PARTIAL COMPACTNESS FOR THE 2-D LANDAU-LIFSHITZ FLOW

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Abstract. Uniform local $C^\infty$-bounds for Ginzburg-Landau type approximations for the Landau-Lifshitz flow on planar domains are proven. They hold outside an energy-concentration set of locally finite parabolic Hausdorff-dimension 2, which has finite times-slices. The approximations subconverge to a global weak solution of the Landau-Lifshitz flow, which is smooth away from the energy concentration set. The same results hold for sequences of global smooth solutions of the 2-d Landau-Lifshitz flow.

1. Introduction

The Ginzburg-Landau approximations $u_\epsilon : \Omega \times \mathbb{R}_+ \to \mathbb{R}^3$ to the Landau-Lifshitz flow are solutions of

$$
\gamma_1 \partial_t u_\epsilon - \gamma_2 u_\epsilon \times \partial_t u_\epsilon - \Delta u_\epsilon = -\frac{1}{\epsilon^2} f(u_\epsilon) \quad \text{in } \Omega \times \mathbb{R}_+ \tag{1.1}
$$

$$
u_\epsilon = u_0 \quad \text{on } (\Omega \times \{0\}) \cup (\partial \Omega \times \mathbb{R}_+) \tag{1.2}
$$

where $\gamma_1 > 0$ and $\gamma_2 \in \mathbb{R}$. "$\times$" denotes the usual vector product in $\mathbb{R}^3$. The domain $\Omega \subset \mathbb{R}^2$ is open, bounded and smooth. The initial and boundary data $u_0$ is always assumed to map a.e. into the standard sphere $S^2 \hookrightarrow \mathbb{R}^3$ or an embedded manifold $N \subset \mathbb{R}^n$ (see below). For the definition of $f_\epsilon$ we distinguish two cases:

Case (I): If $\gamma_2 \neq 0$, the target is $S^2 \hookrightarrow \mathbb{R}^3$ and the right hand side is given by

$$f(u_\epsilon) := -(1 - |u_\epsilon|^2)u_\epsilon = \frac{1}{4} \frac{d}{du}(1 - |u_\epsilon|^2)^2. $$

For small $\epsilon > 0$ the maps $u_\epsilon$ then approximate the Landau-Lifshitz flow

$$
\gamma_1 \partial_t u_\epsilon - \gamma_2 u_\epsilon \times \partial_t u_\epsilon - \Delta u_\epsilon = |\nabla u_\epsilon|^2 u_\epsilon \quad \text{in } \Omega \times \mathbb{R}_+. \tag{1.3}
$$

For sufficiently regular solutions $u : \overline{\Omega} \times \mathbb{R} \to S^2$ equation (1.3) is equivalent to

$$
\partial_t u = -\alpha u \times (u \times \Delta u) + \beta u \times \Delta u \quad \text{in } \Omega \times \mathbb{R}_+, \tag{1.4}
$$

where $\alpha := \frac{\gamma_1}{\gamma_1 + \gamma_2} > 0$ and $\beta := \frac{\gamma_2}{\gamma_1 + \gamma_2} \in \mathbb{R}$. This is the usual form of the Landau-Lifshitz equations known in physics. (Compare [22] and [15], [16], [17].)

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Case (II): If \( \gamma_2 = 0 \), the target is a smooth, closed, isometrically embedded manifold \( N \hookrightarrow \mathbb{R}^n \). For small \( \epsilon > 0 \), the map \( u_{\epsilon} : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^n \) will then be an approximation of a harmonic map flow (compare [36]) and is defined to be a solution of
\[
\partial_t u_{\epsilon} - \Delta u_{\epsilon} = -\frac{1}{2\epsilon^2} \frac{d}{du} \chi(\text{dist}^2(u_{\epsilon}, N)) \quad \text{in } \Omega \times \mathbb{R}^+ \tag{1.5}
\]
\[
u_{\epsilon} = u_0 \quad \text{on } (\Omega \times \{0\}) \cup (\partial \Omega \times \mathbb{R}^+) \,. \tag{1.6}
\]
That is, for the function \( f(u_{\epsilon}) \) in (1.1), we choose
\[
f(u_{\epsilon}) := \frac{1}{2} \frac{d}{du} \chi(\text{dist}^2(u_{\epsilon}, N)) \,.
\]
The cut-off function \( \chi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is smooth, non decreasing and satisfies \( \chi(t) = t \) for \( 0 \leq t \leq \delta_N^2 \) and \( \chi(t) \equiv 2\delta_N^2 \) for \( t \geq 4\delta_N^2 \). The parameter \( \delta_N > 0 \) is chosen in such a way that the nearest neighbour projection \( \bar{U} \ni x \mapsto \pi_N(x) \in N \) is defined and smooth on a tubular neighborhood \( U \subset \mathbb{R}^n \) of \( N \) with uniform radius radius \( 2\delta_N > 0 \). (Such a \( \delta_N > 0 \) always exists if \( N \) is closed. Compare [36] and [29, Section 2.12.3 p.42].)

For fixed \( \epsilon > 0 \), smooth solutions of (1.1)-(1.2) or (1.5)-(1.6) on \( \Omega \times \mathbb{R}^+ \) exist and if \( u_0 \in H^{1,2}(\Omega; N) \cap H^{3/2,2}(\partial \Omega; N) \), they are unique in
\[
H^{1,2}_{\text{loc}} (H^{1,2}) := H^{1,2}_{\text{loc}}(\Omega \times \mathbb{R}^+; \mathbb{R}^n) \cap L^\infty (\mathbb{R}^+; H^{1,2}(\Omega; \mathbb{R}^n))
\]
(Compare [3,36]). Existence is obtained by Galerkin’s method, regularity (\( C^\infty \)) follows from a standard bootstrap argument and uniqueness may be proven as for the two dimensional harmonic map flow (see [30] or [31] (5°) p.234 in the proof of Theorem 6.6).

The total energy of the flow at time \( t \geq 0 \) is defined by
\[
G_\epsilon(u_{\epsilon}(t)) := \int_\Omega g_\epsilon(u_{\epsilon})(x,t) dx \tag{1.7}
\]
where
\[
g_\epsilon(u_{\epsilon}) := \frac{1}{2} |\nabla u_{\epsilon}|^2 + \frac{1}{4\epsilon^2} (1 - |u_{\epsilon}|^2)^2 \quad \text{if } \gamma_2 \neq 0 ,
\]
\[
g_\epsilon(u_{\epsilon}) := \frac{1}{2} |\nabla u_{\epsilon}|^2 + \frac{1}{2\epsilon^2} \chi(\text{dist}^2(u_{\epsilon}, N)) \quad \text{if } \gamma_2 = 0 .
\]
While the total energy of the ”\( \epsilon \)-approximations” always decreases (see Lemma 2.1 below), the local energy given by
\[
G_\epsilon(u_{\epsilon}(t) , B_R^\Omega(x_0)) := \int_{B_R(x_0) \cap \Omega} g_\epsilon(u_{\epsilon})(x,t) dx . \tag{1.8}
\]
may concentrate at space-time points \( (x_0,t_0) \) as \( \epsilon \searrow 0 \) either for fixed \( t = t_0 \) or for variable \( t \neq t_0 \) or \( t \searrow t_0 \). It characterizes the local ”asymptotic regularity behaviour” of the flow. Here asymptotic refers to the limit \( \epsilon \searrow 0 \).

We will show that all the derivatives of the family of maps \( \{u_{\epsilon}\}_{\epsilon > 0} \) are locally uniformly bounded on a regular set \( \text{Reg}(\{u_{\epsilon}\}_{\epsilon > 0}) \) consisting of all points
\[
z_0 = (x_0,t_0) \in \overline{\Omega} \times ]0,\infty[
\]
for which there is \( R_0 = R_0(z_0) > 0 \), such that
\[
\limsup_{\epsilon \searrow 0} \sup_{t_0 - R_0^2 < t < t_0} G_\epsilon(u_{\epsilon}(t), B_{R_0}^\Omega(x_0)) < \epsilon_0, \tag{1.9}
\]
for a constant $\epsilon_0 > 0$ that will be determined later. The complement

$$S\{\{u_\epsilon\}_{\epsilon>0}\} := (\Omega \times \mathbb{R}_+) \setminus \text{Reg}\{\{u_\epsilon\}_{\epsilon>0}\}$$

is referred to as the energy-concentration set. It is closed, has locally finite parabolic Hausdorff dimension two and finite slices at fixed time. The limits of converging subsequences $\{u_{\epsilon_j}\}_j$ are distributional solutions of the Landau-Lifshitz flow (or harmonic map flow if $\gamma_2 = 0$) on all $\Omega \times [0, \infty[$ in $H^{1,2}_{\text{loc}} \cap L^\infty(H^{1,2})$.

Bubbling phenomena of the $\epsilon$-approximations as $\epsilon \searrow 0$ either for fixed $t = t_0$ or for variable $t \nearrow t_0$ or $t \searrow t_0$ as described in [10] will be presented in [18]. Strong subconvergence of the harmonic map flow penalty-approximations in

$$W^{1,0}_{2,\text{loc}}\left(\text{Reg}\{\{u_\epsilon\}_{\epsilon>0}\}; \mathbb{R}^n\right)$$

to a global distributional $H^{1,2}_{\text{loc}} \cap L^\infty(H^{1,2})$-solution of the harmonic map flow was already proved by M. Struwe and Y. Chen in [36] for the case of a closed domain manifold $\Omega = M$ with $\dim M = m \geq 2$ or $M = \mathbb{R}^m$. ($W^{1,0}_{2,\text{loc}}$ refers to functions $f$, whose restriction to any closed ball (in space-time) lies in $L^2$ as well as the restriction of the space-gradient $\nabla f$.)

Struwe and Chen provided uniform local $L^\infty$-bounds for $g_\epsilon(u_\epsilon)$ on $\text{Reg}\{\{u_\epsilon\}_{\epsilon>0}\}$. Their result was extended to compact domains with boundary by Chen and Lin in [7]. The energy-concentration set $S\{\{u_\epsilon\}_{\epsilon>0}\}$ is known to have locally finite $m(= \dim M)$-dimensional Hausdorff measure in the case of the harmonic map flow (see [36]).

X. Cheng investigates in [8] weak(*) $H^{1,2}_{\text{loc}} \cap L^\infty(H^{1,2})$-limits $u_\ast$ of sequences of smooth solutions of the harmonic map flow on the domain $M = \mathbb{R}^m$ and shows that the time slice $S\{\{u_\epsilon\}_0\} \cap (\mathbb{R}^m \times \{t\})$ has finite $(m - 2)$-dimensional Hausdorff measure.

Weak(*)-subconvergence in $H^{1,2}_{\text{loc}} \cap L^\infty(H^{1,2})$ of the Landau-Lifshitz $\epsilon$-approximations from closed surfaces to a distributional $H^{1,2}_{\text{loc}} \cap L^\infty(H^{1,2})$-solution of the Landau-Lifshitz flow was proven by B. Guo and M.C. Hong in [15].

Guo and Ding also studied partial convergence of the two dimensional Landau-Lifshitz penalty-approximations in [9,10] and [11]. Their arguments however contain several gaps and inconsistencies.

2. ENERGY-ESTIMATES

In the case $\gamma_2 = 0$, equation (1.5) is the $L^2$-gradient flow of the functional

$$u \mapsto G_\epsilon(u).$$

(1.1) is not known to be a gradient flow, but the total energy still decreases along the (smooth) flow (1.1)-(1.2).

**Lemma 2.1.** Let $u_\epsilon$ be a solution of (1.1)-(1.2). Then

$$G_\epsilon(u_\epsilon(T)) + \gamma_1 \int_0^T \int_\Omega |\partial_t u_\epsilon|^2 dx dt = G_\epsilon(u_\epsilon(0)) = E(u_0) =: E_0, \quad (2.1)$$

$$G_\epsilon(u_\epsilon(T_2), B^{\Omega}_{R_2}(x_0)) \leq G_\epsilon(u_\epsilon(T_1), B^{\Omega}_{2R}(x_0)) + \frac{C}{\gamma_1 R^2} \int_{T_1}^{T_2} G_\epsilon(u_\epsilon(t), B^{\Omega}_{2R}(x_0)) dt, \quad (2.2)$$

$$\text{for a constant } \epsilon_0 > 0 \text{ that will be determined later.}$$
for $0 \leq T_1 < T_2$. Also for all $\eta > 0$, there exist $T_0 > 0$ and $R_0 > 0$, such that for all $x_0 \in \Omega$ and all $\epsilon > 0$ we have

$$
\sup_{0 \leq t \leq T_0} G_\epsilon(u_\epsilon(t), B_\Omega^\epsilon(x_0)) \leq \eta. \quad (2.3)
$$

Proof. Inequality $(2.1)$ is obtained by multiplying $(1.1)$ with $\partial_\epsilon u_\epsilon$. Inequality $(2.2)$ follows by multiplying $(1.1)$ with $\partial_\epsilon u_\epsilon \phi^2$ for an adequate cut-off function $\phi$ and then integrating by parts and absorbing. Note that $\partial_\epsilon u_\epsilon \equiv 0$ on $\partial \Omega \times \mathbb{R}_+$. Inequality $(2.3)$ follows from $(2.2)$, if we set $T_1 = 0$ and $T_2 = T_0 = \frac{\gamma_2 R_0^2}{\epsilon^2}$ for sufficiently small $R_0 > 0$, such that $G_\epsilon(u_\epsilon(0), B_\Omega^\epsilon(x_0)) = E(u_0, B_\Omega^\epsilon(x_0)) < \eta/2$. \hfill $\square$

The energy estimates imply the penalty-approximations subconverge weak($^*$) in $H^{1,2}_0(\Omega \times \mathbb{R}_+; \mathbb{R}^n) \cap L^\infty(\mathbb{R}_+; H^{1,2}(\Omega; \mathbb{R}^n))$. This was already pointed out by B.Guo and M.C.Hong in section 4 of [15].

3. Partial compactness

In this section we show that, under the uniform smallness condition $(1.9)$ on the local energy, all higher derivatives of $u_\epsilon$ are locally and uniformly bounded. Here “uniform” of course always means uniform in $\epsilon > 0$. In Section 3.1 estimates for linear parabolic systems that can be applied to $(1.1)$ as soon as $\nabla u_\epsilon$ is locally bounded are recalled. In Section 3.2 we show that $\nabla u_\epsilon$ is necessarily locally uniformly bounded, whenever $(1.9)$ holds. In Section 3.3 we derive estimates that will provide bounds for the right hand side of $(1.1)$ and allow to combine the previous estimates into a bootstrap argument.

3.1. Some “standard” parabolic estimates. Equation $(1.1)$ may be written as

$$
L_\epsilon(u_\epsilon) := \partial_\epsilon u_\epsilon - M(u_\epsilon) \Delta u_\epsilon = -\frac{1}{\epsilon^2} M(u_\epsilon)f(u_\epsilon) = f_\epsilon(u_\epsilon). \quad (3.1)
$$

The coefficient-matrix $M(u)$ is smooth with respect to $u$ and also strictly elliptic:

$$
\frac{\gamma_1}{\gamma_1^2 + |\xi|^2} < \xi^2 M(u) \xi = \frac{1}{\gamma_1(\gamma_1^2 + \gamma_2^2 |u|^2)} \left( \gamma_1^2 |\xi|^2 + \gamma_2^2 (u \cdot \xi)^2 \right) < \frac{1}{\gamma_1} |\xi|^2,
$$

for all $\xi \in \mathbb{R}^3$ (See [10] p.12, [15] p.316, [9] p.37). Note that for $\gamma_2 = 0$, we obtain $M(u) = \frac{1}{\gamma_1} \text{Id}$. The results of this section are indeed merely interesting in the case $\gamma_2 \neq 0$, where the left hand side of $(1.1)$ is non-linear. We will therefore restrict ourselves to the case $\gamma_2 \neq 0$.

For fixed $\epsilon > 0$, the solution $u_\epsilon$ of $(1.1)-(1.2)$ is smooth and in particular continuous. $u_\epsilon$ has the same regularity up to the boundary as the boundary data $u_0$. The family of solutions $\{u_\epsilon\}_{\epsilon > 0}$ is also uniformly bounded in $\epsilon > 0$, since $|u_\epsilon(x,t)| \leq 1$ for all $x,t$. This follows from the Maximum Principle applied to the equation obtained by multiplying $(1.1)$ with $(1 - |u_\epsilon|)$. $L_\epsilon$ defines a strongly parabolic system in the sense of Petrovskii (Definition 2, p.599 in [21]) but satisfies as well all the other (not necessarily equivalent) definitions of strong parabolicity for general linear parabolic systems (Definitions 3-6) in [21]. The boundary-data operators also fulfill the required conditions.

First we have estimates in the $W^{2,1}_p$-Sobolev spaces with $p > 1$ (see [21] Chapter IV, Theorem 9.10 p.342 and (10.12) p.355 but also Chapter VII, Theorem 10.4
Let \( f_\epsilon \in L^p(\Omega \times [0,T]; \mathbb{R}^n) \) and \( u_0 \in H^{2,p}(\Omega; \mathbb{R}^n) \). Then for any \( \delta \in ]0,1[ \), \( p > 3/2 \) and for \( t_0 - R^2 > 0 \) a solution of

\[
L_\epsilon(v) = f_\epsilon \quad \text{in} \quad \Omega \times [0,T] \quad \text{and} \quad v = u_0 \quad \text{on} \quad (\Omega \times \{0\}) \cup (\partial \Omega \times [0,T])
\]

satisfies

\[
\|v\|_{W^{2,1}_{p,0}((\Omega \times [0,T]))} \leq C_p(\Omega, T, \omega_u)(\|f_\epsilon\|_{L^p(\Omega \times [0,T])} + \|u_0\|_{H^{2,p}(\Omega)}), \tag{3.2}
\]

\[
\|v\|_{W^{2,1}_{p,0}(P_{\delta R}^\Omega(z_0))} \leq C_p(R, \delta, \Omega, \omega_u)(\|f_\epsilon\|_{L^p(P_{\delta R}^\Omega(z_0))} + \|v\|_{L^\infty(P_{\delta R}^\Omega(z_0))}) \tag{3.3}
\]

with \( 1 \leq q \leq p \). Here

\[
P_{\delta R}^\Omega(z_0) := (B_R(x_0) \times [t_0 - R^2, t_0]) \cap (\Omega \times [0,\infty[)
\]

and \( \delta_{B_R \cap \partial \Omega} = 1 \) if \( B_R \cap \partial \Omega \neq \emptyset \) and 0 otherwise. The trace theorems of course imply

\[
\|u_0\|_{H^{2-(1/p), p}(\partial \Omega)} \leq \|u_0\|_{H^{2,p}(\Omega)}.
\]

The constants \( C_p \) and \( \hat{C}_p \) depend on the indicated quantities and additionally on the uniform lower and upper bounds for the eigenvalues of \( M(u_\epsilon) \), which may be chosen independent of \( \epsilon > 0 \). Note that the constants \( C_p, \hat{C}_p \) also depend on the moduli of continuity of the coefficients of the leading term, i.e., the modulus of continuity \( \omega_u \) of \( u_\epsilon \). The equation can also be written in divergence form,

\[
L_\epsilon(v) := \partial_\epsilon v - div(M(u_\epsilon)\nabla v) + (DM(u_\epsilon)\partial_\epsilon u_\epsilon)\partial_\epsilon v = f_\epsilon.
\]

If we assume in addition

\[
\limsup_{\epsilon \to 0} \sup_{P_{\delta R}^\Omega} |\nabla u_\epsilon| < \infty, \tag{3.4}
\]

then estimates for equations in divergence form imply \( v \in C^{\gamma, \gamma/2}(P_{\delta R}^\Omega; \mathbb{R}^n) \) for some \( \gamma \in ]0,1[ \) and any \( \delta \in ]0,1[ \). (See \[21\] Chapter VII Theorem 3.1 p.582 or Chapter V, Theorem 1.1, p.419.) Indeed if the right hand side \( f_\epsilon \in L^p(P_{R}^\Omega; \mathbb{R}^n) \) with \( p > 2 \), the following estimate for the mixed Hölder-norm of \( v \) on \( P_{\delta R}^\Omega \) holds

\[
\|v\|_{C^{\gamma, \gamma/2}(P_{\delta R}^\Omega)} \leq C(f_\epsilon). \tag{3.5}
\]

(See \[21\] p.7 for the definition of the mixed Hölder-spaces denoted there by \( H^{\gamma, \gamma/2} \)). The bound \( C(f_\epsilon) \) depends on the parabolicity constants, on \( 0 < \delta < 1 \), \( \sup_{P_{\delta R}^\Omega} |u_\epsilon| \), \( \|f_\epsilon\|_{L^p(P_{\delta R}^\Omega)} \), bounds for the coefficients of the equations depending on \( \sup_{P_{\delta R}^\Omega} |\nabla u_\epsilon| \) and also on \( \|u_0\|_{C^{\gamma}(B_R \cap \partial \Omega)} \) if \( B_R \cap \partial \Omega \neq \emptyset \).

Therefore, if \( (3.4) \) holds and \( \|f_\epsilon\|_{L^p(P_{\delta R}^\Omega)} \) or \( \sup_{P_{\delta R}^\Omega} |f_\epsilon| \) are uniformly bounded with respect to \( \epsilon > 0 \), then estimate \( (3.5) \) holds for \( u_\epsilon \) and is uniform in \( \epsilon > 0 \). Now the moduli of continuity of \( u_\epsilon \) on \( P_{\delta R}^\Omega \) is bounded from above (by an increasing function \( h \) with \( \lim_{\delta \to 0} h(t) = 0 \)) independently of \( \epsilon > 0 \). We gain uniform bounds for the modulus of continuity of \( u_\epsilon \) with respect to \( t \geq 0 \) and estimate \( (3.3) \) is now uniform in \( \epsilon > 0 \).

Further by Lemma 3.3 p.80 in Chapter II of \[21\] for \( p > m + 2 (= 4) \) \( m \) being the dimension of the spatial domain, in our case \( m = 2 \), we have

\[
\|\nabla v\|_{C^{\lambda}(P_{\delta R}^\Omega)} \leq C(m, p, \lambda, \Omega)\|v\|_{W^{2,1}_{p,0}(P_{\delta R}^\Omega)} \quad \text{for} \quad \lambda = 1 - (m + 2)/p.
\]
Also if \( (3.4) \) holds and \( \| f_\epsilon \|_{L^p(P_R)} \) is uniformly bounded, then \( (3.3) \) yields \( \epsilon \)-uniform estimates for \( \| \nabla u_\epsilon \|_{C^0(P_R)} \).

### 3.2. The main sup-estimates for the energy-density.

#### 3.2.1. An interior sup-estimate for the Landau-Lifshitz-flow approximations.

In this section we derive an interior sup-estimate for the energy density in the case \( \gamma_2 \neq 0 \), but the proof also works if \( \gamma_2 = 0 \) and the target is \( N \). The proof of the interior estimate is much simpler than in the boundary case and we therefore consider each case separately. The estimate will result from a scaling argument combined to the following higher estimates, that will be proven in the next section. Let

\[
P_R(z_0) := B_R(x_0) \times [t_0 - R^2, t_0] \quad \text{for } z_0 = (x_0, t_0).
\]

**Lemma 3.1.** Let \( u_\epsilon \) be a solution of \( (1.1) \) for each \( \epsilon > 0 \). Assume

\[
\lim_{\epsilon \to 0} \sup_{P_R(z_0)} g_\epsilon(u_\epsilon) \leq C_0
\]

and \( B_R(x_0) \subset \Omega, \, 0 < R^2 < t_0 \). Then for any \( 0 < \delta < 1 \),

\[
\lim_{\epsilon \to 0} \sup_{C^0(P_R(z_0))} \| u_\epsilon \| \leq C_k \quad \text{and} \quad \lim_{\epsilon \to 0} \sup_{C^0(P_R(z_0))} \| \frac{1}{\epsilon^2} (1 - |u_\epsilon|^2) \| \leq \tilde{C}_k
\]

for all \( k \geq 0 \). The constants \( C_k, \tilde{C}_k \) depend on \( C_0, k, R, \delta > 0 \). If \( \gamma_2 = 0 \) and the target is \( N \), they also depend on the geometry of \( N \) (i.e. the metric on \( N \) and its derivatives).

We will now prove the following “\( \epsilon_1 \)-regularity” result.

**Theorem 3.2.** There are constants \( C_1 = C_1(N), \epsilon_1 = \epsilon_1(N) > 0 \), such that if, for some \( 0 < R_0 < \min \{1, \sqrt{T_0}\} \) and \( x_0 \in \Omega \) with \( B_{R_0}(x_0) \subset \Omega \), a solution \( u_\epsilon \) of \( (1.1) \) satisfies

\[
\sup_{t_0 - R_0^2 < t < t_0} \int_{B_{R_0}(x_0)} g_\epsilon(u_\epsilon)(x, t) dx < \epsilon_1,
\]

then

\[
\sup_{P_{R_0}(z_0)} g_\epsilon(u_\epsilon) \leq \frac{C_1}{(1 - \delta)^2 R_0^2}
\]

for any \( \delta \in [0, 1] \).

In the proof we would like to consider points \( z_\epsilon = (x_\epsilon, t_\epsilon) \in \overline{P_R}(z_0) \) such that \( g_\epsilon(z_\epsilon) = \sup_{P_R(z_0)} g_\epsilon \). Difficulties however arise if \( z_\epsilon \in \overline{\partial P_R(z_0)} \), since we then do not have uniform estimates on a neighborhood of \( z_\epsilon \). This is elegantly avoided by considering

\[
\max_{0 \leq \sigma \leq R_0} ((R_0 - \sigma)^2 \sup_{P_\sigma} g_\epsilon).
\]

This trick is initially due to R. Schoen. (See \( [28] \), proof of Theorem 2.2. Schoen’s method was extended to the parabolic context in \( [32], [36] \).)

**Proof of Theorem 3.2.** Without loss of generality, let \( (x_0, t_0) = 0 \). We set \( P_R := P_R(0) \). Since \( u_\epsilon \) is regular, there is some \( \sigma_\epsilon \in [0, R_0] \) such that

\[
(R_0 - \sigma_\epsilon)^2 \sup_{P_{\sigma_\epsilon}} g_\epsilon = \max_{0 \leq \sigma \leq R_0} ((R_0 - \sigma)^2 \sup_{P_\sigma} g_\epsilon).
\]
Moreover, there is some \( z_\varepsilon = (x_\varepsilon, t_\varepsilon) \in \mathcal{P}_{\varepsilon R}, \) such that \( e_\varepsilon = g_\varepsilon(u_\varepsilon(z_\varepsilon)) = \sup_{\mathcal{P}_{\varepsilon R}} g_\varepsilon. \)

Set \( \rho_\varepsilon := \frac{1}{2}(R_0 - \sigma_\varepsilon). \) Since \( P_{\rho_\varepsilon}(z_\varepsilon) \subset P_{\sigma_\varepsilon + \rho_\varepsilon} \subset P_{R_0}, \) we have

\[
\sup_{P_{\rho_\varepsilon}(z_\varepsilon)} g_\varepsilon \leq \frac{1}{(R_0 - (\sigma_\varepsilon + \rho_\varepsilon))^2} (R_0 - (\sigma_\varepsilon + \rho_\varepsilon))^2 \sup_{P_{\sigma_\varepsilon + \rho_\varepsilon}} g_\varepsilon \\
\leq \frac{4}{(R_0 - \sigma_\varepsilon)^2} (R_0 - \sigma_\varepsilon)^2 e_\varepsilon \leq 4e_\varepsilon.
\]

Set \( r_\varepsilon := \sqrt{e_\varepsilon} \rho_\varepsilon \) and consider the rescaled map

\[
v_\varepsilon(y, s) := u(x_\varepsilon + e_\varepsilon^{-1/2} y, t_\varepsilon + e_\varepsilon^{-1} s) \quad \text{for} \quad (y, s) \in P_{r_\varepsilon}.
\]

By definition \( v_\varepsilon \) satisfies (1.1) on \( P_{r_\varepsilon} \) with \( \tilde{\varepsilon} := \sqrt{e_\varepsilon} \) instead of \( \varepsilon \) and

\[
g_\tilde{\varepsilon}(v_\varepsilon)(0, 0) = 1, \sup_{P_{r_\varepsilon}} g_\tilde{\varepsilon}(v_\varepsilon) \leq 4.
\]

Now we claim \( r_\varepsilon \leq 2. \) This will prove the theorem, since by definition of \( r_\varepsilon, \) we then have \((R_0 - \sigma_\varepsilon)^2 \leq 16. \)

Assume \( r_\varepsilon > 2. \) Since \( B_{R_0}(x_0) \subset \Omega, \) all the higher derivatives of \( v_\varepsilon \) are then bounded on \( P_1 \) independently of \( \varepsilon > 0. \) Indeed if \( \liminf_{\varepsilon \to 0} \sqrt{e_\varepsilon} \) exists, \( \sigma_\varepsilon \) and \( \tilde{\varepsilon} \) are then independent of \( \varepsilon \) and \( \sigma_\varepsilon \) is an absolute constant in the sense that it merely depends on the geometry of \( N. \) This lower bound implies

\[
1 = g_\tilde{\varepsilon}(v_\varepsilon)(0, 0) \leq \frac{2}{\theta \sigma^2} \sup_{0 < s < 0} \int_{B_{\theta}} g_\tilde{\varepsilon}(v_\varepsilon)(y, s) \, dy \\
\leq C_* \sup_{t_\varepsilon - r^2 \varepsilon < t < t_\varepsilon} \int_{B_{r^2 \varepsilon - 1/2}} g_\varepsilon(u_\varepsilon)(x, t) \, dx \\
\leq C_* \sup_{t_\varepsilon - r^2 \varepsilon < t < t_\varepsilon} \int_{B_{r^2 \varepsilon - 1/2}} g_\varepsilon(u_\varepsilon)(x, t) \, dx.
\]

Set \( \varepsilon_* := \min\{\frac{1}{2}, \frac{1}{4r_\varepsilon^2}\}. \) Since \( r_\varepsilon = \sqrt{e_\varepsilon} \rho_\varepsilon > 2 > r_0, \) we have \( \frac{r_\varepsilon}{\sqrt{e_\varepsilon}} + \sigma_\varepsilon \leq \rho_\varepsilon + \sigma_\varepsilon \leq R_0 \) and \( \left(\frac{r_\varepsilon}{\sqrt{e_\varepsilon}}\right)^2 + \sigma_\varepsilon^2 \leq (\rho_\varepsilon + \sigma_\varepsilon)^2 \leq R_0^2. \) Then the last estimate yields a contradiction, since the right hand side is smaller than \( \varepsilon_1 \leq \frac{1}{2}. \) Therefore \( r_\varepsilon = \sqrt{e_\varepsilon} \rho_\varepsilon \leq 2 \) and

\[
(1 - \delta)^2 R_0^2 \sup_{P_{r_\varepsilon} \sigma_\varepsilon} g_\varepsilon \leq 16.
\]
3.2.2. A local boundary sup-estimate for the energy density. Local $L^p$-estimates for $\nabla^3 u_\epsilon$ up to the boundary which are uniform in $\epsilon > 0$ cannot be expected, even if $u_0 \in C^\infty(\Omega; S^2)$. Indeed for fixed $\epsilon > 0$, $u_\epsilon$ is smooth up to the boundary and we may thus evaluate (1.1) at $x \in \partial \Omega$ for any $t \geq 0$. This gives $\Delta u_\epsilon = 0$ on $\partial \Omega \times \mathbb{R}_+$. As we will see later, uniform estimates imply the existence of a subsequence $u_{\epsilon_i}$ converging to a map $u_*$, which is a smooth solution of the Landau-Lifshitz or harmonic map flow in $\text{Reg}(\Omega)$.

Theorem 3.4. Consider $u_\epsilon$ be a solution of (1.1), (1.2), with $u_0 \in H^{1/2}(\Omega; S^2) \cap H^{2,p}(\partial \Omega; S^2)$ and $p \geq 2$ for each $\epsilon > 0$. Assume

$$\limsup_{\epsilon \to 0} \sup_{\Omega} \frac{1}{2} \int_{B_{R_0}(x_0)} g_\epsilon(u_\epsilon) dx < c_0 ,$$

then

$$\limsup_{\epsilon \to 0} \sup_{\Omega} g_\epsilon(u_\epsilon) \leq \frac{C_0}{(1 - \delta)^2 R_0^2},$$

for any $\delta \in [0, 1]$. 

\[\text{Lemma 3.3. Let } u_\epsilon \text{ be a solution of (1.1), (1.2), with } u_0 \in H^{1/2}(\Omega; S^2) \cap H^{2,p}(\partial \Omega; S^2) \text{ and } p \geq 2 \text{ for each } \epsilon > 0. \text{ Assume} \]

$$\limsup_{\epsilon \to 0} \frac{1}{2} \int_{B_{R_0}(x_0)} g_\epsilon(u_\epsilon) dx < c_0 ,$$

and $B_R(x_0) \cap \partial \Omega \neq \emptyset$, $0 < R^2 < t_0$. Then for any $\delta \in [0, 1]$, we have

$$\limsup_{\epsilon \to 0} \frac{1}{2} \int_{B_{R_0}(x_0)} g_\epsilon(u_\epsilon) dx < c_0 ,$$

where the constant $C_1$ depends on $C_0, p, R, \delta$ and $\Omega$. Further we have for any $\delta \in [0, 1]$

$$\limsup_{\epsilon \to 0} \frac{1}{2} \int_{B_{R_0}(x_0)} g_\epsilon(u_\epsilon) dx < c_0 ,$$

where $\epsilon \to o_\delta(\epsilon)$ is a function that depends on $\delta \in [0, 1]$ and $\lim_{\epsilon \to 0} \epsilon^{-k} o_\delta(\epsilon) = 0$ for all $k \in \mathbb{N}$. All the constants also depend on the parabolicity constants. If $\gamma_2 = 0$ and the target is $N$, they also depend on the geometry of $N$. 

\[\text{We now prove the following result.}\]

\[\text{Theorem 3.4. Consider } u_0 \in H^{1/2}(\Omega; S^2) \cap C^2(\partial \Omega; S^2). \text{ Let } u_\epsilon \text{ be a solution of (1.1), (1.2) for each } \epsilon > 0. \text{ There are constants } C_0 = C_0(\|u_0\|_{C^2(\partial \Omega)}, E_0, \Omega) \text{ and } c_0 = c_0(\|u_0\|_{C^2(\partial \Omega)}, E_0, \Omega) > 0, \text{ such that if for some } z_0 = (x_0, t_0) \text{ and } R_0 \in [0, \min\{1, \sqrt{t_0}\}]

\[\limsup_{\epsilon \to 0} \sup_{t_0 - R_0^2 < t < t_0} \int_{B_{R_0}(x_0) \cap \Omega} g_\epsilon(u_\epsilon) dx < c_0 ,\]

then

$$\limsup_{\epsilon \to 0} \sup_{\Omega} g_\epsilon(u_\epsilon) \leq \frac{C_0}{(1 - \delta)^2 R_0^2},$$

for any $\delta \in [0, 1]$.\]
Again for $\rho \nu$ $v$, By construction $z$ EJDE-2004/90 PARTIAL COMPACTNESS 9

Indeed if $\liminf_{\epsilon \to 0} \sqrt{\epsilon} C = \liminf_{\epsilon \to 0} \tilde{C}(\epsilon) = 0$.

Proof. Without loss of generality let $(x_0, t_0) = 0$. Since $u_0 \in C^2(\partial \Omega)$ admits an extension $w_0 \in C^2(\Omega)$ and since $t_0 - R^2 > 0$, we may assume $u_0 \in C^2(\Omega)$. We have $u_\epsilon \in C^{1,0}(\Omega \times R^+; S^2)$ for any $0 < \alpha < 1$ and so there are $\sigma_\epsilon \in [0, R_0]$ and $z_\epsilon = (x_\epsilon, t_\epsilon) \in \mathbb{R}^n$, such that

$$(R_0 - \sigma_\epsilon)^2 \sup_{P_{\rho_\epsilon}} g_\epsilon = \max_{0 \leq r \leq R_0} \left( (R_0 - \rho)^2 \sup_{P_{\rho}} g_\epsilon \right),$$

$e_\epsilon = g_\epsilon(u_\epsilon(z_\epsilon)) = \sup_{P_{\rho_\epsilon}} g_\epsilon.$

Again for $\rho_\epsilon := \frac{1}{\epsilon}(R_0 - \sigma_\epsilon)$, we have $\sup_{P_{\rho_\epsilon}} g_\epsilon \leq 4 \epsilon$. Consider the rescaled map

$v_\epsilon(y, s) := u(x_\epsilon + \epsilon^{-1/2} y, t_\epsilon + \epsilon^{-1} s).$

By construction $v_\epsilon$ satisfies

$$\gamma_1 \partial v_\epsilon \epsilon \gamma_2 v_\epsilon \times \partial v_\epsilon - \Delta v_\epsilon = \frac{1}{\epsilon^2}(1 - |v_\epsilon|^2) v_\epsilon$$ on $P_{r_\epsilon}^{\Omega},$

with $\epsilon := \sqrt{c_\epsilon} \epsilon$, $r_\epsilon := \sqrt{c_\epsilon} \rho_\epsilon$, $\Omega_\epsilon := \sqrt{c_\epsilon} (\Omega - x_\epsilon)$ and

$$P_{r_\epsilon}^{\Omega_\epsilon} := (B_{r_\epsilon} \cap \Omega_\epsilon) \times [-r_\epsilon^2, 0[.$$

Further by construction,

$$g_\epsilon(v_\epsilon)(0, 0) = 1 \text{ and } \sup_{P_{\rho_\epsilon}} g_\epsilon(v_\epsilon) \leq 4.$$ (3.7)

The boundary data are also rescaled. Set $v_{\epsilon, 0}(y) := u_0(x_\epsilon + \epsilon^{-1/2} y)$. Then

$$v_{\epsilon}(y, s) = v_{\epsilon, 0}(y) \text{ on } (\partial \Omega_\epsilon \cap B_{r_\epsilon}) \times [-r_\epsilon^2, 0[$$

and

$$\sup_{P_{\rho_\epsilon}^{\Omega_\epsilon}} |\nabla v_{\epsilon, 0}| \leq \epsilon^{-1/2} \sup_{P_{\rho_\epsilon}^{\Omega_\epsilon}} |\nabla u_0|, \quad \sup_{P_{\rho_\epsilon}^{\Omega_\epsilon}} |\nabla^2 v_{\epsilon, 0}| \leq \epsilon^{-1} \sup_{P_{\rho_\epsilon}^{\Omega_\epsilon}} |\nabla^2 u_0|.$$

Now we claim that for sufficiently small $\epsilon > 0$, we have

$$r_\epsilon \leq C_0 := \max\{2, \tilde{C}(\Omega, \|u_0\|_{C^{2}(\Omega)})\},$$

where $\tilde{C}(\cdot) > 0$ will be specified later. Again by definition of $r_\epsilon$, this will prove the theorem.

Assume by contradiction $r_\epsilon > C_0 \geq 2$ for small $\epsilon > 0$. Then

$$e_\epsilon^{-1/2} = \rho_\epsilon/r_\epsilon < R_0/(2C_0) \leq 1/(2C_0),$$

since $0 < R_0 < 1$. First we claim that

$$\liminf_{\epsilon \to 0} \sqrt{\epsilon} C = \liminf_{\epsilon \to 0} \tilde{C}(\epsilon) = 0.$$ (3.6)

Indeed if $\liminf_{\epsilon \to 0} \sqrt{\epsilon} C > 0$, the right hand side of (3.6) is uniformly bounded in $\epsilon = \sqrt{c_\epsilon} \epsilon > 0$ and together with (3.7) we obtain uniform bounds in $C^{\infty}(P_{r_\epsilon}^{\Omega_\epsilon})$. This however leads to a contradiction as in the proof of Theorem 3.2 if $\epsilon_0$ is smaller than $\epsilon_1$. Further if

$$\limsup_{\epsilon \to 0} \left( \sqrt{\epsilon} \dist(x_\epsilon, \partial \Omega) \right) = \limsup_{\epsilon \to 0} \dist(0, \partial \Omega_\epsilon) \geq \frac{1}{2},$$

...
we can also use uniform interior estimates in $C^\infty(P_t^{\Omega/4})$ and proceed as in the proof of Theorem 3.2 to get a contradiction, if we choose $\epsilon_0$ sufficiently small. So far the required upper bound on $\epsilon_0$ is universal in the sense that it only depends on the geometry of $N$ and the parabolicity constants. We therefore have

$$
\lim_{\epsilon \searrow 0} \text{dist}(0, \partial \Omega_\epsilon) < 1/2,
$$

and in the sequel we consider sufficiently small $\epsilon > 0$, such that $\text{dist}(0, \partial \Omega_\epsilon) < 1/2$.

Lemma 3.3 combined to the embedding $W^{2,1}_p(P_t) \hookrightarrow C^1(P_t)$ for $p > 4$, implies

$$
sup_{P_t^\Omega_\epsilon} |\nabla v_\epsilon|^2 
\leq C\|v_\epsilon\|^2_{W^{2,1}_p(P_t^\Omega_\epsilon)} 
\leq C(p, \Omega_\epsilon \cap B_2) \left( \frac{1}{\epsilon^2} |\nabla v_\epsilon|^2_{L^p(P_t^\Omega_\epsilon)} + \|v_\epsilon\|^2_{L^2(P_t^\Omega_\epsilon)} + \|v_\epsilon, 0\|^2_{H^2, p(P_t^\Omega_\epsilon)} \right) 
$$

Note that $\Omega_\epsilon \cap B_2$ has uniformly bounded curvature and so

$$
0 < C(p, \Omega_\epsilon \cap B_2) < C(p, \Omega) .
$$

Since $1 - |v_\epsilon|^2 \leq 1$ and $\sup_{P_t^\Omega_\epsilon} g_\epsilon(v_\epsilon) \leq 4$, Lemma 3.3 implies

$$
\|\nabla v_\epsilon\|^2_{L^p(P_t^\Omega_\epsilon)} \leq C_p \left( \epsilon_0 o(\epsilon) + o(\epsilon) \right),
$$

where $C_p = C(p, E_0)$ and $o(\tau)$ denotes a generic function that satisfies

$$
\lim_{\tau \searrow 0} o(\tau) = 0 .
$$

A Poincaré inequality on $P_t^\Omega_\epsilon$ leads to

$$
\|v_\epsilon\|^2_{L^2(P_t^\Omega_\epsilon)} \leq 2 \left( \|v_\epsilon, 0\|^2_{L^2(P_t^\Omega_\epsilon)} + \|v_\epsilon - v_\epsilon, 0\|^2_{L^2(P_t^\Omega_\epsilon)} \right) 
\leq 2 \|v_\epsilon, 0\|^2_{L^2(P_t^\Omega_\epsilon)} + C(\Omega) \left( \|\nabla v_\epsilon, 0\|^2_{L^2(P_t^\Omega_\epsilon)} + \|\nabla v_\epsilon\|^2_{L^2(P_t^\Omega_\epsilon)} \right) .
$$

Again $\|\nabla v_\epsilon\|^2_{L^2(P_t^\Omega_\epsilon)} \leq \epsilon_0(\epsilon_0)$. Of course

$$
\|v_\epsilon, 0\|_{H^1, 2(B_{2\epsilon}^\Omega)} \leq C(p) \|v_\epsilon, 0\|_{H^{1, p}(B_{2\epsilon}^\Omega)} 
\leq C(p) \|v_\epsilon, 0\|_{H^{2, p}(B_{2\epsilon}^\Omega)} 
\leq C(p, \Omega) \|v_\epsilon, 0\|_{C^2(B_{2\epsilon}^\Omega)}^2
$$

and we still need to estimate $\|v_\epsilon, 0\|_{C^2(B_{2\epsilon}^\Omega)}^2$.

For each $\epsilon > 0$ we may choose coordinates for the target such that $v_\epsilon, 0(0) = 0$. Then

$$
\sup_{B_{2\epsilon}^\Omega} |v_\epsilon, 0| \leq 4 \sup_{B_{2\epsilon}^\Omega} |\nabla v_\epsilon, 0| ,
$$

$$
\|v_\epsilon, 0\|_{C^2(B_{2\epsilon}^\Omega)} \leq C \epsilon_\epsilon^{-1/2} \sup_{B_{\epsilon}^\Omega} |\nabla u_0| + \epsilon_\epsilon^{-1} \sup_{B_{\epsilon}^\Omega} |\nabla^2 u_0| .
$$
The above estimates combined to the one for $\|\frac{1}{\varepsilon^2}(1-|v_\varepsilon|^2)\|_{L^\infty(\mu_\varepsilon)}$ in Lemma 3.3 yield

$$1 \leq \sup_{\mu_\varepsilon} g_\varepsilon(v_\varepsilon)$$

$$\leq \sup_{\mu_\varepsilon} \frac{1}{2} \frac{1}{\varepsilon^2} |\nabla v_\varepsilon|^2 + \varepsilon^2 \sup_{\mu_\varepsilon} \left( \frac{1}{\varepsilon^2}(1-|v_\varepsilon|^2)^2 \right)$$

$$\leq C_1 \left( o(\varepsilon) + o(\varepsilon) + \varepsilon^{-1} \|\nabla u_0\|^2_{C^1(B_{\varepsilon0})} \right),$$

where $C_1 = C_1(\Omega, E_0)$. Now if both $o(\varepsilon) < (1/4)C_1^{-1}$ and $o(\varepsilon) < (1/4)C_1^{-1}$, this leads to

$$\varepsilon \leq 2C_1 \|\nabla u_0\|^2_{C^1(B_{\varepsilon0})},$$

which is in contradiction with $r_\varepsilon > C_0 := \max\{2, 2C_1 \|u_0\|_{C^2(\Omega)}\}$ and $\sqrt{\varepsilon} > 2C_0$.

Thus $r_\varepsilon \leq C_0$ and by definition of $r_\varepsilon$ also

$$\frac{1}{4}(R_0 - \delta R_0)^2 \sup_{\mu_\varepsilon} g_\varepsilon \leq C_0^2 = C(\Omega, E_0, \|u_0\|_{C^2(\Omega)}).$$

Since $t_0 - R_0 > 0$, we could replace $u_0$ in the above by any $w_0 \in C^2(\Omega)$ with $w_0 = u_0$ on $\partial\Omega \cap B_{R_0}$. Therefore the above constants merely depend on $\|u_0\|_{C^2(\partial\Omega)}$. \qed

3.3. Higher estimates. In this section, we prove Lemmata 3.1 and 3.3, for which the following uniform estimates will be needed.

3.3.1. Uniform estimates in $\varepsilon > 0$. The “distance-to-the-target-function” $\rho_\varepsilon := 1 - |v_\varepsilon|^2$ satisfies

$$\gamma_1 \partial_t \rho_\varepsilon - \Delta \rho_\varepsilon + \frac{2}{\varepsilon^2} \rho_\varepsilon = 2|\nabla u_\varepsilon|^2 + \frac{2}{\varepsilon^2} \rho_\varepsilon^2. \quad (3.8)$$

Since $\gamma_1 > 0$, we may assume $\gamma_1 = 1$ without loss of generality. We will now derive uniform a priori estimates for this equation. Lemma 3.5 extends a comparison argument from [1] (Lemma 2, p.130) to the time dependent case and to non-positive solutions.

The parabolic boundary of $P_R := B_R(0) \times ]-R^2, 0]$ is denoted as

$$\partial P_R := (B_R(0) \times \{-R^2\}) \cup (\partial B_R(0) \times [-R^2, 0]).$$

Lemma 3.5. Let $a > 0$, $R \in [0, \frac{1}{4}]$, $\varepsilon \in [0, 1]$ and $g \in C_0^0(P_R)$ with $\varepsilon^2 \sup_{P_R} |g| \leq a$.

Let $f \in C_0^0(P_R) \cap C^2(P_R)$ be a solution of

$$(\partial_t f - \Delta f) + \frac{1}{\varepsilon^2} f = g \text{ in } P_R,$$

$$|f| \leq a \text{ on } \partial P_R.$$

Then for any $\delta \in [0, 1]$, we have

$$\frac{1}{\varepsilon^2} |f| \leq \sup_{P_R} |g| + \frac{2a}{\varepsilon^2} \frac{\varepsilon^2}{(1-\delta^2)^2} \|R^4\|_{P_{5R}}.$$

Proof. Consider $\omega(x,t) = 2ae^{-\frac{1}{2}(R^2-|x|^2)(R^2+t)}$. Then

$$\varepsilon^2 (\partial_t \omega - \Delta \omega) + \omega > 0 \text{ in } P_R,$$

$$\omega = 2a \text{ on } \partial P_R.$$

□
For \( f_1 := f - \epsilon^2 \sup_{P_R} |g| \) and \( f_2 := f + \epsilon^2 \sup_{P_R} |g| \), we have
\[
|f_1| \leq 2a \quad \text{and} \quad |f_2| \leq 2a \quad \text{on} \quad \partial P_R,
\]
and hence
\[
f_1 - \omega \leq 0, \quad f_2 + \omega \geq 0 \quad \text{on} \quad \partial P_R.
\]
Moreover
\[
\epsilon^2 (\partial_t f_1 - \Delta f_1) + f_1 \leq 0, \quad \epsilon^2 (\partial_t f_2 - \Delta f_2) + f_2 \geq 0 \quad \text{in} \quad P_R.
\]
The Maximum Principle now implies \( f_1 - \omega \leq 0 \) and \( f_2 + \omega \geq 0 \) on \( P_R \), that is
\[
-\omega - \epsilon^2 \sup_{P_R} |g| \leq f \leq \omega + \epsilon^2 \sup_{P_R} |g|.
\]

The above lemma will yield interior estimates. If \( B_R \cap \Omega \neq \emptyset \) and \( f \equiv 0 \) on \( B_R \cap \partial \Omega \), we still obtain a local estimate up to the boundary, i.e. on \( P^\Omega_{\delta R} = (B_{\delta R} \cap \Omega) \times (-\delta R)^2, 0] \).

**Corollary 3.6.** Consider a smooth domain \( \Omega \subset \mathbb{R}^2, a > 0, R \in [0, \frac{1}{4}], \epsilon \in [0, 1] \) and \( g \in C^0(\overline{P_R}) \) with \( \epsilon^2 \sup_{P_R} |g| \leq a \). Let \( f \in C^0(\overline{P_R}) \cap C^2(P^\Omega_{\delta R}) \) be a solution of
\[
(\partial_t f - \Delta f) + \frac{1}{\epsilon^2} f = g \quad \text{in} \quad P^\Omega_{\delta R},
\]
\[
|f| \leq a \quad \text{on} \quad \partial P_R \cap \Omega,
\]
\[
f = 0 \quad \text{on} \quad \partial \Omega \cap P_R.
\]
Then for any \( \delta \in [0, 1] \), we have
\[
\frac{1}{\epsilon^2} |f| \leq \sup_{P^\Omega_{\delta R}} |g| + \frac{2a}{\epsilon^2} e^{-\frac{1}{2} (1 - \delta^2)^2 R^2} \quad \text{on} \quad P^\Omega_{\delta R}.
\]

The proof of Lemma 3.5 also applies in this case. The next interior-estimate-version of Lemma 3.5 deals with the case \( B_R(x_0) \cap \Omega \neq \emptyset \) and \( f \neq 0 \) on \( \partial B_R(x_0) \cap \Omega \). The estimate then also depends on \( \text{dist}(x, \partial \Omega) \). We formulate the following lemma in such a way that it readily extends to the case \( \Omega = M \) is a manifold.

**Corollary 3.7.** Let \( U \subset \mathbb{R}^2 \) be an open smooth neighborhood of \( 0 \) with \( \text{diam} U \leq 1 \) and set \( P_{R,U} := U \times [-R^2, 0] \). Consider \( a > 0, R \in [0, \frac{1}{4}], \epsilon \in [0, \frac{1}{4}] \) and \( g \in C^0(\overline{P_R}) \) with \( \epsilon^2 \sup_{P_{R,U}} |g| \leq a \). Let \( f \in C^0(\overline{P_R}) \cap C^2(P_{R,U}) \) be a solution of
\[
(\partial_t f - \Delta f) + \frac{1}{\epsilon^2} f = g \quad \text{in} \quad P_{R,U},
\]
\[
|f| \leq a \quad \text{on} \quad \partial P_{R,U}.
\]
Then there is a constant \( C = C(U) > 0 \), such that for any \( \delta \in [0, 1] \) we have
\[
\frac{1}{\epsilon^2} |f(x, t)| \leq \sup_{P_{R,U}} |g| + \frac{2a}{\epsilon^2} e^{-\frac{C^2}{2} (1 - \delta^2) \text{dist}^2(x, \partial U)} \quad \text{on} \quad P_{\delta R,U}.
\]

Of course
\[
\partial P_{R,U} := (U \times \{-R^2\}) \cup (\partial U \times [-R^2, 0]).
\]
Proof. Set $d(x) := \text{dist}(x, \partial U)$, $C = C(U) := \max\{1, \|\Delta d^2\|_{L^\infty(U)}, \|\nabla d^2\|_{L^\infty(U)}\}$. Note that $d(x) \leq 1$ on $U$ since $\text{diam} U \leq 1$. We claim that
\[ \omega(x, t) := 2ae^{-\frac{1}{2}d^2(x)(R^2+t)} \]
is a supersolution of equation (3.9), if $0 < R < \frac{1}{4}$ and $0 < \epsilon < \frac{1}{4}$. Indeed
\[ e^2(\partial_t - \Delta)\omega + \omega = \omega \left[ 1 - \frac{\epsilon}{C}d^2 + \frac{\epsilon}{C}(R^2 + t)\Delta d^2 - \frac{\epsilon}{C^2}(R^2 + t)^2|\nabla d^2|^2 \right] \geq \omega \left[ 1 - \epsilon R^2 - R^4 \right] \geq \frac{1}{4} \omega > 0 \] on $P_{R,U}$.

The claim now follows just as in the proof of Lemma 3.5. \qed

We will also need a priori $L^p$-estimates for the above equation.

**Lemma 3.8.** Consider a smooth domain $\Omega \subset \mathbb{R}^2$, $g \in L^1(\Omega \times [0, T])$ and $\epsilon > 0$. Let $f \in C^1(\Omega \times [0, T]) \cap C^2(\Omega \times [0, T])$ be a solution of
\[ (\partial_t f - \Delta f) + \frac{1}{\epsilon^2} f = g \quad \text{in} \quad \Omega \times [0, T], \]
\[ f = 0 \quad \text{on} \quad \Omega \times \{0\} \cup \partial \Omega \times [0, T]. \]

For $f \geq 0$, we only need to assume
\[ (\partial_t f - \Delta f) + \frac{1}{\epsilon^2} f \leq g \quad \text{in} \quad \Omega \times [0, T], \]
\[ f = 0 \quad \text{on} \quad \Omega \times \{0\} \cup \partial \Omega \times [0, T]. \]

Then
\[ \|f\|_{L^1(\Omega \times [0, T])} \leq \|g\|_{L^1(\Omega \times [0, T])}. \tag{3.10} \]

and for any $R, \rho > 0$ and $z_0 = (x_0, t_0) \in \Omega \times [0, T]$ with $R^2 + \rho^2 < t_0$,
\[ \int_{P_{R}^{\rho}(z_0)} \frac{1}{\epsilon^2} |f| \, dz \leq \int_{P_{R}^{\rho}(z_0)} (|g| + \frac{C}{\rho^2} |f|) \, dz. \tag{3.11} \]

Proof. (i) Multiplication of the equation for $f$ by $\frac{f}{\sqrt{f^2 + \delta^2}}$ leads to
\[ \frac{\partial_t |f|}{\sqrt{f^2 + \delta^2}} + \frac{|\nabla f|^2}{\sqrt{f^2 + \delta^2}} (1 - \frac{f^2}{f^2 + \delta^2}) + \frac{1}{\epsilon^2} \frac{f^2}{\sqrt{f^2 + \delta^2}} = \frac{gf}{\sqrt{f^2 + \delta^2}} + \Delta \frac{\sqrt{f^2 + \delta^2}}{f^2 + \delta^2}. \]

Now integrate over $\Omega \times [0, T]$ for any $t \in [0, T]$ and let $\delta \to 0$ to obtain
\[ \sup_{0 \leq t \leq T} \int_{\Omega} |f(x, t)| \, dx + \int_{0}^{T} \int_{\Omega} \frac{1}{\epsilon^2} |f| \leq \int_{0}^{T} \int_{\Omega} |g| \, dx \, dt. \]

(ii) We multiply the equation by $f$ with
\[ \left( \frac{f}{\sqrt{f^2 + \delta^2}} \right)(x, t)\phi(x)\eta(t). \]

The cut-off function $\phi$ satisfies $0 \leq \phi \in C^\infty_c(\mathbb{R}^2)$ with $\text{spt} \phi \subset B_{R+\rho}(x_0)$ and $\phi \equiv 1$ on $B_R(x_0)$, whereas $\eta \in C^\infty(\mathbb{R}^2)$ with $0 \leq \eta(t) \leq 1$, $\eta(t_0 - R^2 - \rho^2) = 0$ and $\eta(t) \equiv 1$ if $t \geq t_0 - R^2$. We may assume
\[ |\nabla \phi| \leq \frac{C}{\rho}, \quad |\nabla^2 \phi| \leq \frac{C}{\rho^2} \quad \text{and} \quad |\partial_t \eta| \leq \frac{C}{\rho^2}. \]
This leads to
\[
\frac{\partial_t (|f|^2 \delta \eta)|f|}{\sqrt{f^2 + \delta^2}} + \frac{|\nabla f|^2 \delta^2 \eta}{\sqrt{f^2 + \delta^2}} + \frac{1}{\varepsilon^2} \sqrt{f^2 + \delta^2} + \frac{f^2 \delta \eta}{\sqrt{f^2 + \delta^2}} = \frac{g \delta \eta}{\sqrt{f^2 + \delta^2}} + \text{div} \left( \nabla f \frac{f \delta \eta}{\sqrt{f^2 + \delta^2}} \right) + \frac{f^2 \delta \eta}{2 \sqrt{f^2 + \delta^2}} - 2 \eta \phi \nabla \phi \sqrt{f^2 + \delta^2}.
\]

Of course
\[
\int_{\Omega} \phi \nabla \eta \nabla \sqrt{f^2 + \delta^2} dx = -\int_{\Omega} \eta \sqrt{f^2 + \delta^2} (\phi \nabla^2 \phi + |\nabla \phi|^2) dx.
\]

After integrating \((4.2)\) and letting \(\delta \to 0\), we obtain
\[
\sup_{t_0 - (R^2 + \varepsilon^2) < t < t_0} \int_{B_R^\Omega} \frac{1}{\varepsilon^2} |f| dx + \int_{P_\rho^\Omega} \frac{1}{\varepsilon^2} |f| dz \leq \int_{P_\rho^\Omega} \left( |g| + \frac{C}{\rho^2} |f| \right) dz.
\]

\(\square\)

**Lemma 3.9.** Consider a smooth domain \(\Omega \subset \mathbb{R}^2\), \(g \in L^1(\Omega)\) and \(p \geq 2\) and \(\varepsilon > 0\). Let \(f \in C^1(\Omega \times [0, T]) \cap C^2(\Omega \times [0, T])\) be a solution of
\[
(\partial_t f - \Delta f) + \frac{1}{\varepsilon^2} f = g \quad \text{in} \ \Omega \times [0, T],
\]
\[f = 0 \quad \text{on} \ \partial \Omega \times [0, T].\]

For \(f \geq 0\), we only need to assume
\[
(\partial_t f - \Delta f) + \frac{1}{\varepsilon^2} f \leq g \quad \text{in} \ \Omega \times [0, T],
\]
\[f = 0 \quad \text{on} \ \partial \Omega \times [0, T].\]

(i) For any \(\delta \in [0, 1[\) and \(z_0 = (x_0, t_0) \in \Omega \times [0, T]\) with \(0 < R^2 < t_0\), we have
\[
\left\| \frac{1}{\varepsilon^2} f \right\|_{L^p(P^\Omega_R(z_0))} \leq C_1 \|g\|_{L^p(P^\Omega_R(z_0))} + \varepsilon^{2/p} C_2,
\]
where \(C_1 = C_1(p)\) and \(C_2 = C_2(\|g\|_{L^p(P^\Omega_R(z_0))}, \|f\|_{L^2(P^\Omega_R(z_0))}, p, \delta, R)\).

(ii) The same bound as in (i) holds for
\[
\left\| \left( \frac{1}{\varepsilon^2} \right)^{1-1/p} f \right\|_{L^{2p}(P^\Omega_{R/2}(z_0))}, \left\| \left( \frac{1}{\varepsilon^2} \right)^{1-1/p} f \right\|_{L^\infty(\{t_0 - R^2, t_0\}; L^p(B^\Omega_{R/2}(x_0)))}
\]
and
\[
\left\| \frac{1}{\varepsilon} \nabla f \right\|_{L^2(P^\Omega_{R/2}(z_0))}.
\]

**Proof.** We multiply the equation for \(f\) by \(f \left| f \right|^{2s-2}(x, t) \phi^2(x) \eta(t)\), where \(s \geq 1\). The cut-off functions \(\phi\) and \(\eta\) are the same as in the proof of Lemma 3.8. Then
\[
\frac{1}{2s} \partial_t (|f|^{2s}(x, t) \phi^2(x) \eta(t)) + \frac{2s - 1}{2s} \left| \nabla |f| \right|^2 \phi^2 \eta + \frac{1}{\varepsilon^2} \left| f \right|^{2s} \phi \eta
\]
\[
= -\text{div} \left( \nabla f \left| f \right|^{2s-2} \phi^2 \eta \right) + g f \left| f \right|^{2s-2} \phi^2 \eta + \frac{1}{2s} \left| f \right|^{2s} \phi^2 \delta t \eta - \nabla f \left| f \right|^{2s-1} 2 \nabla \phi \phi \eta,
\]
for any \(s \geq 1\). By Young’s inequality \(ab \leq \delta^{-p} a^p + \delta^p b^q\) for \(p, q > 1\) with \(\frac{1}{p} + \frac{1}{q} = 1\) and \(a, b, \delta > 0\), we have
\[
\|g\| f^{2s-1} \leq \frac{1}{2s} \left( \frac{2s - 1}{2s} \right) \left| f \right|^{2s} + \left( \frac{2s - 1}{2s} - 1 \right) \frac{1}{2s} \left| f \right|^{2s}
\]
and

\[ |\nabla |f||f|^{2s-1}2\nabla \phi \eta| = 2 \left( \frac{1}{s} \right) f^s \phi (|f|^s \nabla \phi) \eta \]

\[ \leq \frac{2s-1}{2s^2} |\nabla |f|^s \phi |^2 \phi^2 \eta + \frac{2}{2s-1} |f|^{2s} |\nabla \phi|^2 \eta . \]

This leads to

\[ \frac{1}{2s} \partial_t (|f|^{2s}(x,t) \phi^2(x) \eta(t)) \leq \frac{2s-1}{2s^2} |\nabla |f|^s \phi |^2 \phi^2 \eta + \frac{1}{2s} |f|^{2s} \phi^2 \eta \]

\[ \leq - \text{div}(\nabla f f|f|^{2s-2} \phi^2 \eta) + (2e^2)^{2s-1} \frac{1}{2s} |g|^{2s} \phi^2 \eta \]

\[ + \frac{2}{2s-1} |f|^{2s} (|\nabla \phi|^2 \eta + \phi^2 |d_t \eta|) . \]

(3.12)

For notational ease we relabel the domain as \( \Omega \times [0, T] \) and assume that \( z_0 = (0,0) \in \Omega \times [0, T] \) and \( 0 < R^2 + \rho^2 < T \). As always \( P_R := P_R(0) \).

(i) Set \( p = 2s \). After multiplying (3.12) with \((\frac{1}{2s})^{p-1}\) and integrating, we get, for \( p \geq 2, \)

\[ \sup_{t \geq -R^2 - \rho^2} \int_{B_{R^2}} \left( \frac{1}{e^2} \right)^{p-1} |f|^{p}(x,t) \phi^2(x) \eta(t) dx \]

\[ + \int_{P_R^0} \left( \frac{1}{e^2} \right)^{p-1} |\nabla f|^{2p-2} \phi^2 \eta dz \]

\[ + \int_{P_R^0} \left( \frac{1}{e^2} \right)^{p} |f|^{p} \phi^2 \eta dz \]

\[ \leq C(p) \left( \int_{P_R^0} |g|^{p} \phi^2 \eta dz + \left( \frac{1}{e^2} \right)^{p-1} \int_{P_R^0} |f|^{p} (|\nabla \phi|^2 \eta + \phi^2 |d_t \eta|) dz \right) . \]

(3.13)

In particular we have

\[ \int_{P_R^0} \left( \frac{1}{e^2} \right)^{p} |f|^{p} dz \leq C(p) \left( \int_{P_R^0} |g|^{p} dz + \frac{2}{e^2} C \int_{P_R^0} \left( \frac{1}{e^2} \right)^{p} f^{p} dz \right) . \]

(3.14)

Let \( p = \frac{k}{2} + 1 \) for \( k \in \mathbb{N} \). Hölder’s inequality for \( q_1 = (2p - 1)/(2p - 2) \) and \( q_2 = 2p - 1 \) implies

\[ \int_{P_R^0} \left( \frac{1}{e^2} |f| \right)^{p-1} |f| dz \leq \left( \frac{1}{e^2} \right)^{p-1} \left( \frac{1}{e^2} \right)^{p-1} \|f\|_{L^{(p-1)/2}} \|f\|_{L^{2p-1}} . \]

Now (3.13) leads to

\[ \|f\|_{L^{p}(P_R^0)} \leq C(p) \left( \|g\|_{L^{p}(P_R^0)} + \frac{C}{\rho^2} \int_{P_R^0} \left( \frac{1}{e^2} \right)^{p} f^{p} dz \right) . \]

An iteration combined either to (3.11) from Lemma 3.8 after the \( k \)-th step yields an estimate of the form

\[ \|f\|_{L^{p}(P_R^0)} \leq C(\|g\|_{L^{p}(P_R^0)}^{(p-1)/4}, \|f\|_{L^{2p-1}(P_R^0)}^{(p-1)/2}, p, \rho, R) . \]

(3.15)

If we set \( \rho := \frac{\delta R}{k+1} \) and insert (3.15) into (3.14), we obtain for any \( \delta > 0 \)

\[ \|f\|_{L^{p}(P_R^0)} \leq C(p) \|g\|_{L^{p}(P_R^0)}^{(p-1)/4} + \frac{C}{\rho^2} \int_{P_R^0} \left( \frac{1}{e^2} \right)^{p} f^{p} dz , \]

(3.16)
where
\[ C_2 = C_2(\|g\|_{L^p(P^{\alpha}_{(1+\delta)R})}, \|f\|_{L^2(P^{\alpha}_{(1+\delta)R})}, p, \delta, R). \]

Claim (i) follows by setting \( R_{\text{new}} = (1 + \delta)R \) and \( \delta_{\text{new}} = \frac{1}{1+\delta} \), i.e. \( \delta_{\text{new}}R_{\text{new}} = R \).

(ii) By applying the estimate
\[ \left( \int_{[a,b]} \int_{B_R} |u|^4 dxdt \right)^{\frac{1}{4}} \leq C \left( \max_{t \in [a,b]} \int_{B_R} |u|^2 dxdt + \int_{[a,b]} \int_{B_R} |\nabla u|^2 dxdt \right) \]
(see Theorem 6.9 p.110 in [23]) to \( u := f^{\eta/2} \phi \sqrt{\eta} \), we find that the expression
\[ \left( \frac{1}{\epsilon^2} \right)^{p-1} \left( \int_{P^\delta_R} |f|^{2p} \phi^4 \eta^2 dz \right)^{\frac{1}{p}} + \int_{P^\delta_R} \frac{1}{\epsilon^2} \left| f \right|^{p/2} \eta^{2} dz \] (3.17)
admits the same bound as (3.13) with a different constant \( C(p) > 0 \). By combining (3.17) with (3.13), we see that the same bounds as in (3.16) also holds for
\[ \left( \frac{1}{\epsilon^2} \right)^{p-1} \left( \sup_{-R^2 < t < 0} \int_{B^\rho_R} |f|^{p} (x, t) dx + \int_{P^\delta_R} \left| \nabla f \right|^{p/2} dz \right) \]
and
\[ \left( \frac{1}{\epsilon^2} \right)^{p-1} \left( \int_{P^\delta_R} |f|^{2p} dz \right)^{\frac{1}{p}}. \]

\[ \square \]

3.3.2. Higher estimates. By considering the flow equation (1.1) and equation (3.8) for \( \rho \epsilon := 1 - |u|^2 \) as a coupled system, the uniform estimates from the previous paragraph and parabolic estimates for (1.1) can be combined in a standard bootstrap argument to prove Lemma 3.1 and 3.3.

In the case of the harmonic map flow \( d_\epsilon := \text{dist}(u_\epsilon, N) \) replaces \( \rho \epsilon \). The corresponding equation is then
\[ \partial_t d_\epsilon - \Delta d_\epsilon + |\nabla u|^2 d_\epsilon + \frac{1}{\epsilon^2} \chi'(d^2_\epsilon) d_\epsilon = \Delta v_\epsilon \cdot \nu_\epsilon \leq C|\nabla u|^2, \]
whenever \( u_\epsilon \in U \). Here we decomposed \( u_\epsilon = v_\epsilon + d_\epsilon \nu_\epsilon \), where \( v_\epsilon := \pi_N(u_\epsilon) \) and \( \nu_\epsilon := \nu(u_\epsilon) \) is the unit normal in \( (T_{\pi_N(u_\epsilon)N})^\perp \), whereas \( d_\epsilon := \text{dist}(u_\epsilon, N) \). Remember that \( \pi_N : U \to N \) denotes the nearest neighbour projection from a tubular neighbourhood \( U \subset \mathbb{R}^n \) of \( N \) onto \( N \).

4. Towards characterizing the limits

We start with alternative characterisations of the “regular set” \( \text{Reg}\{u_\epsilon\} \). Remember that by Lemma 2.1, we have for any \( 0 \leq s < t \)
\[ G_\epsilon(u_\epsilon(t), B^\Omega_R(x_0)) \leq G_\epsilon(u_\epsilon(s), B^\Omega_R(x_0)) + \frac{C(t-s)E_0}{\gamma t R^2}. \]
(4.1)

Set \( \delta_0 := \frac{\min_{x \in \partial N} \text{dist}(x, N)}{C E_0} \), where \( \epsilon_0 \) is the constant from Theorem 3.3. After increasing \( C \) if necessary, we may assume \( 0 < \delta_0 < 1 \).

Lemma 4.1. Let \( u_\epsilon \) be a solution of (1.1)-(1.2) with \( u_0 \in H^{1,2}(\Omega; S^2) \cap C^2(\partial \Omega; S^2) \) for each \( \epsilon > 0 \). Then the following assertions are equivalent:

(i) \( x_0 = (x_0, t_0) \in \text{Reg}\{u_\epsilon\} \).
(ii) \( \exists \delta, R > 0 : \limsup_{\epsilon \searrow 0} \sup_{t_0-\delta < t < t_0} G_\epsilon(u_\epsilon(t), B^\Omega_R(x_0)) < \epsilon_0. \)
(iii) \( \exists \delta > 0 : \lim_{R \searrow 0} \limsup_{\epsilon \searrow 0} \sup_{t_0-\delta < t < t_0} G_\epsilon(u_\epsilon(t), B^\Omega_R(x_0)) = 0. \)
On the other hand by (4.1), we have for 

\[(iv) \exists R > 0 : \limsup_{\epsilon \to 0} \frac{1}{R^2} \int_{t_0 - R^2}^{t_0} \int_{R^2} g_\epsilon(u_\epsilon) \, dx \, dt < \frac{1}{4} \delta_0 \epsilon_0.\]

\[(v) \exists R > 0 : \limsup_{\epsilon \to 0} \sup_{t_0 - \delta < t < t_0 + \delta} G_\epsilon(u_\epsilon(t), B^0_R(x_0)) < \epsilon_0.\]

Proof. “(i) ⇔ (ii)” is obvious.

“(ii) ⇒ (iii)” follows from Theorem 3.4 in Section 3.2.2

“(iii) ⇒ (iv)” is obvious.

“(iv) ⇒ (ii)”: Assume (iv) holds. By (4.1) and the above choice of \( \delta_0 \), we have for sufficiently small \( \epsilon > 0 \),

\[
\sup_{t_0 - (1/2)\delta_0 R^2 < t < t_0} G_\epsilon(u_\epsilon(t), B^0_R(x_0))
\]

\[
\leq \inf_{t_0 - \delta_0 R^2 < s < t_0 - (1/2)\delta_0 R^2} G_\epsilon(u_\epsilon(s), B^0_R(x_0)) + \frac{C\delta_0 R^2 E_0}{\gamma_1 R^2}
\]

\[
\leq \frac{2}{\delta_0 R^2} \int_{t_0 - (1/2)\delta_0 R^2}^{t_0} G_\epsilon(u_\epsilon(t), B^0_R(x_0)) \, dt + \frac{1}{2} \epsilon_0
\]

\[
< \frac{2}{\delta_0} \frac{1}{4} \delta_0 \epsilon_0 + \frac{1}{2} \epsilon_0 < \epsilon_0.
\]

“(v) ⇒ (ii)” is obvious.

“(iii) ⇒ (v)”: Assume (iii) holds. Then there are \( R, \delta > 0 \), such that

\[
\limsup_{\epsilon \to 0} \sup_{t_0 - \delta \leq t \leq t_0} \int_{B_R(x_0) \cap \Omega} g_\epsilon(u_\epsilon(x,t)) \, dx < \epsilon_0/2.
\]

On the other hand by (4.1), we have for \( \delta_{\text{new}} := \frac{\epsilon_0^2 \gamma_1 R^2}{2C E_0} = \delta_0 R^2 \),

\[
\sup_{t_0 \leq t \leq t_0 + \delta_{\text{new}}} \int_{B^{1/2}_R(x_0) \cap \Omega} g_\epsilon(u_\epsilon(x,t)) \, dx \leq \int_{B_R(x_0) \cap \Omega} g_\epsilon(u_\epsilon(x,t_0)) \, dx + \frac{\delta_{\text{new}} CE_0}{\gamma_1 R^2}.
\]

Now (v) holds for \( \frac{1}{2} R \) and min\{\( \delta, \delta_{\text{new}} \)\}. \( \square \)

**Corollary 4.2.** Let \( u_\epsilon \) be a solution of (1.1) - (1.2) with \( u_0 \) in \( H^{1,2}(\Omega; S^2) \cap C^2(\partial \Omega; S^2) \) for each \( \epsilon > 0 \). Let \( \{ \epsilon_i \} \) be a sequence with \( \epsilon_i \searrow 0 \) as \( i \to \infty \). Then the following holds:

(i) \( \text{Reg}(\{u_\epsilon\}) \) and \( \text{Reg}(\{u_\epsilon, \epsilon_i\}) \) are open in \( \overline{\Omega} \times \mathbb{R}_+ \).

(ii) There is some \( T_0 > 0 \), such that \( \overline{\Omega} \times [0, T_0] \subset \text{Reg}(\{u_\epsilon\}) \).

**Proof.** (i) follows from Lemma 4.1 (v).

(ii) The existence of \( T_0 \) immediately follows from Lemma 2.1 [2.3]. \( \square \)

Set

\[ Q_R(z) := B_R(x) \times [t - R^2, t + R^2] \quad \text{for} \quad z = (x,t). \]

and let \( S^2 \) denote the 2-dimensional parabolic Hausdorff measure.

**Proposition 4.3.** Let \( u_\epsilon \) be a solution of (1.1) - (1.2) with \( u_0 \) in \( H^{1,2}(\Omega; S^2) \cap C^2(\partial \Omega; S^2) \) for each \( \epsilon > 0 \). Then the following holds:

(i) \( S(\{u_\epsilon\}) \) has locally finite two dimensional parabolic Hausdorff-measure. More precisely there is a constant \( K_1 = K_1(E_0, \epsilon_0) > 0 \), such that for any compact interval \( I \subset \mathbb{R}_+ \)

\[ S^2(\{u_\epsilon\} \cap (\overline{\Omega} \times I)) \leq K_1 |I|. \]

(ii) There is a constant \( K_2 = K_2(E_0, \epsilon_0) > 0 \), such that for any \( t > 0 \) the set \( S^1(\{u_\epsilon\} \cap (\overline{\Omega} \times \{t\})) \) consists of at most \( K_2 \) points.
Proof: (i) By (iv) of Lemma 4.1, we have for any \( z_0 = (x_0, t_0) \in S(\{u_\epsilon\}_\epsilon) \), any \( R > 0 \) and sufficiently small \( 0 < \epsilon \leq \epsilon(z_0) \)

\[
\frac{1}{R^2} \int_{t_0}^{t_0+R^2} \int_{B_R^2(x_0)} g_\epsilon(u_\epsilon) \, dx \, dt \geq \frac{1}{4} \delta_0 \epsilon_0 \, .
\]

(4.2)

Fix a compact interval \( I \subset \mathbb{R}^+ \) and \( \delta > 0 \). By compactness and Vitali’s Covering Theorem any covering of \( S(\{u_\epsilon\}_\epsilon) \cap (\Omega \times I) \) by parabolic cylinders \( Q^0_R(z) \) with \( 0 < R^2 < \delta \) and \( z \in S(\{u_\epsilon\}_\epsilon) \cap (\Omega \times I) \) contains a finite covering \( \bigcup_j Q^0_R(z_j) \supseteq S(\{u_\epsilon\}_\epsilon) \cap (\Omega \times I) \), such that the cylinders \( Q^0_R(z_j) \) are pairwise disjoint. By (4.2) and the energy estimate, we obtain for \( 0 < \epsilon \leq \min_j \{ \epsilon(z_j) \} \)

\[
\sum_j \omega_2(5R_j)^2 \leq 25 \frac{4\omega_2}{\delta_0 \epsilon_0} \sum_j \int_{t_j-R^2}^{t_j} \int_{B^2_{R_j}(x_j)} g_\epsilon(u_\epsilon) \, dx \, dt < \frac{100\omega_2}{\delta_0 \epsilon_0} (|I| + \delta) E_0 \, .
\]

By letting \( \delta \searrow 0 \), we find

\[
\mathcal{F}_2^I(S(\{u_\epsilon\}_\epsilon) \cap I) \leq \frac{100\omega_2 E_0}{\delta_0 \epsilon_0} |I| \, .
\]

(ii) Pick any \((x_1, T), \ldots, (x_k, T) \in S(\{u_\epsilon\}_\epsilon)\). By assumption we have for

\[
\forall R, \gamma > 0, \exists \epsilon \in [0, \gamma]: \sup_{T - \delta < t < T} \int_{B^2_R(x)} g_\epsilon(u_\epsilon(x, t)) \, dx \geq \frac{\epsilon_0}{2} \text{ for } 1 \leq l \leq k \, .
\]

We may choose \( R > 0 \), such that the \( B^2_R(x_l) (1 \leq l \leq k) \) are pairwise disjoint. Choose \( \delta \in [0, \frac{\sqrt{2\epsilon_0 R^2}}{4CE_0}] \), where \( C \) is the constant from Lemma 2.1 and \( \epsilon \in [0, \gamma] \) as above. Since \( t \mapsto \int_{B^2_R(x)} g_\epsilon(u_\epsilon(x, t)) \, dx \) is continuous, we may find \( t^l_0 \in ]T - \delta, T[ \) such that

\[
\int_{B^2_R(x_l)} g_\epsilon(u_\epsilon(x, t^l_0)) \, dx \geq \frac{\epsilon_0}{2} \text{ for } 1 \leq l \leq k \, .
\]

The energy estimate and the local energy inequality, Lemma 2.1 and (2.2) now imply

\[
E_0 \geq \sum_{l=1}^{k} \int_{B^2_R(x_l)} g_\epsilon(u_\epsilon(x, T - \delta)) \, dx \geq \sum_{l=1}^{k} \left( \int_{B^2_R(x_l)} g_\epsilon(u_\epsilon(x, t^l_0)) \, dx - \frac{C}{\gamma_1 R^2} \int_{T - \delta}^{T} \int_{B^2_R(x_l)} |\nabla u_\epsilon(x, t)|^2 \, dx \, dt \right) \, .
\]

Thus \( E_0 \geq k \left( \frac{\epsilon_0}{2} - \frac{C\epsilon_0}{\gamma_1 R^2} \delta \right) \). Now since \( \delta < \frac{R^2 \epsilon_0}{4CE_0} \), this implies \( k \leq \frac{8E_0}{\epsilon_0} =: K_2 \).

(Compare [32] and [31] (1°) of the proof of Theorem 6.6 p.229 for a similar argument in the case of the harmonic map flow.)

\[ \square \]

**Theorem 4.4.** Let \( u_\epsilon \) be a solution of [1.1]-[1.2] with \( u_0 \) in \( H^{1,2}(\Omega; S^2) \cap H^{3/2,2}(\partial \Omega; S^2) \) for each \( \epsilon > 0 \). Then the following holds:

There is at least one sequence \( \{\epsilon_i\}_i \), with \( \epsilon_i \to 0 \) as \( i \to \infty \) and

\[
u_* \in H^{1,2}_{loc}(\Omega \times \mathbb{R}^+; S^2) \cap L^∞(\mathbb{R}^+; H^{1,2}(\Omega; S^2)),
\]

such that \( u_{\epsilon_i} \rightarrow u_* \) weakly in \( H^{1,2}_{loc}(\Omega \times \mathbb{R}^+; S^2) \) and weak* in \( L^∞(\mathbb{R}^+; H^{1,2}(\Omega; S^2)) \).

In addition: (i) For any such sequence \( \{u_{\epsilon_i}\}_i \), we have

\[
\lim_{i \to \infty} u_{\epsilon_i} = u_* \text{ in } C^∞(\text{Reg}(\{u_{\epsilon_i}\}_i) \cap (\Omega \times \mathbb{R}^+; S^2)).
\]
and \( \frac{1}{\epsilon}
abla |u_\epsilon|^2 \to |\nabla u_*|^2 \) in \( C_\infty(\text{Reg}(\{u_\epsilon\}) \cap (\Omega \times \mathbb{R}_+)) \).

(ii) \( u_* \) is a smooth solution of \( (1.3) \) in \( \text{Reg}(\{u_\epsilon\}) \cap (\Omega \times \mathbb{R}_+) \) and a distributional solution in

\[
H^1_\text{loc}(\overline{\Omega} \times \mathbb{R}_+) \cap L^\infty(\mathbb{R}_+; H^{1,2}(\Omega; \mathbb{R}^n))
\]
on all \( \Omega \times \mathbb{R}_+ \). Further \( \lim_{t \to 0} u_\epsilon(\cdot, t) = u_0 \) in \( H^{1,2}(\Omega; \mathbb{R}^3) \) and

\[
u_\epsilon(\cdot, t)|_{\partial \Omega} = u_0|_{\partial \Omega}
\]
as a \( H^{2,2}(\Omega; \mathbb{R}^3) \)-trace for a.e. \( t > 0 \).

(iii) If \( u_* \) is regular at \( z_0 = (x_0, t_0) \in \overline{\Omega} \times \mathbb{R}_+ \) in the sense that

\[
\lim_{R \searrow 0} \sup_{R^2 \leq t \leq t_0} \int_{B_R(x_0)} |\nabla u_*|^2 \, dx = 0
\]
and if \( z_0 \) is parabolically isolated for \( \{u_\epsilon\}_\epsilon \), i.e.

\[ B_{R_0}(x_0) \times [t_0 - R_0^2, t_0] \subset \text{Reg}(\{u_\epsilon\}) \text{ for some } R_0 > 0, \]

then \( z_0 \in \text{Reg}(\{u_\epsilon\}) \). In particular, \( u_* \) cannot (backward) concentrate energy and (backward) bubble at \( z_0 \) as \( t \searrow t_0 \). Compare [17, 18].

Proof. (i) The convergence statements follow from the energy estimate (Lemma 2.1, Theorem 3.2 and Lemma 3.1). (ii) For the case \( \gamma_2 = 0 \) and \( f(u_\epsilon) = \frac{1}{2} \frac{d}{d\alpha} \chi(\text{dist}^2(u_\epsilon, N)) \), this is proven in [36] p.95. We will prove it in the case \( \gamma_2 \neq 0 \). If we apply \( "u_\epsilon \times \gamma_1" \) from the left to \( (1.1) \) and pass to the limit \( \epsilon_1 \to 0 \) on \( \text{Reg}(\{u_\epsilon\}) \cap (\Omega \times \mathbb{R}_+) \), we obtain

\[
\gamma_1 u_* \times \partial_t u_* - \gamma_2 u_* \times (u_* \times \partial_t u_*) - u_* \Delta u_* = 0. \tag{4.3}
\]

Since \( (1 - |u_\epsilon|^2) \to 0 \) smoothly, we also have

\[
|u_\epsilon(x, t)| = 1 \quad \text{in } \text{Reg}(\{u_\epsilon\}) \cap (\Omega \times \mathbb{R}_+).
\]

Now we use \( a \times (b \times c) = (ac)b - (ab)c \) and \( |a| \equiv 1 \) while applying \( "u_\epsilon \times \gamma_1" \) from the left to \( (4.3) \), to obtain

\[
\gamma_1 \partial_t u_* - \gamma_2 u_* \times \partial_t u_* - \Delta u_* = |\nabla u_*|^2 u_* \quad \text{in } \Omega \times \mathbb{R}_+. \tag{4.4}
\]

In particular, since the left side of \( (1.1) \) converges to the left side of \( (4.4) \), we have

\[
\frac{1}{\epsilon_1^2} (1 - |u_\epsilon|^2) \to |\nabla u_*|^2 \text{ in } C_\infty(\text{Reg}(\{u_\epsilon\}) \cap (\Omega \times \mathbb{R}_+)).
\]

We now prove that \( u_* \) is a distributional \( H^1_\text{loc} \cap L^\infty(H^{1,2}) \)-solution of \( (1.3) \) on all \( \Omega \times \mathbb{R}_+ \). Note that the sequence \( \{u_\epsilon\}_\epsilon \) converges weakly in \( H^{1,2}(\Omega \times \mathbb{R}_+; S^2) \) and smoothly on \( \text{Reg}(\{u_\epsilon\}) \cap (\Omega \times \mathbb{R}_+) \). Further since \( S^2(\{u_\epsilon\}) := S(\{u_\epsilon\}) \cap (\Omega \times \{t\}) \) is finite for all \( t \geq 0 \), we have both \( u_\epsilon \to u_* \) pointwise a.e. in \( \Omega \times \mathbb{R}_+ \) and \( u_\epsilon(\cdot, t) \to u_\epsilon(\cdot, t) \) pointwise a.e. in \( \Omega \) for all \( t \in \mathbb{R}_+ \). Since \( \int_0^\infty \int_{\Omega} |\partial_t u_*|^2 \, dx \, dt \leq E_0 \), by Fatou’s Lemma the complement of

\[
A := \{ t \geq 0 : \liminf_{\epsilon_0 \to 0} \int_\Omega |\partial_t u_\epsilon|^2(\cdot, t) \, dx < \infty \}
\]
has measure 0. Pick \( t_0 \in A \). Then there is a subsequence still denoted by \( u_\epsilon \), such that \( \partial_t u_\epsilon(\cdot, t_0) \to \partial_t u_\epsilon(\cdot, t_0) \) weakly in \( L^2(\Omega; \mathbb{R}^3) \). By the local energy estimate, we may assume that, for the same subsequence, we also have \( u_\epsilon(\cdot, t_0) \to u_\epsilon(\cdot, t_0) \) weakly in \( H^{1,2}(\Omega; S^2) \). By pointwise a.e. uniqueness of the limit, the whole sequence converges. Also

\[
u_\epsilon(\cdot, t_0) \in H^{1,2}(\Omega; S^2) \quad \text{and} \quad \partial_t u_\epsilon(\cdot, t_0) \in L^2(\Omega; \mathbb{R}^3) \quad \text{for all } t_0 \in A.
\]
follows from $E$ for any $\phi$ and $R$ integrable on $[0, \infty)$. Therefore we may multiply the equation with $\phi \in C_c^\infty([0, \infty[)$ and so $\nabla u_*(x,t) \phi(x)$ has support in $Reg \{u_0\}$. After passing to the limit $k \to \infty$, we find that for any $t \geq 0$,

$$
\int_\Omega \gamma_1 \partial_t u_*(x,t) \phi(x) - \gamma_2 (u_* \times \partial_t u_*) \phi(x) + \nabla u_*(x,t) \nabla \phi(x) dx
$$

$$
= \int_\Omega (|\nabla u_*|^2 u_*)(x,t) \phi(x) dx.
$$

This equation holds for a.e. $t \geq 0$. On the other hand, we have $u_* \in H^{1,2}(\Omega \times [0, T]; S^2)$ for any $T > 0$ and so both sides of the above equation are locally integrable on $\mathbb{R}_+$. Therefore we may multiply the equation with $\psi \in C_c^\infty([0, \infty[)$ and integrate over $\mathbb{R}_+$. Moreover linear combinations $\sum_k a_k \phi_k(x) \psi_k(t)$ with $\phi_k \in C_c^\infty(\Omega)$ and $\psi_k \in C_c^\infty([0, \infty[)$ are dense in $C_c^\infty(\Omega \times [0, \infty[)$ and so

$$
\int_0^\infty \int_\Omega \gamma_1 \partial_t u_*(x,t) \phi(x,t) - \gamma_2 (u_* \times \partial_t u_*) \phi(x,t) + \nabla u_*(x,t) \nabla \phi(x,t) dx dt
$$

$$
= \int_0^\infty \int_\Omega (|\nabla u_*|^2 u_*)(x,t) \phi(x,t) dx dt,
$$

for any $\phi \in C_c^\infty(\Omega \times [0, \infty[)$. Finally $\lim_{t \to 0} u_*(.,t) = u_0$ in $H^{1,2}(\Omega; S^2)$ immediately follows from $E(u_*(.,t)) \leq E(u_0)$, since we have weak convergence as $t \to 0$.

(iii): By assumption there is $R > 0$, such that

$$
\sup_{t_0 - R^2 \leq t \leq t_0} \int_{B_R(x_0) \cap \Omega} \frac{1}{2} |\nabla u_*(x,t)|^2 dx < \frac{\epsilon_0}{4}.
$$

Set $\delta := \min \{ R^2, \frac{2kR^2e_0}{2C_0} \}$ and $s_0 := t_0 - \frac{1}{2} \delta$. We may assume we have for the same $R > 0$ $P_2R^1(\Omega) \setminus \{z_0\} \subset Reg \{u_0\}$. Then Theorem 3.2 and 3.4 and Lemma 3.1 and 3.3 imply

$$
\lim_{i \to \infty} \int_{B_R(x_0) \cap \Omega} g_k(u_{e_i}(x,s_0)) dx = \int_{B_R(x_0) \cap \Omega} \frac{1}{2} |\nabla u_*(x,s_0)|^2 dx < \frac{\epsilon_0}{4}.
$$
Now by Lemma 2.1 (2.2), we have
\[
\sup_{s_0 \leq t \leq s_0 + \delta} \int_{B_R(x_0) \cap \Omega} g_{\epsilon_i}(u_{\epsilon_i}(x,t)) \, dx \leq \int_{B_R(x_0) \cap \Omega} g_{\epsilon_i}(u_{\epsilon_i}(x,s_0)) \, dx + \frac{\delta C E_0}{\gamma_1 R^2} < \frac{\epsilon_0}{2} + \frac{\epsilon_0}{2},
\]
for \( i \) sufficiently large. Since by construction \( t_0 \in [s_0, s_0 + \delta] \), the claim follows. \( \square \)

By Theorem 4.4, Corollary 4.2 (ii) and uniqueness of smooth solutions, we obtain the following.

**Remark 4.5.** There is \( T_0 > 0 \), such that
\[
\lim_{\epsilon \to 0} u_\epsilon = u_* \quad \text{in} \quad C^\infty(\Omega \times [0, T_0]; S^2),
\]
where \( u_* \) is the unique smooth solution of (1.3) with initial and boundary data \( u_0 \).
(Compare [17].)

If the energy of a (sub-)limit \( u_* \) was everywhere decreasing, A.Freire’s uniqueness result [13] would imply that \( u_* \) is (globally) the Struwe-solution. However all we can say about the energy of sublimits \( u_* \) is the following Lemma 4.6. In particular extension \( u_* \) after the maximal smooth existence time \( T_0 \) with backward bubbling cannot be excluded. (See [18].)

**Lemma 4.6.** Let \( u_\epsilon \) be a solution of (1.1) - (1.2) for fixed \( \epsilon > 0 \) and assume \( u_* = \text{weak-}H^{1,2}, \lim_{i \to \infty} u_{\epsilon_i} \) for a sequence \( 0 < \epsilon_i \searrow 0 \). If \( s < t \) and \( S^s(\{u_{\epsilon_i}\}) := (\Omega \times \{s\}) \cap S(\{u_{\epsilon_i}\}_i) = \emptyset \) and \( S^t(\{u_{\epsilon_i}\}) \neq \emptyset \), then
\[
\int_{\Omega} \frac{1}{2} \nabla u_*^2(x,s) \, dx \geq \int_{\Omega} \frac{1}{2} \nabla u_*^2(x,\tau) \, dx \quad \forall \tau > s,
\]
and
\[
\int_{\Omega} \frac{1}{2} \nabla u_*^2(x,s) \, dx \geq \int_{\Omega} \frac{1}{2} \nabla u_*^2(x,t) \, dx + \epsilon_0,
\]
where \( \epsilon_0 > 0 \) is the constant from Theorem 3.4.

**Proof.** Set \( \overline{x} := (x_1, \ldots, x_K) \) if \( S^t(\{u_{\epsilon_i}\}) = \{x_1, \ldots, x_K\} \) and \( B_R(\overline{x}) := \bigcup_{j=1}^K B_R(x_j) \). Then
\[
E(u_*(s), \Omega) := \int_{\Omega} \frac{1}{2} |\nabla u_*|^2(x,s) \, dx
\]
= \( \lim_{i \to \infty} \int_{\Omega} g_{\epsilon_i}(u_{\epsilon_i})(x,s) \, dx \)
\[
\geq \limsup_{i \to \infty} \int_{\Omega} g_{\epsilon_i}(u_{\epsilon_i})(x,\tau) \, dx \quad \forall \tau > s \quad \text{(by Lemma 2.1)}
\]
\[
\geq \int_{\Omega} \frac{1}{2} |\nabla u_*|^2(x,\tau) \, dx \quad \forall \tau > s \quad \text{(by weak lower semi-continuity)}
\]
Also
\[
E(u_*(s), \Omega) \geq \limsup_{i \to \infty} \left( \int_{\Omega \setminus B_R(\overline{x})} g_{\epsilon_i}(u_{\epsilon_i})(x,\tau) \, dx + \int_{B_R(\overline{x})} g_{\epsilon_i}(u_{\epsilon_i})(x,\tau) \, dx \right),
\]
for all \( \tau \in [s, t] \). Now for any \( \delta \in [0, 1] \) and \( R > 0 \), there are sequences \( s < t_i \not\nearrow t \) and \( 0 < \delta_i \searrow 0 \), such that

\[
\int_{B_R(\tau)} g_{\delta_i}(u_{\tau_i})(x, t_i) \, dx = \sup_{\delta_i < \tau < t} \int_{B_R(\tau)} g_{\delta_i}(u_{\tau_i})(x, \tau) \, dx \geq \delta \varepsilon_0
\]

and so

\[
E(u_\tau(s)) \geq \limsup_{i \to \infty} \int_{\Omega \setminus B_R(\tau)} g_{\delta_i}(u_{\tau_i})(x, t_i) \, dx + \delta \varepsilon_0
\]

\[
\geq \int_{\Omega \setminus B_R(\tau)} \frac{1}{2} | \nabla u_\tau |^2(x, t) \, dx + \delta \varepsilon_0 \quad \forall R > 0, \quad \delta \in [0, 1].
\]

Since the last inequality holds for any \( R > 0 \) and \( \delta \in [0, 1] \), the claim follows. \( \square \)

Theorem 4.4 provides an alternative version of the construction of the “Struwe-solution” (see [17]).

**Corollary 4.7.** Let \( u_0 \in H^{1,2}(\Omega; S^2) \). Then there is a global distributional solution \( u \in H^{1,2}_{\text{loc}}(\Omega \times ]0, \infty[; S^2) \cap L^\infty(\Omega \times ]0, \infty[; H^{1,2}(S^2)) \) with \( \partial_t u \in L^2(\Omega \times ]0, \infty[; \mathbb{R}^3) \) of (1.3) with initial and boundary data \( u_0 \), which is smooth on \( \Omega \times ]0, \infty[ \) except at finitely many points and has decreasing and right continuous energy. If in addition \( u_0 \in H^{3/2,2}(\partial\Omega; S^2) \), then \( u \) is unique among the solutions of (1.3) with initial and boundary data \( u_0 \) which are smooth except for isolated singular points and with \( \lim_{t \searrow 0} E(u(t)) < E(u(s)) + \varepsilon_0 \) for all \( s \geq 0 \). (It is also unique among the \( H^{1,2}_{\text{loc}} \)-solutions with decreasing energy by Freire’s result.)

**Proof.** By Theorem 4.4 the \( \varepsilon \)-approximation scheme provides a smooth short time solution

\[
u \in C^\infty(\Omega \times [0, T_0]; S^2)
\]

to (1.3) with boundary data \( u_0 \) and \( \lim_{t \searrow T_0} u(\cdot, t) = u_0 \) in \( H^{1,2}(\Omega; \mathbb{R}^3) \). Also there are \( \{x_1, \ldots, x_K\} \subset \Omega \), such that

\[
\lim_{t_i \nearrow T_0} u(\cdot, t_i) = u(\cdot, T_0) \quad \text{in} \quad C^\infty(\Omega \setminus \{x_1, \ldots, x_K\}, \mathbb{R}^3)
\]

and

\[
\| \nabla u(\cdot, T_0) \|_{L^2(\Omega)}^2 \leq \liminf_{t_i \nearrow T_0} \| \nabla u(\cdot, t_i) \|_{L^2(\Omega)}^2 \leq 2E_0.
\]

In particular \( u(\cdot, T_0) \in H^{1,2}(\Omega) \). If we now set \( \tilde{u}_0 := u(\cdot, T_0) \) and repeat the same procedure with \( \tilde{u}_0 \) instead of \( u_0 \), we obtain step by step a global solution with point singularities. To see that \( \partial_t u \in L^2(\Omega \times \mathbb{R}^+; \mathbb{R}^3) \), we sum up the energy inequalities of each time interval \( [t_k, t_{k+1}] \) on which \( u \) is regular and use that the energy is right continuous, whereas

\[
\limsup_{t_i \nearrow t_{k+1}} E(u(t_i)) \geq E(u(t_{k+1})) + \varepsilon_0
\]

by Lemma 4.6. This yields

\[
\int_0^\infty \int_\Omega | \partial_t u |^2 \, dx \, dt \leq E_0 - \sum_k \varepsilon_0
\]

and also shows that there can only be finitely many “singular times” \( t_k \).

Now assume we have two solutions \( u_1 \) and \( u_2 \) of (1.3) with initial and boundary data \( u_0 \) and both with finitely many point singularities and \( \lim_{t \searrow 0} E(u(t)) < E(u(s)) + \varepsilon_0 \) for all \( s \geq 0 \). By Remark 4.5 we have \( u_1 = u_2 \) on \( \Omega \times [0, T_1] \), where
$T_1$ is the maximal common smooth existence time, i.e. either $u_1$ or $u_2$ has point singularities at $T_1$. However by Corollary 4 on the existence of smooth extensions in [17], if $u_1$ admits a smooth extension up to $T_1$, then so does $u_2$ and conversely. Moreover, since the criterion for the existence of a smooth extension is local, both solutions have the same singularities $x_1, \ldots, x_K$ at time $T_1$ and $u_1(\cdot, T_1) = u_2(\cdot, T_1)$ on $\Omega \setminus \{x_1, \ldots, x_K\}$. By Theorem 6 in [17], and the assumption on the energy, the extension of $u_1$ and $u_2$ after $T_1$ is again unique “for a short time” and an iteration of the previous argument leads to the claimed uniqueness. □

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**REFERENCES**


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