing η small enough and using the inequality $|\mathbf{u}(t)|_H^2 \le |\mathbf{u}_0|_H + \int_0^t |\mathbf{v}(s)|_H^2\, ds$, yields

$$\begin{aligned} &|\mathbf{v}(t)|_H^2 + \frac{\delta^2}{2}\int_0^t \|\mathbf{v}\|_V^2 ds + \delta^2\|\mathbf{u}(t)\|_V^2 + \frac{1}{\varepsilon}\int_{\Gamma_C} ((u_n(t)-g)_+)^2 d\Gamma \\ &\le C(\mathbf{u}_0, \mathbf{v}_0) + C\int_0^t \|\mathbf{f}\|_{\mathcal{V}'}^2 ds + C\int_0^t |\mathbf{v}(s)|_H^2\, ds. \end{aligned}$$

An application of Gronwall's inequality gives

$$
\begin{aligned}
&|\mathbf{v}(t)|_H^2 + \frac{\delta^2}{2}\int_0^t \|\mathbf{v}\|_V^2 ds + \delta^2\|\mathbf{u}(t)\|_V^2 + \frac{1}{\varepsilon}\int_{\Gamma_C} ((u_n(t)-g)_+)^2 d\Gamma \\
&\le C(\mathbf{u}_0,\mathbf{v}_0) + C\int_0^T \|\mathbf{f}\|_{\mathcal{V}'}^2 ds = C.
\end{aligned}
\tag{3.14}
$$

Now, let $\mathbf{w} \in L^2(0,T;H_0^1(\Omega)^N)$ and apply (3.2) to it, thus,

$$
\langle \mathbf{v}', \mathbf{w}\rangle + \langle M\mathbf{v}, \mathbf{w}\rangle + \langle L\mathbf{u}, \mathbf{w}\rangle = (\mathbf{f}_B, \mathbf{w})_{\mathcal{H}}. \tag{3.15}
$$

It follows from estimate (3.14) that $\mathbf{v}'$ is bounded in $L^2(0,T;H^{-1}(\Omega)^N)$, independently of ε. We conclude that there exists a constant C , which is independent of ε, such that

$$
|\mathbf{v}(t)|_H^2 + \int_0^t \|\mathbf{v}\|_V^2 ds + \|\mathbf{u}(t)\|_V^2 + \frac{1}{\varepsilon}\int_{\Gamma_C} ((u_n(t)-g)_+)^2 d\Gamma + \|\mathbf{v}'\|_{L^2(0,T;H^{-1}(\Omega)^N)} \le C. \tag{3.16}
$$

Next, recall that $\Omega_T \equiv [0,T]\times\Omega$, and we have the following result.

Lemma 3.4. *Let* $(\mathbf{u},\mathbf{v})$ *be a solution of Problem* $\mathcal{P}_\varepsilon$*. Then, there exists a constant* C*, independent of* ε*, such that*

$$
\|\mathbf{v}' - \operatorname{Div}(A\varepsilon(\mathbf{u}) + B\varepsilon(\mathbf{v}))\|_{L^2(\Omega_T)} \le C. \tag{3.17}
$$

In (3.17) measurable representatives are used whenever appropriate.

Proof. Let $\phi \in C_c^\infty(\Omega_T;\mathbb{R}^N)$, then by (3.2),

$$
\int_0^T\int_\Omega -\mathbf{v}\cdot\phi_t + (A\varepsilon(\mathbf{u}) + B\varepsilon(\mathbf{v}))\cdot\varepsilon(\phi)\,dx\,dt = \int_0^T\int_\Omega \mathbf{f}_B\cdot\phi\,dx\,dt.
$$

Therefore,

$$
|(\mathbf{v}' - \operatorname{Div}(A\varepsilon(\mathbf{u}) + B\varepsilon(\mathbf{v})))(\phi)| \le \|\mathbf{f}_B\|_{L^2(0,T;H)}\|\phi\|_{L^2(0,T;H)}
$$

holds in the sense of distributions, which establishes (3.17).

Note that nothing is being said about $\mathbf{v}'$ or $\operatorname{Div}(A\varepsilon(\mathbf{u}) + B\varepsilon(\mathbf{v}))$ separately. This estimate holds because the term that involves ε relates to the boundary and so is irrelevant when we deal with $\phi \in C_c^\infty(\Omega_T;\mathbb{R}^N)$ which vanishes near the boundary, and such functions are dense in $L^2(\Omega_T;\mathbb{R}^N)$.

The proof of the following lemma is straightforward.

Lemma 3.5. *If* $\mathbf{v},\mathbf{u} \in \mathcal{V}$, $\mathbf{v} = \mathbf{u}'$ *and* $\mathbf{v}' - \operatorname{Div}(A\varepsilon(\mathbf{u}) + B\varepsilon(\mathbf{v})) \in L^2(\Omega_T)$*, then for* $\phi \in W_0^{1,\infty}(0,T)$ *and* $\psi \in W_0^{1,\infty}(\Omega)$*,*

$$
\begin{aligned}
&\int_{\Omega_T} (v_iv_i - (A_{ijkl}\varepsilon(\mathbf{u})_{kl} + B_{ijkl}\varepsilon(\mathbf{v})_{kl})\varepsilon(\mathbf{u})_{ij})\psi\phi\,dx\,dt \\
&= \int_{\Omega_T} (-v_i + (A_{ijlk}\varepsilon(\mathbf{u})_{kl} + B_{ijkl}\varepsilon(\mathbf{v}))_{,j})u_i\psi\phi\,dx\,dt \\
&\quad - \int_{\Omega_T} (u_iv_i\psi\phi' - A_{ijkl}\varepsilon(\mathbf{u})_{kl}u_i\psi_{,j}\phi - B_{ijkl}\varepsilon(\mathbf{v})_{kl}u_i\psi_{,j}\phi)\,dx\,dt.
\end{aligned}
\tag{3.18}
$$

We now denote the solution of Problem $\mathcal{P}_\varepsilon$ by $\mathbf{u}^\varepsilon$ and let $\mathbf{u}^{\varepsilon\prime} = \mathbf{v}^\varepsilon$. We deduce from the estimates (3.14) and (3.17) and from Theorems 3.1 and 3.2 that there exists a subsequence, still indexed by ε, such that as $\varepsilon \to 0$, the following convergences take place:

$$\mathbf{u}^\varepsilon \to \mathbf{u} \text{ weak} * \text{ in } L^\infty(0,T;V), \tag{3.19}$$
$$\mathbf{u}^\varepsilon \to \mathbf{u} \text{ strongly in } C([0,T];U), \tag{3.20}$$
$$\mathbf{v}^\varepsilon \to \mathbf{v} \text{ weakly in } \mathcal{V}, \tag{3.21}$$
$$\mathbf{v}^\varepsilon \to \mathbf{v} \text{ weak} * \text{ in } L^\infty(0,T;H), \tag{3.22}$$
$$\mathbf{u}^\varepsilon(T) \to \mathbf{u}(T) \text{ weakly in } V, \tag{3.23}$$
$$\mathbf{v}^\varepsilon \to \mathbf{v} \text{ strongly in } L^2(0,T;U), \tag{3.24}$$
$$\mathbf{v}^\varepsilon(\mathbf{x},t) \to \mathbf{v}(\mathbf{x},t) \text{ pointwise a.e. on } \Gamma_C \times [0,T]\,, \tag{3.25}$$
$$\mathbf{v}^{\varepsilon\prime} \to \mathbf{v}' \text{ weak} * \text{ in } L^2(0,T;H^{-1}(\Omega)^N), \tag{3.26}$$
$$\mathbf{v}^{\varepsilon\prime} - \operatorname{Div}(A\varepsilon(\mathbf{u}^\varepsilon) + B\varepsilon(\mathbf{v}^\varepsilon)) \to \mathbf{v}' - \operatorname{Div}(A\varepsilon(\mathbf{u}) + B\varepsilon(\mathbf{v})) \text{ weakly in } \mathcal{H}, \tag{3.27}$$

where a measurable representative is being used in (3.25). Moreover, we have

$$|\mathbf{v}^\varepsilon(T)|_H \le C. \tag{3.28}$$

The following is a fundamental result which will, ultimately, make it possible to pass to the limit in Problem $\mathcal{P}_\varepsilon$.

Lemma 3.6. *Let $\{\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon\}$ be the sequence, found above, of solutions of Problem $\mathcal{P}_\varepsilon$. Then,*

$$\begin{aligned}&\limsup_{\varepsilon\to 0} \int_{\Omega_T} (v_i^\varepsilon v_i^\varepsilon - (A_{ijkl}\varepsilon(\mathbf{u}^\varepsilon)_{kl} + B_{ijkl}\varepsilon(\mathbf{v}^\varepsilon)_{kl})\varepsilon(\mathbf{u}^\varepsilon)_{ij})\,dx\,dt\\ &\le \int_{\Omega_T} (v_i v_i - (A_{ijkl}\varepsilon(\mathbf{u})_{kl} + B_{ijkl}\varepsilon(\mathbf{v})_{kl})\varepsilon(\mathbf{u})_{ij})\,dx\,dt.\end{aligned}$$

In terms of the abstract operators,

$$\begin{aligned}&\limsup_{\varepsilon\to 0} \int_0^T -\langle M\mathbf{v}^\varepsilon, \mathbf{u}^\varepsilon\rangle dt + \int_0^T (\mathbf{v}^\varepsilon, \mathbf{v}^\varepsilon)_H dt - \int_0^T \langle L\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon\rangle dt\\ &\le \int_0^T -\langle M\mathbf{v}, \mathbf{u}\rangle dt + \int_0^T (\mathbf{v}, \mathbf{v})_H dt - \int_0^T \langle L\mathbf{u}, \mathbf{u}\rangle dt.\end{aligned} \tag{3.29}$$

Proof. Let $\eta > 0$ be given. Let ϕ_δ be a piecewise linear and continuous function such that for small $\delta > 0$, $\phi_\delta(t) = 1$ on $[\delta, T-\delta]$, $\phi_\delta(0) = 0$ and $\phi_\delta(T) = 0$. Also, let $\psi_\delta \in C_c^\infty(\Omega)$ be such that $\psi_\delta(\mathbf{x}) \in [0,1]$ for all $\mathbf{x}$, and also,

$$\operatorname{meas}(\Omega \setminus [\psi_\delta = 1]) \equiv m(Q_\delta) < \delta, \tag{3.30}$$
$$\frac{1}{2\delta}\int_\Omega B_{ijkl}\varepsilon(\mathbf{u}_0)_{kl}\varepsilon(\mathbf{u}_0)_{ij}(1-\psi_\delta) < \eta, \tag{3.31}$$
$$\int_0^T \int_{Q_\delta} v_i v_i < \eta, \quad \int_0^\delta \int_\Omega v_i v_i < \eta, \quad \int_{T-\delta}^T \int_\Omega v_i v_i < \eta. \tag{3.32}$$

Now by (3.19)-(3.27) and formula (3.18),

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\Omega_T}(v_i^\varepsilon v_i^\varepsilon - (A_{ijkl}\varepsilon(\mathbf{u}^\varepsilon)_{kl} + B_{ijkl}\varepsilon(\mathbf{v}^\varepsilon)_{kl})\varepsilon(\mathbf{u}^\varepsilon)_{ij})\phi_\delta\psi_\delta\,dx\,dt \\
&= \int_{\Omega_T}(-v_i + (A_{ijlk}\varepsilon(\mathbf{u})_{kl} + B_{ijkl}\varepsilon(\mathbf{v}))_{,j})u_i\psi_\delta\phi_\delta\,dx\,dt \\
&\quad - \int_{\Omega_T}(u_iv_i\psi\phi' - A_{ijkl}\varepsilon(\mathbf{u})_{kl}u_i\psi_{,j}\phi - B_{ijkl}\varepsilon(\mathbf{v})_{kl}u_i\psi_{\delta,j}\phi_\delta)\,dx\,dt,
\end{aligned}
\tag{3.33}
$$

which equals

$$
\int_{\Omega_T}(v_iv_i - (A_{ijkl}\varepsilon(\mathbf{u})_{kl} + B_{ijkl}\varepsilon(\mathbf{v})_{kl})\varepsilon(\mathbf{u})_{ij})\psi_\delta\phi_\delta\,dx\,dt,
$$

by Lemma 3.5. Since ψ_δ and ϕ_δ are not identically equal to one, we have to consider the integrals

$$
\begin{aligned}
I_1 &\equiv \int_{\Omega_T} v_i^\varepsilon v_i^\varepsilon(1-\psi_\delta\phi_\delta)\,dx\,dt, \\
I_2 &\equiv -\int_{\Omega_T} B_{ijkl}\varepsilon(\mathbf{v}^\varepsilon)_{kl}\varepsilon(\mathbf{u}^\varepsilon)_{ij}(1-\psi_\delta\phi_\delta)\,dx\,dt, \\
I_3 &\equiv -\int_{\Omega_T} A_{ijkl}\varepsilon(\mathbf{u}^\varepsilon)_{kl}\varepsilon(\mathbf{u}^\varepsilon)_{ij}(1-\psi_\delta\phi_\delta)\,dx\,dt.
\end{aligned}
\tag{3.34}
$$

It is clear that $I_3 \le 0$. Next,

$$
0 \le I_1 \le \int_0^T\int_{Q_\delta} v_i^\varepsilon v_i^\varepsilon\,dx\,dt + \int_0^\delta\int_\Omega v_i^\varepsilon v_i^\varepsilon\,dx\,dt + \int_{T-\delta}^T\int_\Omega v_i^\varepsilon v_i^\varepsilon\,dx\,dt.
$$

Now, $v_i^\varepsilon v_i^\varepsilon \to v_iv_i$ in $L^1(\Omega_T)$ by (3.22), and so it follows from (3.32), for ε small enough,

$$
\int_0^T\int_{Q_\delta} v_i^\varepsilon v_i^\varepsilon\,dx\,dt < \eta, \int_0^\delta\int_\Omega v_i^\varepsilon v_i^\varepsilon\,dx\,dt < \eta, \quad \int_{T-\delta}^T\int_\Omega v_i^\varepsilon v_i^\varepsilon\,dx\,dt \quad < \eta.
$$

Thus, $I_1 \le 3\eta$ when ε is small enough. It remains to consider I_2 for small ε. Integrating I_2 by parts one obtains

$$
\begin{aligned}
I_2 &= -\frac{1}{2}\int_\Omega B_{ijkl}\varepsilon(\mathbf{u}^\varepsilon(T))_{kl}\varepsilon(\mathbf{u}^\varepsilon(T))_{ij}\,dx + \frac{1}{2}\int_\Omega B_{ijkl}\varepsilon(\mathbf{u}_0)_{kl}\varepsilon(\mathbf{u}_0)_{ij}\,dx \\
&\quad - \frac{1}{2\delta}\int_\Omega\int_0^\delta B_{ijkl}\varepsilon(\mathbf{u}^\varepsilon)_{kl}\varepsilon(\mathbf{u}^\varepsilon)_{ij}\psi_\delta\,dtdx \\
&\quad + \frac{1}{2\delta}\int_\Omega\int_{T-\delta}^T B_{ijkl}\varepsilon(\mathbf{u}^\varepsilon)_{kl}\varepsilon(\mathbf{u}^\varepsilon)_{ij}\psi_\delta\,dtdx \\
&\le \frac{1}{2\delta}\int_\Omega\int_{T-\delta}^T(B_{ijkl}(\varepsilon(\mathbf{u}^\varepsilon)_{kl}\varepsilon(\mathbf{u}^\varepsilon)_{ij} - \varepsilon(\mathbf{u}^\varepsilon(T))_{kl}\varepsilon(\mathbf{u}^\varepsilon(T))_{ij}))dt\,dx \\
&\quad + \frac{1}{2\delta}\int_\Omega\int_0^\delta(B_{ijkl}(\varepsilon(\mathbf{u}_0)_{kl}\varepsilon(\mathbf{u}_0)_{ij} - \varepsilon(\mathbf{u}^\varepsilon)_{kl}\varepsilon(\mathbf{u}^\varepsilon)_{ij}))\psi_\delta dtdx \\
&\quad + \frac{1}{2\delta}\int_\Omega B_{ijkl}\varepsilon(\mathbf{u}_0)_{kl}\varepsilon(\mathbf{u}_0)_{ij}(1-\psi_\delta)\,dx.
\end{aligned}
\tag{3.35}
$$

It follows from (3.31) that (3.35) is less than η. Consider now the second term on the right-hand side, we have

$$
\begin{aligned}
&\Big|\frac{1}{2\delta}\int_\Omega\int_0^\delta (B_{ijkl}(\varepsilon(\mathbf{u}_0)_{kl}\varepsilon(\mathbf{u}_0)_{ij}-\varepsilon(\mathbf{u}^\varepsilon)_{kl}\varepsilon(\mathbf{u}^\varepsilon)_{ij}))\psi_\delta dtdx\Big|\\
&\le \frac{1}{2\delta}\int_\Omega\int_0^\delta |B_{ijkl}(\varepsilon(\mathbf{u}_0)_{kl}\varepsilon(\mathbf{u}_0)_{ij}-\varepsilon(\mathbf{u}^\varepsilon)_{kl}\varepsilon(\mathbf{u}^\varepsilon)_{ij})|\,dtdx\\
&\le \frac{1}{2\delta}\int_\Omega\int_0^\delta\int_0^t \Big|\frac{d}{dt}(B_{ijkl}\varepsilon(\mathbf{u}^\varepsilon)_{kl}\varepsilon(\mathbf{u}^\varepsilon)_{ij})\Big|\,dsdtdx\\
&\le \frac{1}{\delta}\int_\Omega\int_0^\delta\int_0^t |B_{ijkl}\varepsilon(\mathbf{v}^\varepsilon)_{kl}\varepsilon(\mathbf{u}^\varepsilon)_{ij}|\,ds\,dt\,dx\\
&= \frac{1}{\delta}\int_0^\delta\int_0^t\int_\Omega |B_{ijkl}\varepsilon(\mathbf{v}^\varepsilon)_{kl}\varepsilon(\mathbf{u}^\varepsilon)_{ij}|\;dx\,ds\,dt\\
&\le \frac{C}{\delta}\int_0^\delta\int_0^\delta \|\mathbf{v}^\varepsilon\|_V\|\mathbf{u}^\varepsilon\|_V dsdt \le C\sqrt{\delta} < \eta,
\end{aligned}
$$

whenever δ is sufficiently small. Formula (3.35) is estimated similarly and this shows that, for the choice of a sufficiently small δ, we have $I_2 < 3\eta$. Below, we choose such a δ and then

$$
\begin{aligned}
&\limsup_{\varepsilon\to 0}\int_{\Omega_T}(v_i^\varepsilon v_i^\varepsilon - (A_{ijkl}\varepsilon(\mathbf{u}^\varepsilon)_{kl}+B_{ijkl}\varepsilon(\mathbf{v}^\varepsilon)_{kl})\varepsilon(\mathbf{u}^\varepsilon)_{ij})\,dx\,dt\\
&\le \limsup_{\varepsilon\to 0}\int_{\Omega_T}(v_i^\varepsilon v_i^\varepsilon - (A_{ijkl}\varepsilon(\mathbf{u}^\varepsilon)_{kl}+B_{ijkl}\varepsilon(\mathbf{v}s^\varepsilon)_{kl})\varepsilon(\mathbf{u}^\varepsilon)_{ij})(1-\psi_\delta\phi_\delta)\,dx\,dt\\
&\quad+\limsup_{\varepsilon\to 0}\int_{\Omega_T}(v_i^\varepsilon v_i^\varepsilon - (A_{ijkl}\varepsilon(\mathbf{u}^\varepsilon)_{kl}+B_{ijkl}\varepsilon(\mathbf{v}^\varepsilon)_{kl})\varepsilon(\mathbf{u}^\varepsilon)_{ij})\psi_\delta\phi_\delta\,dx\,dt\\
&\le 6\eta+\int_{\Omega_T}(v_iv_i-(A_{ijkl}\varepsilon(\mathbf{u})_{kl}+B_{ijkl}\varepsilon(\mathbf{v})_{kl})\varepsilon(\mathbf{u})_{ij})\,dx\,dt,
\end{aligned}
$$

and since η was arbitrary, the conclusion of the lemma follows. □

4. Existence

We prove our first main result, Theorem 2.1, which guarantees the existence of a weak solution for Problem (2.14) - (2.21). We recall Problem $\mathcal{P}_\varepsilon$ and restate it here for the sake of convenience.

Problem $\mathcal{P}_\varepsilon$: Find $\mathbf{u}^\varepsilon, \mathbf{v}^\varepsilon \in \mathcal{V}$ such that, for $\mathbf{u}_0 \in V$ and $\mathbf{v}^\varepsilon(0) = \mathbf{v}_0 \in H$, there hold

$$
\mathbf{v}^{\varepsilon\prime} + M\mathbf{v}^\varepsilon + L\mathbf{u}^\varepsilon + \frac{1}{\varepsilon}P(\mathbf{u}^\varepsilon) + \gamma_T^*\xi^\varepsilon = \mathbf{f} \quad \text{in} \quad \mathcal{V}', \tag{4.1}
$$

$\mathbf{u}^\varepsilon(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}^\varepsilon(s)ds$, and for all $\mathbf{z} \in \mathcal{V}$,

$$
\langle\gamma_T^*\xi^\varepsilon, \mathbf{z}\rangle \le \int_0^T\int_{\Gamma_C} |(\mathcal{R}\sigma^\varepsilon)_n|\,\mu(|\mathbf{v}_T^\varepsilon - \mathbf{v}_F|)(|\mathbf{v}_T^\varepsilon - \mathbf{v}_F + \mathbf{z}_T| - |\mathbf{v}_T^\varepsilon - \mathbf{v}_F|)d\Gamma dt. \tag{4.2}
$$

We note that the boundedness of μ implies $\xi^\varepsilon \in [-K\,|(\mathcal{R}\sigma^\varepsilon)_n|\,, K\,|(\mathcal{R}\sigma^\varepsilon)_n|]$ a.e. on Γ_C, (3.8). Also, assumption (2.11) implies that there exists a constant C such that

$$
\|(\mathcal{R}\sigma^\varepsilon)_n\|_{L^2(0,t;L^2(\Gamma_c))} \le C\|\mathbf{v}^\varepsilon\|_{L^2(0,t;H)}. \tag{4.3}
$$

Therefore, ξ^ε is bounded in $L^2(0,T;L^2(\Gamma_C)^N)$ and so we may take a further subsequence and assume, in addition to the above convergences, that

$$\xi^\varepsilon \to \xi \text{ weakly in } L^2(0,T;L^2(\Gamma_C)^N). \tag{4.4}$$

In addition, we may assume, after taking a suitable subsequence and using the fact that L and M are linear, that

$$L\mathbf{u}^\varepsilon \to L\mathbf{u}, \quad M\mathbf{v}^\varepsilon \to M\mathbf{v} \text{ in } \mathcal{V}'. \tag{4.5}$$

It follows from (3.14) and (3.20) that $\int_{\Gamma_C}((u_n(t)-g)_+)^2 d\Gamma = 0$ for each $t \in [0,T]$, and so $P(\mathbf{u}) = 0$. Now, we recall that $\mathcal{K}$ and $\mathcal{K}_{\mathbf{u}}$ are given in (2.22) and (2.29), respectively. We multiply (4.1) by $\mathbf{u}^\varepsilon - \mathbf{w}$, with $\mathbf{w} \in \mathcal{K}_{\mathbf{u}}$ and integrate over $[0,T]$. Then,

$$\frac{1}{\varepsilon}\int_0^T \langle P(\mathbf{u}^\varepsilon), \mathbf{u}^\varepsilon - \mathbf{w}\rangle dt \geq 0,$$

and thus

$$\begin{aligned}
&(\mathbf{v}^\varepsilon(T), \mathbf{u}^\varepsilon(T) - \mathbf{w}(T))_H - (\mathbf{v}_0, \mathbf{u}_0 - \mathbf{w}(0))_H + \int_0^T \langle M\mathbf{v}^\varepsilon, \mathbf{u}^\varepsilon - \mathbf{w}\rangle dt \\
&\quad + \int_0^T \langle L\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon - \mathbf{w}\rangle dt + \int_0^T \langle \gamma_T^* \xi^\varepsilon, \mathbf{u}^\varepsilon - \mathbf{w}\rangle dt \\
&\leq \int_0^T (\mathbf{v}^\varepsilon, \mathbf{v}^\varepsilon - \mathbf{w}')_H dt + \int_0^T \langle \mathbf{f}, \mathbf{u}^\varepsilon - \mathbf{w}\rangle dt.
\end{aligned} \tag{4.6}$$

From (3.14), (3.28), and (3.20) we find,

$$\lim_{\varepsilon\to 0}(\mathbf{v}^\varepsilon(T), \mathbf{u}^\varepsilon(T) - \mathbf{w}(T))_H = 0, \tag{4.7}$$

and also

$$\int_0^T \langle \gamma_T^* \xi^\varepsilon, \mathbf{u}^\varepsilon - \mathbf{w}\rangle dt = \int_0^T \int_{\Gamma_C} \xi^\varepsilon (\mathbf{u}_T^\varepsilon - \mathbf{w}_T) d\Gamma dt.$$

Now, (3.20) and (4.4) show that this term converges to

$$\int_0^T \langle \gamma_T^* \xi, \mathbf{u} - \mathbf{w}\rangle dt = \int_0^T \int_{\Gamma_C} \xi (\mathbf{u}_T - \mathbf{w}_T) d\Gamma dt. \tag{4.8}$$

Together with inequality (4.6) these imply

$$\begin{aligned}
&(\mathbf{v}^\varepsilon(T), \mathbf{u}^\varepsilon(T) - \mathbf{w}(T))_H - (\mathbf{v}_0, \mathbf{u}_0 - \mathbf{w}(0))_H + \int_0^T \langle M\mathbf{v}^\varepsilon, \mathbf{u}^\varepsilon\rangle dt \\
&- \int_0^T (\mathbf{v}^\varepsilon, \mathbf{v}^\varepsilon)_H dt + \int_0^T \langle L\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon\rangle dt + \int_0^T \langle \gamma_T^* \xi^\varepsilon, \mathbf{u}^\varepsilon - \mathbf{w}\rangle dt \\
&\leq \int_0^T \langle L\mathbf{u}^\varepsilon, \mathbf{w}\rangle dt + \int_0^T \langle M\mathbf{v}^\varepsilon, \mathbf{w}\rangle dt + \int_0^T (\mathbf{v}^\varepsilon, \mathbf{w}')_H dt + \int_0^T \langle \mathbf{f}, \mathbf{u}^\varepsilon - \mathbf{w}\rangle dt.
\end{aligned} \tag{4.9}$$

We take the lim inf of both sides of (4.9) as $\varepsilon \to 0$ and use Lemma 3.6 with (4.8), (4.5) and (4.7) to conclude

$$\begin{aligned} &-(\mathbf{v}_0, \mathbf{u}_0 - \mathbf{w}(0))_H + \int_0^T \langle M\mathbf{v}, \mathbf{u}\rangle dt - \int_0^T (\mathbf{v}, \mathbf{v})_H dt \\ &+ \int_0^T \langle L\mathbf{u}, \mathbf{u}\rangle dt + \int_0^T \langle \gamma_T^* \xi, \mathbf{u} - \mathbf{w}\rangle dt \\ &\le \int_0^T \langle L\mathbf{u}, \mathbf{w}\rangle dt + \int_0^T \langle M\mathbf{v}, \mathbf{w}\rangle dt + \int_0^T (\mathbf{v}, \mathbf{w}')_H dt + \int_0^T \langle \mathbf{f}, \mathbf{u} - \mathbf{w}\rangle dt. \end{aligned} \tag{4.10}$$

This, in turn, implies

$$\begin{aligned} &-(\mathbf{v}_0, \mathbf{u}_0 - \mathbf{w}(0))_H + \int_0^T \langle M\mathbf{v}, \mathbf{u} - \mathbf{w}\rangle dt - \int_0^T (\mathbf{v}, \mathbf{v} - \mathbf{w}')_H dt \\ &+ \int_0^T \langle L\mathbf{u}, \mathbf{u} - \mathbf{w}\rangle dt + \int_0^T \langle \gamma_T^* \xi, \mathbf{u} - \mathbf{w}\rangle dt \\ &\le \int_0^T \langle \mathbf{f}, \mathbf{u} - \mathbf{w}\rangle dt. \end{aligned} \tag{4.11}$$

It only remains to verify that for all $\mathbf{z} \in \mathcal{V}$,

$$\langle \gamma_T^* \xi, \mathbf{z}\rangle \le \int_0^T \int_{\Gamma_C} |(\mathcal{R}\sigma)_n|\, \mu(|\mathbf{v}_T - \mathbf{v}_F|)(|\mathbf{v}_T - \mathbf{v}_F + \mathbf{z}_T| - |\mathbf{v}_T - \mathbf{v}_F|) d\Gamma dt. \tag{4.12}$$

It follows from (3.24), (3.25) and (2.11) that

$$|(\mathcal{R}\sigma^\varepsilon)_n| \to |(\mathcal{R}\sigma)_n| \quad \text{in} \quad L^2(0, T; L^2(\Gamma_C)),$$

and also $\mathbf{v}_T^\varepsilon \to \mathbf{v}_T$ in $L^2(0, T; L^2(\Gamma_C)^N)$, while $\mu(|\mathbf{v}_T^\varepsilon - \mathbf{v}_F|) \to \mu(|\mathbf{v}_T - \mathbf{v}_F|)$ pointwise in $\Gamma_C \times [0, T]$, and is bounded uniformly. This allows us to pass to the limit $\varepsilon \to 0$ in the inequality (4.2) and obtain (4.12). The proof of Theorem 2.1 is now complete.

5. Discontinuous friction coefficient

In this section we consider a discontinuous, set-valued friction coefficient μ, depicted in Fig. 2, which represents a sharp drop from the static value μ_0 to a dynamic value $\mu_s(0)$, when relative slip commences.

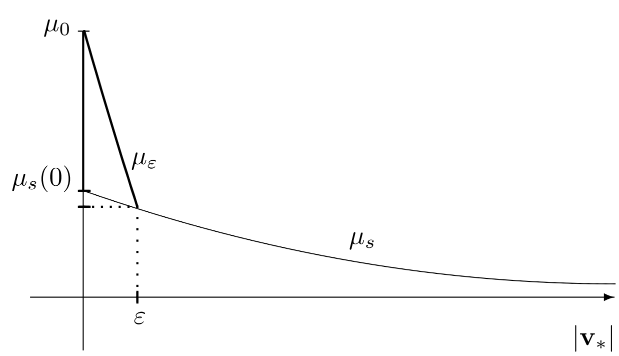

Figure 2. The graph of μ vs. the slip rate $|\mathbf{v}_*|$, and its approximation μ_ε.

The jump in the friction coefficient μ when slip begins is given by the vertical segment $[\mu_s(0), \mu_0]$. Thus,

$$\mu(v) = \begin{cases} [\mu_s(0), \mu_0] & v = 0, \\ \mu_s(v) & v > 0, \end{cases}$$

where μ_s is a Lipschitz, bounded, and positive function which describes the dependence of the coefficient on the slip rate. The function μ_ε, shown in Fig. 2, is a Lipschitz continuous approximation of the set-valued function for $0 \le v \le \varepsilon$.

It follows from Theorem 2.1 that the problem obtained from (2.14)–(2.21) by replacing μ with μ_ε has a weak solution. For convenience we list the conclusion of this theorem with appropriate modifications.

Theorem 5.1. *There exist* $\mathbf{u}$ *in* $C([0,T];U) \cap L^\infty(0,T;V)$, $\mathbf{u}$ *in* $\mathcal{K}$ *and* $\mathbf{v}$ *in* $L^2(0,T;V) \cap L^\infty(0,T;H)$, $\mathbf{v}(0) = \mathbf{v}_0 \in H$, *such that* $\mathbf{v}' \in L^2(0,T;H^{-1}(\Omega)^N)$, $\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}(s)ds$, *and*

$$\begin{aligned} &-(\mathbf{v}_0, \mathbf{u}_0 - \mathbf{w}(0))_H + \int_0^T \langle M\mathbf{v}, \mathbf{u} - \mathbf{w}\rangle dt - \int_0^T (\mathbf{v}, \mathbf{v} - \mathbf{w}')_H dt \\ &+ \int_0^T \langle L\mathbf{u}, \mathbf{u} - \mathbf{w}\rangle dt + \int_0^T \langle \gamma_T^* \xi, \mathbf{u} - \mathbf{w}\rangle dt \\ &\le \int_0^T \langle \mathbf{f}, \mathbf{u} - \mathbf{w}\rangle dt, \end{aligned} \tag{5.1}$$

which holds for all $\mathbf{w} \in \mathcal{K}_{\mathbf{u}}$. *Moreover, for all* $\mathbf{z} \in \mathcal{V}$,

$$\langle \gamma_T^* \xi, \mathbf{z}\rangle \le \int_0^T \int_{\Gamma_C} |(\mathcal{R}\sigma)_n| \, \mu_\varepsilon(|\mathbf{v}_T - \mathbf{v}_F|)(|\mathbf{v}_T - \mathbf{v}_F + \mathbf{z}_T| - |\mathbf{v}_T - \mathbf{v}_F|) d\Gamma dt. \tag{5.2}$$

Here, $\mathcal{K}$ and $\mathcal{K}_{\mathbf{u}}$ are given in (2.22) and (2.29), respectively. We refer to $\{\mathbf{u}_\varepsilon, \mathbf{v}_\varepsilon\}$ as a solution which is guaranteed by the theorem. We point out that here ε is not related to the penalization parameter used earlier, but indicates the extent to which μ_ε approximates μ (Fig. 2). In the proof of Theorem 2.1 the Lipschitz constant for μ was not used in the derivation of the estimates, therefore, there exists a constant C, which is independent of ε, such that

$$|\mathbf{v}_\varepsilon(t)|_H^2 + \int_0^t \|\mathbf{v}_\varepsilon\|_V^2 ds + \delta^2 \|\mathbf{u}_\varepsilon(t)\|_V^2 + \|\mathbf{v}_\varepsilon'\|_{L^2(0,T;H^{-1}(\Omega)^N)} \le C. \tag{5.3}$$

This estimate together with (4.3) and (3.8) yield

$$\|\xi_\varepsilon\|_{L^2(0,T;L^2(\Gamma_C)^N)} \le C, \tag{5.4}$$

where here and below C is a generic positive constant independent of ε. Therefore, there exists a subsequence $\varepsilon \to 0$ for which

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weak} * \text{ in } L^\infty(0,T;V), \tag{5.5}$$
$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ strongly in } C([0,T];U), \tag{5.6}$$
$$\mathbf{v}_\varepsilon \to \mathbf{v} \text{ weakly in } \mathcal{V}, \tag{5.7}$$
$$\mathbf{v}_\varepsilon \to \mathbf{v} \text{ weak} * \text{ in } L^\infty(0,T;H), \tag{5.8}$$
$$\mathbf{u}_\varepsilon(T) \to \mathbf{u}(T) \text{ weakly in } V, \tag{5.9}$$
$$\mathbf{v}_\varepsilon \to \mathbf{v} \text{ strongly in } L^2(0,T;U), \tag{5.10}$$
$$\mathbf{v}_\varepsilon(\mathbf{x},t) \to \mathbf{v}(\mathbf{x},t) \text{ pointwise a.e. on } \Gamma_C \times [0,T]\,, \tag{5.11}$$
$$\mathbf{v}'_\varepsilon \to \mathbf{v}' \text{ weak} * \text{ in } L^2(0,T;H^{-1}(\Omega)^N), \tag{5.12}$$
$$|\mathbf{v}_\varepsilon(T)|_H \le C. \tag{5.13}$$

The result of Lemma 3.4 still holds, and so there exists a further subsequence such that

$$\mathbf{v}'_\varepsilon - \operatorname{Div}(A\varepsilon(\mathbf{u}_\varepsilon) + B\varepsilon(\mathbf{v}_\varepsilon)) \to \mathbf{v}' - \operatorname{Div}(A\varepsilon(\mathbf{u}) + B\varepsilon(\mathbf{v})) \text{ weakly in } \mathcal{H}, \tag{5.14}$$

and, thus, the conclusion of Lemma 3.6 also holds,

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \int_0^T -\langle M\mathbf{v}^\varepsilon, \mathbf{u}^\varepsilon\rangle dt + \int_0^T (\mathbf{v}^\varepsilon, \mathbf{v}^\varepsilon)_H dt - \int_0^T \langle L\mathbf{u}^\varepsilon, \mathbf{u}^\varepsilon\rangle dt \\
&\le \int_0^T -\langle M\mathbf{v}, \mathbf{u}\rangle dt + \int_0^T (\mathbf{v}, \mathbf{v})_H dt - \int_0^T \langle L\mathbf{u}, \mathbf{u}\rangle dt.
\end{aligned} \tag{5.15}$$

As above, (5.10) implies

$$|(\mathcal{R}\sigma_\varepsilon)_n| \to |(\mathcal{R}\sigma)_n| \quad \text{ in } L^2(0,T;L^2(\Gamma_C)), \tag{5.16}$$

and we may take a further subsequence, if necessary, such that $\xi_\varepsilon \to \xi$ weakly in $L^2(0,T;L^2(\Gamma_C)^N)$. Passing now to the limit we obtain

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}(s)ds, \quad \mathbf{u}_0 \in V, \quad \mathbf{v}(0) = \mathbf{v}_0 \in H.$$

Letting $\mathbf{w} \in \mathcal{K}$, it follows from Theorem 5.1 that

$$\begin{aligned}
&(\mathbf{v}_\varepsilon(T), \mathbf{u}_\varepsilon(T) - \mathbf{w}(T))_H - (\mathbf{v}_0, \mathbf{u}_0 - \mathbf{w}(0))_H + \int_0^T \langle M\mathbf{v}_\varepsilon, \mathbf{u}_\varepsilon - \mathbf{w}\rangle dt \\
&- \int_0^T (\mathbf{v}_\varepsilon, \mathbf{v}_\varepsilon - \mathbf{w}')_H dt + \int_0^T \langle L\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon - \mathbf{w}\rangle dt + \int_0^T \langle \gamma_T^* \xi_\varepsilon, \mathbf{u}_\varepsilon - \mathbf{w}\rangle dt \\
&\le \int_0^T \langle \mathbf{f}, \mathbf{u}_\varepsilon - \mathbf{w}\rangle dt.
\end{aligned} \tag{5.17}$$

The strong convergence in (5.6) allows us to pass to the limit

$$\lim_{\varepsilon\to 0} \int_0^T \langle \gamma_T^* \xi_\varepsilon, \mathbf{u}_\varepsilon - \mathbf{w}\rangle dt = \int_0^T \langle \gamma_T^* \xi, \mathbf{u} - \mathbf{w}\rangle dt.$$

It follows now from the boundedness of $|\mathbf{v}_\varepsilon(T)|_H$, the weak convergence of $u_\varepsilon(T)$ to $u(T)$ in V and the compactness of the embedding of V into H that

$$\lim_{\varepsilon\to 0} (\mathbf{v}_\varepsilon(T), \mathbf{u}_\varepsilon(T) - \mathbf{w}(T))_H = 0.$$

Rewriting (5.17), taking the lim sup of both sides and using (5.15) yields

$$\begin{aligned}
&-(\mathbf{v}_0, \mathbf{u}_0 - \mathbf{w}(0))_H - \int_0^T \langle L\mathbf{u}, \mathbf{w}\rangle dt + \int_0^T (\mathbf{v}, \mathbf{w}')_H dt \\
&- \int_0^T \langle M\mathbf{v}, \mathbf{w}\rangle dt + \int_0^T \langle \gamma_T^* \xi, \mathbf{u} - \mathbf{w}\rangle dt \\
&\le \int_0^T -\langle M\mathbf{v}, \mathbf{u}\rangle \, dt + \int_0^T (\mathbf{v}, \mathbf{v})_H dt - \int_0^T \langle L\mathbf{u}, \mathbf{u}\rangle dt + \int_0^T \langle \mathbf{f}, \mathbf{u} - \mathbf{w}\rangle dt,
\end{aligned} \tag{5.18}$$

which implies

$$\begin{aligned}
&-(\mathbf{v}_0, \mathbf{u}_0 - \mathbf{w}(0))_H + \int_0^T \langle L\mathbf{u}, \mathbf{u} - \mathbf{w}\rangle dt - \int_0^T (\mathbf{v}, \mathbf{v} - \mathbf{w}')_H dt \\
&+ \int_0^T \langle M\mathbf{v}, \mathbf{u} - \mathbf{w}\rangle dt + \int_0^T \langle \gamma_T^* \xi, \mathbf{u} - \mathbf{w}\rangle dt \\
&\le \int_0^T \langle \mathbf{f}, \mathbf{u} - \mathbf{w}\rangle dt.
\end{aligned} \tag{5.19}$$

It only remains to examine (5.2), without ε. It follows from (5.10) that $\mathbf{v}_\varepsilon \to \mathbf{v}$ strongly in $L^2(0,T;U)$ and so $\mathbf{v}_{\varepsilon T} \to \mathbf{v}_T$ strongly in $L^2(0,T;L^2(\Gamma_C)^N)$. Taking a measurable representative, we may assume $\mathbf{v}_{\varepsilon T}(\mathbf{x},t) \to \mathbf{v}_T(\mathbf{x},t)$ pointwise a.e. and in $L^2(\Gamma_C \times [0,T])$. Taking a further subsequence we may assume, in addition, that $\mu_\varepsilon(|\mathbf{v}_{\varepsilon T} - \mathbf{v}_F|) \to q$ weak* in $L^\infty(E)$, where

$$E = \{(\mathbf{x},t) : |\mathbf{v}_T(\mathbf{x},t) - \mathbf{v}_F(\mathbf{x},t)| = 0\}.$$

Let $\varepsilon_0 > 0$ be given. Then, for a.e $(\mathbf{x},t) \in E$, we have

$$\mu_\varepsilon(|\mathbf{v}_{\varepsilon T} - \mathbf{v}_F|\,(\mathbf{x},t)) \in [\mu_s(0+) - h(\varepsilon_0), \mu_0]\,,$$

whenever ε is small enough. Therefore, $q(\mathbf{x},t) \in [\mu_s(0+) - h(\varepsilon_0), \mu_0]$, a.e., where $h(\varepsilon) = \mu_s(0) - \mu_s(\varepsilon)$ is the function measuring the approach to the graph. Since ε_0 is arbitrary, we obtain $q(\mathbf{x},t) \in [\mu_s(0+), \mu_0]$, a.e. on Γ_C. If $|\mathbf{v}_T(\mathbf{x},t) - \mathbf{v}_F(\mathbf{x},t)| > 0$, we find that for a.e. $(\mathbf{x},t)$ and for ε small enough,

$$\mu_\varepsilon(|\mathbf{v}_{\varepsilon T} - \mathbf{v}_F|\,(\mathbf{x},t)) = \mu_s(|\mathbf{v}_{\varepsilon T} - \mathbf{v}_F|\,(\mathbf{x},\mathbf{t})),$$

and this converges to $\mu_s(|\mathbf{v}_T - \mathbf{v}_F|\,(\mathbf{x},t))$. From (5.2) we obtain

$$\begin{aligned}
\langle \gamma_T^* \xi_\varepsilon, \mathbf{z}\rangle &\le \int_0^T \int_{\Gamma_C} |(\mathcal{R}\sigma_\varepsilon)_n|\, \mu_\varepsilon(|\mathbf{v}_{\varepsilon T} - \mathbf{v}_F|)(|\mathbf{v}_{\varepsilon T} - \mathbf{v}_F + \mathbf{z}_T| - |\mathbf{v}_{\varepsilon T} - \mathbf{v}_F|) d\Gamma dt \\
&= \int_E |(\mathcal{R}\sigma_\varepsilon)_n|\, \mu_\varepsilon(|\mathbf{v}_{\varepsilon T} - \mathbf{v}_F|)(|\mathbf{v}_{\varepsilon T} - \mathbf{v}_F + \mathbf{z}_T| - |\mathbf{v}_{\varepsilon T} - \mathbf{v}_F|) d\Gamma dt \\
&\quad + \int_{E^C} |(\mathcal{R}\sigma_\varepsilon)_n|\, \mu_\varepsilon(|\mathbf{v}_{\varepsilon T} - \mathbf{v}_F|)(|\mathbf{v}_{\varepsilon T} - \mathbf{v}_F + \mathbf{z}_T| - |\mathbf{v}_{\varepsilon T} - \mathbf{v}_F|) d\Gamma dt,
\end{aligned}$$

and it follows from (5.10) and (5.16) that we may pass to the limit, thus,

$$\begin{aligned}
\langle \gamma_T^* \xi, \mathbf{z}\rangle &\le \int_E |(\mathcal{R}\sigma)_n|\, q(|\mathbf{v}_T - \mathbf{v}_F + \mathbf{z}_T| - |\mathbf{v}_T - \mathbf{v}_F|) d\Gamma dt \\
&\quad + \int_{E^C} |(\mathcal{R}\sigma)_n|\, \mu_s(|\mathbf{v}_T - \mathbf{v}_F|)(|\mathbf{v}_T - \mathbf{v}_F + \mathbf{z}_T| - |\mathbf{v}_T - \mathbf{v}_F|) d\Gamma dt.
\end{aligned}$$

Since $q(\mathbf{x},t) \in [\mu_s(0+), \mu_0]$ a.e., it shows that there exists a measurable function k_μ, with the property that $k_\mu(\mathbf{x},t) \in \mu(|\mathbf{v}_T - \mathbf{v}_F|\,(\mathbf{x},t))$ a.e., such that

$$\langle \gamma_T^* \xi, \mathbf{z} \rangle \le \int_0^T \int_{\Gamma_C} |(\mathcal{R}\sigma)_n|\, k_\mu(|\mathbf{v}_{\varepsilon T} - \mathbf{v}_F \mathbf{z}_T| - |\mathbf{v}_{\varepsilon T} - \mathbf{v}_F|) d\Gamma dt.$$

Thus, we have established the second main result in this work, an existence theorem which extends Theorem 2.1 to the case when the friction coefficient is discontinuous.

Theorem 5.2. *There exist* $\mathbf{u} \in C((0,T];U) \cap L^\infty(0,T;V) \cap \mathcal{K}$ *and* $\mathbf{v} \in L^2(0,T;V) \cap L^\infty(0,T;H)$, $\mathbf{v}' \in L^2(0,T;H^{-1}(\Omega)^N)$, *and*

$$\mathbf{u}(t) = \mathbf{u}_0 + \int_0^t \mathbf{v}(s) ds, \quad \mathbf{u}_0 \in V, \quad \mathbf{v}(0) = \mathbf{v}_0 \in H, \tag{5.20}$$

such that

$$\begin{aligned} &-(\mathbf{v}_0, \mathbf{u}_0 - \mathbf{w}(0))_H + \int_0^T \langle M\mathbf{v}, \mathbf{u} - \mathbf{w} \rangle dt - \int_0^T (\mathbf{v}, \mathbf{v} - \mathbf{w}')_H dt \\ &+ \int_0^T \langle L\mathbf{u}, \mathbf{u} - \mathbf{w} \rangle dt + \int_0^T \langle \gamma_T^* \xi, \mathbf{u} - \mathbf{w} \rangle dt \\ &\le \int_0^T \langle \mathbf{f}, \mathbf{u} - \mathbf{w} \rangle dt, \end{aligned} \tag{5.21}$$

where

$$\langle \gamma_T^* \xi, \mathbf{z} \rangle \le \int_0^T \int_{\Gamma_C} |(\mathcal{R}\sigma)_n|\, k_\mu(|\mathbf{v}_T - \mathbf{v}_F + \mathbf{z}_T| - |\mathbf{v}_T - \mathbf{v}_F|) d\Gamma dt, \tag{5.22}$$

for all $\mathbf{z} \in \mathcal{V}$. *Here,* k_μ *is a function in* $L^\infty(\Gamma_C \times (0,T])$ *with the property that* $k_\mu(\mathbf{x},t) \in \mu(|\mathbf{v}_T - \mathbf{v}_F|)$ *for a.e.* $(\mathbf{x},t) \in \Gamma_C \times [0,T]$.

6. Conclusions

We considered a model for dynamic frictional contact between a deformable body and a moving rigid foundation. The contact was modelled with the Signorini condition and friction with a general nonlocal law in which the friction coefficient depended on the slip velocity between the surface and the foundation. We have shown that there exists a weak solution to the problem when the friction coefficient is a Lipschitz function of the slip rate, or when it is a graph with a jump from the static to the dynamic value at the onset of sliding.

The existence of weak solutions for these problems was obtained by using the theory of set-valued pseudomonotone maps of [17]. The regularization of the contact stress was introduced in Section 2, and it remains an open problem either to justify it from the surface microstructure considerations, or to eliminate it. Whereas the uniqueness of the weak solutions for the problem with Lipschitz friction coefficient is unknown, and seems unlikely, there are uniqueness results for problems with the normal compliances condition. Moreover, the Signorini condition has a very low regularity ceiling, since once the surface comes into contact with the rigid foundation the velocity is discontinuous, which means that the acceleration is a measure or a distribution. On the other hand, the normal compliance condition lead to a much better regularity [20].

The thermoviscoelastic contact problem with Signorini's contact condition is of some interest, and will be investigated in the future.

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