Electronic Journal of Differential Equations, Vol. 2004(2004), No. 91, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

SOLUTION MATCHING FOR A THREE-POINT BOUNDARY-VALUE PROBLEM ON A TIME SCALE

MARTIN EGGENSPERGER, ERIC R. KAUFMANN, NICKOLAI KOSMATOV

Abstract. Let $\mathbb{T}$ be a time scale such that $t_1, t_2, t_3 \in \mathbb{T}$. We show the existence of a unique solution for the three-point boundary value problem

$$y^{\Delta\Delta\Delta}(t) = f(t, y(t), y^{\Delta}(t), y^{\Delta\Delta}(t)), \quad t \in [t_1, t_3] \cap \mathbb{T},$$
$$y(t_1) = y_1, \quad y(t_2) = y_2, \quad y(t_3) = y_3 \,.$$

We do this by matching a solution to the first equation satisfying a two-point boundary conditions on $[t_1, t_2] \cap \mathbb{T}$ with a solution satisfying a two-point boundary conditions on $[t_2, t_3] \cap \mathbb{T}$.

1. Introduction

Bailey, Shampine and Waltman [2] were the first to use solution matching techniques to obtain solutions of two-point boundary value problems for the second order equation $y'' = f(x, y, y')$ by matching solutions of initial value problems. Since then, many authors have used this technique on three-point boundary value problems on an interval $[a, c]$ for an n^{th} order differential equation by piecing together solutions of two-point boundary value problems on $[a, b]$, where $b \in (a, c)$ is fixed, with solutions of two-point boundary value problems on $[b, c]$; see for example, Barr and Sherman [3], Das and Lalli [6], Henderson [7, 8], Henderson and Taunton [9], Lakshmikantham and Murty [12], Moorti and Garner [13], and Rao, Murty and Rao [14].

All the above cited works considered boundary value problems for differential equations. In this work, we will use the solution matching technique to obtain a solution to a three-point boundary value problem for a Δ-differential equation on a time scale. The theory of time scales was introduced by Stephan Hilger, [10], as a means of unifying theories of differential equations and difference equations. Three excellent sources about dynamic systems on time scales are the books by Bohner and Peterson [4], Bohner and Peterson [5], and Kaymakcalan *et. al.*, [11]. The definitions below can be found in [4].

A *time scale* $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$. For $t < \sup \mathbb{T}$ and $r > \inf \mathbb{T}$, we define the *forward jump operator*, σ, and the *backward jump operator*, ρ,

2000 *Mathematics Subject Classification.* 34B10, 34B15, 34G20.
Key words and phrases. Time scale; boundary-value problem; solution matching.

Submitted May 14, 2004. Published July 8, 2004.

respectively, by

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\} \in \mathbb{T},$$
$$\rho(r) = \sup\{\tau \in \mathbb{T} : \tau < r\} \in \mathbb{T}.$$

If $\sigma(t) > t$, t is said to be *right scattered*, and if $\sigma(t) = t$, t is said to be *right dense*. If $\rho(t) < t$, t is said to be *left scattered*, and if $\rho(t) = t$, t is said to be *left dense*.

If $\mathbb{T}$ has a left-scattered maximum at M, then we define $\mathbb{T}^\kappa = \mathbb{T}\backslash\{M\}$. Otherwise we define $\mathbb{T}^\kappa = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum at m, then we define $\mathbb{T}_\kappa = \mathbb{T} \setminus \{m\}$. Otherwise we define $\mathbb{T}_\kappa = \mathbb{T}$.

We say that the function x has a *generalized zero (g.z.)* at t if $x(t) = 0$ or if $x(\sigma(t)) \cdot x(t) < 0$. In the latter case, we would say the generalized zero is in the real interval $(t, \sigma(t))$.

For $x : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$, (assume t is not left scattered if $t = \sup \mathbb{T}$), we define the *delta derivative* of $x(t)$, $x^\Delta(t)$, to be the number (when it exists), with the property that, for each $\varepsilon > 0$, there is a neighborhood, U, of t such that

$$\left|x(\sigma(t)) - x(s) - x^\Delta(t)(\sigma(t) - s)\right| \le \varepsilon|\sigma(t) - s|,$$

for all $s \in U$.

For $x : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$, (assume t is not right scattered if $t = \inf \mathbb{T}$), we define the *nabla derivative* of $x(t)$, $x^\nabla(t)$, to be the number (when it exists), with the property that, for each $\varepsilon > 0$, there is a neighborhood, U, of t such that

$$\left|x(\rho(t)) - x(s) - x^\nabla(t)(\rho(t) - s)\right| \le \varepsilon|\rho(t) - s|,$$

for all $s \in U$.

Remarks: If $\mathbb{T} = \mathbb{R}$, then $x^\Delta(t) = x^\nabla(t) = x'(t)$. If $\mathbb{T} = \mathbb{Z}$, then $x^\Delta(t) = x(t+1) - x(t)$ is the forward difference operator while $x^\nabla(t) = x(t) - x(t-1)$ is the backward difference operator.

Let $\mathbb{T}$ be a time scale such that $t_1, t_2, t_3 \in \mathbb{T}$. We consider the existence of solutions of the three-point boundary value problem

$$y^{\Delta\Delta\Delta}(t) = f(t, y(t), y^\Delta(t), y^{\Delta\Delta}(t)),\ t \in (t_1, t_3) \cap \mathbb{T}, \tag{1.1}$$

$$y(t_1) = y_1,\ y(t_2) = y_2,\ y(t_3) = y_3. \tag{1.2}$$

We obtain solutions by matching a solution of (1.1) satisfying two-point boundary conditions on $[t_1, t_2] \cap \mathbb{T}$ to a solution of (1.1) satisfying two-point boundary conditions on $[t_2, t_3] \cap \mathbb{T}$. In particular, we will give sufficient conditions such that if $y_1(t)$ is the solution of (1.1) satisfying the boundary conditions $y(t_1) = y_1, y(t_2) = y_2, y^{\Delta^j}(t_2) = m$, ($j = 1$ or 2) and $y_2(t)$ is $y(t_2) = y_2, y^{\Delta^j}(t_2) = m$, $y(t_3) = y_3$, (using the same j), then the solution of (1.1), (1.2) is

$$y(t) = \begin{cases} y_1(t), & t \in [t_1, t_2] \cap \mathbb{T}, \\ y_2(t), & t \in [t_2, t_3] \cap \mathbb{T}\,. \end{cases}$$

We will assume that $f : \mathbb{T} \times \mathbb{R}^3 \to \mathbb{R}$ is continuous and that solutions of initial value problems for (1.1) exist and are unique on $[t_1, t_3] \cap \mathbb{T}$. Moreover, we require that $t_2 \in \mathbb{T}$ is dense and fixed throughout. In addition to these hypotheses, we suppose that there exists a function $g : \mathbb{T} \times \mathbb{R}^3 \to \mathbb{R}$ such that:

(A) For each $v_3, u_3 \in \mathbb{R}$ the function f satisfies

$$f(t, v_1, v_2, v_3) - f(t, u_1, u_2, u_3) > g(t, v_1 - u_1, v_2 - u_2, v_3 - u_3)$$

when $t \in (t_1, t_2] \cap \mathbb{T}$, $u_1 - v_1 \geq 0$, and $u_2 - v_2 < 0$, or when $t \in [t_2, t_3) \cap \mathbb{T}$, $u_1 - v_1 \leq 0$, and $u_2 - v_2 < 0$

(B) There exists $\varepsilon_1 > 0$ such that, for each $0 < \varepsilon < \varepsilon_1$, the initial value problem

$$y^{\Delta\Delta\Delta}(t) = g(t, y(t), y^{\Delta}(t), y^{\Delta\Delta}(t)), \quad t \in [t_1, t_3] \cap \mathbb{T},$$
$$y(t_2) = 0, \quad y^{\Delta\Delta}(t_2) = 0, \quad y^{\Delta}(t_2) = \varepsilon,$$

has a solution z such that z^{Δ} does not change sign on $[t_1, t_3] \cap \mathbb{T}$

(C) There exists $\varepsilon_2 > 0$ such that, for each $0 < \varepsilon < \varepsilon_2$, the initial value problem

$$y^{\Delta\Delta\Delta}(t) = g(t, y(t), y^{\Delta}(t), y^{\Delta\Delta}(t)), \quad t \in [t_1, t_3] \cap \mathbb{T},$$
$$y(t_2) = 0, \quad y^{\Delta}(t_2) = 0, \quad y^{\Delta\Delta}(t_2) = \varepsilon(-\varepsilon)$$

has a solution z on $[t_2, t_3] \cap \mathbb{T}, ([t_1, t_2] \cap \mathbb{T})$, such that $z^{\Delta\Delta}$ does not change sign on $[t_2, t_3] \cap \mathbb{T}$, $([t_1, t_2] \cap \mathbb{T})$

(D) For each $w \in \mathbb{R}$, the function g satisfies $g(t, v_1, v_2, w) \geq g(t, u_1, u_2, w)$ when $t \in (t_1, t_2] \cap \mathbb{T}$, $u_1 - v_1 \geq 0$ and $v_2 > u_2 \geq 0$, or when $t \in [t_2, t_3) \cap \mathbb{T}$, $u_1 - v_1 \leq 0$ and $v_2 > u_2 \geq 0$

We will need also the following two theorems due to Atici and Guseinov, (Theorems 2.5 and 2.6 in [1, pg. 79]).

Theorem 1.1. *If $f : \mathbb{T} \to \mathbb{C}$ is Δ-differentiable on $\mathbb{T}^{\kappa}$ and if f^{Δ} is continuous on $\mathbb{T}^{\kappa}$, then f is ∇-differentiable on $\mathbb{T}_{\kappa}$ and*

$$f^{\nabla}(t) = f^{\Delta}(\rho(t))$$

for all $t \in \mathbb{T}_{\kappa}$.

Theorem 1.2. *If $f : \mathbb{T} \to \mathbb{C}$ is ∇-differentiable on $\mathbb{T}_{\kappa}$ and if f^{∇} is continuous on $\mathbb{T}_{\kappa}$, then f is Δ-differentiable on $\mathbb{T}^{\kappa}$ and*

$$f^{\Delta}(t) = f^{\nabla}(\sigma(t))$$

for all $t \in \mathbb{T}^{\kappa}$.

2. Existence and Uniqueness of Solutions

Consider the boundary conditions,

$$y(t_1) = y_1, \quad y(t_2) = y_2, \quad y^{\Delta^j}(t_2) = m \tag{2.1}$$

for $j = 1, 2$, and

$$y(t_2) = y_2, \quad y^{\Delta^j}(t_2) = m, \quad y(t_3) = y_3, \tag{2.2}$$

for $j = 1, 2$, where $y_1, y_2, y_3, m \in \mathbb{R}$. In this section, the solution of (1.1), (2.1), $(j = 1, 2)$ is matched with the solution of (1.1), (2.2), $(j = 1, 2)$ to obtain a unique solution of (1.1), (1.2). Our first theorem states that solutions of (1.1), (2.1), $j = 1, 2$, and (1.1), (2.2), $j = 1, 2$, are unique.

Theorem 2.1. *Let $y_1, y_2, y_3 \in \mathbb{R}$, and assume that conditions (A) through (D) are satisfied. Then, given $m \in \mathbb{R}$, each of the boundary value problems* (1.1),(2.1), *$j = 1, 2$, and* (1.1)(2.2), *$j = 1, 2$, has at most one solution.*

Proof. We will consider only the proof for (1.1), (2.1) with with $j = 1$; the arguments for the other cases is similar.

Let us assume that there are distinct solutions α and β of (1.1), (2.1) (with $j = 1$). Define $w \equiv \alpha - \beta$. Then $w(t_1) = w(t_2) = w^{\Delta}(t_2) = 0$. By uniqueness of solutions of initial value problems for (1.1) we know that $w^{\Delta\Delta}(t_2) \neq 0$. Without loss of generality, we let $w^{\Delta\Delta}(t_2) < 0$.

Since $w(t_1) = 0$ and since t_2 is dense, there exists an $r_1 \in (t_1, t_2) \cap \mathbb{T}$ such that $w^{\Delta\Delta}(t)$ has a g.z. at r_1, $w^{\Delta}(t) > 0$ on $[r_1, t_2) \cap \mathbb{T}$, $w(t) < 0$ on $(r_1, t_2] \cap \mathbb{T}$, and $w^{\Delta\Delta}(t) < 0$ on $[r_1, t_2) \cap \mathbb{T}$. From the definition of a generalized zero, we have either $w^{\Delta\Delta}(r_1) = 0$ or $w^{\Delta\Delta}(r_1) \cdot w^{\Delta\Delta}(\sigma(r_1)) < 0$. If r_1 is right dense, then $w^{\Delta\Delta}(r_1) = 0$. If r_1 is right scattered and $w^{\Delta\Delta}(r_1) \neq 0$, then $w^{\Delta\Delta}(r_1) \cdot w^{\Delta\Delta}(\sigma(r_1)) < 0$. Since $w^{\Delta\Delta}(t) < 0$ on $(r_1, t_2] \cap \mathbb{T}$, $w^{\Delta\Delta}(r_1) > 0$. Thus $w^{\Delta\Delta}(r_1) \geq 0$.

Now let $0 < \varepsilon < \frac{1}{2}\min\{\varepsilon_2, -w^{\Delta\Delta}(t_2)\}$ and let z_ε satisfy the criteria of hypothesis (C) relative to the interval $[t_1, t_2] \cap \mathbb{T}$; that is

$$z_\varepsilon^{\Delta\Delta\Delta}(t) = g(t, z_\varepsilon(t), z_\varepsilon^{\Delta}(t), z_\varepsilon^{\Delta\Delta}(t)), \quad t \in [t_1, t_3] \cap \mathbb{T},$$
$$z_\varepsilon(t_2) = z_\varepsilon^{\Delta}(t_2) = 0, \quad z_\varepsilon^{\Delta\Delta}(t_2) = -\varepsilon$$

and $z_\varepsilon^{\Delta\Delta}$ does not change sign in $[t_1, t_2] \cap \mathbb{T}$.

Set $Z \equiv w - z_\varepsilon$. Then $Z(t_2) = Z^{\Delta}(t_2) = 0$, and $Z^{\Delta\Delta}(t_2) < 0$. Moreover, $Z^{\Delta\Delta}(r_1) = w^{\Delta\Delta}(r_1) - z_\varepsilon^{\Delta\Delta}(r_1) > 0$, and $Z^{\Delta\Delta}(t_2) < 0$ imply that there exists an $r_2 \in [r_1, t_2) \cap \mathbb{T}$ such that $Z^{\Delta\Delta}$ has a g.z. at r_2 and $Z^{\Delta\Delta}(t) < 0$ on $(r_2, t_2] \cap \mathbb{T}$. As above, since $Z^{\Delta\Delta}$ has a g.z. at r_2, $Z^{\Delta\Delta}(r_2) \geq 0$. Also, $Z^{\Delta}(t) > 0$ and $Z(t) < 0$ on $[r_2, t_2) \cap \mathbb{T}$.

When $\sigma(r_2) > r_2$,

$$Z^{\Delta\Delta\Delta}(r_2) = \frac{Z^{\Delta\Delta}(\sigma(r_2)) - Z^{\Delta\Delta}(r_2)}{\sigma(r_2) - r_2} < 0\,.$$

When $\sigma(r_2) = r_2$,

$$Z^{\Delta\Delta\Delta}(r_2) = \lim_{t \to r_2^+} \frac{Z^{\Delta\Delta}(t)}{t - r_2} < 0\,.$$

Regardless of wether r_2 is right dense or right scattered we have, from the definition of the delta derivative, that $Z^{\Delta\Delta\Delta}(r_2) < 0$.

From conditions (A) and (D) we have

$$\begin{aligned} Z^{\Delta\Delta\Delta}(r_2) &= w^{\Delta\Delta\Delta}(r_2) - z_\varepsilon^{\Delta\Delta\Delta}(r_2) \\ &> g(r_2, w(r_2), w^{\Delta}(r_2), w^{\Delta\Delta}(r_2)) - g(r_2, z_\varepsilon(r_2), z_\varepsilon^{\Delta}(r_2), z_\varepsilon^{\Delta\Delta}(r_2)) \\ &\geq 0. \end{aligned}$$

That is, $Z^{\Delta\Delta\Delta}(r_2) > 0$, which is a contradiction. Our assumption must be wrong and consequently (1.1) (2.1) has at most one solution. □

Theorem 2.2. *Assume that hypotheses (A) through (D) are satisfied. Then* (1.1), (1.2) *has at most one solution.*

Proof. Assume that there exist two distinct solutions α and β of (1.1), (1.2). Define $w = \alpha - \beta$. Then $w(t_1) = w(t_2) = w(t_3) = 0$. From Theorem 2.1, $w^{\Delta}(t_2) \neq 0$ and $w^{\Delta\Delta}(t_2) \neq 0$. Without loss of generality let $w^{\Delta}(t_2) = \alpha^{\Delta}(t_2) - \beta^{\Delta}(t_2) > 0$. By Theorem 1.2 we have $w^{\nabla}(t_2) = w^{\Delta}(t_2) > 0$. Then there exist points $r_1 \in (t_1, t_2) \cap \mathbb{T}$ and $r_2 \in (t_2, t_3) \cap \mathbb{T}$ such that w^{Δ} has a g.z. at r_1 and r_2 and $w^{\Delta}(t) > 0$ on $(r_1, r_2) \cap \mathbb{T}$.

Let $\varepsilon = \frac{1}{2}\min\{\varepsilon_1, w^{\Delta}(t_2)\}$ and let z_ε be the solution of the initial value problem $z_\varepsilon^{\Delta\Delta\Delta}(t) = g(t, z_\varepsilon(t), z_\varepsilon^{\Delta}(t), z_\varepsilon^{\Delta\Delta}), t \in [t_1, t_3] \cap \mathbb{T}, z_\varepsilon(t_2) = 0, z^{\Delta}(t_2) = \varepsilon, z_\varepsilon(t_2) = 0$. By condition (B), z_ε^{Δ} does not change sign on $[t_1, t_3] \cap \mathbb{T}$.

Define $Z \equiv w - z_\varepsilon$. Then $Z(t_2) = 0, Z^{\Delta}(t_2) > 0$, and $Z^{\Delta\Delta}(t_2) = w^{\Delta\Delta}(t_2) \neq 0$. There are two cases to consider.

Case 1: $Z^{\Delta\Delta}(t_2) < 0$**.** Recall that w^{Δ} has a g.z. at r_1. If r_1 is right dense, then $w^{\Delta}(r_1) = 0$. If r_1 is right scattered, then either $w^{\Delta}(r_1) = 0$ or $w^{\Delta}(\sigma(r_1)) \cdot w^{\Delta}(r_1) < 0$. In the latter case since $w^{\Delta}(t) > 0$ on $(r_1, r_2) \cap \mathbb{T}$, we have $w^{\Delta}(r_1) < 0$. Regardless of wether r_1 is right dense or right scattered we have $Z^{\Delta}(r_1) = w^{\Delta}(r_1) - z_\varepsilon^{\Delta}(r_1) \leq 0$.

Since $Z^{\Delta}(r_1) \leq 0$ and $Z^{\Delta\Delta}(t_2) < 0$, there exists an $r_3 \in (r_1, t_2] \cap \mathbb{T}$ such that $Z^{\Delta\Delta}$ has a g.z. at r_3 and $Z^{\Delta\Delta}(t) < 0$ on $(r_3, t_2] \cap \mathbb{T}$.

On the one hand, if $\sigma(r_3) > r_3$, then

$$Z^{\Delta\Delta\Delta}(r_3) = \frac{Z^{\Delta\Delta}(\sigma(r_3)) - Z^{\Delta\Delta}(r_3)}{\sigma(r_3) - r_3} < 0\,.$$

If $\sigma(r_3) = r_3$, then

$$Z^{\Delta\Delta\Delta}(r_3) = \lim_{t \to r_3^+} \frac{Z^{\Delta\Delta}(t)}{t - r_3} < 0\,.$$

Regardless of wether r_3 is right dense or right scattered we have, from the definition of the delta derivative, that $Z^{\Delta\Delta\Delta}(r_3) < 0$.

On the other hand, from conditions (A) and (D) we have

$$\begin{aligned} Z^{\Delta\Delta\Delta}(r_3) &= w^{\Delta\Delta\Delta}(r_3) - z_\varepsilon^{\Delta\Delta\Delta}(r_3) \\ &> g(r_3, w(r_3), w^{\Delta}(r_3), w^{\Delta\Delta}(r_3)) - g(r_3, z_\varepsilon(r_3), z_\varepsilon^{\Delta}(r_3), z_\varepsilon^{\Delta\Delta}(r_3)) \\ &\geq 0\,. \end{aligned}$$

That is, conditions (A) and (D) imply that $Z^{\Delta\Delta\Delta}(r_3) > 0$ which is a contradiction. Consequently, $Z^{\Delta\Delta}(t_2) \not< 0$.

Case 2: $Z^{\Delta\Delta}(t_2) > 0$**.** Again, we know that w^{Δ} has a g.z. at r_2. If $\sigma(r_2) = r_2$, then $w^{\Delta}(r_2) = 0$. If $\sigma(r_2) > r_2$, then either $w^{\Delta}(r_2) = 0$ or $w^{\Delta}(r_2) > 0$ and $w^{\Delta}(\sigma(r_2)) < 0$ or $w^{\Delta}(r_2) < 0$ and $w^{\Delta}(\rho(r_2)) > 0$. Consequently, either $Z^{\Delta}(r_2) < 0$ or $Z^{\Delta}(\sigma(r_2)) < 0$.

Since $Z^{\Delta}(r^*) < 0$, (where $r^* = r_2$ or $r^* = \sigma(r_2)$), and since $Z^{\Delta\Delta}(t_2) > 0$, there exists $r_4 \in (t_2, r^*)$ such that $Z^{\Delta\Delta}$ has a g.z. at r_4, $Z^{\Delta\Delta}(t) > 0$ on $[t_2, r_4) \cap \mathbb{T}$, and $Z^{\Delta\Delta}$ does not have a g.z. in $[t_2, r_4) \cap \mathbb{T}$.

We now obtain a contradiction. On the one hand, we can use the definition of the Δ-derivative to calculate $Z^{\Delta\Delta\Delta}(r_4)$. If $\rho(r_4) = r_4$, then by Theorem 1.1 we have

$$Z^{\Delta\Delta\Delta}(r_4) = Z^{\Delta\Delta\nabla}(r_4) = \lim_{t \to r_4^-} \frac{Z^{\Delta\Delta}(t) - 0}{t - r_4} < 0.$$

If $\rho(r_4) < r_4$, then either $\sigma(r_4) = r_4$ or $\sigma(r_4) > r_4$. If $\sigma(r_4) = r_4$, then

$$Z^{\Delta\Delta\Delta}(r_4) = \lim_{t \to r_4^+} \frac{Z^{\Delta\Delta}(t)}{t - r_4} < 0.$$

If $\sigma(r_4) > r_4$, then

$$Z^{\Delta\Delta\Delta}(r_4) = \frac{Z^{\Delta\Delta}(\sigma(r_4)) - Z^{\Delta\Delta}(r_4)}{\sigma(r_4) - r_4} < 0.$$

In any case, we have, by definition of the Δ-derivative, that $Z^{\Delta\Delta\Delta}(r_4) < 0$.

On the other hand, we have from conditions (A) and (D),

$$\begin{aligned} Z^{\Delta\Delta\Delta}(r_4) &= w^{\Delta\Delta\Delta}(r_4) - z_\varepsilon^{\Delta\Delta\Delta}(r_4) \\ &> g(r_4, w(r_4), w^{\Delta}(r_4), w^{\Delta\Delta}(r_4)) - g(r_4, z_\varepsilon(r_4), z_\varepsilon^{\Delta}(r_4), z_\varepsilon^{\Delta\Delta}(r_4)) \\ &\geq 0. \end{aligned}$$

Conditions (A) and (D) imply $Z^{\Delta\Delta\Delta}(r_4) > 0$ which is a contradiction. Thus $Z^{\Delta\Delta}(t_2) \not> 0$.

Since $Z^{\Delta\Delta}(t_2) \neq 0$ and $Z^{\Delta\Delta}(t_2) < 0$ and $Z^{\Delta\Delta}(t_2) > 0$ lead to contradictions, our original assumption must be false. As such, the boundary value problem (1.1), (1.2) has at most one solution and the theorem is proved. □

Now given $m \in \mathbb{R}$, let $\alpha(x,m), \beta(x,m), u(x,m)$ and $v(x,m)$ denote the solutions, when they exist, of the boundary value problems for (1.1),(2.1) and (1.1),(2.2), $j = 1, 2$, respectively.

Theorem 2.3. *Suppose that (A) through (D) are satisfied and that, for each $m \in \mathbb{R}$, there exist solutions of* (1.1), (2.1) *and* (1.1), (2.2), $j = 1, 2$. *Then $u^{\Delta}(t_2, m)$ and $\alpha^{\Delta\Delta}(t_2, m)$ are strictly increasing functions of m whose range is $\mathbb{R}$, and $v^{\Delta}(t_2, m)$ and $\beta^{\Delta\Delta}(t_2, m)$ are strictly decreasing functions of m with ranges all of $\mathbb{R}$.*

Proof. The "strictness" of the conclusion arises from Theorem 2.1. We will prove the theorem with respect to the solution $\alpha(t,m)$. Let $m_1 > m_2$ and let $w(t) \equiv \alpha(t, m_1) - \alpha(t, m_2)$. Then when $w(t_1) = w(t_2) = 0, w^{\Delta}(t_2) > 0$, and $w^{\Delta\Delta}(t_2) \neq 0$.

Assume that $w^{\Delta\Delta}(t_2) < 0$. Then there exists an $r_1 \in (t_1, t_2) \cap \mathbb{T}$ such that w^{Δ} has a g.z. at r_1 and $w^{\Delta}(t) > 0$ on $(r_1, t_2] \cap \mathbb{T}$. By continuity, there exists an $r_2 \in (r_1, t_2) \cap \mathbb{T}$ such that $w^{\Delta\Delta}$ has a g.z. at r_2 and $w^{\Delta\Delta}(t) < 0$ on $(r_2, t_2] \cap \mathbb{T}$. Note that $w(t) < 0$ on $[r_2, t_2) \cap \mathbb{T}$.

Let $0 < \varepsilon < \min\{\varepsilon_2, -w^{\Delta\Delta}(t_2)\}$ and let z_ε be the solution of the initial value problem satisfying conditions of (C), and set $Z \equiv w - z_\varepsilon$. Then $Z(t_2) = 0, Z^{\Delta}(t_2) = w^{\Delta}(t_2) > 0$, and $Z^{\Delta\Delta}(t_2) < 0$. Furthermore $Z^{\Delta\Delta}(r_2) \geq 0$. Thus there exist $r_3 \in (r_2, t_2) \cap \mathbb{T}$ such that $Z^{\Delta\Delta}(r_3) = 0$ and $Z^{\Delta\Delta}(t) < 0$ on $(r_3, t_2]$. Then $Z^{\Delta}(t) > 0$ and $Z(T) < 0$ on $[r_3, t_2)$. As in the proofs of Theorems 2.1 and 2.2, we can then argue that $Z^{\Delta\Delta\Delta}(r_3) < 0$ and $Z^{\Delta\Delta\Delta}(r_3) > 0$, which is again a contradiction. Thus $w^{\Delta\Delta}(t_2) > 0$ and consequently, $\alpha^{\Delta\Delta}(t_2, m)$ is strictly increasing as a function of m.

We now show that $\{\alpha^{\Delta\Delta}(t_2, m) \big| m \in \mathbb{R}\} = \mathbb{R}$. Let $k \in \mathbb{R}$ and consider the solution $u(x,k)$ of the (1.1), (2.1) (with $j = 2$) with u as specified above. Consider also the solution $\alpha(x, u^{\Delta}(t_2, k))$, of (1.1), (2.1) (with $j = 1$). Then $\alpha(x, u^{\Delta}(t_2, k))$ and $u(x,k)$ are solutions of (1.1), (2.1). Hence, by Theorem 2.1, $\alpha(x, u^{\Delta}(t_2, k)) \equiv u(x,k)$. Therefore, $\alpha^{\Delta\Delta}(t_2, u^{\Delta}(t_2, k)) = k$ and so $\{\alpha^{\Delta\Delta}(t_2, m) : m \in \mathbb{R}\} = \mathbb{R}$. The other three parts are established in a similar manner and the proof is complete. □

Theorem 2.4. *Assume the hypothesis of Theorem 2.3. Then* (1.1), (1.2) *has a unique solution.*

Proof. By Theorem 2.3, there exists a unique m_0 such that $u^{\Delta}(t_2, m_0) = v^{\Delta}(t_2, m_0)$. Also $u^{\Delta\Delta}(t_2, m_0) = m_0 = v^{\Delta\Delta}(t_2, m_0)$. Then,

$$y(t) = \begin{cases} u(t, m_0) = y_1(t), & t_1 \leq t \leq t_2, \\ v(t, m_0) = y_2(t), & t_2 \leq t \leq t_3, \end{cases}$$

is a solution of (1.1), (1.2). By Theorem 2.2, $y(t)$ is the unique solution. □

References

[1] F. M. Atici and G. Sh. Guseinov, On Green's Functions and Positive Solutions for Boundary Value Problems on Time Scales, *J. Comput. Appl. Math.* **141** (2002), 75-99.
[2] P. Bailey, L. Shampine, and P. Waltman, *Nonlinear Two Point Boundary Value Problems*, Academic Press, New York, 1968.
[3] D. Barr and T. Sherman, Existence and uniqueness of solutions of three-point boundary value problems, *J. Differential Equations* **13**(1973), 197-212.
[4] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales, An introduction with Applications*, Birkhäuser, Boston, 2001.
[5] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.
[6] K. M. Das and B. S. Lalli, Boundary value problems for $y''' = f(x, y, y')$, *J. Math. Anal. Appl.* **81** (1981), 300-307.
[7] J. Henderson, Three-point boundary value problems for ordinary differential equations, *Nonlinear Anal.* **7** (1983), 411-417.
[8] J. Henderson, Solution Matching for boundary value problems for linear equations, *Internat. J. Math. & Math. Sci.* **12** No. 4 (1989), 713-720.
[9] J. Henderson and R. D. Taunton, Solutions of boundary value problems by matching methods, *Appl. Anal.* **49** (1993), No. 3-4, 235-246.
[10] S. Hilger, Analysis on measure chains - a unified approach to continuous and discrete calculus, *Results Math.* **18**(1990), 18-56.
[11] B. Kaymakcalan, V. Lakshmikantham, and S. Sivasundaram, *Dynamical Systems on Measure Chains*, Kluwer Academic Publishers, Boston, 1996.
[12] V. Lakshmikantham and K. N. Murty, Theory of differential inequalities and three-point boundary value problems, *PanAm. Math. J.* **1** (1991), 1-9.
[13] V. R. G. Moorti and J. B. Garner, Existence -uniqueness theorems for three-point boundary value problems for nth-order nonlinear differential equations, *J. Differential Equations* **29** (1978), 205-213.
[14] D.R.K.S. Rao, K.N. Murty, A.S. Rao, On three-point boundary value problems associated with third order differential equations, *Nonlinear Anal.* **5** (1981), 669-673.

Martin Eggensperger
General Studies, Southeast Arkansas College, Pine Bluff, Arkansas, USA
E-mail address: meggensperger@seark.edu

Eric R. Kaufmann
Department of Mathematics and Statistics, University of Arkansas at Little Rock, Little Rock, Arkansas 72204-1099, USA
E-mail address: erkaufmann@ualr.edu

Nickolai Kosmatov
Department of Mathematics and Statistics, University of Arkansas at Little Rock, Little Rock, Arkansas 72204-1099, USA
E-mail address: nxkosmatov@ualr.edu