

EXPONENTIAL DECAY OF SOLUTIONS TO A VISCOELASTIC EQUATION WITH NONLINEAR LOCALIZED DAMPING

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ABSTRACT. In this paper we consider the nonlinear viscoelastic equation

$$u_{tt} - \Delta u + \int_0^t g(t - \tau) \Delta u(\tau) d\tau + a(x)|u_t|^m u_t + b|u|^\gamma u = 0,$$

in a bounded domain. Without imposing geometry restrictions on the boundary, we establish an exponential decay result, under weaker conditions than those in [3].

1. INTRODUCTION

Cavalcanti *et al* [3] studied the equation

$$\begin{aligned} u_{tt} - \Delta u + \int_0^t g(t - \tau) \Delta u(\tau) d\tau + a(x)u_t + |u|^\gamma u &= 0, \quad \text{in } \Omega \times (0, \infty) \\ u(x, t) &= 0, \quad x \in \partial\Omega, t \geq 0 \\ u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain of \mathbb{R}^n ($n \geq 1$) with a smooth boundary $\partial\Omega$, $\gamma > 0$, g is a positive function, and $a : \Omega \rightarrow \mathbb{R}^+$ is a function, which may be null on a part of Ω . Under the condition that $a(x) \geq a_0 > 0$ on $\omega \subset \Omega$, with ω satisfying some geometry restrictions and

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \quad t \geq 0,$$

such that $\|g\|_{L^1((0, \infty))}$ is small enough, the authors obtained an exponential rate of decay. This work extended the result of Zuazua [19], in which he considered (1.1) with $g = 0$ and the linear damping is localized. Cavalcanti *et al* [4] considered the equation

$$u_{tt} - k_0 \Delta u + \int_0^t \operatorname{div}[a(x)g(t - \tau) \nabla u(\tau)] d\tau + b(x)h(u_t) + f(u) = 0, \quad \text{in } \Omega \times (0, \infty),$$

under similar conditions on the relaxation function g and $a(x) + b(x) \geq \delta > 0$, for all $x \in \Omega$. They improved the result in [3] by establishing exponential stability for g decaying exponentially and h linear and polynomial stability for g decaying polynomially and h nonlinear. Their proof, based on the use of piecewise multipliers,

2000 *Mathematics Subject Classification*. 35B35, 35L20, 35L70.

Key words and phrases. Exponential decay; global existence; nonlinear localized damping.

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Submitted May 17, 2004. Published June 29, 2004.

is similar to the one in [3]. Another problem, where the damping induced by the viscosity is acting on the domain and a part of the boundary, was also discussed by Cavalcanti *et al* [5] and existence and uniform decay rate results were established. In the same direction, Cavalcanti *et al* [2] have also studied, in a bounded domain, the equation

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)d\tau - \gamma\Delta u_t = 0,$$

with $x \in \Omega$, $t > 0$, $\rho > 0$. They proved a global existence result for $\gamma \geq 0$ and an exponential decay for $\gamma > 0$. This last result has been extended to a situation, where a source term is competing with the strong damping mechanism and the one induced by the viscosity, by Messaoudi and Tatar [16]. There, the authors combined well known methods with perturbation techniques to show that a solution with positive but small energy exist globally and decay to the rest state exponentially. Messaoudi [17] considered the equation

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau + au_t|u_t|^m = b|u|^\gamma u, \quad \text{in } \Omega \times (0, \infty)$$

and showed, under suitable conditions on g , that solutions with negative energy blow up in finite time if $\gamma > m$, and continue to exist if $m \geq \gamma$. We also should mention the work of Kavashima and Shibata [9], in which a global existence and exponential stability of small solutions to a nonlinear viscoelastic problem has been established.

In the absence of the viscoelastic term ($g = 0$), the problem has been extensively studied and many results concerning global existence and nonexistence have been proved. For instance, for the problem

$$\begin{aligned} u_{tt} - \Delta u + au_t|u_t|^m &= b|u|^\gamma u, & \text{in } \Omega \times (0, \infty) \\ u(x, t) &= 0, & x \in \partial\Omega, t \geq 0 \\ u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), & x \in \Omega, \end{aligned} \tag{1.2}$$

with $m, \gamma \geq 0$, it is well known that, for $a = 0$, the source term $b|u|^\gamma$, ($\gamma > 0$) causes finite time blow up of solutions with negative initial energy (see [1, 8]) and for $b = 0$, the damping term $au_t|u_t|^m$ assures global existence for arbitrary initial data (see [7, 10]). The interaction between the damping and the source terms was first considered by Levine [11, 12] in the linear damping case ($m = 0$). He showed that solutions with negative initial energy blow up in finite time. Georgiev and Todorova [6] extended Levine's result to the nonlinear damping case ($m > 0$). In their work, the authors introduced a different method and determined suitable relations between m and γ , for which there is global existence or alternatively finite time blow up. Precisely; they showed that solutions with negative energy continue to exist globally 'in time' if $m \geq \gamma$ and blow up in finite time if $\gamma > m$ and the initial energy is sufficiently negative. Without imposing the condition that the initial energy is sufficiently negative, Messaoudi [15] extended the blow up result of [6] to solutions with negative initial energy only. For results of same nature, we refer the reader to Levine and Serrin [13] and Levine and Park [14], Vitillaro [18].

In the present work, we are concerned with

$$\begin{aligned} u_{tt} - \Delta u + \int_0^t g(t - \tau) \Delta u(\tau) d\tau + a(x) u_t |u_t|^m + |u|^\gamma u &= 0, \quad \text{in } \Omega \times (0, \infty) \\ u(x, t) &= 0, \quad x \in \partial\Omega, t \geq 0 \\ u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \end{aligned} \quad (1.3)$$

for $m \geq 0$. We will prove an exponential decay result under weaker conditions on both a and g . In fact we will allow a to vanish on any part of Ω (including Ω itself). As a consequence, the geometry restriction imposed on a part of $\partial\Omega$ by Cavalcanti *et al* [3] is dropped. Although this present work and [4] both improve [3], they have different nature and use different approaches. Our method of proof is based on the use of the perturbed energy technique. Our choice of the Lyapunov functional made our proof easier than the one in [3, 4]. This paper is organized as follows. In Section 2, We present some notation and material needed for our work and we state the global existence theorem in [3]. Section 3 contains the statement and the proof of our main result.

2. PRELIMINARIES

In this section, we shall prepare some material needed in the proof of our result and state, without proof, a global existence result, which may be proved by repeating the argument of [3]. We use the standard Lebesgue space $L^p(\Omega)$ and Sobolev space $H_0^1(\Omega)$ with their usual scalar products and norms. The symbols ∇ and Δ will stand for the gradient and the Laplacian respectively and the subscript t will denote the time differentiation.

For the relaxation function $g(t)$ we assume

(G1) $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a bounded C^1 function such that $g(0) > 0$ and

$$1 - \int_0^\infty g(s) ds = l > 0.$$

(G2) There exists a positive constant ξ such that $g'(t) \leq -\xi g(t)$, for $t \geq 0$.

Proposition 2.1. *Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. Assume that g satisfies (G1) and*

$$\begin{aligned} 0 \leq \gamma \leq \frac{2}{n-2}, \quad n \geq 3 \\ \gamma \geq 0, \quad n = 1, 2. \end{aligned} \quad (2.1)$$

Then problem (1.3) has a unique global solution,

$$\begin{aligned} u &\in C([0, \infty); H_0^1(\Omega)) \\ u_t &\in C([0, \infty); L^2(\Omega)) \cap L_a^{m+2}(\Omega \times (0, \infty)), \end{aligned} \quad (2.2)$$

where L_a^{m+2} is the weighted Lebesgue space.

Remark 2.2. Condition (2.1) is needed so that the nonlinearity is Lipschitz from $H^1(\Omega)$ to $L^2(\Omega)$. Condition (G1) is necessary to guarantee the hyperbolicity of the system (1.3).

Now, we introduce the energy

$$\mathcal{E}(t) := \frac{1}{2} \left(1 - \int_0^t g(s) ds \right) \|\nabla u(t)\|_2^2 + \frac{1}{2} \|u_t\|_2^2 + \frac{1}{2} (g \circ \nabla u)(t) + \frac{1}{\gamma+2} \|u\|_{\gamma+2}^{\gamma+2}, \quad (2.3)$$

where

$$(g \circ v)(t) = \int_0^t g(t - \tau) \|v(t) - v(\tau)\|_2^2 d\tau. \quad (2.4)$$

Remark 2.3. Multiplying equation (1.3) by u_t and integrating over Ω , then using integration by parts and hypotheses (G1) and (G2), we obtain, after some manipulations,

$$\begin{aligned} \mathcal{E}'(t) &\leq - \left(\int_{\Omega} a(x) |u_t|^{m+2} dx - \frac{1}{2} (g' \circ \nabla u)(t) + \frac{1}{2} g(t) \|\nabla u(t)\|^2 \right) \\ &\leq - \int_{\Omega} a(x) |u_t|^{m+2} dx + \frac{1}{2} (g' \circ \nabla u)(t) \leq 0. \end{aligned} \quad (2.5)$$

This implies that “modified” energy is uniformly bounded (by $\mathcal{E}(0)$) and is decreasing in t .

We will also use the embedding $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for $2 \leq q \leq 2n/(n-2)$ if $n \geq 3$ or $q \geq 2$ if $n = 1, 2$ and $L^r(\Omega) \hookrightarrow L^q(\Omega)$, for $q < r$. We will use the same embedding constant denoted by C_p ; i.e.

$$\|v\|_q \leq C_p \|\nabla v\|_2, \quad \|v\|_q \leq C_p \|v\|_r. \quad (2.6)$$

3. EXPONENTIAL DECAY

Before we state and prove our main result, we prove the following lemma.

Lemma 3.1. *Let $m \leq 2/(n-2)$, for $n \geq 3$. Then there exists a constant C depending on C_p , $\|a\|_{\infty}$, $\mathcal{E}(0)$, and m only, such that the solution (2.2) satisfies*

$$\int_{\Omega} a(x) |u|^{m+2} dx \leq C (\|\nabla u\|_2^2 + \|u\|_{\gamma+2}^{\gamma+2}) \quad (3.1)$$

Proof. If $m \leq \gamma$ then we have two cases either $\|u\|_{m+2} \leq 1$, in which case

$$\int_{\Omega} a(x) |u|^{m+2} dx \leq \|a\|_{\infty} \|u\|_{m+2}^{m+2} \leq \|a\|_{\infty} \|u\|_{m+2}^2 \leq C_p^2 \|a\|_{\infty} \|\nabla u\|_2^2; \quad (3.2)$$

or $\|u\|_{m+2} > 1$, in which case

$$\int_{\Omega} a(x) |u|^{m+2} dx \leq \|a\|_{\infty} \|u\|_{m+2}^{m+2} \leq \|a\|_{\infty} \|u\|_{m+2}^{\gamma+2} \leq C_p^{\gamma+2} \|a\|_{\infty} \|u\|_{\gamma+2}^{\gamma+2}. \quad (3.3)$$

If $m > \gamma$ then

$$\begin{aligned} \int_{\Omega} a(x) |u|^{m+2} dx &\leq C_p^{m+2} \|a\|_{\infty} \|\nabla u\|_2^{m+2} \leq C_p^{m+2} \|a\|_{\infty} \|\nabla u\|_2^2 \left(\frac{2\mathcal{E}(t)}{l} \right)^{m/2} \\ &\leq C_p^{m+2} \|a\|_{\infty} \left(\frac{2\mathcal{E}(0)}{l} \right)^{m/2} \|\nabla u\|_2^2 \end{aligned}$$

Combining (3.2), (3.3) with the above inequality, we complete the proof. \square

Theorem 3.2. *Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. Assume that g satisfies (G1) and (G2), such that*

$$0 \leq \max\{m, \gamma\} \leq \frac{2}{n-2}, \quad n \geq 3.$$

Then there exist positive constants k and K , such that the solution given by (2.2) satisfies $\mathcal{E}(t) \leq K e^{-kt}$ for all $t \geq 0$.

Proof. We define the function

$$F(t) := \mathcal{E}(t) + \varepsilon_1 \Psi(t) + \varepsilon_2 \chi(t) \quad (3.4)$$

where ε_1 and ε_2 are positive constants to be specified later and

$$\begin{aligned} \Psi(t) &:= \int_{\Omega} u u_t \, dx \\ \chi(t) &:= - \int_{\Omega} u_t \int_0^t g(t-\tau)(u(t) - u(\tau)) \, d\tau \, dx. \end{aligned}$$

It is straightforward to see that for ε_1 and ε_2 small, we have

$$\alpha_1 F(t) \leq \mathcal{E}(t) \leq \alpha_2 F(t), \quad (3.5)$$

holds for two positive constants α_1 and α_2 . In fact

$$\begin{aligned} F(t) &\leq \mathcal{E}(t) + (\varepsilon_1/2) \int_{\Omega} |u_t|^2 \, dx + (\varepsilon_1/2) \int_{\Omega} |u|^2 \, dx \\ &\quad + (\varepsilon_2/2) \int_{\Omega} |u_t|^2 \, dx + (\varepsilon_2/2) \int_{\Omega} \left(\int_0^t g(t-\tau)(u(t) - u(\tau)) \, d\tau \right)^2 \, dx \\ &\leq \mathcal{E}(t) + (\varepsilon_1/2) \int_{\Omega} |u_t|^2 \, dx + (\varepsilon_1/2) C_p \int_{\Omega} |\nabla u|^2 \, dx \\ &\quad + (\varepsilon_2/2) \int_{\Omega} |u_t|^2 \, dx + (\varepsilon_2/2) C_p (1-l) (g \circ \nabla u)(t) \\ &\leq \frac{1}{\alpha_1} \mathcal{E}(t) \end{aligned} \quad (3.6)$$

where $\mathcal{E}(t)$ is the energy, and

$$\begin{aligned} F(t) &\geq \mathcal{E}(t) - (\varepsilon_1/2) \int_{\Omega} |u_t|^2 \, dx - (\varepsilon_1/2) \int_{\Omega} |u|^2 \, dx \\ &\quad - (\varepsilon_2/2) \int_{\Omega} |u_t|^2 \, dx - (\varepsilon_2/2) C_p (1-l) (g \circ \nabla u)(t) \\ &\geq \frac{1}{2} l \|\nabla u(t)\|_2^2 + \frac{1}{2} \|u_t\|_2^2 + \frac{1}{2} (g \circ \nabla u)(t) + \frac{1}{\gamma+2} \|u\|_{\gamma+2}^{\gamma+2} \\ &\quad - \frac{\varepsilon_1 + \varepsilon_2}{2} \int_{\Omega} |u_t|^2 \, dx - \left(\frac{\varepsilon_1}{2}\right) C_p \int_{\Omega} |\nabla u|^2 \, dx - \left(\frac{\varepsilon_2}{2}\right) C_p (1-l) (g \circ \nabla u)(t) \\ &\geq \frac{1}{\alpha_1} \mathcal{E}(t) \end{aligned}$$

for ε_1 and ε_2 small enough. Using equation (1.3), we easily see that

$$\begin{aligned} \Psi'(t) &= \int_{\Omega} (u u_{tt} + u_t^2) \, dx \\ &= \int_{\Omega} u_t^2 \, dx - \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} \nabla u(t) \int_0^t g(t-\tau) \nabla u(\tau) \, d\tau \, dx \\ &\quad - \int_{\Omega} |u|^{\gamma+2} \, dx - \int_{\Omega} a(x) |u_t|^m u_t \, dx \end{aligned} \quad (3.7)$$

We now estimate the third term in the right-hand side of (3.7) as follows:

$$\int_{\Omega} \nabla u(t) \cdot \int_0^t g(t-\tau) \nabla u(\tau) \, d\tau \, dx$$

$$\begin{aligned} &\leq \frac{1}{2} \int_{\Omega} |\nabla u(t)|^2 dx + \frac{1}{2} \int_{\Omega} \left(\int_0^t g(t-\tau) |\nabla u(\tau)| d\tau \right)^2 dx \\ &\leq \frac{1}{2} \int_{\Omega} |\nabla u(t)|^2 dx + \frac{1}{2} \int_{\Omega} \left(\int_0^t g(t-\tau) (|\nabla u(\tau) - \nabla u(t)| + |\nabla u(t)|) d\tau \right)^2 dx. \end{aligned}$$

Using Cauchy-Schwarz and Young's inequality, and $\int_0^t g(\tau) d\tau \leq \int_0^\infty g(\tau) d\tau = 1-l$, we obtain that for any $\eta > 0$,

$$\begin{aligned} &\int_{\Omega} \left(\int_0^t g(t-\tau) (|\nabla u(\tau) - \nabla u(t)| + |\nabla u(t)|) d\tau \right)^2 dx \\ &\leq \int_{\Omega} \left(\int_0^t g(t-\tau) (|\nabla u(\tau) - \nabla u(t)|) d\tau \right)^2 dx + \int_{\Omega} \left(\int_0^t g(t-\tau) |\nabla u(t)| d\tau \right)^2 dx \\ &\quad + 2 \int_{\Omega} \left(\int_0^t g(t-\tau) (|\nabla u(\tau) - \nabla u(t)|) d\tau \right) \left(\int_0^t g(t-\tau) |\nabla u(t)| d\tau \right) dx \\ &\leq (1+\eta) \int_{\Omega} \left(\int_0^t g(t-\tau) |\nabla u(t)| d\tau \right)^2 dx \\ &\quad + (1+\frac{1}{\eta}) \int_{\Omega} \left(\int_0^t g(t-\tau) (|\nabla u(\tau) - \nabla u(t)|) d\tau \right)^2 dx \\ &\leq (1+\frac{1}{\eta}) \int_{\Omega} \int_0^t g(t-\tau) d\tau \int_0^t g(t-\tau) |\nabla u(\tau) - \nabla u(t)|^2 d\tau dx \\ &\quad + (1+\eta) \int_{\Omega} |\nabla u(t)|^2 \left(\int_0^t g(t-\tau) d\tau \right)^2 dx \\ &\leq (1+\eta)(1-l)^2 \int_{\Omega} |\nabla u(t)|^2 dx \\ &\quad + (1+\frac{1}{\eta})(1-l) \int_{\Omega} \int_0^t g(t-\tau) |\nabla u(\tau) - \nabla u(t)|^2 d\tau dx. \end{aligned}$$

For the fifth term of the right-hand side of (3.7), we use Young's inequality and Lemma 3.1 to get

$$\begin{aligned} \int_{\Omega} a(x) |u_t|^m u_t u dx &\leq \delta \int_{\Omega} a(x) |u|^{m+2} dx + c(\delta) \int_{\Omega} a(x) |u_t|^{m+2} dx \\ &\leq c(\delta) \int_{\Omega} a(x) |u_t|^{m+2} dx + \delta C \{ \|\nabla u\|_2^2 + \|u\|_{\gamma+2}^{\gamma+2} \} \end{aligned} \quad (3.8)$$

By combining (3.7)–(3.8), we have

$$\begin{aligned} &\Psi'(t) \\ &\leq \int_{\Omega} u_t^2 dx - \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} |u|^{\gamma+2} dx + \frac{1}{2} \int_{\Omega} |\nabla u(t)|^2 dx \\ &\quad + \frac{1}{2} (1+\eta)(1-l)^2 \int_{\Omega} |\nabla u(t)|^2 dx + c(\delta) \int_{\Omega} a(x) |u_t|^{m+2} dx + \delta C_p \|\nabla u\|_2^2 \\ &\quad + \delta C_p \|u\|_{\gamma+2}^{\gamma+2} \} + \frac{1}{2} (1+\frac{1}{\eta})(1-l) \int_{\Omega} \int_0^t g(t-\tau) |\nabla u(\tau) - \nabla u(t)|^2 d\tau dx \\ &\leq \int_{\Omega} u_t^2 dx - \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} |u|^{\gamma+2} dx + \frac{1}{2} [1 + (1+\eta)(1-l)^2] \int_{\Omega} |\nabla u(t)|^2 dx \end{aligned}$$

$$+ \frac{1}{2} \left(1 + \frac{1}{\eta}\right) (1-l) (g \circ \nabla u)(t) + c(\delta) \int_{\Omega} a(x) |u_t|^{m+2} dx + \delta C \{ \|\nabla u\|_2^2 + \|u\|_{\gamma+2}^{\gamma+2} \}$$

By choosing $\eta = l/(1-l)$ and $\delta = l/4C$, the above inequality becomes

$$\begin{aligned} \Psi'(t) &\leq \int_{\Omega} u_t^2 dx - \frac{l}{4} \int_{\Omega} |\nabla u|^2 dx - \frac{4-l}{4} \int_{\Omega} |u|^{\gamma+2} dx + \frac{1-l}{2l} (g \circ \nabla u)(t) \\ &\quad + c(\delta) \int_{\Omega} a(x) |u_t|^{m+2} dx. \end{aligned} \quad (3.9)$$

Next we estimate

$$\begin{aligned} \chi'(t) &= - \int_{\Omega} u_{tt} \int_0^t g(t-\tau) (u(t) - u(\tau)) d\tau dx \\ &\quad - \int_{\Omega} u_t \int_0^t g'(t-\tau) (u(t) - u(\tau)) d\tau dx - \left(\int_0^t g(s) ds \right) \int_{\Omega} u_t^2 dx \\ &= \int_{\Omega} \nabla u(t) \cdot \left(\int_0^t g(t-\tau) (\nabla u(t) - \nabla u(\tau)) d\tau \right) dx \\ &\quad - \int_{\Omega} \left(\int_0^t g(t-\tau) \nabla u(\tau) d\tau \right) \cdot \left(\int_0^t g(t-\tau) (\nabla u(t) - \nabla u(\tau)) d\tau \right) dx \quad (3.10) \\ &\quad + \int_{\Omega} a(x) u_t(t) \int_0^t g(t-\tau) (u(t) - u(\tau)) d\tau dx \\ &\quad + \int_{\Omega} |u|^{\gamma} u \int_0^t g(t-\tau) (u(t) - u(\tau)) d\tau dx \\ &\quad - \int_{\Omega} u_t \int_0^t g'(t-\tau) (u(t) - u(\tau)) d\tau dx - \left(\int_0^t g(s) ds \right) \int_{\Omega} u_t^2 dx \end{aligned}$$

Similarly to (3.7), we estimate the right-hand side terms of the above inequality. So for $\delta > 0$, we have: For the first term,

$$- \int_{\Omega} \nabla u(t) \cdot \left(\int_0^t g(t-\tau) (\nabla u(t) - \nabla u(\tau)) d\tau \right) dx \leq \delta \int_{\Omega} |\nabla u|^2 dx + \frac{1-l}{4\delta} (g \circ \nabla u)(t). \quad (3.11)$$

For the second term,

$$\begin{aligned} &\int_{\Omega} \left(\int_0^t g(t-s) \nabla u(s) ds \right) \left(\int_0^t g(t-s) (\nabla u(t) - \nabla u(s)) ds \right) dx \\ &\leq \delta \int_{\Omega} \left| \int_0^t g(t-s) \nabla u(s) ds \right|^2 dx + \frac{1}{4\delta} \int_{\Omega} \left| \int_0^t g(t-s) (\nabla u(t) - \nabla u(s)) ds \right|^2 dx \\ &\leq \delta \int_{\Omega} \left(\int_0^t g(t-s) (|\nabla u(t) - \nabla u(s)| + |\nabla u(t)|) ds \right)^2 dx \\ &\quad + \frac{1}{4\delta} \left(\int_0^t g(t-s) ds \right) \int_{\Omega} \int_0^t g(t-s) |\nabla u(t) - \nabla u(s)|^2 ds dx. \\ &\leq \delta \int_{\Omega} \left(\int_0^t g(t-s) |\nabla u(t) - \nabla u(s)| ds \right)^2 dx \\ &\quad + 2\delta(1-l)^2 \int_{\Omega} |\nabla u|^2 dx + \frac{1}{4\delta} (1-l) (g \circ \nabla u)(t) \\ &\leq (2\delta + \frac{1}{4\delta}) (1-l) (g \circ \nabla u)(t) + 2\delta(1-l)^2 \int_{\Omega} |\nabla u|^2 dx. \end{aligned}$$

For the third term,

$$\int_{\Omega} a(x)u_t(t) \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx \leq \delta \|a\|_{\infty} \int_{\Omega} u_t^2 dx + \frac{C_p(1-l)}{4\delta} (g \circ \nabla u)(t).$$

For the fourth term,

$$\begin{aligned} & \int_{\Omega} |u|^{\gamma} u \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx \\ & \leq \delta \int_{\Omega} |u|^{2(\gamma+1)} dx + \frac{1}{4\delta} \int_{\Omega} \left(\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau \right)^2 dx \end{aligned} \quad (3.12)$$

We use (2.1) (2.3) and (2.5) to obtain

$$\int_{\Omega} |u|^{2(\gamma+1)} dx \leq C_p \|\nabla u\|_2^{2(\gamma+1)} \leq C_p \left(\frac{\mathcal{E}(0)}{l} \right)^{2\gamma} \|\nabla u\|_2^2 \quad (3.13)$$

By inserting (3.13) in (3.12), we get

$$\int_{\Omega} |u|^{\gamma} u \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx \leq \delta C_p \left(\frac{\mathcal{E}(0)}{l} \right)^{2\gamma} \|\nabla u\|_2^2 + \frac{C_p(1-l)}{4\delta} (g \circ \nabla u)(t).$$

For the fifth term,

$$\begin{aligned} & - \int_{\Omega} u_t \int_0^t g'(t-\tau)(u(t)-u(\tau))d\tau dx \\ & \leq \delta \int_{\Omega} |u_t|^2 dx + \frac{g(0)}{4\delta} C_p \int_{\Omega} \int_0^t -g'(t-s) |\nabla u(t) - \nabla u(s)|^2 ds dx. \end{aligned} \quad (3.14)$$

Combining (3.10)–(3.14) yields

$$\begin{aligned} & \chi'(t) \\ & \leq \delta \{1 + 2(1-l)^2 + C_p \left(\frac{\mathcal{E}(0)}{l} \right)^{2\gamma}\} \|\nabla u\|_2^2 \left[\frac{1-l}{2\delta} + 2\delta(1-l) + \frac{C_p(1-l)}{2\delta} \right] (g \circ \nabla u)(t) \\ & \quad + \frac{g(0)}{4\delta} C_p (-g' \circ \nabla u)(t) + \left[\delta(1 + \|a\|_{\infty}) - \int_0^t g(s) ds \right] \int_{\Omega} u_t^2 dx. \end{aligned} \quad (3.15)$$

Since $g(0) > 0$ then there exists $t_0 > 0$ such that

$$\int_0^t g(s) ds \geq \int_0^{t_0} g(s) ds = g_0 > 0, \quad \forall t \geq t_0. \quad (3.16)$$

Using (3.4), (3.9), (3.15), and (3.16), we obtain

$$\begin{aligned} & F'(t) \\ & \leq -(1 - \varepsilon_1 c(\delta)) \int_{\Omega} a(x) |u_t|^{m+2} dx \\ & \quad - [\varepsilon_2 \{g_0 - \delta(1 + \|a\|_{\infty})\} - \varepsilon_1] \int_{\Omega} u_t^2 dx - \varepsilon_1 \frac{4-l}{4} \int_{\Omega} |u|^{\gamma+2} dx \\ & \quad - \left[\frac{\varepsilon_1 l}{4} - \varepsilon_2 \delta \{1 + 2(1-l)^2 + C_p \left(\frac{\mathcal{E}(0)}{l} \right)^{2\gamma}\} \right] \|\nabla u\|_2^2 \\ & \quad + \left[\frac{1}{2} - \frac{\varepsilon_1(1-l)}{2\xi l} - \varepsilon_2 \left\{ \frac{g(0)}{4\delta} C_p + \frac{(1-l)C_p}{2\delta\xi} + \frac{1-l}{2\delta\xi} + \frac{2\delta(1-l)}{\xi} \right\} \right] (g' \circ \nabla u)(t). \end{aligned}$$

Now, we choose δ so small that

$$g_0 - \delta(1 + \|a\|_\infty) > \frac{1}{2}g_0$$

$$\frac{4}{l}\delta\{1 + 2(1-l)^2 + C_p(\frac{\mathcal{E}(0)}{l})^{2\gamma}\} < \frac{1}{4}g_0.$$

Whence δ is fixed, the choice of any two positive constants ε_1 and ε_2 satisfying

$$\frac{1}{4}g_0\varepsilon_2 < \varepsilon_1 < \frac{1}{2}g_0\varepsilon_2 \quad (3.17)$$

will make

$$k_1 = \varepsilon_2\{g_0 - \delta(1 + \|a\|_\infty)\} - \varepsilon_1 > 0$$

$$k_2 = \frac{\varepsilon_1 l}{4} - \varepsilon_2\delta\{1 + 2(1-l)^2 + C_p(\frac{\mathcal{E}(0)}{l})^{2\gamma}\} > 0.$$

We then pick ε_1 and ε_2 so small that (3.5) and (3.17) remain valid and

$$1 - \varepsilon_1 c(\delta) > 0,$$

$$\frac{1}{2} - \frac{\varepsilon_1(1-l)}{2\xi l} - \varepsilon_2\left\{\frac{g(0)}{4\delta}C_p + \frac{(1-l)C_p}{2\delta\xi} + \frac{1-l}{2\delta\xi} + \frac{2\delta(1-l)}{\xi}\right\} > 0$$

Therefore, we arrive at $F'(t) \leq -\beta\mathcal{E}(t)$ for all $t \geq t_0$. This inequality and (3.5) yield $F'(t) \leq -\beta\alpha_1 F(t)$, for all $t \geq t_0$. A simple integration leads to

$$F(t) \leq F(t_0)e^{\beta\alpha_1 t_0}e^{-\beta\alpha_1 t}, \quad \forall t \geq t_0.$$

This inequality and (3.5) yields

$$\mathcal{E}(t) \leq \alpha_2 F(t_0)e^{\beta\alpha_1 t_0}e^{-\beta\alpha_1 t}, \quad \forall t \geq t_0,$$

which completes the proof. \square

Remark 3.3. Note that our result is proved without imposing any restriction on the size of $\|g\|_{L^1}$. Also note that the function a may vanish on the whole domain Ω . In other words, contrary to [3], measure (ω) can be zero. As a consequence, no geometry restriction on the boundary has been assumed.

Acknowledgment. The authors would like to express their sincere thanks to King Fahd University of Petroleum and Minerals for its support. This work has been funded by KFUPM under Project # MS/VISCO ELASTIC/270.

REFERENCES

- [1] Ball J., *Remarks on blow up and nonexistence theorems for nonlinear evolutions equations*, Quart. J. Math. Oxford **28** (1977), 473 - 486.
- [2] Cavalcanti M. M., V. N. Domingos Cavalcanti and J. Ferreira; *Existence and uniform decay for nonlinear viscoelastic equation with strong damping*, Math. Meth. Appl. Sci. **24** (2001), 1043 - 1053.
- [3] Cavalcanti M. M., V. N. Domingos Cavalcanti and J. A. Soriano; *Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping*, Elect. J. Diff. Eqs. Vol. **2002** (2002) 44, 1 - 14.
- [4] Cavalcanti M. M. and Oquendo H. P.; *Frictional versus viscoelastic damping in a semilinear wave equation*, SIAM J. Control Optim. vol **42** # **4** (2003), 1310 - 1324.
- [5] Cavalcanti M. M., V. N. Domingos Cavalcanti, J. S. Prates Filho and J. A. Soriano; *Existence and uniform decay rates for viscoelastic problems with nonlinear boundary damping*, Diff. Integ. Eqs. **14** (2001) 1, 85 - 116.
- [6] Georgiev, V. and G. Todorova, *Existence of solutions of the wave equation with nonlinear damping and source terms*, J. Diff. Eqs. **109** (1994), 295 - 308.

- [7] Haraux, A. and E. Zuazua, *Decay estimates for some semilinear damped hyperbolic problems*, Arch. Rational Mech. Anal. **150** (1988), 191 - 206.
- [8] Kalantarov V. K. and O. A. Ladyzhenskaya, *The occurrence of collapse for quasilinear equations of parabolic and hyperbolic type*, J. Soviet Math. **10** (1978), 53 -70.
- [9] Kawashima S. and Shibata Y., *Global Existence and exponential stability of small solutions to nonlinear viscoelasticity*, Comm. Math. Physics, Vol. **148** # 1(1992), 189 - 208.
- [10] Kopackova M., *Remarks on bounded solutions of a semilinear dissipative hyperbolic equation*, Comment. Math. Univ. Carolin. **30** (1989), 713 - 719.
- [11] Levine, H. A., *Instability and nonexistence of global solutions of nonlinear wave equation of the form $Pu_{tt} = Au + F(u)$* , Trans. Amer. Math. Soc. **192** (1974), 1 - 21.
- [12] Levine H. A., *Some additional remarks on the nonexistence of global solutions to nonlinear wave equation*, SIAM J. Math. Anal. **5** (1974), 138 - 146.
- [13] Levine H. A. and J. Serrin, *A global nonexistence theorem for quasilinear evolution equation with dissipation*, Arch. Rational Mech. Anal., **137** (1997), 341 - 361.
- [14] Levine, H. A. and S. Ro Park, *Global existence and global nonexistence of solutions of the Cauchy problem for a nonlinearly damped wave equation*, J. Math. Anal. Appl. **228** (1998), 181 - 205.
- [15] Messaoudi S. A., *Blow up in a nonlinearly damped wave equation*, Mathematische Nachrichten, **231** (2001), 1 - 7.
- [16] Messaoudi S. A. and Tatar N-E., *Global existence asymptotic behavior for a Nonlinear Viscoelastic Problem*, Math. Meth. Sci. Research J., Vol. **7** # **4** (2003), 136-149.
- [17] Messaoudi S. A., *Blow up and global existence in a nonlinear viscoelastic wave equation*, Mathematische Nachrichten, vol. **260** (2003), 5 8 - 66.
- [18] Vitillaro E., *Global nonexistence theorems for a class of evolution equations with dissipation*, Arch. Rational Mech. Anal., **149** (1999), 155 - 182.
- [19] Zuazua E., *Exponential decay for the semilinear wave equation with locally distributed damping*, Comm. PDE. **15** (1990), 205 - 235.

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