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AN ASYMPTOTIC PROPERTY OF SOLUTIONS TO LINEAR NONAUTONOMOUS DELAY DIFFERENTIAL EQUATIONS

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Abstract. We study first order linear delay differential equations with variable coefficients and constant delays. Using solutions to a characteristic equation, we show asymptotic properties of solutions to the delay equation. To illustrate the hypothesis of the main theorem, we present an example.

1. Introduction

We study the asymptotic behavior of solutions to the delay differential equation

$$x'(t) = a(t)x(t) + \sum_{j=1}^{k} b_j(t)x(t - \tau_j), \quad \text{for } t \geq 0 \tag{1.1}$$

with initial condition

$$x(t) = \phi(t), \quad \text{for } \min_{1 \leq j \leq k} \{-\tau_j\} \leq t \leq 0, \tag{1.2}$$

where the coefficients a and b_j are continuous real-valued functions on $[0, \infty)$, the delays τ_j are positive real numbers ($j = 1, 2, \dots, k$), k is a positive integer, and ϕ is a given continuous function.

The work on this paper is motivated by the publication of the following interesting results. Driver, Sasser and Slater [6] obtained significant results on asymptotic behavior, non-oscillation, and stability of solutions to first order linear delay differential equations with constant coefficients and one constant delay. Driver [4] obtained similar results for first order linear autonomous delay differential equations with infinitely many distributed delays. The results in [6] have been improved and extended by Philos [11] for first order linear delay differential equations with coefficients that are periodic having a common period, and delays that are constants multiples of this period. These results have been extended and improved by Kordonis, Niyianni and Philos [10] for first order linear autonomous neutral delay differential equations. The results in [10, 11] have been extended and slightly improved by Philos and Purnaras in [12]. There the authors study first order linear neutral delay differential equations with periodic coefficients having a common period and constant delays that are multiples of this period. Graef and Qian [7]

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obtained results closely related to the ones above for first order forced delay differential equations. Driver [5] and Arino and Pituk [1] obtained important results for linear differential systems with small delays.

In the present paper, we define a characteristic equation and then utilize its solution to state asymptotic results for solutions of the delay equation. Also we obtain a non-oscillation result, Remark 2.2. Our main result is stated as Theorem 2.3 and proved in the next section. The limit obtained in Theorem 2.3 is found explicitly when the solution to the characteristic equation is a constant. An application of Theorem 2.3 provides a necessary and sufficient condition for all solutions of (1.1) to be bounded, and a necessary and sufficient condition for all solutions of (1.1) to tend to zero at ∞. The last section contains an example and discussions on the results of the paper.

2. Statement of Results

We shall assume that the delays are positive and denote

$$\tau = \max\{\tau_j : 1 \le j \le k\}, \quad \sigma = \min\{\tau_j : 1 \le j \le k\}\,.$$

Let $C([-\tau,0],\mathbb{R})$ denote the set of continuous real-valued functions on $[-\tau,0]$.

By a solution x to the delay differential equation (1.1), we mean a continuous real-valued function, defined on $[-\tau,\infty)$, which is continuously differentiable on $[0,\infty)$ and satisfies (1.1). It is well-known that for each given $\phi \in C([-\tau,0],\mathbb{R})$, problem (1.1)-(1.2) has a unique solution; see for example [2, 8, 9].

With the delay equation (1.1), we associate the integral equation

$$\begin{gathered}\lambda(t) = a(t) + \sum_{j=1}^{k} b_j(t)\exp\Big[-\int_{t-\tau_j}^{t}\lambda(s)ds\Big], \quad \text{for } t \ge 0\,;\\ \lambda(t) = \lambda_0(t), \quad \text{for } -\tau \le t \le 0\end{gathered} \tag{2.1}$$

which is called the (generalized) *characteristic equation* of (1.1). This equation is obtained when looking for solutions of the form

$$x(t) = \phi(0)\exp\Big[\int_0^t \lambda(s)ds\Big]\,.$$

Note that this solution can not change sign; i.e., $x(t)$ is either positive or negative or identically zero.

By a solution λ to the characteristic equation, we mean a continuous real-valued function, defined on $[-\tau,\infty)$, which satisfies (2.1).

Lemma 2.1. *For each λ_0 in $C([-\tau,0],\mathbb{R})$, the characteristic equation has a unique global solution.*

Proof. Let $u(t) = \exp\big[\int_0^t \lambda(s)ds\big]$ for $t \ge 0$, and $w(t) = \exp\big[\int_0^t \lambda_0(s)ds\big]$ for $-\tau \le t \le 0$. Then using the characteristic equation, for $0 \le t \le \sigma$, we obtain the linear differential equation

$$u'(t) = \lambda(t)u(t) = a(t)u(t) + \sum_{j=1}^{k} b_j(t)w(t-\tau_j)$$

with $u(0) = 1$. The solution to this equation is

$$u(t) = \Big[1 + \int_0^t \sum_{j=1}^k b_j(s) \exp\Big[\int_0^{s-\tau_j} \lambda_0(r)\,dr - \int_0^s a(r)\,dr\Big]\,ds\Big] \exp\Big[\int_0^t a(r)\,dr\Big]$$

which allows defining $\lambda(t) = u'(t)/u(t)$ on $[0, \sigma]$. For the next interval, let $w(t) = u(t)$ on $[-\tau, \sigma]$. Then for $\sigma \le t \le 2\sigma$, we obtain the differential equation

$$u'(t) = a(t)u(t) + \sum_{j=1}^k b_j(t) w(t - \tau_j)$$

whose solution is

$$u(t) = \Big[1 + \int_0^t \sum_{j=1}^k b_j(s) \exp\Big[\int_0^{s-\tau_j} \lambda(r)\,dr - \int_0^s a(r)\,dr\Big]\,ds\Big] \exp\Big[\int_0^t a(r)\,dr\Big] \tag{2.2}$$

which allows defining $\lambda(t)$ on $[\sigma, 2\sigma]$. Proceeding in this manner, we define $\lambda(t)$ for all $t \ge -\tau$, which completes the proof. $\square$

Remark 2.2. If the solution to (1.1)-(1.2) does not have zeros on some interval $[t^* - \tau, t^*]$, then the solution does not have zeros on $[t^*, \infty)$; i.e., the solution can not change sign on $[t^*, \infty)$. To show this claim let t^* be the initial time for the characteristic equation and $\lambda_0(t)$ be given implicitly by $x(t) = x(t^*) \exp\big[\int_{t^*}^t \lambda_0(s)\,ds\big]$, with $t^* - \tau \le t \le t^*$. Then, by the uniqueness of solutions to (1.1),

$$x(t) = x(t^*) \exp\Big[\int_{t^*}^t \lambda(s)\,ds\Big], \quad \text{for } t \ge t^*\,.$$

Therefore, $x(t)$ can not have zeros on $[t^*, \infty)$.

Our main result is the following theorem.

Theorem 2.3. *Assume that*

$$\sup_{t \ge \tau} \sum_{j=1}^k |b_j(t)| \tau_j \exp\Big[-\int_{t-\tau_j}^t \lambda(s) ds\Big] < 1\,. \tag{2.3}$$

Then for each solution x of (1.1)-(1.2) there exists a constant L_{ϕ,λ_0} such that

$$\lim_{t\to\infty} x(t) \exp\Big[-\int_0^t \lambda(s) ds\Big] = L_{\phi,\lambda_0}\,,$$

and

$$\lim_{t\to\infty} \Big\{x(t) \exp\Big[-\int_0^t \lambda(s) ds\Big]\Big\}' = 0.$$

Remark 2.4. Under the conditions of Theorem 2.3, a solution to (1.1) can not grow faster than the exponential function determined by the characteristic equation; i.e., there exists a constant M such that

$$\big|x(t)\big| \le M \exp\Big[\int_0^t \lambda(s) ds\Big], \quad \text{for } t \ge 0\,.$$

Remark 2.5. When the solution to (2.1) is a constant λ_0 satisfying (2.3),

$$\lim_{t\to\infty} x(t) \exp(-t\lambda_0) = L_{\phi,\lambda_0}\,.$$

In particular when zero is the solution to (2.1), $\lim_{t\to\infty} x(t) = L_{\phi,0}$.

Note that if λ is a solution of (2.1), then

$$x(t) = \phi(0) \exp\Big[\int_0^t \lambda(s)ds\Big]$$

is a solution of (1.1) with initial function $\phi(t) = \phi(0)\exp\big[\int_0^t \lambda(s)ds\big]$. Then we obtain easily the following results.

Remark 2.6. Under the assumptions of Theorem 2.3, we have:

(1) Every solution of (1.1) is bounded if and only if $\limsup_{t\to\infty} \int_0^t \lambda(s)ds < \infty$.
(2) Every solution of (1.1) tends to zero at ∞ if and only if $\lim_{t\to\infty} \int_0^t \lambda(s)ds = -\infty$.

3. Proof of main result

Proof of Theorem 2.3. For solutions x of (1.1)-(1.2) and λ of (2.1), we define

$$y(t) = x(t) \exp\Big[-\int_0^t \lambda(s)ds\Big], \quad t \ge -\tau.$$

Differentiating in this function, and using (1.1), (2.1), we obtain

$$\begin{aligned} &y'(t) \\ &= \Big(x'(t) - x(t)\lambda(t)\Big) \exp\Big[-\int_0^t \lambda(s)ds\Big] \\ &= \Big(\sum_{j=1}^k b_j(t)x(t-\tau_j) - x(t)\sum_{j=1}^k b_j(t)\exp\Big[-\int_{t-\tau_j}^t \lambda(s)ds\Big]\Big) \exp\Big[-\int_0^t \lambda(s)ds\Big]. \end{aligned}$$

Using that $x(t-\tau_j) = y(t-\tau_j)\exp\big[\int_0^{t-\tau_j} \lambda(s)ds\big]$, the above equality yields

$$y'(t) = -\sum_{j=1}^k b_j(t)[y(t) - y(t-\tau_j)] \exp\Big[-\int_{t-\tau_j}^t \lambda(s)ds\Big] \quad \text{for } t \ge 0\,.$$

From this equation and the fundamental theorem of calculus,

$$y'(t) = -\sum_{j=1}^k b_j(t)\Big[\int_{t-\tau_j}^t y'(s)ds\Big] \exp\Big[-\int_{t-\tau_j}^t \lambda(s)ds\Big] \quad \text{for } t \ge \tau\,. \tag{3.1}$$

If all b_j's are identically zero on $[\tau,\infty)$, from (3.1), $y' = 0$ and y is constant on $[\tau,\infty)$ which would complete the proof. Therefore, we assume that at least one b_j is not identically zero on $[\tau,\infty)$. Let

$$\mu_{\lambda_0} = \sup_{t\ge\tau} \sum_{j=1}^k |b_j(t)|\tau_j \exp\Big[-\int_{t-\tau_j}^t \lambda(s)ds\Big]\,.$$

Then, by (2.3),

$$0 < \mu_{\lambda_0} < 1\,. \tag{3.2}$$

Note that the maximum of $|y'|$ on $[0,\tau]$ depends on x and λ; hence, on the initial functions ϕ and λ_0. Let

$$M_{\phi,\lambda_0} = \max\big\{|y'(t)| : 0 \le t \le \tau\big\}\,.$$

We shall show that M_{ϕ,λ_0} is also a bound of $|y'|$ on the whole interval $[0,\infty)$; i.e.,

$$|y'(t)| \le M_{\phi,\lambda_0} \quad \text{for all } t \ge 0\,. \tag{3.3}$$

On the contrary, assume that there exist $\epsilon > 0$ and $t \ge 0$ such that $|y'(t)| > M_{\phi,\lambda_0} + \epsilon$. Since $|y'(t)| \le M_{\phi,\lambda_0}$ for $0 \le t \le \tau$, by the continuity of y', there exists $t^* > \tau$ such that

$$|y'(t)| < M_{\phi,\lambda_0} + \epsilon\,, \quad \text{for } 0 \le t < t^*, \quad \text{and} \quad |y'(t^*)| = M_{\phi,\lambda_0} + \epsilon\,.$$

Using the definition of μ_{λ_0}, (3.1) and (3.2), we obtain

$$\begin{aligned}
M_{\phi,\lambda_0} + \epsilon &= |y'(t^*)| \\
&\le \sum_{j=1}^{k} |b_j(t^*)| \Big[\int_{t^*-\tau_j}^{t^*} |y'(s)| ds \Big] \exp\Big[- \int_{t^*-\tau_j}^{t^*} \lambda(s) ds \Big] \\
&\le \big(M_{\phi,\lambda_0} + \epsilon\big) \sum_{j=1}^{k} |b_j(t^*)| \tau_j \exp\Big[- \int_{t^*-\tau_j}^{t^*} \lambda(s) ds \Big] \\
&\le \big(M_{\phi,\lambda_0} + \epsilon\big)(\mu_{\lambda_0}) < M_{\phi,\lambda_0} + \epsilon
\end{aligned}$$

which is a contradiction. Therefore, inequality (3.3) holds. If $M_{\phi,\lambda_0} = 0$, from (3.3) it follows that $y' = 0$ and y is constant on $[0, \infty)$, which would complete the proof. Therefore, we assume that $M_{\phi,\lambda_0} > 0$.

In view of (3.1) and (3.3),

$$\begin{aligned}
|y'(t)| &\le \sum_{j=1}^{k} |b_j(t)| \Big[\int_{t-\tau_j}^{t} |y'(s)| ds \Big] \exp\Big[- \int_{t-\tau_j}^{t} \lambda(s) ds \Big] \\
&\le M_{\phi,\lambda_0} \sum_{j=1}^{k} |b_j(t)| \tau_j \exp\Big[- \int_{t-\tau_j}^{t} \lambda(s) ds \Big] \\
&\le M_{\phi,\lambda_0}(\mu_{\lambda_0}) \quad \text{for } t \ge \tau\,.
\end{aligned}$$

Using this inequality, we can show by induction that

$$|y'(t)| \le M_{\phi,\lambda_0}(\mu_{\lambda_0})^n \quad \text{for } t \ge n\tau \quad (n = 0, 1, \dots). \tag{3.4}$$

For an arbitrary $t \ge 0$, we set $n = \lfloor t/\tau \rfloor$ (the greatest integer less than or equal to t/τ). Then $t \ge n\tau$ and $\frac{t}{\tau} - 1 < n$. Thus, by (3.2) and (3.4),

$$|y'(t)| \le M_{\phi,\lambda_0}(\mu_{\lambda_0})^n \le M_{\phi,\lambda_0}(\mu_{\lambda_0})^{\frac{t}{\tau}-1}. \tag{3.5}$$

As $t \to \infty$, we have $n \to \infty$ and, by (3.2), $(\mu_{\lambda_0})^n \to 0$. Therefore, by (3.5),

$$\lim_{t\to\infty} y'(t) = 0$$

which proves the second limit in Theorem 2.3.

To prove that $\lim_{t\to\infty} y(t)$ exists (as a real number), we use the Cauchy convergence criterion. For $t > T \ge 0$, from (3.5), we have

$$\begin{aligned}
|y(t) - y(T)| &\le \int_T^t |y'(s)| ds \le \int_T^t M_{\phi,\lambda_0}(\mu_{\lambda_0})^{\frac{s}{\tau}-1}\, ds \\
&= M_{\phi,\lambda_0} \frac{\tau}{\ln(\mu_{\lambda_0})} \Big[(\mu_{\lambda_0})^{\frac{s}{\tau}-1} \Big]_{s=T}^{s=t} \\
&= M_{\phi,\lambda_0} \frac{\tau}{\ln(\mu_{\lambda_0})} \Big[(\mu_{\lambda_0})^{\frac{t}{\tau}-1} - (\mu_{\lambda_0})^{\frac{T}{\tau}-1} \Big]\,.
\end{aligned}$$

As $T \to \infty$, we have $t \to \infty$ and, by (3.2), the two right-most terms above approach zero. Therefore, $\lim_{T\to\infty} |y(t) - y(T)| = 0$ which by the Cauchy convergence criterion implies the existence of $\lim_{t\to\infty} y(t)$. We call this limit L_{ϕ,λ_0} because it depends on y which in turn depends on the initial functions ϕ and λ_0. This shows the first limit in Theorem 2.3 and completes the proof. $\square$

4. Discussion

To illustrate the hypothesis in Theorem 2.3, we provide an example of a non-autonomous (and non-periodic) delay differential equation of the form (1.1), for which (2.1) has a explicit solution and satisfies (2.3).

Example. Let $k = 1$, $\tau_1 = 2$, and $a(t) = 1/(2(t+3))$, $b_1(t) = 1/(2(t+1))$ for $t \geq 0$. It is easy to verify that

$$\lambda(t) = \frac{1}{t+3}$$

is a solution of (2.1) and satisfies (2.3). Indeed, we can easily check that

$$\sup_{t\geq\tau_1} |b_1(t)|\tau_1 \exp\Big[-\int_{t-\tau_1}^{t} \lambda(s)ds\Big] = \sup_{t\geq 2} \frac{1}{t+3} = \frac{1}{5} < 1.$$

Remark 4.1. Finding conditions on a and b_j that guarantee hypothesis (2.3) remains an open question. To imply this hypothesis, we can use for example the stronger condition

$$\sup_{t\geq\tau} \sum_{j=1}^{k} |b_j(t)| \exp\Big[-\int_{t-\tau_j}^{t} \lambda(s)ds\Big] < \frac{1}{\tau}\,.$$

Furthermore, if, for each $t \geq 0$, it holds $b_j(t) \geq 0$ for all j's or $b_j(t) \leq 0$ for all j's, from the characteristic equation, it follows that

$$|\lambda(t) - a(t)| = \sum_{j=1}^{k} |b_j(t)| \exp\Big[-\int_{t-\tau_j}^{t} \lambda(s)ds\Big], \quad \text{for } t \geq 0\,.$$

Note that this equality is obvious when $k = 1$. Under the above assumptions, the condition (2.3) is implied by $\sup_{t\geq 0} |\lambda(t) - a(t)| < 1/\tau$. This is the strategy in the next lemma.

Lemma 4.2. *Assume that the coefficients a, b_j and the initial function of the characteristic equation satisfy the following conditions for all $t \geq 0$: $b_j(t) \geq 0$ and, for some c with $0 \leq c < \frac{1}{\tau}$,*

$$\begin{aligned}
&\sum_{j\in J(t)} b_j(t) \exp\Big[\int_0^{t-\tau_j} \lambda_0(s)\,ds - \int_0^t a(s)\,ds\Big] \\
&+ \sum_{j\notin J(t)} b_j(t) \exp\Big[\int_0^{t-\tau_j} c\,ds - \int_{t-\tau_j}^t a(s)\,ds\Big] \leq c,
\end{aligned} \tag{4.1}$$

where $J(t)$ consists of those indices j for which $t - \tau_j \leq 0$, $(j = 1, 2, \dots k)$. Then

$$\sup_{t\geq 0} \sum_{j=1}^{k} |b_j(t)|\tau_j \exp\Big[-\int_{t-\tau_j}^{t} \lambda(s)ds\Big] < 1\,,$$

which implies the hypothesis for Theorem 2.3.

Proof. Since $b_j(t) \geq 0$, the definitions of τ and of λ imply
$$\sum_{j=1}^{k} b_j(t)\tau_j \exp\Big[-\int_{t-\tau_j}^{t} \lambda(s)ds\Big] \leq \tau \sum_{j=1}^{k} b_j(t) \exp\Big[-\int_{t-\tau_j}^{t} \lambda(s)ds\Big] = \tau|\lambda(t) - a(t)|\,.$$
The statement of this lemma follows if we show that $|\lambda(t) - a(t)| \leq c$ for $t \geq 0$. As in the proof of Lemma 2.1, let $u(t) = \exp\big[\int_0^t \lambda(s)\,ds\big]$ for $t \geq -\tau$, with λ defined by (2.1). From the characteristic equation,
$$\lambda(t) - a(t) = \sum_{j=1}^{k} b_j(t) \exp\Big[-\int_{t-\tau_j}^{t} \lambda(s)\,ds\Big] = \frac{1}{u(t)} \sum_{j=1}^{k} b_j(t) \exp\Big[\int_0^{t-\tau_j} \lambda(s)\,ds\Big]\,.$$
Since $u(t)$ is the solution given by (2.2), for $t \geq 0$,
$$\lambda(t) - a(t) = \frac{\sum_{j=1}^{k} b_j(t) \exp\big[\int_0^{t-\tau_j} \lambda(s)\,ds\big] \exp\big[-\int_0^t a(s)\,ds\big]}{1 + \int_0^t \sum_{j=1}^{k} b_j(s) \exp\big[\int_0^{s-\tau_j} \lambda(r)\,dr - \int_0^s a(r)\,dr\big]\,ds}\,.$$
Since $b_j(t) \geq 0$, the denominator in the above expression is greater than or equal to 1 and
$$|\lambda(t) - a(t)| \leq \sum_{j=1}^{k} b_j(t) \exp\Big[\int_0^{t-\tau_j} \lambda(s)\,ds - \int_0^t a(s)\,ds\Big]\,. \tag{4.2}$$
When $0 \leq t \leq \sigma$, we have $t - \tau_j \leq 0$, then all j's are in the class $J(t)$ and $t - \tau_j \leq s \leq 0$. So we use $\lambda(s) = \lambda_0(s)$ in (4.2). Therefore, (4.1) implies
$$|\lambda(t) - a(t)| \leq c \quad \text{for all } t \text{ in } [0, \sigma]\,. \tag{4.3}$$
For each fixed t in $[\sigma, 2\sigma]$, we have two possible cases:
Case 1: $j \in J(t)$. Here $t - \tau_j \leq 0$ and $t - \tau_j \leq s \leq 0$; so we use $\lambda(s) = \lambda_0(s)$ in (4.2). Then, for this case, each summand in (4.2) is equal to
$$b_j(t) \exp\Big[\int_0^{t-\tau_j} \lambda_0(s)\,ds - \int_0^t a(s)\,ds\Big]\,.$$
Case 2: $j \notin J(t)$. Here $0 < t - \tau_j \leq \sigma$ and $0 \leq s \leq t - \tau_j \leq \sigma$. Using (4.3), $\lambda(s) \leq a(s) + c$ and, in this case, each summand in (4.2) is bounded by
$$b_j(t) \exp\Big[\int_0^{t-\tau_j} c\,ds - \int_{t-\tau_j}^{t} a(s)\,ds\Big]\,.$$
From the two cases above and (4.1), we have $|\lambda(t) - a(t)| \leq c$ on $[\sigma, 2\sigma]$. Inductively, we can prove the same inequality on $[2\sigma, 3\sigma]$, $[3\sigma, 4\sigma]$, etc. This completes the proof. □

We remark that the class $J(t)$ is non-empty only when $t \leq \tau$. Then the first summation in (4.1) needs to be less than or equal to c only for small t, which is not too restrictive. Meanwhile the class $J(t)$ is empty for $t > \tau$, and the second summation needs to be less than or equal to c for all large t. This is very restrictive. In particular, it requires $\int_{t-\tau_j}^{t} a(s)\,ds \to \infty$ as $t \to \infty$. Note that in the example above a, b_j do not satisfy the conditions of Lemma 4.2.

The real number L_{ϕ,λ_0}, in Theorem 2.3, has been given explicitly in two special cases: For linear autonomous delay differential equations and for linear delay differential equations with periodic coefficients having a common period and constant delays that are multiples of this period. See [4, 6, 7, 11] (and [10, 12] for linear neutral delay differential equations).

The proof of Theorem 2.3 is based on an integral representation of y'. Meanwhile, in the autonomous case, and in the case where the coefficients are periodic with a common period and the delays are multiples of this period, the proof is based on an integral representation of y. See [4, 6, 7, 11] (and [10, 12] for the neutral case).

We would be interested in generalizing our theorem to linear delay differential equations with variable coefficients and variable delays. For variable delays that are bounded, this seems easy to be achieved. However, the general case of variable delays seems to be somewhat difficult. Asymptotic behavior of solutions to differential equations with variable delays and variable coefficients has been studied in [3], using a method that does not use characteristic equations. Furthermore, it would be interesting to generalize our theorem for linear non-autonomous delay differential equations with infinitely many distributed delays. It will be the subject of a future work to extend the present results to linear neutral delay differential equations with variable coefficients and constant delays.

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