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POSITIVE SOLUTIONS OF THREE-POINT BOUNDARY-VALUE PROBLEMS FOR P-LAPLACIAN SINGULAR DIFFERENTIAL EQUATIONS

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Abstract. In this paper we prove the existence of positive solutions for the three-point singular boundary-value problem

$$-[\phi_p(u')]' = q(t)f(t,u(t)), \quad 0 < t < 1$$

subject to

$$u(0) - g(u'(0)) = 0, \quad u(1) - \beta u(\eta) = 0$$

or to

$$u(0) - \alpha u(\eta) = 0, \quad u(1) + g(u'(1)) = 0,$$

where ϕ_p is the p-Laplacian operator, $0 < \eta < 1$; $0 < \alpha, \beta < 1$ are fixed points and g is a monotone continuous function defined on the real line $\mathbb{R}$ with $g(0) = 0$ and $ug(u) \geq 0$. Our approach is a combination of Nonlinear Alternative of Leray-Schauder with the properties of the associated vector field at the (u, u') plane. More precisely, we show that the solutions of the above boundary-value problem remains away from the origin for the case where the nonlinearity is sublinear and so we avoid its singularity at $u = 0$.

1. Introduction

In this note we consider the nonlinear 3-point singular problem

$$-[\phi_p(u')]' = q(t)f(t,u(t)), \quad 0 < t < 1 \tag{1.1}$$

subject to

$$u(0) - g(u'(0)) = 0, \quad u(1) - \beta u(\eta) = 0 \tag{1.2}$$

or to

$$u(0) - \alpha u(\eta) = 0, \quad u(1) + g(u'(1)) = 0 \tag{1.3}$$

where $\phi_p(s) = |s|^{p-2}s$, $(p > 1)$ is the well known p-Laplacian operator, $0 < \eta < 1$; $0 < \alpha, \beta < 1$ are fixed points and g is a monotone continuous function defined on the real line $\mathbb{R}$ with $g(0) = 0$ and $ug(u) \geq 0$. The inhomogeneous term in (1.1) is allowed to be singular at $u = 0$ and $q(t)$ may be singular at $t = 0$ or/and $t = 1$ and finally we suppose that $q(0) > 0$. Further assumptions concerning the nonlinearity $f(t,u)$ will be clarified later.

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Let B be the Banach space $C[0,1]$ endowed with the norm $\|x\| = \max_{t\in[0,1]} |x(t)|$. A solution $u(t)$ of (1.1) subject to (1.2) or (1.3) means that $u(t) \in C^1[0,1]$, is positive on $(0,1)$, $\phi_p(u'(.)) \in C(0,1) \cap L^1[0,1]$ and satisfies the differential equation as well as the corresponding boundary conditions. It is well known that when $p > 1$, $\phi_p(s)$ is strictly increasing on $\mathbb{R}$ and so its inverse ϕ_p^{-1} exists and further $\phi_p^{-1} = \phi_q$, where $1/p + 1/q = 1$.

In [6], Erbe and Wang by using Green's functions and Krasnoselskii's fixed point theorem in cones proved existence of a positive solution of the boundary-value problem studied the Sturm-Liouville boundary-value problem

$$\begin{gathered} x''(t) = -f(t, x(t)), \\ \alpha x(0) - \beta x'(0) = 0, \quad \gamma x(1) + \delta x'(1) = 0, \end{gathered}$$

where $\alpha, \beta, \gamma, \delta \geq 0$ and $\rho := \beta\gamma + \alpha\gamma + \alpha\delta > 0$, mainly under the assumptions:

$$\begin{gathered} f_0 := \lim_{x\to 0+} \max_{0\le t\le 1} \frac{f(t,x)}{x} = 0, \\ f_\infty := \lim_{x\to +\infty} \min_{0\le t\le 1} \frac{f(t,x)}{x} = +\infty \end{gathered}$$

i.e., f is *supelinear* at both ends points $x = 0$ and $x = \infty$ or under

$$\begin{gathered} f_0 := \lim_{x\to 0+} \min_{0\le t\le 1} \frac{f(t,x)}{x} = +\infty \\ f_\infty := \lim_{x\to +\infty} \max_{0\le t\le 1} \frac{f(t,x)}{x} = 0, \end{gathered}$$

i.e., f is *sublinear* at both $x = 0$ and $x = \infty$.

The study of multi-point boundary-value problems was initiated by Il'in and Moiseev in [11, 12]. Many authors since then considered nonlinear 3-point boundary-value problems (see e.g., [3, 7, 8, 10, 13, 18, 19, 21, 22] and the references therein). In particular, Ma in [19] proved the existence of a positive solution to the three-point nonlinear boundary-value problem

$$\begin{gathered} -u''(t) = q(t)f(u(t)), \quad 0 < t < 1, \\ u(0) = 0, \quad \alpha u(\eta) = u(1), \end{gathered}$$

where $0 < \alpha$, $0 < \eta < 1$ and $\alpha\eta < 1$. The results of Ma were complemented in the works of Webb [22], Kaufmann [13], Kaufmann and Kosmatov [14], and Kaufmann and Raffoul [15].

Among the studies on semipositone multi-point boundary-value problems, we mention the papers by Cao and Ma [5] and Liu [16]. Cao and Ma considered the boundary-value problem

$$\begin{gathered} -u''(t) = \lambda q(t)a(t)f(u(t), u'(t)), \quad 0 < t < 1, \\ u(0) = 0, \quad \sum_{i=1}^{m-2} \alpha_i u(\eta_i) = u(1)\,. \end{gathered}$$

They applied the Leray-Schauder fixed point theorem to obtain an interval of eigenvalues for which at least one positive solution exists. Liu applied a fixed point index method to obtain such an interval for

$$-u''(t) = \lambda q(t)a(t)f(u(t)), \quad 0 < t < 1,$$
$$u'(0) = 0, \quad \beta u(\eta) = u(1).$$

In the above papers there are no assumptions for singularity of the nonlilearity f at the point $u = 0$. Zhang and Wang [24] and recently Liu [17] obtained some existence results for a singular nonlinear second order 3-point boundary-value problem, for the case when only singularity of $q(t)$ at $t = 0$ or $t = 1$ is permitted. Also recently, by using the method of fixed point index, Xu [23] studied the problem

$$-u''(t) = f(u(t)), \quad 0 < t < 1, \quad u(0) = 0, \quad \alpha u(\eta) = u(1),$$

where $f(t, u)$ is allowed to have singularity at $u = 0$. Other applications of Krasnosel'skiĭ's fixed point theorem to semipositone problems can, for example, be found in [1]. Further recently interesting results have been proved in [4], [9] or [17].

Finally, Ma and Ge in [20], proved the existence of a positive solution of the 3-point singular boundary-value problem (1.1)-(1.2) under the following assumptions:

- (H1) $q(t) \in C(0, 1) \cap L^1[0, 1]$ with $q(t) \geq 0$ and nondecreasing on $(0, 1)$
- (H2) $f \in C([0, 1] \times (0, +\infty), (0, +\infty))$
- (H3) $0 \leq f(t, y) \leq f_1(y) + f_2(y)$ on $[0, 1] \times (0, +\infty)$ with $f_1 > 0$ continuous nonincreasing on $(0, +\infty)$ and $\int_0^L f_1(u)du < +\infty$ for any fixed $L > 0$; $f_2 \geq 0$ and continuous on $[0, +\infty)$
- (H4) For any $K > 0$ there exists $\psi_K(t) : (0, 1) \to (0, +\infty)$ such that $f(t, y) \geq \psi_K(t),\ t \in (0, 1)$ for any $y(t) \in C[0, 1]$ with $0 \leq y(t) \leq K$
- (H5) $\int_0^\eta \phi_p^{-1}(\int_s^\eta q(r)\psi(r)dr)ds > 0$, $\int_\eta^1 \phi_p^{-1}(\int_\eta^s q(r)\psi(r)dr)ds > 0$ and for any $k_1 > 0$ and $k_2 > 0$, $\int_0^\eta f_1(k_1 s)q(s)ds + \int_\eta^1 f_1(k_2(1 - s))q(s)ds < +\infty$ and mainly
$$\sup_{c>0} \frac{c}{\phi_p^{-1}(I^{-1}[G_0(c)])(\frac{p}{p-1})^{\frac{1}{p}}[\frac{1}{1-\beta}(\int_0^\eta [q(s)]^{\frac{1}{p}}ds + \int_\eta^1 [q(s)]^{\frac{1}{p}}ds)]} > 1,$$
where
$$I(c) := \int_0^c \phi_p^{-1}(z)dz = \frac{p-1}{p}c^{\frac{p}{p-1}} \quad \text{and} \quad G_0(c) = \int_0^c [f_1(u) + f_2(u)]du\ .$$

It is not difficult to prove the next useful properties of $I(c)$:

$$I^{-1}(uv) \leq I^{-1}(u)I^{-1}(v) \quad \text{for } u \geq 0, v \geq 0$$

and whenever $c < 0$, we have $I(-c) = I(c)$. Indeed, since $-c > 0$,

$$I(-c) = \int_0^{-c} \phi_p^{-1}(z)dz = \int_0^{-c} \phi_q(z)dz = \int_0^{-c} |z|^{q-2}zdz = \int_0^{-c} z^{q-1}dz = \frac{(-c)^q}{q}$$

and

$$I(c) = \int_0^c |z|^{q-2}zdz = \int_0^c (-z)^{q-2}zdz = -\int_0^c (-z)^{q-1}d(-z) = \frac{(-c)^q}{q}.$$

In this work, mainly motivated by the above mentioned paper of Ma and Ge [20], we combine the properties of the vector field at the face (u, u') plane and sublinearity of $f(t, u)$ at the origin $u = 0$ with the alternative continuation principle of Leray-Shauder, proving the existence of a positive solution for the boundary-value problem (1.1)-(1.3) and eliminating several of the assumptions (H1)-(H5).

2. Preliminaries

We now proceed with the auxiliaries. Consider the boundary-value problem

$$-[\phi_p(u')]' = q(t)f(t,u(t)), \quad 0<t<1, \tag{2.1}$$

$$u(0)-g(u'(0))=0, \quad \beta u(\eta)=u(1) \tag{2.2}$$

and give two concept-assumptions as follows:

$$f_0 := \lim_{u\to 0}\max_{0\le t\le 1}\frac{f(t,u)}{u}=+\infty \tag{2.3}$$

i.e., f is *sublinear* at the end point 0, and

$$\lim_{u\to 0}\frac{g^{-1}(u)}{u}=\mu\in[0,\frac{1}{2}). \tag{2.4}$$

We further consider that :

- (A1) $q(t)\in C(0,1)\cap L^1[0,1]$ with $q(t)>0$ and nondecreasing on $(0,1)$
- (A2) $f\in C([0,1]\times(0,+\infty),(0,+\infty))$
- (A3) $\int_0^L \max_{0\le t\le 1} f(t,u)du<+\infty$, for any fixed $L>0$.
- (A4) $g\in C(\mathbb{R},\mathbb{R})$, is a nondecreasing function with $ug(u)>0$, $u\neq 0$.

Remar 2.1. Note that the differential equation (2.1) defines a vector field whose properties will be crucial for our study. More specifically, working at the (u,u') *face semi-plane* $(u>0)$, the sign condition on f (see assumption (A2)), immediately gives (since $\phi_p'(u')>0$ for all $u'\in\mathbb{R}$) that $u''<0$. Thus, any trajectory $(u(t),u'(t))$, $t\ge 0$, emanating from the semi-line

$$E:=\{(u,u'):u-g(u')=0, \quad u>0\}$$

"trends" in a natural way, (when $u'(t)>0$) toward the positive u-semi-axis and then (when $u'(t)<0$) turns toward the negative u'-semi-axis. Finally, by setting a certain growth rate on f (say sublinearity) we can control the vector field, so that all trajectories with $u(0)$ small enough satisfy the relation

$$u(1)-\beta u(\eta)\neq 0.$$

So, all solutions of the given boundary-value problem cannot have their initial values arbitrary small, avoiding in this way the singular point $u=0$ of the nonlinearity.

Namely we have the next result.

Lemma 2.2. *Let $0<\beta<1$. If $u\in C[0,1]$ is a solution of (2.1)-(2.2), then $u=u(t)$ is concave. Furthermore for every solution with $u(0)>0$, it follows that*

- (i) *There exists a $t_0\in[0,1)$ such that $u(t_0)=\max_{0\le t\le 1}\|u(t)\|=\|u\|$,*
- (ii) *$u(t)>0$, $t\in[0,1]$ and*
- (iii) *$\inf_{t\in[\eta,1]}u(t)\ge\gamma u(t_0)=\gamma\|u\|$, where $\gamma=\min\{\beta\eta,\ \frac{\beta(1-\eta)}{1-\beta\eta}\}$.*

Proof. Let $u(t)$ be a solution to (2.1)-(2.2). Then, since $[\phi_p(u')]'=-q(t)f(t,u(t))\le 0$, $\phi_p(u')$ is non-increasing. Consequently $u'(t)$ is non-increasing which implies the concavity of $u(t)$.

(i) Since $u(0)>0$, by the first condition in (2.2) and the assumption (A4), we get $u'(0)>0$. If $u'(t)\ge 0$, $t\in[0,1]$ then $u(1)\ge u(\eta)>\beta u(\eta)$, a contradiction. Hence there exists $t_0>0$ such that $u(t_0)=\max_{0\le t\le 1}\|u(t)\|=\|u\|$.

(ii) If $\eta \in (0, t_0)$, then $u(\eta) > u(0) > 0$ and so $u(1) = \beta u(\eta) > 0$. If $\eta \in (t_0, 1)$, then $u(\eta) > u(1)$ and so

$$0 = u(1) - \beta u(\eta) < u(1) - \beta u(1)$$

and hence $u(1) > 0$. Finally, the concavity of $u(t)$ yields $u(t) > 0$, $t \in [0, 1]$.

(iii) The proof follows the concavity of the solution. Indeed, since $u(1) = \beta u(\eta) < u(\eta)$, let first consider the case $t_0 \leq \eta < 1$. Then,

$$\min_{t \in [\eta, 1]} u(t) = u(1).$$

Furthermore, we have

$$u(t_0) \leq u(1) + \frac{u(\eta) - u(1)}{1 - \eta} = u(1)\{1 - \frac{1 - \frac{1}{\beta}}{1 - \eta}\} = u(1)\frac{1 - \beta\eta}{\beta(1 - \eta)}.$$

Consequently,

$$\min_{t \in [\eta, 1]} u(t) \geq \frac{\beta(1 - \eta)}{1 - \beta\eta}\|u\|.$$

Let us now assume that $\eta < t_0 < 1$. Since $u(\eta) > u(1)$, we have again

$$\min_{t \in [\eta, 1]} u(t) = u(1).$$

¿From the concavity of u, we know that

$$\frac{u(\eta)}{\eta} \geq \frac{u(t_0)}{t_0}.$$

Combining the above and the boundary condition $u(1) = \beta u(\eta)$, we conclude that

$$\frac{u(\eta)}{\beta\eta} \geq \frac{u(t_0)}{t_0} \geq u(t_0) = \|u\|,$$

that is $\min_{t \in [\eta, 1]} u(t) \geq \beta\eta\|u\|$. □

3. Existence for the first boundary-value problem

In this section we consider the boundary-value problem (2.1)-(2.2) and prove the next result.

Lemma 3.1. *Suppose that conditions* (2.3)-(2.4) *hold. Then, there exists an* $\eta_0 > 0$ *such that for any* $\eta \leq \eta_0$ *any solution of* (2.1) *with* $u(0) = \frac{\eta}{2}$*, satisfies the inequality*

$$0 < u(t) \leq \eta \leq \eta_0, \quad t \in [0, 1]. \tag{3.1}$$

Proof. By assumption (2.3) it follows that for any $K > 0$ there exists η_0 such that

$$\min_{0 \leq t \leq 1} f(t, u) > Ku, \ 0 < u \leq \eta_0. \tag{3.2}$$

Let $K > \max\{2\mu^2,\ 2\frac{1+2\mu}{\min\{\gamma,1\}}\}$. We examine first the case $p > 2$. Taking into account (2.4), we may chose η_0 small enough so that

$$\frac{g^{-1}(\frac{\eta_0}{2})}{\frac{\eta_0}{2}} \leq 2\mu \quad \text{and} \quad (p-1)\frac{[g^{-1}(\eta)]^{p-2}}{\min_{0 \leq t \leq 1} q(t)} < 1, \quad \eta \in (0, \eta_0] \tag{3.3}$$

(if $\lim_{u \to 0} \frac{g^{-1}(u)}{u} = 0$, then we may find $\mu > 0$ so that (3.3) still holds true).

Assume that the boundary-value problem (2.1)-(2.2) has a solution $u(t)$, $t \in [0,1]$ with initial value $u(0)$ arbitrary small. Then, we may assume that $u(0) = \eta/2$ for some $\eta \in (0, \eta_0]$ with

$$\frac{\min_{0 \le t \le 1} q(t)}{(p-1)[g^{-1}(\frac{\eta}{2})]^{p-2}} \ge 1.$$

We demonstrate first that (3.1) holds true. If not, by Lemma 2.2, there exist $t^* \in (0,1]$ such that $\frac{\eta}{2} \le u(t) < \eta$, $0 \le t < t^*$ and $u(t^*) = \eta$. Then by (3.2) it follows that

$$[\phi_p(u')]' = [\phi_p'(u')]u'' = -q(t)f(t,u(t)) \le -Kq(t)u(t) \le -Kq(t)\frac{\eta}{2},$$

i.e.,

$$\begin{aligned} u''(t) &\le -Kq(t)\frac{\eta}{2}\frac{1}{[\phi_p'(u')]} \\ &\le -K\frac{\eta_0}{2}\frac{1}{[\phi_p'(u')]}\min_{0 \le t \le 1} q(t) \\ &\le -K\frac{\eta}{2}\frac{\min_{0 \le t \le 1} q(t)}{(p-1)[g^{-1}(\frac{\eta}{2})]^{p-2}} \le -K\frac{\eta}{2} \end{aligned}$$

Consequently, by (3.3) and the Taylor formula, we get that for some $t \in [0, t^*]$,

$$\begin{aligned} \eta = u(t^*) &\le \frac{\eta}{2} + t^* g^{-1}(\frac{\eta}{2}) + \frac{(t^*)^2}{2}u''(t) \\ &\le \frac{\eta}{2} + t^* g^{-1}(\frac{\eta}{2}) - \frac{(t^*)^2}{2}K\frac{\eta}{2} \\ &\le \frac{\eta}{2} + t^* 2\mu\frac{\eta}{2} - \frac{(t^*)^2}{2}K\frac{\eta}{2}. \end{aligned}$$

Considering now the map

$$\phi(t^*) := Kt^{*2} - 4\mu t^* + 2,$$

the above inequality yields $\phi(t^*) \le 0$. This is a contradiction, since the above choice of $K > 2\mu^2$, yields $\phi(t) > 0$ for all $t \in [0,1]$. As a result, noticing Lemma 2.2, we obtain $0 < u(t) \le \eta_0$, $t \in [0,1]$.

If now $p \le 2$, then since $\lim_{u \to 0} g^{-1}(u) = 0$, we easily get $u''(t) \le 0$. As a result, $\frac{\eta}{2} < t^* g^{-1}(\frac{\eta}{2})$ and thus $t^* > \frac{1}{2\mu} > 1$ (in view of (2.4)), a contradiction due to the initial choice of $t^* \in [0,1]$. □

Lemma 3.2. *Suppose that conditions* (2.3)-(2.4) *hold. Then for any* $\eta \in (0, \eta_0)$ *(*η_0 *as above) there exists an* $\alpha_0 = \alpha_0(\eta) > 0$ *such that for any (possible) solution* $u = u(t)$ *of* (2.1)-(2.2)*, with* $u(0) = \frac{\eta}{2}$*, the following inequality holds:*

$$u(t) \ge \alpha_0, \quad t \in [0,1].$$

Proof. By the concavity of $u(t)$, it is obvious that

$$\min_{t \in [0,1]} u(t) = \min\{u(0),\ u(1)\}.$$

However, in view of Lemma 2.2, $\min_{t \in [\eta,1]} u(t) \ge \gamma u(t_0) = \gamma\|u\| \ge \gamma u(0) = \gamma\frac{\eta}{2}$ and so

$$u(t) \ge \min_{t \in [0,1]} u(t) \ge \frac{\eta}{2}\min\{1, \gamma\} := \alpha_0(\eta).$$

□

Proposition 3.3. *Suppose that conditions* (2.3)-(2.4) *hold. Then, there exists an* $\eta_0^* > 0$ *such that any solution of* (2.1)-(2.2) *satisfies the inequality* $u(0) \geq \eta_0^*$ *and so, by the previous Lemma,*

$$u(t) \geq \alpha_0(\eta_0^*) := \alpha_0^*, \quad t \in [0,1],$$

α_0^* *being a positive constant.*

Proof. Supposing the opposite, we may find a solution $u(t)$ of (2.1)-(2.2), such that $u(0) = \frac{\eta}{2}$, η being the same as in (3.1), i.e. η is arbitrarily small.

Then, noticing (3.2), as above by Taylor formula we can find a $t \in [0,1]$ such that

$$\begin{aligned} u(1) - \beta u(\eta) &\leq \frac{\eta}{2} + g^{-1}(\frac{\eta}{2}) + \frac{1}{2}u''(t) - \beta u(\eta) \\ &< \frac{\eta}{2} + g^{-1}(\frac{\eta}{2}) - \frac{1}{2}Ku(t) \\ &\leq \frac{\eta}{2} + g^{-1}(\frac{\eta}{2}) - \frac{K}{2}\min\{\alpha_0, \frac{\eta}{2}\} \\ &\leq \frac{\eta}{2} + g^{-1}(\frac{\eta}{2}) - \frac{K}{2}\frac{\eta}{2}\min\{1, \gamma\} < 0, \end{aligned}$$

due to the choice $K > 2\frac{1+2\mu}{\min\{\gamma,1\}}$. This contradiction completes the proof. □

We give now an existence principle, which is crucial for the proof of our results.

Lemma 3.4. *(Nonlinear Alternative of Leray-Shauder Type)* [2] *Let* V *be a Banach space and* $C \subset V$ *a convex set. Assume that* U *is a relative open subset of* C *with* $u_0 \in U$ *and* $T : \bar{U} \to C$ *a completely continuous (continuous and compact) map. Then either*

(I) T *has a fixed point, or*

(II) *there exists* $u \in \partial U$ *and* $\lambda \in (0,1)$ *with* $u = \lambda T(u) + (1-\lambda) u_0$.

Theorem 3.5. *Assume (A1)-(A4) hold and*

$$\sup_{c>0,\, 0\leq t\leq 1} \frac{c^p}{\int_0^c f(t,u)du} > \frac{p}{p-1}\Big[\frac{1}{1-\beta}\int_\eta^1 [q(t)]^{1/p} dt + \int_0^\eta [q(t)]^{1/p} dt\Big]^p. \tag{3.4}$$

Then the 3-point boundary-value problem (2.1)-(2.2) *has at least a positive solution.*

Proof. In order to show that (2.1)-(2.2) has a solution, we consider the boundary-value problem

$$\begin{gathered} -[\phi_p(u')]' = q(t)F(t,u(t)), \quad 0 < t < 1, \\ u(0) - g(u'(0)) = 0, \quad u(1) - \beta u(\eta) = 0, \end{gathered} \tag{3.5}$$

where

$$F(t,u(t)) = \begin{cases} f(t,u), & u \geq \alpha_0^* \\ f(t,\alpha_0^*), & u < \alpha_0^* \end{cases}$$

and α_0^* is given in Proposition 3.3. Then clearly $F \in C([0,1] \times [0,+\infty), [0,+\infty))$. Consider further the family of problems

$$\begin{gathered} -[\phi_p(u')]' = q(t)\lambda F(t,u(t)), \quad 0 < t < 1,\ 0 < \lambda < 1 \\ u(0) - g(u'(0)) = 0, \quad u(1) - \beta u(\eta) = 0. \end{gathered} \tag{3.6}$$

If $u = u(t)$, $t \in [0,1]$ is a solution of (3.6), then again by Proposition 3.3, $u(t) \geq \alpha_0^*$, $t \in [0,1]$. We are going to prove the existence of another constant $A_0^* > \alpha_0^*$ such that $u(0) \leq A_0^*$. Indeed, setting $(u(0), u'(0)) = (u_0, u_0') \in E$ and since

$$u'(t) = \phi_p^{-1}[\phi_p(u_0') - \lambda \int_0^t q(s)F(s,u(s))ds],$$
$$u(t) = u_0 + \int_0^t \phi_p^{-1}[\phi_p(u_0') - \lambda \int_0^t q(s)F(s,u(s))ds]dt,$$

the initial values must be chosen so that

$$\begin{aligned} Q(u_0') &:= u(1) - \beta u(\eta) \\ &= g(u_0') + \int_0^1 \phi_p^{-1}[\phi_p(u_0') - \lambda \int_0^t q(s)F(s,u(s))ds]dt \\ &\quad - \beta[g(u_0') + \int_0^\eta \phi_p^{-1}[\phi_p(u_0') - \lambda \int_0^t q(s)F(s,u(s))ds]dt] = 0. \end{aligned} \tag{3.7}$$

By Proposition 3.3 and its proof, there is an $\eta > 0$ such that $Q(g^{-1}(\frac{\eta}{2})) < 0$, and moreover by the definition of Q,

$$Q\{\phi_p^{-1}\big(\lambda \int_0^t q(s)F(s,u(s))ds\big)\} > 0.$$

Hence u_0' is upper bounded and similarly $u_0 = g(u_0')$, i.e. $(u_0, u_0') \in E_0 \subset E$, E_0 being a compact subset of $\mathbb{R}^2$.

We consider now the Banach space $B = C[0,1]$ and for any $x \in B$, let $u = u(t)$ be a solution of the boundary-value problem

$$-[\phi_p(u')]' = q(t)F(t,x(t)), \quad 0 < t < 1,$$
$$u(0) - g(u'(0)) = 0, \quad u(1) - \beta u(\eta) = 0.$$

By the monotonicity of functions g and Q for each solution $u(t)$ of this boundary-value problem, its initial value (u_0, u_0') is uniquely determined and furthermore the map

$$x \to \phi_p[g^{-1}(u_0)]$$

is continuous (see [20]).

Consider now the operator

$$T_\lambda x(t) = u_0 + \int_0^t \phi_p^{-1}\big[\phi_p[g^{-1}(u_0)\big] - \lambda \int_0^s q(r)F(r,x(r))dr]ds, \; x \in B,$$

where u_0 is the unique constant corresponding to function $x(t)$ and satisfying (3.7). It is easily verified that $u(t)$ is a solution to (3.6) if and only if u is a fixed point of T_1 in $C[0,1]$.

(I) We shall prove that $T = T_1 : B \to B$ is completely continuous. By continuity of the map $x \to \phi_p[g^{-1}(u_0)]$ it is not difficult to be proved that T is continuous. So we must only prove that T is compact i.e. it maps every bounded subset of B into a relatively compact set. Consider the closed ball $\Sigma = \{x \in B : \|x\| \leq R\}$. Since $\alpha_0^* \leq u(0) \leq A_0^*$,

$$\begin{aligned} \|(Tx)(t)\| &\leq A_0^* + \phi_p^{-1}[\phi_p(g^{-1}(A_0^*)) + \max_{x\in\Sigma, s\in[0,1]} F(s,x) \int_0^t q(s)ds] \\ &\leq A_0^* + \phi_p^{-1}(M_1) \end{aligned}$$

and

$$\|(Tx)'(t)\| = \phi_p^{-1}[\phi_p(g^{-1}(A_0^*)) + \int_0^t q(s)F(s,x(s))ds] \le \phi_p^{-1}(M_1).$$

Hence the Arzela-Ascoli Theorem guarantees the compactness of T.

(II) We will show that there exists a $M > 0$ such that $\|u\| \le M$ for any solution of (3.6). We set

$$G(c) = \int_0^c \max_{0\le t\le 1} f(t,u)ds$$

Noting (3.4), we may indeed find a $M > 0$ such that

$$\frac{M}{\left(\int_0^M \sup_{0\le t\le 1} f(t,u)du\right)^{\frac{1}{p}} (\frac{p}{p-1})^{\frac{1}{p}} \left[\frac{1}{1-\beta}\int_\eta^1 [q(t)]^{1/p}dt + \int_0^\eta [q(t)]^{1/p}dt\right]} > 1.$$

Also by Proposition 3.3, any solution of the boundary-value problem (3.6) is convex and there exists a point $t_0 \in (0,1)$ such that

$$u'(t) \ge 0,\ t\in[0,t_0),\ u'(t_0) = 0 \quad \text{and} \quad u'(t) \le 0,\ t \in (t_0,1].$$

Working in the interval $[t_0,t] \subset [t_0,1]$, we have

$$0 \le -(\phi_p(u'))' = \lambda q(t)F(t,u) \le q(t)\max_{0\le t\le 1} f(t,u).$$

Multiplying by $-u' > 0$, we get

$$(\phi_p(u'))'\phi_p^{-1}(\phi_p(u')) \le q(t)\max_{0\le t\le 1} f(t,u), \quad t\in[t_0,1]$$

and then integrating on $[t_0,t]$, we obtain

$$\begin{aligned}\int_0^{\phi_p(u'(t))} \phi_p^{-1}(u'(t))u'(t)dt &\le q(t)\int_{u(t)}^{u(t_0)} \max_{0\le t\le 1} f(t,u)du \\ &\le q(t)\int_0^{u(t_0)} \max_{0\le t\le 1} f(t,u)du = q(t)G(u(t_0)),\end{aligned}$$

hence

$$I(-\phi_p(u'(t))) = I(\phi_p(u'(t))) \le q(t)G(u(t_0))$$

and so

$$0 \le -u'(t) \le \phi_p^{-1}\{[I^{-1}(q(t))]I^{-1}(G(u(t_0)))\}, \quad t\in[t_0,t]. \tag{3.8}$$

If $\eta \in (t_0,1]$, an integration over $[\eta,1]$ yields

$$u(\eta) - u(1) \le \phi_p^{-1}\{I^{-1}[(G(u(t_0)))]\}\int_\eta^1 \phi_p^{-1}\{I^{-1}((q(t))\}.$$

If $\eta \in (0,t_0]$, we integrate over $[t_0,1]$ to obtain

$$\begin{aligned}u(t_0) - u(1) &\le \phi_p^{-1}\{I^{-1}(G(u(t_0)))\}\int_{t_0}^1 \phi_p^{-1}[I^{-1}(q(t))]dt \\ &\le \phi_p^{-1}\{I^{-1}(G(u(t_0)))\}\int_\eta^1 \phi_p^{-1}[I^{-1}(q(t))]dt.\end{aligned}$$

Then clearly it follows that

$$u(\eta) - u(1) \le u(t_0) - u(1) \le \phi_p^{-1}\{I^{-1}(G(u(t_0)))\}\int_\eta^1 \phi_p^{-1}[I^{-1}(q(t))]dt.$$

Moreover, since $u(1)=\beta u(\eta)$, we get

$$u(1) \le \frac{\beta}{1-\beta}\phi_p^{-1}\{I^{-1}(G(u(t_0)))\}\int_\eta^1 \phi_p^{-1}[I^{-1}(q(t))]dt$$

and so a new integration from t_0 to 1 of (3.8) yields

$$\begin{aligned}
u(t_0) &= u(1) + \phi_p^{-1}\{I^{-1}(G(u(t_0)))\}\int_{t_0}^1 \phi_p^{-1}[I^{-1}(q(t))]dt \\
&\le u(1) + \phi_p^{-1}\{I^{-1}(G(u(t_0)))\}\int_0^1 \phi_p^{-1}[I^{-1}(q(t))]dt \\
&\le \phi_p^{-1}\{I^{-1}(G(u(t_0)))\}[\frac{\beta}{1-\beta}\int_\eta^1 \phi_p^{-1}[I^{-1}(q(t))]dt + \int_0^1 \phi_p^{-1}[I^{-1}(q(t))]dt] \\
&\le \phi_p^{-1}\{I^{-1}(G(u(t_0)))\}\big[\frac{1}{1-\beta}\int_\eta^1 \phi_p^{-1}[I^{-1}(q(t))]dt + \int_0^\eta \phi_p^{-1}[I^{-1}(q(t))]dt\big] \\
&= \phi_p^{-1}\{I^{-1}(G(u(t_0)))\}(\frac{p}{p-1})^{\frac{1}{p}}\big[\frac{1}{1-\beta}\int_\eta^1 q^{\frac{1}{p}}(t)dt + \int_0^\eta q^{\frac{1}{p}}(t)dt\big].
\end{aligned}$$

Consequently

$$\frac{u(t_0)}{\phi_p^{-1}\{I^{-1}(G(u(t_0)))\}(\frac{p}{p-1})^{\frac{1}{p}}\big[\frac{1}{1-\beta}\int_\eta^1 q^{\frac{1}{p}}(t)dt + \int_0^\eta q^{\frac{1}{p}}(t)dt\big]} < 1$$

which by the assumption (3.4) implies that $u(t_0) < M$. Finally in view of Lemma 3.4, we may set

$$C := \{u \in B = C[0,1] : \|u\| \le M\} \quad \text{and } U := \{u \in C : \|u\| < M\}.$$

Then, the second part of the nonlinear Alternative of Leray-Shauder Type is ruled out and so we conclude that there exists a fixed point of the operator

$$Tx(t) = T_1x(t) = u_0 + \int_0^t \phi_p^{-1}\Big[\phi_p[g^{-1}(u_0)] - \int_0^s q(r)F(r,x(r))dr\Big]ds.$$

This of course yields a solution $u=u(t)$ of (3.5) and noting Proposition 3.3 and the definition of the modification $F, u(t)$ is actually a solution of our original boundary-value problem (2.1)-(2.2). □

4. Existence for the second boundary-value problem

In the following we will study the boundary-value problem (1.1)-(1.3). For this purpose, we give the next result concerning the boundary-value problem

$$-[\phi_p(y'(s))]' = q(s)f(s,y(s)), \quad 0<t<1, \tag{4.1}$$

$$y(0) - g(y'(0)) = 0, \quad y(1) - \alpha y(1-\eta) = 0\,. \tag{4.2}$$

Theorem 4.1. *Assume that (A1), (A3) and (A4) hold. Instead of (A2) we assume*

(A2*) $f \in C([0,1]\times(-\infty,0),(-\infty,0))$.

Also assume that

$$\sup_{c<0,\,0\le t\le 1}\frac{(-c)^p}{\int_c^0 f(t,u)du} > \frac{p}{p-1}\Big[\frac{1}{1-\alpha}\int_{1-\eta}^1 [q(t)]^{1/p}dt + \int_0^{1-\eta}[q(t)]^{1/p}dt\Big]^p. \tag{4.3}$$

Then, the 3-point boundary-value problem (4.1)-(4.2) *has at least one negative solution.*

To prove the above Theorem, we give some lemmas symmetrical to the previous case, with similar proofs which are partly omitted.

Note that the differential equation (4.1) defines also a vector field. So if we focus on the (y, y') face semi-plane $(y < 0)$, then by (4.3), we see that $y'' > 0$. Thus, any trajectory $(y(t), y'(t))$, $t \geq 0$, emanating from the semi-line

$$E^* := \{(y, y') : y - g(y') = 0, \quad y < 0\}$$

"trends" in a natural way, (when $y'(t) < 0$) toward the negative y-semi-axis and then (when $y'(t) > 0$) trends toward the positive y'-semi-axis. As a result, we may control the vector field, so that $y(1) + \alpha y(\eta) = 0$.

Lemma 4.2. *Let $0 < \alpha < 1$. If $y \in C[0, 1]$ is a solution of the boundary-value problem* (4.1)-(4.2)*, then y is convex. Furthermore for every solution with $y(0) < 0$, it follows that*

(i) $y(t) < 0$*, $t \in [0, 1]$*
(ii) *There exists a $t_0 \in [0, 1)$ such that*
$$\sup_{t\in[\eta,1]} y(t) \leq \delta y(t_0) = -\delta\|y\|,$$
where $\delta = \min\{\alpha(1 - \eta), \frac{\alpha(1-\eta)}{1-\alpha(1-\eta)}\}$
(iii) *There exists a $t_0 \in [0, 1)$ such that $y(t_0) = -\max_{0\leq t\leq 1} y(t) = -\|y\|$*

Proof. Let $y(t)$ be a solution of (4.1)-(4.2). Then since,

$$[\phi_p(y')]' = -q(t)f(t, y(t)) \geq 0,$$

$[\phi_p(y')]$ is nondecreasing and so is $y'(t)$, a fact that implies the convexity of $y(t)$.

(i) Since $y(0) < 0$, by the first condition in (4.2) we get $y'(0) < 0$. If $y'(t) \leq 0$, $t \in [0, 1]$, then $y(1) \leq y(\eta) < \alpha y(\eta)$, a contradiction. Hence, there exists a $t_0 > 0$ such that $y(t_0) = \min_{0\leq t\leq 1} y(t) = -\|y\|$.

(ii) If $1 - \eta \in (0, t_0)$, then $y(1 - \eta) < y(0) < 0$ and so $y(1) = \alpha y(1 - \eta) < 0$. If $1 - \eta \in (t_0, 1)$ then $y(1 - \eta) < y(1)$ and so

$$0 = y(1) - \alpha y(1 - \eta) > y(1) - \alpha y(1).$$

Hence, $y(1) < 0$. Finally, the convexity of $y(t)$ yields $y(t) < 0$, $t \in [0, 1]$.

(iii) The proof follows by the convexity of the solution and since it is analogous to the given one at Lemma 2.2, we omit it. □

Lemma 4.3. *Suppose that conditions* (2.3) *and* (2.4) *hold. Then, there exists an $\eta_0 < 0$ such that for any $\eta \in (\eta_0, 0)$, any solution of* (4.1) *with $y(0) = \eta/2$, satisfies the inequality*

$$\eta_0 \leq \eta \leq y(t) < 0, \quad t \in [0, 1].$$

Furthermore, there exists an $\alpha_0 = \alpha_0(\eta) < 0$ such that any (possible) solution $y = y(t)$ of (4.1)-(4.2) *with $y(0) = \frac{\eta}{2}$, satisfies the inequality*

$$y(t) \leq \alpha_0(\eta), \quad t \in [0, 1]. \tag{4.4}$$

Proof. By the sublinearity of the function $f(t, y)$ at the point $y = 0$, for every $K > \max\{2\mu^2, 2\frac{1+2\mu}{\min\{\delta,1\}}\}$ there exists an $\eta_0 < 0$ such that

$$\max_{0\leq t\leq 1} f(t, y) < Ky, \quad \eta_0 \leq y < 0. \tag{4.5}$$

Consider a solution $y = y(t)$ of (4.1) with $y(0) = \frac{\eta}{2}$, where $\eta \in (\eta_0, 0)$ is chosen small enough so that
$$\frac{\min_{0\le t\le 1} q(t)}{(p-1)\eta^{p-2}} > 1.$$
We shall prove first that $\eta \le y(t) < 0$, $t \in [0,1]$. If not, by Lemma 4.2, there exists a $t^* \in (0,1]$ such that $\eta \le y(t) < \frac{\eta}{2}$, $0 \le t < t^*$ and $y(t^*) = \eta$. Then, by (4.5) follows that
$$[\phi_p(y')]' = [\phi_p'(y')]y'' = -q(t)f(t,y) \ge -Kq(t)y(t) \ge -Kq(t)\frac{\eta}{2},$$
i.e.
$$\begin{aligned} y''(t) &\ge -Kq(t)\frac{\eta}{2}\frac{1}{[\phi_p'(y')]} \\ &\ge -K\frac{\eta}{2}\frac{1}{[\phi_p'(y')]}\max_{0\le t\le 1} q(t) \\ &\ge -K\frac{\eta}{2}\frac{\max_{0\le t\le 1} q(t)}{M_3} \ge -K\frac{\eta}{2} > 0, \end{aligned}$$
where, noticing the monotonicity of $\phi_p'(s) = (p-1)(-s)^{p-2} > 0$, $s < 0$ and of $y'(t)$, $0 \le t < 1$,
$$M_3 = \max\big\{\phi_p'(y') = \begin{cases} \phi_p'(g^{-1}[\eta/2]), & \text{if } p > 2 \\ \phi_p'(g^{-1}[\eta]), & \text{if } p \in (1,2) \end{cases}\big\} > 0$$
Consequently, by (3.3) and Taylor's formula we conclude that for some $t \in [0,t^*]$,
$$\begin{aligned} \eta = y(t^*) &= \frac{\eta}{2} + t^*g^{-1}(\frac{\eta}{2}) + \frac{(t^*)^2}{2}y''(t) \\ &\ge \frac{\eta}{2} + t^*g^{-1}(\frac{\eta}{2}) - \frac{(t^*)^2}{2}K\frac{\eta}{2} \\ &\ge \frac{\eta}{2} + t^*2\mu\frac{\eta}{2} - \frac{(t^*)^2}{2}K\frac{\eta}{2}. \end{aligned}$$
Considering now the map
$$\phi(t^*) := Kt^{*2} + 4\mu t^* + 2,$$
the above inequality yields $\phi(t^*) \le 0$, given that $\eta < 0$. This is a contradiction, since the above choice of $K > 2\mu^2$, yields $\phi(t) > 0$ for all $t \in [0,1]$. Consequently, noticing Lemma 4.2, we obtain $\eta \le y(t) < 0$, $t \in [0,1]$.

We proceed now with the proof of inequality (4.4). By the convexity of $y(t)$, it is obvious that
$$\max_{t\in[0,1]} y(t) = \max\{y(0),\ y(1)\}.$$
However, in view of the same Lemma 4.2,
$$\sup_{t\in[\eta,1]} y(t) \le \delta y(t_0) = -\delta\|y\| \le \delta y(0) = \delta\frac{\eta}{2}$$
and so
$$y(t) \le \sup_{t\in[\eta,1]} y(t) \le \frac{\eta}{2}\min\{1,\delta\} := \alpha_0(\eta) < 0.$$
$\square$

Proposition 4.4. *Suppose that conditions* (2.3) *and* (2.4) *hold. Then there exists an* $\eta_0^* < 0$ *such that any solution of* (4.1)-(4.2) *satisfies the inequality* $y(0) \le \eta_0^*$ *and furthermore,*

$$y(t) \le \alpha_0(\eta_0^*) := \alpha_0^*, \quad t \in [0,1],$$

α_0^* *being a negative constant.*

Proof. Supposing the opposite, we may find a solution $y(t)$ of the boundary-value problem (4.1)-(4.2), such that $y(0) = \frac{\eta}{2}$, with $\eta \in (\eta_0, 0)$, η_0 being the same as in the previous Lemma. Then, noticing (4.5), we may find a $t \in [0,1]$ such that

$$\begin{aligned} y(1) - \alpha y(1-\eta) &\ge \frac{\eta}{2} + g^{-1}(\frac{\eta}{2}) + \frac{1}{2} y''(t) - \alpha y(1-\eta) \\ &> \frac{\eta}{2} + g^{-1}(\frac{\eta}{2}) - \frac{1}{2} K y(t) \\ &\ge \frac{\eta}{2} + g^{-1}(\frac{\eta}{2}) - \frac{K}{2} \min\{\alpha_0(\eta), \frac{\eta}{2}\} \\ &\ge \frac{\eta}{2} + g^{-1}(\frac{\eta}{2}) - \frac{K}{2}\frac{\eta}{2} \min\{1, \delta\} > 0, \end{aligned}$$

due to the choice $K > 2\frac{1+2\mu}{\min\{\delta,1\}}$ and since $y(1-\eta) < 0$. This contradiction completes the proof. $\square$

Proof of Theorem. 4.1. To show that (4.1)-(4.2) has a solution, we consider the problem

$$\begin{gathered} -[\phi_p(y')]' = q(t) F(t, y(t)), \quad 0 < t < 1, \\ y(0) - g(y'(0)) = 0, \quad y(1) - \alpha y(1-\eta) = 0, \end{gathered} \tag{4.6}$$

where

$$F(t, y(t)) = \begin{cases} f(t,y), & \text{if } y \le \alpha_0^* \\ f(t, f(t, \alpha_0^*)), & \text{if } y > \alpha_0^*. \end{cases}$$

Then, clearly $F \in C([0,1] \times (-\infty, 0], (-\infty, 0))$. Consider now the family of problems

$$\begin{gathered} -[\phi_p(y')]' = q(t) \lambda F(t, y(t)), \quad 0 < t < 1, \ 0 < \lambda < 1 \\ y(0) - g(y'(0)) = 0, \quad y(1) - \alpha y(1-\eta) = 0. \end{gathered} \tag{4.7}$$

Let $y = y(t)$, $t \in [0,1]$, be a solution of (4.7). By Proposition 4.4, there is a (fixed) $\alpha_0^* < 0$ such that $y(t) \le \alpha_0^*$, $t \in [0,1]$. We are going to prove the existence of another constant $A_0^* < \alpha_0^*$ such that $y(0) \ge A_0^*$. This is the case since, setting $(y(0), y'(0)) = (y_0, y_0') \in E^*$ we obtain

$$\begin{gathered} y'(t) = \phi_p^{-1}\Big[\phi_p(y_0') - \lambda \int_0^t q(s) F(s, y(s)) ds\Big], \\ y(t) = y_0 + \int_0^t \phi_p^{-1}\Big[\phi_p(y_0') - \lambda \int_0^t q(s) F(s, y(s)) ds\Big] dt. \end{gathered} \tag{4.8}$$

The initial values must be chosen so that

$$\begin{aligned} Q^*(y_0') := y(1) - \alpha y(1-\eta) = g(y_0') + \int_0^1 \phi_p^{-1}\Big[\phi_p(y_0') - \lambda \int_0^t q(s) F(s, y(s)) ds\Big] dt \\ - \alpha\Big[g(y_0') + \int_0^{1-\eta} \phi_p^{-1}\Big[\phi_p(y_0') - \lambda \int_0^t q(s) F(s, y(s)) ds\Big] dt\Big] = 0. \end{aligned}$$

By Proposition 4.4 and its proof, there exists an $\eta \in (\eta_0, 0)$ such that $Q^*(g^{-1}(\frac{\eta}{2})) > 0$ and moreover by the definition of Q^* and the fact that $g(y_0') < 0$,

$$Q^*\Big\{\phi_p^{-1}\Big(\lambda \int_0^t q(s)F(s, y(s))ds\Big)\Big\} < 0.$$

Hence y_0' is lower bounded and similar $y_0 = g(y_0')$, i.e. $(y_0, y_0') \in E_0^*$ with E_0^* a compact subset of $\mathbb{R}^2$. Furthermore, by the monotonicity of the functions g and Q^*, for each solution $y(t)$ of the boundary-value problem (4.7), its initial value (y_0, y_0') is uniquely determined and continuous.

Consider now the operator

$$T_\lambda y(t) = y_0 + \int_0^t \phi_p^{-1}\Big[\phi_p[g^{-1}(y_0)] - \lambda \int_0^s q(r)F(r, y(r))dr\Big]ds,$$

where y_0 is the unique constant corresponding to the function $y(t)$ satisfying (4.8). It is easily verified that $y(t)$ is a solution of (4.6) if and only if y is a fixed point of T_1 in $C[0,1]$.

(i) We consider the Banach space $B = C[0,1]$ and we may easily, as above, prove that $T := T_1 : B \to B$ is completely continuous.

(ii) We will show that there exists a $M > 0$ such that $\|y\| \le M$ for any solution of (4.7). We set

$$G(c) = \int_c^0 \min_{0\le t\le 1} f(t, y)ds > 0, \quad c \le 0.$$

Noting (4.3), we may find a $M > 0$ such that

$$\frac{-M}{\big(\int_{-M}^0 \min_{0\le t\le 1} f(t, y)dy\big)^{\frac{1}{p}} (\frac{p}{p-1})^{\frac{2}{p}} \big[\frac{1}{1-\alpha}\int_{1-\eta}^1 [q(t)]^{1/p}dt + \int_0^{1-\eta}[q(t)]^{1/p}dt\big]} < -1.$$

Also by the previous, any solution of (4.7) being convex, satisfies

$$y(t) \le \alpha_0(\eta_0^*) := \alpha_0^* < 0, \quad t \in [0,1],$$

and further there is a $t_0 \in (0,1)$ such that

$$y'(t) \le 0, \quad t \in [0, t_0), \quad y'(t_0) = 0 \quad \text{and} \quad y'(t) \ge 0, \ t \in (t_0, 1].$$

Working in the interval $[t_0, t] \subset [t_0, 1]$, we have

$$-(\phi_p(y'))' = \lambda q(t)F(t, y) \ge q(t) \min_{t\in[0,1]} F(t, y).$$

Multiplying by $-y' < 0$, integrating on $[t_0, t]$, and given that $q(t)$ is non-increasing, we obtain

$$\begin{aligned}
\int_0^{\phi_p[y'(t)]} \phi_p^{-1}(z)dz &\le -q(t)\int_{y(t_0)}^{y(t)} \min_{0\le t\le 1} F(t, y)dy \\
&\le -q(t)\int_{y(t_0)}^{0} \min_{0\le t\le 1} F(t, y)dy \\
&= -q(t)G(y(t_0)) < 0.
\end{aligned}$$

Hence,

$$I(\phi_p^{-1}(y'(t))) \le -q(t)G(y(t_0)) \le q(t)G(y(t_0))$$

and so

$$0 \le y'(t) \le \phi_p^{-1}\{I^{-1}(q(t))I^{-1}[G(y(t_0))]\}, \quad t \in [t_0, t]. \tag{4.9}$$

If $1-\eta \in (t_0, 1]$, an integration over $[1-\eta, 1]$ yields

$$y(1) - y(1-\eta) \le \phi_p^{-1}[I^{-1}(G(y(t_0)))] \int_{1-\eta}^{1} \phi_p^{-1}[I^{-1}(q(t))]dt.$$

If $1-\eta \in (0, t_0]$, we integrate over $[t_0, 1]$ to obtain

$$\begin{aligned} y(1) - y(t_0) &\le \phi_p^{-1}[I^{-1}(G(y(t_0)))] \int_{t_0}^{1} \phi_p^{-1}[I^{-1}(q(t))]dt \\ &\le \phi_p^{-1}[I^{-1}(G(y(t_0)))] \int_{1-\eta}^{1} \phi_p^{-1}[I^{-1}(q(t))]dt. \end{aligned}$$

Since $y(t_0) \le y(1-\eta)$, it follows that

$$y(1) - y(1-\eta) \le y(1) - y(t_0) \le \phi_p^{-1}[I^{-1}(G(y(t_0)))] \int_{1-\eta}^{1} \phi_p^{-1}[I^{-1}(q(t))]dt.$$

Moreover, since $y(1) = \alpha y(1-\eta)$, we get

$$y(1) \ge \frac{\alpha}{\alpha - 1} \phi_p^{-1}[I^{-1}(G(y(t_0)))] \int_{1-\eta}^{1} \phi_p^{-1}[I^{-1}(q(t))]dt,$$

and so a new integration from t_0 to 1 of (4.9) yields

$$\begin{aligned} &y(t_0) \\ &= y(1) - \phi_p^{-1}[I^{-1}(G(y(t_0)))] \int_{t_0}^{1} \phi_p^{-1}[I^{-1}(q(t))]dt \\ &\ge y(1) - \phi_p^{-1}[I^{-1}(G(y(t_0)))] \int_{0}^{1} \phi_p^{-1}[I^{-1}(q(t))]dt \\ &\ge \phi_p^{-1}[I^{-1}(G(y(t_0)))][\frac{\alpha}{\alpha-1} \int_{1-\eta}^{1} \phi_p^{-1}[I^{-1}(q(t))]dt - \int_0^1 \phi_p^{-1}[I^{-1}(q(t))]dt] \\ &= -\phi_p^{-1}[I^{-1}(G(y(t_0)))][\frac{1}{1-\alpha} \int_{1-\eta}^{1} \phi_p^{-1}[I^{-1}(q(t))]dt + \int_0^{1-\eta} \phi_p^{-1}[I^{-1}(q(t))]dt] \\ &= -\phi_p^{-1}[I^{-1}(G(y(t_0)))](\frac{p}{p-1})^{\frac{1}{p}}[\frac{1}{1-\alpha} \int_{1-\eta}^{1} [q(t)]^{\frac{1}{p}}(t)dt + \int_0^{1-\eta} [q(t)]^{\frac{1}{p}}(t)dt]. \end{aligned}$$

Consequently, (recall that $y(t_0) < 0$)

$$\frac{y(t_0)}{[G(y(t_0))]^{\frac{1}{p}} (\frac{p}{p-1})^{\frac{1}{p}} [\frac{1}{1-\alpha} \int_{1-\eta}^{1} [q(t)]^{\frac{1}{p}}(t)dt + \int_0^{1-\eta} [q(t)]^{\frac{1}{p}}(t)dt]} > -1,$$

which in turn, by the assumption (4.3) and the choice of M, implies $y(t_0) > -M$. Hence we obtain $\|y\| \le M$.

Finally, in view of Lemma 3.4, we may set

$$C := \{y \in B = C[0,1] : \|y\| \le M\} \quad \text{and} \quad U := \{y \in C : \|y\| < M\}.$$

Then, the second part of the nonlinear Alternative of Leray-Shauder Type is ruled out and so we conclude that there exists a fixed point of the operator

$$Ty(t) = T_1 y(t) = y_0 + \int_0^t \phi_p^{-1}\Big[\phi_p[g^{-1}(y_0)] - \int_0^s q(r)F(r, y(r))dr\Big]ds.$$

This of course yields a solution $y = y(t)$ of the boundary-value problem (4.6) and noting Proposition 3.3 and the definition of the modification $F, y(t)$ is actually a solution of our boundary-value problem (4.1)-(4.2). $\square$

Consider now the boundary-value problem

$$-[\phi_p(u'(s))]' = q^*(s)f^*(s, u\ (s)), \quad 0 < t < 1, \tag{4.10}$$

$$u(0) - \alpha u(\eta) = 0, \quad u(1) - g(u'(1)) = 0\,. \tag{4.11}$$

Theorem 4.5. *Under the assumptions of Theorem 3.5, the boundary-value problem* (4.10)–(4.11) *has at least one positive solution.*

Proof. We make the transformation $u(s) = -y(1-s)$, $s \in (0, 1)$, where $y = y(t)$ is a solution of the boundary-value problem (4.1)-(4.2) (it exists by Theorem 4.1). Then, clearly

$$u'(s) = y'(1-s), \quad \phi_p(u'(s)) = \phi_p(y'(1-s))$$

and

$$\begin{aligned}-(\phi_p(u'(s)))' &= -(\phi_p(y'(1-s)))' = q(1-s)f(1-s, y(1-s)) \\ &= q(1-s)f(1-s, -u(s)) := -q^*(s)f^*(s, u(s)),\end{aligned}$$

where $f^*(s, u(s)) := f(1-s, -u(s))$ and $q^*(s) := q(1-s)$, $s \in (0, 1)$. Consequently, the function $u = u(t)$ is a solution of the boundary-value problem

$$[\phi_p(u'(t))]' = -q^*(t)f^*(t, u(t)), \quad 0 < t < 1.$$

Moreover, since $y = y(t)$ satisfies the boundary conditions (4.2), we obtain

$$u(1) + g(u'(1)) = 0, \quad u(0) - \alpha u(\eta) = 0,$$

that is the function $u(s) = -y(1-s)$, $s \in (0, 1)$, is actually the required solution of (4.1)-(4.2). $\square$

5. An example

Consider the boundary-value problem

$$\begin{gathered}' = -\frac{a}{\sqrt{(1-t)}}[u^{-\frac{1}{2}} + \sin^2 u^{-\frac{1}{4}}], \quad 0 < t < 1, \\ u(0) = [u'(0)]^{1/3}, \quad u(1) = \frac{1}{2}y(\frac{1}{3}),\end{gathered} \tag{5.1}$$

where $\phi_p(s) = |s|^{p-2}s$, $p = 3$ and $a > 0$ is a constant.

Comparing to Theorem 3.5, we have chosen $g(v) = v^{1/3}$, $q(t) = a/\sqrt{(1-t)}$, $\beta = 1/2$ and $\eta = 1/3$. It is trivial to verify that assumptions (A1)-(A4) hold true for the system (5.1). Furthermore, since

$$\frac{c^3}{\int_0^c [u^{-1/2} + \sin^2 u^{-1/4}]} \geq \frac{c^3}{c + 2\sqrt{c}}$$

and

$$\frac{3}{3-1}\Big[\frac{1}{1-1/2}\int_{1/3}^1 \frac{a^{1/3}}{(\sqrt{1-t})^{1/3}}dt + \int_0^{1/3} \frac{a^{1/3}}{(\sqrt{1-t})^{1/3}}dt\Big] = \frac{18}{5}a^{1/3},$$

it follows that (3.4) is fulfilled for every $a > 0$. Hence the boundary-value problem (5.1) admits a positive solution.

Remar 5.1. The results in [20] can not be applied to (5.1), since the assumption (H5) is not satisfied. Indeed,

$$\int_{\eta}^{1} f_1(k_2(1-s))q(s)ds = \int_{1/3}^{1} \frac{2a}{k_2^{1/2}(1-s)^{1/2}(1-s)^{1/2}} = +\infty.$$

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