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FIXED POINT THEOREM AND ITS APPLICATION TO PERTURBED INTEGRAL EQUATIONS IN MODULAR FUNCTION SPACES

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Abstract. In this paper, we present a modular version of Krasnoselskii's fixed point theorem. Then this result is applied to the existence of solutions to perturbed integral equations in modular function spaces.

1. Introduction

Using the same argument as in [1], we present a modular version of Krasnoselskii's fixed point theorem, result that is well known in Banach spaces. The modular ρ considered here is convex, satisfies the Fatou property, and satisfies the Δ_2-condition. We are interested in the existence of a fixed point for the application $S : B \to B$; where B is a convex, closed, and bounded subset of X_ρ; $S = T + U$ with $T : B \to B$ that satisfies a contraction type hypothesis (see [1]); and $U : B \to B$ is ρ-completely continuous.

Since ρ satisfies the Δ_2-condition, U being ρ-completely continuous is equivalent to the condition U, $\|\cdot\|_\rho$-completely continuous, where $\|\cdot\|_\rho$ is the Luxemburg norm. On the other hand if T is ρ-contraction, then T is not necessarily $\|\cdot\|_\rho$-contraction (see counterexample in [5, page 945, Ex. 2.15]).

We apply our main theorem to the study of solutions to the perturbed integral equation

$$u(t) = \exp(-t) f_0 + \int_0^t \exp(s-t)(T+h)u(s)ds \tag{1.1}$$

in the modular space $C^\varphi = C([0, b], L^\varphi)$, where L^φ is the Musielak-Orlicz space, f_0 is a fixed element in L^φ. Some hypotheses on the operators T and h are stated below. Also, we present an example of this class of equations.

For more details about modular spaces, we refer the reader to the books edited by Musielak [9] and by Kozlowski [6]. Now recall some definitions.

Let X be an arbitrary vector space over K ($K = \mathbb{R}$ or $K = \mathbb{C}$).

(a) A functional $\rho : X \to [0, +\infty]$ is called modular if

- (i) $\rho(x) = 0$ implies $x = 0$.
- (ii) $\rho(-x) = \rho(x)$ for all x in X in the case of X being real. $\rho(e^{it}x) = \rho(x)$ for any real t in the case of X being complex.

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(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.

If in place of (iii) there holds

(iii') $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ for $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$,

then the modular ρ is called convex.

(b) If ρ is a modular in X, then the set $X_\rho = \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}$ is called a modular space.

(c) (i) If ρ is a modular in X, then $|x|_\rho = \inf\{u > 0, \rho(\frac{x}{u}) \leq u\}$ is a F-norm.

(ii) If ρ is a convex modular, then $\|x\|_\rho = Inf\{u > 0, \rho(\frac{x}{u}) \leq 1\}$ is called the Luxemburg norm.

Let X_ρ be a modular space. (a) A sequence $(x_n)_{n\in\mathbb{N}}$ in X_ρ is said to be

(i) ρ-convergent to x, denoted by $x_n \xrightarrow{\rho} x$, if $\rho(x_n - x) \to 0$ as $n \to +\infty$.

(ii) ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n, m \to +\infty$.

(b) X_ρ is ρ-complete if any ρ-Cauchy sequence is ρ-convergent.

(c) A subset B of X_ρ is said to be ρ-closed if for any sequence $(x_n)_{n\in\mathbb{N}} \subset B$, such that $x_n \xrightarrow{\rho} x$, then $x \in B$. Here $\overline{B}^\rho$ denotes the closure of B in the sense of ρ.

We say that the subset A of X_ρ is ρ-bounded if:

$\sup_{x,y\in A} \rho(x - y) < +\infty$, and let the ρ-diameter of A, denoted by $\delta_\rho(A)$, to be

$$\delta_\rho(A) = \sup_{x,y\in A} \rho(x - y).$$

Recall also that ρ has the Fatou property if $\rho(x-y) \leq \liminf \rho(x_n - y_n)$, whenever $x_n \xrightarrow{\rho} x$ and $y_n \xrightarrow{\rho} y$.

We say that ρ satisfies the Δ_2-condition if:

$\rho(2x_n) \to 0$ as $n \to +\infty$ whenever $\rho(x_n) \to 0$ as $n \to +\infty$, for any sequence $(x_n)_{n\in\mathbb{N}}$ in X_ρ.

2. Main result

Theorem 2.1. *Let ρ be a convex modular that satisfies the Δ_2-condition, X_ρ be a ρ-complete modular space and B be a convex, ρ-closed, ρ-bounded subset of X_ρ. Assume that U and T are two applications from B into B such that U is ρ-completely continuous and there exist real numbers $k > 0$, and $c > \max(1, k)$ that satisfy $\rho(c(Tx - Ty)) \leq k\rho(x - y)$ for any x, y in B. And $T(B) + U(B) \subset B$. Then the operator $S = T + U$ has a fixed point.*

Remark 2.2. Since an operator ρ-Lipschitz is not necessarily $\|.\|_\rho$-Lipschitz (see counterexample in [5, page 945, Ex. 2.15]), then the result above gives a modular version of Krasnoselskii's fixed point theorem.

We need the following lemma for proving Theorem 2.1.

Lemma 2.3. *Let ρ be a convex modular and X_ρ be a modular space. If a subset B of X_ρ is ρ-bounded then B is $\|.\|_\rho$-bounded.*

Proof. Suppose that B is not $\|.\|_\rho$-bounded. So there exist sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in B such that $\|x_n - y_n\|_\rho \to +\infty$ as $n \to +\infty$. Hence for any $A > 1$ there exists $N \in \mathbb{N}$, such that if $n > N$, then $\|x_n - y_n\|_\rho > A$ i.e. $\|\frac{x_n - y_n}{A}\|_\rho > 1$ whenever $n > N$. This implies $\rho(\frac{x_n - y_n}{A}) \geq \|\frac{x_n - y_n}{A}\|_\rho > 1$ (see [9, p.8]). Hence, $1 < \rho(\frac{x_n - y_n}{A}) \leq \frac{1}{A}\rho(x_n - y_n)$ whenever $n > N$. So $A < \rho(x_n - y_n)$ for any $n > N$. This shows that B is not ρ-bounded. Hence, the lemma is established. $\square$

Proof of Theorem 2.1. Firstly, we show that the operator $I-T$ is a bijection from B into $U(B)$ (where I is the identity function). Let x in B, and consider the following sequence defined by $y_{n+1} = Ty_n + Ux$, with y_0 a fixed element in B. Then the sequence $(y_n)_{n\in\mathbb{N}}$ is Cauchy. Indeed,

$$\begin{aligned}\rho(y_{n+m} - y_n) &= \rho(Ty_{m+n-1} - Ty_{n-1})\\ &= \rho(\frac{1}{c}(c(Ty_{n+m-1} - Ty_{n-1})))\\ &\le \frac{k}{c}\rho(y_{m+n-1} - y_{n-1}),\end{aligned}$$

by induction, we have

$$\rho(y_{m+n} - y_n) \le (\frac{k}{c})^n \rho(y_m - y_0)$$

and by hypothesis, B is ρ-bounded, then we have $\rho(y_m - y_0) \le \delta_\rho(B) < \infty$ for any $m \in \mathbb{N}$, which implies

$$\rho(y_{m+n} - y_n) \le (\frac{k}{c})^n \delta_\rho(B),$$

and by hypothesis $c > \max(1, k)$ we have $(\frac{k}{c})^n \to 0$ as $n \to +\infty$. Therefore, $\rho(y_{m+n} - y_n) \to 0$ as $n, m \to +\infty$. Which implies that the sequence $(y_n)_{n\in\mathbb{N}}$ is ρ-Cauchy. Since X_ρ is ρ-complete, B is closed and T is continuous then the sequence $(y_n)_{n\in\mathbb{N}}$ is convergent to an element $y \in B$ and $y = Ty + Ux$. Indeed,

$$\begin{aligned}\rho(\frac{y - Ty - U(x)}{2}) &= \rho(\frac{y - y_n + y_n - Ty - U(x)}{2})\\ &= \rho(\frac{y - y_n + Ty_{n-1} - Ty}{2})\\ &\le \rho(y - y_n) + \rho(Ty_{n-1} - Ty),\end{aligned}$$

which implies that $y - Ty = U(x)$.

Then it follows that for any $x \in B$, there exists $y \in B$ such that $(I-T)y = Ux$. Therefore, we get that $(I-T)(B) \subset U(B)$ (Indeed, if we suppose that there exists $y \in B$ such that $y - Ty \notin U(B)$ i.e., for any $x \in B$, we have $y - Ty \ne U(x)$ which is absurd), and $I - T$ is a surjective operator from B into $U(B)$.

Let y_1, y_2 in B such that $(I-T)y_1 = (I-T)y_2$, then $y_1 - y_2 = Ty_1 - Ty_2$; therefore, $\rho(y_1 - y_2) \le \frac{k}{c}\rho(y_1 - y_2)$, and since $c > \max(1, k)$ it follows that $\rho(y_1 - y_2) = 0$ and $y_1 = y_2$. Which shows that $I - T$ is injective operator. Therefore, $I - T$ is a bijection operator from B into $U(B)$.

Secondly, we show that $(I-T)^{-1}$ is continuous. Let $(x_n)_{n\in\mathbb{N}} \subset U(B)$ be a convergent sequence to $x \in U(B)$, and consider the sequence defined by $z_n = (I-T)^{-1}(x_n)$, then $(z_n)_{n\in\mathbb{N}}$ is ρ-Cauchy. Indeed,

$$\begin{aligned}z_{n+m} - z_n &= z_{m+n} - Tz_{m+n} + Tz_{m+n} - Tz_n + Tz_n - z_n\\ &= x_{m+n} + Tz_{m+n} - Tz_n - x_n\\ &= x_{m+n} - x_n + Tz_{m+n} - Tz_n;\end{aligned}$$

therefore, if we take α such that $\frac{1}{\alpha} + \frac{1}{c} = 1$, then

$$\begin{aligned}\rho(z_{m+n} - z_n) &= \rho(\frac{1}{c}(c(Tz_{m+n} - Tz_n)) + \frac{1}{\alpha}\alpha(x_{m+n} - x_n))\\ &\le \frac{k}{c}\rho(z_{m+n} - z_n) + \frac{1}{\alpha}\rho(\alpha(x_{m+n} - x_n)).\end{aligned}$$

Then,

$$\rho(z_{m+n}-z_n)\le \frac{c}{c-k}\frac{1}{\alpha}\rho(\alpha(x_{m+n}-x_n)).$$

And since $\rho(x_{m+n}-x_n)\to 0$ as $m,n\to+\infty$, then by the Δ_2-condition $\rho(\alpha(x_{m+n}-x_n))\to 0$ as $m,n\to+\infty$. Therefore, $\rho(z_{m+n}-z_n)\to 0$ as $m,n\to+\infty$, and by hypothesis X_ρ is ρ-complete, then the sequence $(z_n)_{n\in\mathbb{N}}$ is convergent to an element $z\in B$. On the other hand, $x_n=z_n-T(z_n)$ is convergent to $x=z-T(z)$. Indeed,

$$\rho(\frac{z_n-T(z_n)-z+T(z)}{2})\le \rho(z_n-z)+\rho(T(z_n)-T(z)).$$

Since $\rho(z_n-z)\to 0$ as $n\to+\infty$ and T is continuous, $\rho(\frac{z_n-T(z_n)-z+T(z)}{2})\to 0$ as $n\to+\infty$, and by Δ_2-condition we have $\rho(z_n-T(z_n)-(z-T(z))\to 0$ as $n\to+\infty$. Therefore, $(I-T)^{-1}(x_n)$ converges to $(I-T)^{-1}(x)$, which implies that $(I-T)^{-1}$ is continuous.

Finally, we consider the function f defined by

$$f(x)=(I-T)^{-1}U(x).$$

Since U is ρ-completely continuous and $(I-T)^{-1}$ is ρ-continuous, it follows by the Δ_2-condition that U is $\|\cdot\|_\rho$- completely continuous and $(I-T)^{-1}$ is $\|.\|_\rho$-continuous. Which implies that f is $\|.\|_\rho$-completely continuous from B into B. By the Δ_2-condition, B is $\|.\|_\rho$-closed. Then, using Lemma 2.3 and Schauder's fixed point theorem, f has a fixed point. Let x_0 be such that $f(x_0)=x_0$, then we have $x_0=f(x_0)=(I-T)^{-1}U(x_0)$ which implies that $x_0=(T+U)(x_0)$. Therefore, S has a fixed point , which completes the proof. □

The next section presents an application of Theorem 2.1. We study the existence of solutions in the modular space $C^\varphi=C([0,b],L^\varphi)$. For details about the spaces C^φ and L^φ, we refer the reader to [1] and to books edited by Musielak [9] and Kozlowski [6].

3. Perturbed integral equations

In this section, we study the existence of solutions to perturbed integral equations on the Musielak-Orlicz space L^φ. For this, we begin by setting the functional framework of this integral equation.

Functional framework. Let L^φ be the Musielak-Orlicz space. Then both the modular ρ and its associated F-norm satisfy the Fatou property. Hence forth, we assume that ρ is convex and satisfies the Δ_2-condition (the F-norm becomes the Luxemburg norm [4]). Therefore, we have

$$\|x_n-x\|_\rho\to 0\Longleftrightarrow \rho(x_n-x)\to 0$$

as $n\to+\infty$ on L^φ. This implies that the topologies generated by $\|.\|_\rho$ and ρ are equivalent. Note that, under such conditions on ρ, $(L^\varphi(\Omega),\|.\|_\rho)$ is a Banach space, where $\Omega=[0,b]$.

We denote by $C^\varphi=C([0,b],L^\varphi)$ the space of all ρ-continuous functions from $[0,b]$ to L^φ, endowed with the modular ρ_a defined by $\rho_a(u)=\sup_{t\in[0,b]}\exp(-at)\rho(u(t))$, where $a\ge 0$. On the space C^φ one can consider the three topologies associated with the modular ρ_a (see [9] and [2]), the Luxemburg norm $\|.\|_{\rho_a}$, and the norm $|.|_\infty$ defined by $|u|_\infty=\sup_{t\in[0,b]}\|u(t)\|_\rho$.

We note that the three topologies above are equivalent in the following sense $\rho_a(x_n - x) \to 0 \Leftrightarrow \|x_n - x\|_{\rho_a} \to 0 \Leftrightarrow |x_n - x|\infty \to 0$ as $n \to +\infty$. Indeed, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in C^φ such that $|x_n - x|_\infty \to 0$ as $n \to +\infty$ and with $x \in C^\varphi$, hence for all $0 < \epsilon < 1$ there exists $N \in \mathbb{N}$ such that for any $n > N$ we have

$$\sup_{t\in[0,b]} \|x_n(t) - x(t)\|_\rho \le \epsilon < 1.$$

On the other hand, $\|x_n(t) - x(t)\|_\rho \le \epsilon < 1$ for all $t \in [0, b]$ implies $\rho(x_n(t) - x(t)) \le \epsilon < 1$ for all $t \in [0, b]$. Then

$$\sup_{t\in[0,b]} \exp(-at)\rho(x_n(t) - x(t)) \le \epsilon$$

for all $n \ge N$. This implies $\rho_a(x_n - x) \to 0$ as $n \to +\infty$. By the Δ_2-condition we have $\|x_n - x\|_{\rho_a} \to 0$ as $n \to +\infty$.

Conversely, by letting $u > 0$ be such that $\sup_{t\in[0,b]} \exp(-at)\rho(\frac{x_n(t)-x(t)}{u}) \le 1$, we have

$$e^{-ab}\rho(\frac{x_n(t) - x(t)}{u}) \le e^{-at}\rho(\frac{x_n(t) - x(t)}{u}) \le 1$$

for all $t \in [0, b]$. This implies

$$e^{-ab}\rho(\frac{x_n(t) - x(t)}{u}) \le \sup_{t\in[0,b]} \exp(-at)\rho(\frac{x_n(t) - x(t)}{u}) \le 1.$$

Therefore,

$$A := \{u > 0; \sup_{t\in[0,b]} \exp(-at)\rho(\frac{x_n(t) - x(t)}{u}) \le 1\}$$
$$\subset B := e^{-ab}\{u > 0; \rho(\frac{x_n(t) - x(t)}{u}) \le 1\}.$$

Hence, $\inf(A) \ge \inf(B)$, which implies

$$\|x_n - x\|_{\rho_a} \ge e^{-ab}\|x_n(t) - x(t)\|_\rho$$

for all $t \in [0, b]$. Hence,

$$e^{ab}\|x_n - x\|_{\rho_a} \ge \sup_{t\in[0,b]} \|x_n(t) - x(t)\|_\rho = |x_n - x|_\infty.$$

Therefore, $|x_n - x|_\infty \to 0$ as $n \to +\infty$ is equivalent to $\|x_n - x\|_{\rho_a} \to 0$ as $n \to +\infty$.

To study the integral equation (1.1). we set the following hypotheses:

(H1) Let B be a convex, ρ-closed, ρ-bounded subset of L^φ, and $0 \in B$.
(H2) Let $T : B \to B$ be an application for which there exists a real number $k > 0$ such that $\rho(Tx - Ty) \le k\rho(x - y)$ for all $x, y \in B$. Also let $h : B \to B$ be an application ρ-completely continuous such that $T(B) + h(B) \subseteq B$.
(H3) Let f_0 be a fixed element of B.

Theorem 3.1. *Under these hypotheses and for any $b > 0$, the integral equation* (1.1) *has a solution $u \in C^\varphi = C([0, b], L^\varphi)$.*

When we restrict our attention to the Banach space $(L^\varphi, \|.\|_\rho)$, Equation (1.1) can be written as

$$u'(t) + (I - (T + h))u(t) = 0.$$

When $h \equiv 0$, Equation (1.1) becomes

$$u(t) = \exp(-t) f_0 + \int_0^t \exp(s-t) T u(s) ds.$$

The equation above has been studied in [1] and [3]. The proof of Theorem 3.1 is based on Lemma 2.3 and the next lemma.

Lemma 3.2. *If a family $M \subset C^{\varphi}$ is equicontinuous in the sense of $\|.\|_\rho$, then M is equicontinuous in the sense of ρ.*

Proof. Recall that if $\|x\|_\rho < 1$, then $\rho(x) \leq \|x\|_\rho$ (see [9, p.2]). Let $0 < \epsilon < 1$, there exists $\delta > 0$ such that if $|t - \bar{t}| < \delta$ then $\|f(t) - f(\bar{t})\|_\rho \leq \epsilon < 1$ for all $f \in M$. Hence, $\rho(f(t) - f(\bar{t})) \leq \|f(t) - f(\bar{t})\|_\rho \leq \epsilon$ for any $f \in M$ whenever $|t - \bar{t}| < \delta$. This implies that M is ρ-equicontinuous and the proof is complete. $\square$

Proof of Theorem 3.1. Let $a > 0$ and ρ_a be a modular in $D = C([0,b], B)$ defined by $\rho_a(u) = \sup_{t \in [0,b]} \exp(-at) \rho(u(t))$ for $u \in D$ (see [1]).

By [1, Prop. 2.1 (3)], D is convex, ρ_a-closed and since B is ρ-bounded, then D is ρ_a-bounded .

Claim: D is invariant under the operator S given by

$$Su(t) = \exp(-t) f_0 + \int_0^t \exp(s-t)(T+h) u(s) ds.$$

First, we prove that Su is continuous from $[0,b]$ into $(L^{\varphi}, \|.\|_\rho)$. Let t_n, $t_0 \in [0,b]$ such that $t_n \to t_0$ as $n \to +\infty$. Since T and h are ρ-continuous, then $(T+h)u$ is ρ-continuous at t_0. Indeed,

$$\begin{aligned} &\rho((T+h)u(t_n) - (T+h)u(t_0)) \\ &\leq \frac{1}{2}\rho(2(Tu(t_n) - Tu(t_0)) + \frac{1}{2}\rho(2(hu(t_n) - hu(t_0))). \end{aligned}$$

By the Δ_2-condition, we have $\rho((T+h)u(t_n) - (T+h)u(t_0)) \to 0$ as $n \to +\infty$. Again by Δ_2-condition, $(T+h)u$ is $\|.\|_\rho$-continuous at t_0. Hence Su is $\|.\|_\rho$-continuous at t_0.

Next, we prove that $Su(t) \in B$, for any $t \in [0,b]$. It is well known that in Banach space $(L^{\varphi}, \|.\|_\rho)$,

$$\begin{aligned} &\int_0^t \exp(s-t)(T+h)u(s) ds \\ &\in (\int_0^t \exp(s-t) ds) \overline{\mathrm{co}}^{\|\cdot\|_\rho}\{(T+h)u(s), \quad 0 \leq s \leq t\}, \end{aligned}$$

where $\overline{\mathrm{co}}^{\|\cdot\|_\rho}$ denotes the closure of the convex hull in the sense of $\|.\|_\rho$. Since $(T+h)(B) \subseteq B$, $\int_0^t \exp(s-t)(T+h)u(s) ds \in (1 - \exp(-t)) \overline{\mathrm{co}}^{\|\cdot\|_\rho}(B)$. But B is convex and ρ-closed. Thus $\overline{\mathrm{co}}^{\|\cdot\|_\rho}(B) = \overline{B}^{\|\cdot\|_\rho} \subset \overline{B}^{\rho} = B$. Therefore, $Su(t) \in \exp(-t)B + (1 - \exp(-t))B \subseteq B$ for all $t \in [0,b]$. Hence, D is invariant by S.

Now consider the operators: $T_1 u(t) = \exp(-t) f_0 + \int_0^t \exp(s-t) T u(s) ds$ and $h_1 u(t) = \int_0^t \exp(s-t) h u(s) ds$. Observe that $S = T_1 + h_1$. Next, we show that T_1 and h_1 satisfy the hypotheses of Theorem 2.1.

(1) We note that, by the same argument in the proof of fixed point theorem (see [1]), we show that D is invariant under h_1 and T_1 and there exists $c > \max(1, k_0)$ such that
$$\rho_a(c(T_1u - T_1v)) \leq k_0\rho_a(u - v), \quad \forall u, v \in D,$$
where $1 < c \leq \frac{e^b}{e^b-1}$, $k_0 = c\frac{k}{1+a}$ and $a \geq k$. The same techniques used in the proof of $S(D) \subset D$ are used to establish $T_1(D) + h_1(D) \subset D$: By taking the hypothesis $T(B) + h(B) \subset B$, which gives $T_1u(t) + h_1v(t) \in \exp(-t)B + (1 - \exp(-t))B \subset B$ for any $t \in [0, b]$ and $u, v \in D$.
(2) Claim: h_1 is ρ_a-completely continuous. Let $M \subset D$, then $h_1(M)$ is equicontinuous in the sense of $\|.\|_\rho$. Indeed, let $u \in M$, we have
$$\begin{aligned}
&h_1u(t) - h_1u(\bar{t})\\
&= \int_0^t \exp(s - t)hu(s)ds - \int_0^{\bar{t}} \exp(s - \bar{t})hu(s)ds\\
&= e^{-t}\int_0^t e^s hu(s)ds - e^{-\bar{t}}\int_0^{\bar{t}} e^s hu(s)ds\\
&= e^{-t}\int_0^t e^s hu(s)ds - e^{-\bar{t}}\int_0^t e^s hu(s)ds + e^{-\bar{t}}\int_0^t e^s hu(s)ds - e^{-\bar{t}}\int_0^{\bar{t}} e^s hu(s)ds\\
&= (e^{-t} - e^{-\bar{t}})\int_0^t e^s hu(s)ds + e^{-\bar{t}}\int_{\bar{t}}^t e^s hu(s)ds.
\end{aligned}$$
Hence,
$$\begin{aligned}
\|h_1u(t) - h_1u(\bar{t})\|_\rho &\leq |e^{-t} - e^{-\bar{t}}|be^b\delta_{\|.\|_\rho}(B) + \delta_{\|.\|_\rho}(B)|\int_{\bar{t}}^t e^s ds|\\
&\leq |e^{-t} - e^{-\bar{t}}|be^b\delta_{\|.\|_\rho}(B) + \delta_{\|.\|_\rho}(B)|e^t - e^{\bar{t}}|
\end{aligned}$$
On the other hand, the functions $t \mapsto e^{-t}$ and $t \mapsto e^t$ are uniformly continuous on the compact $[0, b]$. Hence for $\epsilon > 0$, there exists $\eta_1 > 0$ such that if $|t - \bar{t}| < \eta_1$ then $|e^{-t} - e^{-\bar{t}}| \leq \frac{\epsilon}{2be^b\delta_{\|.\|_\rho}(B)}$, and there exists $\eta_2 > 0$ such that if $|t - \bar{t}| < \eta_2$ then $|e^t - e^{\bar{t}}| \leq \frac{\epsilon}{2\delta_{\|.\|_\rho}(B)}$.

Hence, there exists $\eta = \min(\eta_1, \eta_2)$ such that if $|t - \bar{t}| < \eta$ then $\|h_1u(t) - h_1u(\bar{t})\|_\rho \leq \epsilon$ for any $u \in M$. Therefore, $h_1(M)$ is equicontinuous in the sense of $\|.\|_\rho$, and by Lemma 3.2, $h_1(M)$ is ρ-equicontinuous. Otherwise,
$$\begin{aligned}
h_1u(t) = \int_0^t \exp(s - t)hu(s)ds &\in (1 - \exp(-t))\overline{\text{co}}^{\|\cdot\|_\rho}\{hu(s),\ 0 \leq s \leq t\}\\
&\subset (1 - \exp(-t))\overline{\text{co}}^{\|\cdot\|_\rho}(h(B)).
\end{aligned}$$
Hence $h_1(M(t)) \subset (1-\exp(-t))\overline{\text{co}}^{\|\cdot\|_\rho}(h(B))$ for all $t \in [0, b]$. But $h(B)$ is ρ-compact and by Δ_2-condition $\underline{h(B)}$ is $\|.\|_\rho$ compact, which implies that $\overline{\text{co}}^{\|\cdot\|_\rho}(h(B))$ is compact. Therefore, $\overline{h_1(M(t))}$ is $\|.\|_\rho$ compact for all $t \in [0, b]$, and by Ascoli's theorem $\overline{h_1(M)}^{|\cdot|_\infty}$ is compact. Hence, by the equivalence of three topologies considered in functional framework, $\overline{h_1(M)}$ is ρ_a-compact. Using the standard techniques [10, proof of the Theorem 3 page 103], we show that h_1 is $\|.\|_{\rho_a}$-continuous then h_1 is ρ_a-continuous. Hence, h_1 is ρ_a-completely continuous. It then follows from Theorem 2.1 that S has a fixed point which is a solution of the equation (1.1). $\square$

3.1. **Example of equation** (1.1)**.** In this example, we study the existence of a solution of the integral equation

$$\begin{aligned} u(t) = \exp(-t)f_0 + \int_0^t \exp(s-t)\Big(\int_0^b \exp(-\xi)g_2(s,\xi,u(\xi))d\xi\Big)ds \\ + \int_0^t \exp(s-t)\Big(\int_0^b \exp(-\xi)g_1(s,\xi,u(\xi))d\xi\Big)ds/ \end{aligned} \tag{3.1}$$

under the hypotheses stated below. Let X_ρ be a finite dimensional vector subspace of L^φ, and ρ be a convex modular on L^φ, satisfying the Δ_2-condition. Let B be a convex, ρ-closed, ρ-bounded subset of X_ρ and $0 \in B$. Let $b > 0$ very small, $g_1,\ g_2$ be functions from $[0,b]\times[0,b]\times B$ into B, $\gamma : [0,b]\times[0,b]\times[0,b] \to \mathbb{R}^+$ and $\beta : [0,b]\times[0,b] \to \mathbb{R}^+$ be measurable functions such that:

- (H1') (i) $g_i(t,.,x) : s \mapsto g_i(t,s,x)$ where $i \in \{1,2\}$ are measurable functions on $[0,b]$ for each $x \in B$ and for almost all $t \in [0,b]$.
  (ii) $g_i(t,s,.) : x \mapsto g_i(t,s,x)$, where $i \in \{1,2\}$, are ρ-continuous on B for almost all $t,s \in [0,b]$.
- (H2') For any $i \in \{1,2\}$, $\rho(g_i(t,s,x) - g_i(\tau,s,x)) \le \gamma(t,\tau,s)$ for all (t,s,x) and (τ,s,x) in $[0,b]\times[0,b]\times B$ and $\lim_{t\to\tau}\int_0^b \gamma(t,\tau,s)ds = 0$ uniformly for $\tau \in [0,b]$.
- (H3') $\rho(g_2(t,s,x) - g_2(t,s,y)) \le \rho(x-y)$ for all (t,s,x) and (t,s,y) in $[0,b]\times[0,b]\times B$.

These hypotheses have been used by Martin [8].

Now, assume that f_0 is a fixed element of B, and that h, T are the Uryshon operators on $C([0,b],B)$ defined by:

$$\begin{aligned} [hu](t) &= \int_0^b \exp(-s)g_1(t,s,u(s))ds, \\ [Tu](t) &= \int_0^b \exp(-s)g_2(t,s,u(s))ds, \end{aligned}$$

for $t \in [0,b]$ and $u \in (C([0,b],B),\rho_a)$ with $(a > 0)$.

Proposition 3.3. *(1) Under the hypotheses (H1')–(H3'), the operator T is ρ_a-Lipschitz from $C([0,b],B)$ into $C([0,b],B)$.*
(2) Under the hypotheses (H1')–(H2'), the operator h is ρ_a-completely continuous from $C([0,b],B)$ into $C([0,b],B)$.

Proof. (1) We show that $C([0,b],B)$ is invariant by T. (i) Note that $(X_\rho, \|.\|_\rho)$ is a Banach space with finite dimension. By hypothesis (H1')(i), $g_2(t,.,u(.)) : s \mapsto g_2(t,s,u(s))$ is measurable, and since B is ρ-bounded, $g_2(t,.,u(.)) : s \mapsto g_2(t,s,u(s))$ is an integrable function from $[0,b]$ into $(X_\rho,\|.\|_\rho)$. Then for $u \in C([0,b],B)$, we have

$$\begin{aligned} [Tu](t) &\in \int_0^b \exp(-s)ds\,\overline{\mathrm{co}}^{\|\cdot\|_\rho}\{g_2(t,s,u(s)), s \in [0,b]\} \\ &\subset (1-\exp(-b))\overline{\mathrm{co}}^{\|\cdot\|_\rho}(B). \end{aligned}$$

But B is convex and ρ-closed thus $\overline{\mathrm{co}}^{\|\cdot\|_\rho}(B) = \overline{B}^{\|\cdot\|_\rho} \subset \overline{B}^\rho = B$. Since $0 \in B$ and $0 < 1-\exp(-b) < 1$, we have $[Tu](t) \in B$ for all $t \in [0,b]$.

(ii) Let $u \in C([0,b],B)$ then Tu is continuous from $[0,b]$ into (B,ρ). Indeed, let $(t_n)_{n\in\mathbb{N}}$ be a sequence and r in $[0,b]$ such that $t_n \to r$ as $n \to +\infty$ and we have

$$[Tu](t_n) - [Tu](r) = \int_0^b \exp(-s)(g_2(t_n,s,u(s)) - g_2(r,s,u(s)))ds.$$

Let $K = \{s_0, s_1, \dots, s_m\}$ be a subdivision of $[0,b]$. Then $\sum_{i=0}^{m-1}(s_{i+1}-s_i)e^{-s_i}x(s_i)$ is $\|.\|_\rho$-convergent. Thus ρ-converges to $\int_0^b \exp(-s)x(s)ds$ in X_ρ when $|K| = sup\{|s_{i+1}-s_i|, i=0,\dots,m-1\} \to 0$ as $m \to +\infty$. Since

$$\begin{aligned}&\int_0^b \exp(-s)(g_2(t,s,u(s)) - g_2(\tau,s,u(s)))ds\\ &= \lim \sum_{i=0}^{m-1}(s_{i+1}-s_i)\exp(-s_i)(g_2(t,s_i,u(s_i)) - g_2(\tau,s_i,u(s_i))),\end{aligned}$$

and $\sum_{i=0}^{m-1}(s_{i+1}-s_i)\exp(-s_i) \le \int_0^b \exp(-s)ds = 1-\exp(-b) < 1$, then by the Fatou property we have:

$$\begin{aligned}&\rho([Tu](t_n) - [Tu](r))\\ &\le \liminf \sum_{i=0}^{m-1}(s_{i+1}-s_i)\exp(-s_i)\rho(g_2(t_n,s_i,u(s_i)) - g_2(r,s_i,u(s_i)))\\ &\le \liminf \sum_{i=0}^{m-1}(s_{i+1}-s_i)\exp(-s_i)\gamma(t_n,r,s_i)\\ &\le \int_0^b \exp(-s)\gamma(t_n,r,s)ds\\ &\le \int_0^b \gamma(t_n,r,s)ds\end{aligned}$$

Hence by hypothesis (H2') Tu is ρ-continuous at r.
(2) We show that T is ρ_a-Lipschitz. Let u, v in $C([0,b],B)$, we have.

$$\begin{aligned}&\rho([Tu](t) - [Tv](t))\\ &\le \liminf \sum_{i=0}^{m-1}(s_{i+1}-s_i)(\exp(-s_i))\rho(g_2(t,s_i,u(s_i)) - g_2(t,s_i,v(s_i)))\\ &\le \liminf \sum_{i=0}^{m-1}(s_{i+1}-s_i)\exp(-s_i)\rho(u(s_i) - v(s_i))\\ &\le \liminf \sum_{i=0}^{m-1}(s_{i+1}-s_i)\exp(as_i)\rho_a(u-v).\end{aligned}$$

Therefore,

$$\begin{aligned}\exp(-at)\rho([Tu](t) - [Tv](t)) &\le \exp(-at)(\int_0^b \exp(as)ds) \quad \rho_a(u-v)\\ &\le \frac{e^{ba}-1}{a}\rho_a(u-v).\end{aligned}$$

Hence,

$$\rho_a([Tu]-[Tv]) \le \frac{e^{ba}-1}{a}\rho_a(u-v).$$

(3) Using the same argument of (1), we show that $C([0,b],B)$ is invariant by h.
(4) Now, we claim that $h(C([0,b],B))$ is equicontinuous in the sense of ρ, and ρ_a-compact. We have:

$$[hu](t)-[hu](\tau) = \int_0^b \exp(-s)(g_1(t,s,u(s)) - g_1(\tau,s,u(s)))ds.$$

We easily obtain

$$\rho([hu](t)-[hu](\tau)) \le \int_0^b \gamma(t,\tau,s)ds,$$

by using again the same argument in (1). And since, $\lim_{t\to\tau}\int_0^b \gamma(t,\tau,s)ds = 0$ uniformly for $\tau \in [0,b]$, then $h(C([0,b],B))$ is ρ-equicontinuous. On the other hand, since B is ρ-bounded then, $h(C([0,b],B))$ is ρ_a-bounded subset of $C([0,b],B)$. Indeed, let u,v in $C([0,b],B)$, we have

$$[hu](t)-[hv](t) = \int_0^b \exp(-s)(g_1(t,s,u(s)) - g_1(t,s,v(s)))ds.$$

Again from (1), we obtain

$$\begin{aligned}
&\rho([hu](t)-[hv](t))\\
&\le \liminf \sum_{i=0}^{m-1}(s_{i+1}-s_i)\exp(-s_i)\rho(g_1(t,s_i,u(s_i)) - g_1(t,s_i,v(s_i)))\\
&\le \liminf \sum_{i=0}^{m-1}(s_{i+1}-s_i)\exp(-s_i)\delta_\rho(B)\\
&\le (\int_0^b \exp(-s)ds) \quad \delta_\rho(B).
\end{aligned}$$

Hence,

$$\rho_a([hu]-[hv]) \le (1-e^{-b})\delta_\rho(B) < \infty$$

Therefore, $h(C([0,b],B))$ is a ρ_a-bounded subset of $C([0,b],B)$ and by Lemma 2.3, it is $\|.\|_{\rho_a}$-bounded subset of $C([0,b],B)$. On the other hand, since $(X_\rho, \|.\|_\rho)$ is a Banach space with finite dimensional , then for each $t \in [0,b]$ we have $\overline{h(C([0,b],B))(t)}$ is $\|.\|_\rho$-compact. Thus, by Ascoli's theorem we have $\overline{h(C([0,b],B))}$ is $\|.\|_{\rho_a}$-compact, then $\overline{h(C([0,b],B))}$ is ρ_a-compact. Hence for any $M \subset C([0,b],B)$, we have $\overline{h(M)}$ is ρ_a-compact. Using the standard techniques [10, Theorem 3 page 103], we that h is $\|.\|_{\rho_a}$-continuous then h is ρ_a-continuous. So h is ρ_a-completely continuous. □

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