Electronic Journal of Differential Equations, Vol. 2005(2005), No. 116, pp. 1–43.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

PSEUDODIFFERENTIAL OPERATORS WITH GENERALIZED SYMBOLS AND REGULARITY THEORY

CLAUDIA GARETTO, TODOR GRAMCHEV, MICHAEL OBERGUGGENBERGER

Abstract. We study pseudodifferential operators with amplitudes $a_\varepsilon(x,\xi)$ depending on a singular parameter $\varepsilon \to 0$ with asymptotic properties measured by different scales. We prove, taking into account the asymptotic behavior for $\varepsilon \to 0$, refined versions of estimates for classical pseudodifferential operators. We apply these estimates to nets of regularizations of exotic operators as well as operators with amplitudes of low regularity, providing a unified method for treating both classes. Further, we develop a full symbolic calculus for pseudodifferential operators acting on algebras of Colombeau generalized functions. As an application, we formulate a sufficient condition of hypoellipticity in this setting, which leads to regularity results for generalized pseudodifferential equations.

1. Introduction

This paper is devoted to pseudodifferential equations of the form

$$A_\epsilon(x,D)u_\epsilon(x) = f_\epsilon(x),$$

where $x \in \Omega \subset \mathbb{R}^n$, depending on a small parameter $\epsilon > 0$. Equations of this type arise, e. g., in the study of singularly perturbed partial differential equations, in semiclassical analysis, or when regularizing partial differential operators with non-smooth coefficients or pseudodifferential operators with irregular symbols. We take the point of view of asymptotic analysis: the regularity of the right hand side and of the solution as well as the mapping properties of the operator will be described by means of asymptotic estimates in terms of the parameter $\epsilon \to 0$. We will develop a full pseudodifferential calculus in this setting, with formal series expansions of symbols, construction of parametrices and deduction of regularity results. Our investigations will naturally lead us to introducing different scales of growth in the parameter ϵ, rapid decay signifying negligibility and new classes of ϵ-dependent amplitudes, symbols and operators acting on algebras of generalized functions. As

2000 *Mathematics Subject Classification.* 35S50, 35S30, 46F10, 46F30, 35D10.
Key words and phrases. Pseudodifferential operators; small parameter; slow scale net; algebras of generalized functions.

Submitted June 13, 2005. Published October 21, 2005.
C. Garetto was supported by INDAM–GNAMPA, Italy.
T. Gramchev was supported by INDAM–GNAMPA, Italy and by grant PST.CLG.979347 from NATO.
M. Oberguggenberger was supported by project P14576-MAT from FWF, Austria.

another motivation we mention the recent results on the calculus for generalized functions and their applications in geometry and physics, cf. [15, 16, 17].

Before going into a detailed description of the contents of the paper and its relation to previous research, we wish to exhibit some of the essential effects by means of a number of motivating examples.

Example 1.1. *Singularly perturbed differential equations.* The appearance of scales of growth and decay can be seen from two very simple equations on $\mathbb{R}$,

$$\Big(-\epsilon^2\frac{d^2}{dx^2}+1\Big)u_\epsilon = f \tag{1.1}$$

and

$$\Big(-\epsilon^2\frac{d^2}{dx^2}-1\Big)v_\epsilon = f. \tag{1.2}$$

Suppose, for simplicity, that $f\in\mathcal{E}'(\mathbb{R})$ is a distribution with compact support and that we want to solve the equations in $\mathscr{S}'(\mathbb{R})$, the space of tempered distributions. Let

$$U_\epsilon(x)=\frac{1}{2\epsilon}e^{-|x/\epsilon|}\,,\quad V_\epsilon(x)=\frac{1}{\epsilon}\sin\Big(\frac{x}{\epsilon}\Big)H(x)$$

where H denotes the Heaviside function. The (unique) solution of (1.1) in $\mathscr{S}'(\mathbb{R})$ is given by

$$u_\epsilon(x)=U_\epsilon * f(x),$$

while (1.2) has the solutions

$$v_\epsilon(x)=V_\epsilon * f(x)+C_1\sin\Big(\frac{x}{\epsilon}\Big)+C_2\cos\Big(\frac{x}{\epsilon}\Big).$$

The basic asymptotic scale - growth in powers of $\frac{1}{\epsilon}$ - enters the picture, when we regularize a given distribution $f\in\mathcal{E}'(\mathbb{R})$ by means of convolution:

$$f_\epsilon(x)=f*\varphi_\epsilon(x), \tag{1.3}$$

where $\varphi_\epsilon\in\mathcal{C}^\infty_c(\mathbb{R})$ is a mollifier of the form

$$\varphi_\epsilon(x)=\frac{1}{\epsilon}\varphi\Big(\frac{x}{\epsilon}\Big), \tag{1.4}$$

with $\int\varphi(x)dx=1$. Then the family of smooth, compactly supported functions $(f_\epsilon)_{\epsilon\in(0,1]}$ satisfies an asymptotic estimate of the type

$$\forall\alpha\in\mathbb{N},\ \exists N\in\mathbb{N}:\sup_{x\in\mathbb{R}}|\partial^\alpha f_\epsilon(x)|=O(\epsilon^{-N}). \tag{1.5}$$

If we replace the right hand sides in (1.1) and (1.2) by a family of smooth functions f_ϵ enjoying the asymptotic property (1.5) then an estimate of the same type (1.5) holds for the solutions u_ϵ and v_ϵ.

On the other hand, a family of smooth functions $(f_\epsilon)_{\epsilon\in(0,1]}$ satisfying an estimate of the type

$$\forall\alpha\in\mathbb{N},\ \forall q\in\mathbb{N},\ \sup_{x\in\mathbb{R}}|\partial^\alpha f_\epsilon(x)|=O(\epsilon^{q}) \tag{1.6}$$

as $\epsilon\to 0$, will be considered as asymptotically negligible. Clearly, if f_ϵ as right hand side in (1.1) or (1.2) is asymptotically negligible, so are the solutions u_ϵ and v_ϵ (with $C_1=C_2=0$ in the latter case). The condition

$$\exists N\in\mathbb{N}:\ \forall\alpha\in\mathbb{N},\ \sup_{x\in\mathbb{R}}|\partial^\alpha f_\epsilon(x)|=O(\epsilon^{-N}) \tag{1.7}$$

signifies a regularity property of the family $(f_\epsilon)_{\epsilon\in(0,1]}$; it is known [34] that if the regularizations (1.3) of a distribution f satisfy (1.7) then f actually is an infinitely differentiable function.

Now assume the right hand sides in (1.1) and (1.2) are given by compactly supported smooth functions satisfying the regularity property (1.7). We ask whether the corresponding solutions will inherit this property. This is true of the solution u_ϵ to (1.1), as can be seen by Fourier transforming the equation. It is not true of the solutions v_ϵ to (1.2); already the homogeneous part $C_1 \sin\frac{x}{\epsilon} + C_2\cos\frac{x}{\epsilon}$ destroys the property.

However, let us consider equation (1.2) with a different scaling in ϵ, say

$$\Big(-\omega(\epsilon)^2\frac{d^2}{dx^2} - 1\Big)v_\epsilon = f_\epsilon \tag{1.8}$$

with $\omega(\epsilon)\to 0$. If v_ϵ is a solution, we may express the higher derivatives by means of its 0-th derivative and the derivatives of the right hand side:

$$\begin{aligned}
-\frac{d^2}{dx^2}v_\epsilon &= \frac{1}{\omega(\epsilon)^2}f_\epsilon + \frac{1}{\omega(\epsilon)^2}v_\epsilon,\\
-\frac{d^4}{dx^4}v_\epsilon &= \frac{1}{\omega(\epsilon)^2}\frac{d^2}{dx^2}f_\epsilon + \frac{1}{\omega(\epsilon)^2}\frac{d^2}{dx^2}v_\epsilon\\
&= \frac{1}{\omega(\epsilon)^2}\frac{d^2}{dx^2}f_\epsilon - \frac{1}{\omega(\epsilon)^4}f_\epsilon - \frac{1}{\omega(\epsilon)^4}v_\epsilon,
\end{aligned}$$

and so on. Thus if f_ϵ satisfies the regularity property (1.7) and the net $(\omega(\epsilon))_{\epsilon\in(0,1]}$ satisfies

$$\forall p\geq 0: \Big(\frac{1}{\omega(\epsilon)}\Big)^p = O\Big(\frac{1}{\epsilon}\Big) \tag{1.9}$$

as $\epsilon\to 0$ then every solution $(v_\epsilon)_{\epsilon\in(0,1]}$ satisfies (1.7) as well. We shall refer to property (1.9) by saying that $1/\omega(\epsilon)$ forms a *slow scale net*.

This example not only shows the appearance of different asymptotic scales, but also that regularity results in terms of property (1.7) depend on lower order terms in the equation and/or the scales used to describe the asymptotic behavior as $\epsilon\to 0$.

Example 1.2. *Regularity of distributions expressed in terms of asymptotic estimates on the regularizations.* Let $f\in\mathscr{S}'(\mathbb{R}^n)$, $s\in\mathbb{R}$ and φ_ϵ a regularizer as in (1.4). The following assertions about Sobolev regularity hold:

(a) If $f\in H^s(\mathbb{R}^n)$ then

$$\forall\alpha\in\mathbb{N}^n: \|\partial^\alpha f*\varphi_\epsilon\|_{L^2(\mathbb{R}^n)} = O\big(\epsilon^{-(|\alpha|-s)_+}\big) \tag{1.10}$$

where $(\cdot)_+$ denotes the positive part of a real number.

(b) Conversely, if $f\in\mathcal{E}'(\mathbb{R}^n)$ and (1.10) holds, then $f\in H^t(\mathbb{R}^n)$ for all $t < s-n/2$. In addition, f belongs to $H^s(\mathbb{R}^n)$ in case s is a nonnegative integer.

Indeed, it is readily seen that f belongs to $L^2(\mathbb{R}^n)$ if and only if $\|f*\varphi_\epsilon\|_{L^2(\mathbb{R}^n)} = O(1)$. Part (a), for $s<0$, follows easily by Fourier transform, while for $s=k+\tau$ with $k\in\mathbb{N}, 0\leq\tau<1$ the observation that f belongs to $H^s(\mathbb{R}^n)$ if and only if $\partial^\alpha f$ is in $L^2(\mathbb{R}^n)$ for $|\alpha|\leq k$ and in $H^{\tau-1}(\mathbb{R}^n)$ for $|\alpha| = k+1$ may be used. Part (b) for $s<0$ is derived along the lines of [34, Thm. 25.2] by showing that $(1+|\xi|)^s$ times the Fourier transform $\widehat{f}(\xi)$ is bounded. For $s\geq 0$, a similar observation as above concludes the argument.

An analogous characterization for the Zygmund classes $\mathcal{C}^s_*(\mathbb{R}^n)$ has been proven by Hörmann [22]. Further, given a distribution $f \in \mathcal{D}'(\Omega)$, it was already indicated above that f is a smooth function if and only if the regularizations $f * \varphi_\epsilon$ satisfy property (1.7) (suitably localized with the supremum taken on compact sets of Ω).

Example 1.3. *Regularization of operators with non-smooth coefficients.* Consider a linear partial differential operator

$$A(x,D) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha.$$

If its coefficients are distributions, we may form the regularized operator

$$A_\epsilon(x,D) = \sum_{|\alpha|\le m} a_\alpha * \varphi_\epsilon(x) D^\alpha.$$

The regularized coefficients will satisfy an estimate of type (1.5), at least locally on compact sets, and the action of $A_\epsilon(x,D)$ on nets $(u_\epsilon)_{\epsilon\in(0,1]}$ preserves the asymptotic properties (1.5) and (1.6); that is, if $(u_\epsilon)_{\epsilon\in(0,1]}$ enjoys either of these properties, so does $(A_\epsilon(\cdot,D)u_\epsilon)_{\epsilon\in(0,1]}$.

However, the regularity property (1.7) will not be preserved in general, unless the regularization of the coefficients is performed with a slow scale mollifier, that is, by convolution with $\varphi_{\omega(\epsilon)}$ where $\omega(\epsilon)^{-1}$ is a slow scale net. For example, consider the multiplication operator

$$M_\epsilon(x,D)u(x) = \varphi_{\omega(\epsilon)} u(x).$$

Then $(M_\epsilon(x,D))_{\epsilon\in(0,1]}$ maps the space of nets enjoying regularity property (1.7) into itself if and only if $\omega(\epsilon)^{-1}$ is a slow scale net. Indeed, the sufficiency of the slow scale condition is quite clear. To prove its necessity, take a fixed smooth function u identically equal to one near $x = 0$. Then the derivatives $\partial^\alpha(M_\epsilon u)$ have a uniform asymptotic bound $O(\epsilon^{-N})$ independently of $\alpha \in \mathbb{N}^n$ if and only if $\omega(\epsilon)^{-|\alpha|} = O(\epsilon^{-N})$ for all α; that is, if and only if $\omega(\epsilon)^{-1}$ is a slow scale net.

Example 1.4. *L^2-estimates for pseudodifferential operators in exotic classes.* Consider first a symbol $a(x,\xi)$ in the Hörmander class $S^0_{1,0}(\mathbb{R}^{2n})$ (for simplicity, we restrict our discussion to global zero order symbols here). It is well known that the corresponding operator $a(x,D)$ maps $L^2(\mathbb{R}^n)$ continuously into itself, with operator norm depending on a finite number of derivatives of $a(x,\xi)$. More precisely, an estimate of the following form holds (see e. g. [27, Sect. 2.4, Thm. 4.1] and [19, Sect. 18.1], see also [26] and the references therein):

$$\|a(x,D)u\|^2_{L^2(\mathbb{R}^n)} \le c_0^2\|u\||^2_{L^2(\mathbb{R}^n)} + c_1^2 p_l^2(a)\|u\||^2_{L^2(\mathbb{R}^n)} \tag{1.11}$$

where c_0 is a strict upper bound for the L^∞-norm of the symbol a on $\mathbb{R}^{2n}$ and p_l signifies the norm

$$p_l(a) = \max_{|\alpha+\beta|\le l} \sup_{(x,\xi)\in\mathbb{R}^{2n}} |\partial^\alpha_\xi \partial^\beta_x a(x,\xi)|\langle\xi\rangle^{|\alpha|}.$$

In estimate (1.11), l is an integer depending on the type of symbol, but generically is strictly greater than zero. However, if $a(x,\xi)$ is positively homogeneous of order zero with respect to ξ the L^2-continuity holds provided $\partial^\alpha_\xi a(x,\xi)$ is bounded in $\mathbb{R}^n \times \mathbb{S}^{n-1}$, cf. [5].

If the symbol $a(x,\xi)$ belongs to the exotic class $S^0_{1,1}(\mathbb{R}^{2n})$ then $a(x,D)$ will not map $L^2(\mathbb{R}^n)$ into itself, in general [4, Thm. 9], [38]. However, if we regularize by convolution in the x-variable,

$$a_\epsilon(x,\xi) = a(\cdot,\xi) * \varphi_\epsilon(x),$$

we get a family of symbols each of which belongs to the class $S^0_{1,0}(\mathbb{R}^{2n})$, thus maps $L^2(\mathbb{R}^n)$ continuously into itself, but with an operator norm that behaves asymptotically like ϵ^{-N} where N is some integer less or equal to l in (1.11); note that the convolution with the mollifier φ_ϵ does not increase the constant c_0.

Classically, there is a pseudodifferential calculus (including e. g. composition) for symbols in $S^0_{1,0}(\mathbb{R}^{2n})$, but not for exotic classes, like $S^0_{1,1}(\mathbb{R}^{2n})$, in general. The regularization approach bridges this gap: we will develop a full pseudodifferential calculus for classes of regularized symbols in this paper. Estimate (1.11) remains valid with uniform finite bounds in ϵ for symbols $a_\epsilon(x,\xi)$ obtained by convolution from symbols in $S^0_{1,0}(\mathbb{R}^{2n})$, but we will have to face asymptotic growth as $\epsilon \to 0$ in exchange for the lack of L^2-continuity in the case of exotic symbols.

Remark 1.5. *Algebras of generalized functions.* The families of smooth functions $(u_\epsilon)_{\epsilon\in(0,1]}$ satisfying estimate (1.5), globally or possibly only on compact sets, form a differential algebra; the nets $(u_\epsilon)_{\epsilon\in(0,1]}$ of negligible elements form a differential ideal therein. The space of distributions can be embedded into the corresponding factor algebra by means of cut-off and convolution, with a consistent notion of derivatives. The fact that for smooth functions f, the net $(f - f * \varphi_\epsilon)_{\epsilon\in(0,1]}$ is negligible for suitably chosen regularizers φ_ϵ was discovered by Colombeau [6, 7]; thus the multiplication in the factor algebra is also consistent with the product of smooth functions. The factor algebras of nets satisfying (1.5) modulo negligible nets is a suitable framework for studying families of pseudodifferential operators and the asymptotic behavior of their action on functions or generalized functions. We note that a condition similar to the asymptotic negligibility (1.6) was considered by Maslov et al. [29, 30] earlier in the context of asymptotic solutions to partial differential equations.

In introducing factor spaces of families of amplitudes and symbols (modulo negligible ones) as well, we will succeed in this paper to establish a full symbolic calculus of operators acting on generalized functions. This is a new contribution to the field of non-smooth operators. Our essential tools for describing the mapping properties and regularity results will be asymptotic estimates and scales of growth.

We now describe the contents of the paper in more detail. Section 2 serves to introduce the basic notions - asymptotic properties defining the algebras of generalized functions on which our operators will act, the notion of regularity intrinsic to these algebras (the so-called $\mathcal{G}^\infty$-regularity, indicated in (1.7)), some new technical results needed, and a basic theory of integral operators with generalized kernels. In Section 3 we start our theory by studying oscillatory integrals with smooth phase functions and generalized symbols, introduced as equivalence classes of certain nets of smooth symbols modulo negligible ones. Section 4 employs these techniques to introducing and studying pseudodifferential operators with generalized amplitudes, their mapping properties, pseudo-locality with respect to the notion of $\mathcal{G}^\infty$-regularity mentioned above, and their kernels in the sense of the algebras of generalized functions. The full symbolic calculus of our class of generalized pseudodifferential operators is developed in Section 5. It starts with formal series and asymptotic expansions of equivalence classes of symbols, proceeds with the construction of symbols for (generalized) pseudodifferential operators, their transposes and their compositions. The paper culminates in the regularity theory presented in Section 6. We generalize the notion of hypoellipticity to our class of symbols and construct parametrices for these symbols. We show that the solutions to the corresponding pseudodifferential equations are $\mathcal{G}^\infty$-regular in those regions where the right hand sides are $\mathcal{G}^\infty$-regular. Here the importance of different scales of asymptotic growth becomes apparent.

What concerns previous literature on the subject, we mention that $\mathcal{G}^\infty$-regularity was introduced in [34] where it was already applied to prove regularity results for solutions to classical constant coefficient partial differential equations. Completely new effects arise when the coefficients are allowed to be generalized constants, depending on the parameter $\epsilon > 0$. These effects and a regularity theory for such operators was developed in [24], see also [31], and extended to the case of partial differential operators with generalized, non-constant coefficients in [25]. The study of pseudodifferential operators in the setting of algebras of generalized functions was started in [33], developed in a rudimentary version in [32, 36]. A full version with nets of symbols and a full symbolic calculus, albeit for global symbols and in the algebra of tempered generalized functions is due to [11]. Our contribution is the first in the literature containing a full local symbolic calculus of generalized pseudodifferential operators, equivalence classes of symbols, strong $\mathcal{G}^\infty$-regularity results and the incorporation of different scales (the necessity of which was demonstrated in [24]). For microlocal notions of $\mathcal{G}^\infty$-regularity we refer to [20, 21, 23, 32, 39]. Motivating examples from semiclassical analysis can be found in [3, 37]. Further studies of kernel operators in Colombeau algebras including topological investigations are carried out in [9, 12, 13].

2. Basic notions

In this section we recall the definitions and results needed from the theory of Colombeau generalized functions. For details of the constructions we refer to [1, 7, 8, 16, 32, 34]. In the sequel we denote by $\mathcal{E}[\Omega]$, Ω an open subset of $\mathbb{R}^n$, the algebra of all the sequences $(u_\epsilon)_{\epsilon\in(0,1]}$ (for short, $(u_\epsilon)_\epsilon$) of smooth functions $u_\epsilon \in \mathcal{C}^\infty(\Omega)$.

Definition 2.1. $\mathcal{E}_M(\Omega)$ is the differential subalgebra of the elements $(u_\epsilon)_\epsilon \in \mathcal{E}[\Omega]$ such that for all $K \Subset \Omega$, for all $\alpha \in \mathbb{N}^n$ there exists $N \in \mathbb{N}$ with the following property:

$$\sup_{x\in K} |\partial^\alpha u_\epsilon(x)| = O(\epsilon^{-N}) \quad \text{as } \epsilon \to 0.$$

Definition 2.2. We denote by $\mathcal{N}(\Omega)$ the differential subalgebra of the elements $(u_\epsilon)_\epsilon$ in $\mathcal{E}[\Omega]$ such that for all $K \Subset \Omega$, for all $\alpha \in \mathbb{N}^n$ and $q \in \mathbb{N}$ the following property holds:

$$\sup_{x\in K} |\partial^\alpha u_\epsilon(x)| = O(\epsilon^{q}) \quad \text{as } \epsilon \to 0.$$

The elements of $\mathcal{E}_M(\Omega)$ and $\mathcal{N}(\Omega)$ are called moderate and negligible, respectively.

The factor algebra $\mathcal{G}(\Omega) := \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega)$ is the algebra of generalized functions on Ω. As shown e. g. in [34], suitable regularizations and the sheaf properties of $\mathcal{G}(\Omega)$, allow us to define an embedding $\imath$ of $\mathcal{D}'(\Omega)$ into $\mathcal{G}(\Omega)$ extending the constant embedding $\sigma : f \to (f)_\epsilon + \mathcal{N}(\Omega)$ of $\mathcal{C}^\infty(\Omega)$ into $\mathcal{G}(\Omega)$. In the computations of this paper, the following characterization of $\mathcal{N}(\Omega)$ as a subspace of $\mathcal{E}_M(\Omega)$, proved in Theorem 1.2.3 of [16], will be very useful.

Proposition 2.3. *$(u_\epsilon)_\epsilon \in \mathcal{E}_M(\Omega)$ is negligible if and only if*

$$\forall K \Subset \Omega,\ \forall q \in \mathbb{N}, \quad \sup_{x\in K} |u_\epsilon(x)| = O(\epsilon^q) \quad as \quad \epsilon \to 0.$$

We consider now some particular subalgebras of $\mathcal{G}(\Omega)$.

Definition 2.4. Let $\mathbb{K}$ be the field $\mathbb{R}$ or $\mathbb{C}$. We set

$$\mathcal{E}_M = \{(r_\epsilon)_\epsilon \in \mathbb{K}^{(0,1]} : \exists N \in \mathbb{N} : |r_\epsilon| = O(\epsilon^{-N}) \text{ as } \epsilon \to 0\},$$
$$\mathcal{N} = \{(r_\epsilon)_\epsilon \in \mathbb{K}^{(0,1]} : \forall q \in \mathbb{N} : \quad |r_\epsilon| = O(\epsilon^{q}) \text{ as } \epsilon \to 0\}.$$

$\widetilde{\mathbb{K}} := \mathcal{E}_M/\mathcal{N}$ is called the ring of generalized numbers.

In the case of $\mathbb{K} = \mathbb{R}$ we get the algebra $\widetilde{\mathbb{R}}$ of real generalized numbers and for $\mathbb{K} = \mathbb{C}$ the algebra $\widetilde{\mathbb{C}}$ of complex generalized numbers. $\widetilde{\mathbb{R}}$ can be endowed with the structure of a partially ordered ring (for $r, s \in \widetilde{\mathbb{R}}$, $r \le s$ if and only if there are representatives $(r_\epsilon)_\epsilon$ and $(s_\epsilon)_\epsilon$ with $r_\epsilon \le s_\epsilon$ for all $\epsilon \in (0,1]$). $\widetilde{\mathbb{C}}$ is naturally embedded in $\mathcal{G}(\Omega)$ and it can be considered as the ring of constants of $\mathcal{G}(\Omega)$ if Ω is connected. Moreover, using $\widetilde{\mathbb{C}}$, we can define a concept of generalized point value for the generalized functions of $\mathcal{G}(\Omega)$. In the sequel we recall the crucial steps of this construction, referring to Section 1.2.4 in [16] and to [35] for the proofs.

Definition 2.5. On

$$\Omega_M = \{(x_\epsilon)_\epsilon \in \Omega^{(0,1]} : \exists N \in \mathbb{N}, \quad |x_\epsilon| = O(\epsilon^{-N}) \quad \text{as } \epsilon \to 0\},$$

we introduce an equivalence relation given by

$$(x_\epsilon)_\epsilon \sim (y_\epsilon)_\epsilon \ \Leftrightarrow\ \forall q \in \mathbb{N},\ |x_\epsilon - y_\epsilon| = O(\epsilon^q) \quad \text{as } \epsilon \to 0$$

and denote by $\widetilde{\Omega} := \Omega_M/\sim$ the set of generalized points. Moreover, if $[(x_\epsilon)_\epsilon]$ is the class of $(x_\epsilon)_\epsilon$ in $\widetilde{\Omega}$ then the set of compactly generalized points is

$$\widetilde{\Omega}_c = \{\tilde{x} = [(x_\epsilon)_\epsilon] \in \widetilde{\Omega} : \exists K \Subset \Omega,\ \exists \eta > 0 : \forall \epsilon \in (0,\eta],\ x_\epsilon \in K\}.$$

Obviously if the $\widetilde{\Omega}_c$-property holds for one representative of $\tilde{x} \in \widetilde{\Omega}$ then it holds for every representative. Also, for $\Omega = \mathbb{R}$ we have that the factor $\mathbb{R}_M/\sim$ is the usual algebra of real generalized numbers.

In the following $(u_\epsilon)_\epsilon$ and $(x_\epsilon)_\epsilon$ are arbitrary representatives of $u \in \mathcal{G}(\Omega)$ and $\tilde{x} \in \widetilde{\Omega}_c$, respectively. It is clear that the generalized point value of u at $\tilde{x}$,

$$u(\tilde{x}) := (u_\epsilon(x_\epsilon))_\epsilon + \mathcal{N} \tag{2.1}$$

is a well-defined element of $\widetilde{\mathbb{C}}$. An interesting application of this notion is the characterization of generalized functions by their generalized point values.

Proposition 2.6. *Let $u \in \mathcal{G}(\Omega)$. Then $u = 0$ if and only if $u(\tilde{x}) = 0$ for all $\tilde{x} \in \widetilde{\Omega}_c$.*

We continue now our study of $\mathcal{G}(\Omega)$ with the notions of support and generalized singular support.

Definition 2.7. We denote by $\mathcal{E}_{c,M}(\Omega)$ the set of all the elements $(u_\epsilon)_\epsilon \in \mathcal{E}_M(\Omega)$ such that there exists $K \Subset \Omega$ with $\operatorname{supp} u_\epsilon \subseteq K$ for all $\epsilon \in (0,1]$.

Definition 2.8. We denote by $\mathcal{N}_c(\Omega)$ the set of all the elements $(u_\epsilon)_\epsilon \in \mathcal{N}(\Omega)$ such that there exists $K \Subset \Omega$ with $\operatorname{supp} u_\epsilon \subseteq K$ for all $\epsilon \in (0,1]$.

$\mathcal{G}_c(\Omega) := \mathcal{E}_{c,M}(\Omega)/\mathcal{N}_c(\Omega)$ is the algebra of compactly supported generalized functions. Since the map $l : \mathcal{G}_c(\Omega) \to \mathcal{G}(\Omega) : (u_\epsilon)_\epsilon + \mathcal{N}_c(\Omega) \to (u_\epsilon)_\epsilon + \mathcal{N}(\Omega)$ is injective, $\mathcal{G}_c(\Omega)$ is a subalgebra of $\mathcal{G}(\Omega)$ containing $\mathcal{E}'(\Omega)$ as a subspace and $\mathcal{C}^\infty_c(\Omega)$ as a subalgebra. Recalling that for $u \in \mathcal{G}(\Omega)$ and Ω' an open subset of Ω, $u|_{\Omega'}$ is the generalized function in $\mathcal{G}(\Omega')$ having as representative $(u_\epsilon|_{\Omega'})_\epsilon$, it is possible to define the support of u, setting

$$\Omega \setminus \operatorname{supp} u = \{x \in \Omega :\ \exists V(x) \subset \Omega,\ \text{open}, x \in V(x) :\ u|_{V(x)} = 0\}.$$

The map l identifies $\mathcal{G}_c(\Omega)$ with the set of generalized functions in $\mathcal{G}(\Omega)$ with compact support. It is sufficient to observe that if $u \in \mathcal{G}(\Omega)$ with $\operatorname{supp} u \Subset \Omega$, and $\psi \in \mathcal{C}^\infty_c(\Omega)$ is identically equal to 1 in a neighborhood of $\operatorname{supp} u$ then $\psi u := (\psi u_\epsilon)_\epsilon + \mathcal{N}_c(\Omega)$ belongs to $\mathcal{G}_c(\Omega)$ and $u = l(\psi u)$. If we consider an open subset Ω' of Ω, the map $\mathcal{G}_c(\Omega') \to \mathcal{G}_c(\Omega)$: $(u_\epsilon)_\epsilon + \mathcal{N}_c(\Omega') \to (u_\epsilon)_\epsilon + \mathcal{N}_c(\Omega)$ allows us to embed $\mathcal{G}_c(\Omega')$ into $\mathcal{G}_c(\Omega)$.

A generalized function $u \in \mathcal{G}(\Omega)$ can be integrated over a compact subset of Ω, using the definition

$$\int_K u(x)dx := \Big(\int_K u_\epsilon(x)dx\Big)_\epsilon + \mathcal{N};$$

in particular, a generalized function $u \in \mathcal{G}_c(\Omega)$ can be integrated over Ω by means of the prescription

$$\int_\Omega u(x)dx := \Big(\int_V u_\epsilon(x)dx\Big)_\epsilon + \mathcal{N},$$

where V is any compact set containing $\operatorname{supp} u$ in its interior.

Definition 2.9. We denote by $\mathcal{E}^\infty_M(\Omega)$ the set of all the elements $(u_\epsilon)_\epsilon \in \mathcal{E}[\Omega]$ such that for all $K \Subset \Omega$ there exists $N \in \mathbb{N}$ with the following property:

$$\forall \alpha \in \mathbb{N}^n :\quad \sup_{x\in K} |\partial^\alpha u_\epsilon(x)| = O(\epsilon^{-N}) \quad as \quad \epsilon \to 0.$$

$\mathcal{G}^\infty(\Omega) := \mathcal{E}^\infty_M(\Omega)/\mathcal{N}(\Omega)$ is the algebra of regular generalized functions. Theorem 25.2 in [34] shows that $\mathcal{G}^\infty(\Omega) \cap \mathcal{D}'(\Omega) = \mathcal{C}^\infty(\Omega)$. Finally, if $\mathcal{E}^\infty_{c,M}(\Omega) := \mathcal{E}^\infty_M(\Omega) \cap \mathcal{E}_{c,M}(\Omega)$, $\mathcal{G}^\infty_c(\Omega) := \mathcal{E}^\infty_{c,M}(\Omega)/\mathcal{N}_c(\Omega)$ is the algebra of regular compactly supported generalized functions, and $\mathcal{G}^\infty_c(\Omega) \cap \mathcal{E}'(\Omega) = \mathcal{C}^\infty_c(\Omega)$.

As above, it is possible to define the generalized singular support of $u \in \mathcal{G}(\Omega)$ setting

$$\Omega \setminus \operatorname{sing\,supp}_{\mathrm{g}} u = \{x \in \Omega : \exists V(x) \subset \Omega,\ \text{open},\ x \in V(x) :\ u|_{V(x)} \in \mathcal{G}^\infty(V(x))\}.$$

Using the sheaf properties of $\mathcal{G}^\infty(\Omega)$ we can identify the algebra of regular generalized functions with the set of generalized functions in $\mathcal{G}(\Omega)$ having empty generalized singular support. In the same way $\mathcal{G}^\infty_c(\Omega)$ is the set of generalized functions in $\mathcal{G}(\Omega)$ with compact support and empty generalized singular support.

Definition 2.10. Let $\mathscr{S}[\mathbb{R}^n] := \mathscr{S}(\mathbb{R}^n)^{(0,1]}$. The elements of

$$\mathcal{E}_{\mathscr{S}}(\mathbb{R}^n) = \big\{(u_\epsilon)_\epsilon \in \mathscr{S}[\mathbb{R}^n] :\ \forall \alpha,\beta \in \mathbb{N}^n,\ \exists N \in \mathbb{N} :\ \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta u_\epsilon(x)| = O(\epsilon^{-N}) \text{ as } \epsilon \to 0\big\}$$

are called $\mathscr{S}$-moderate. The elements of

$$\begin{aligned}&\mathcal{E}^\infty_{\mathscr{S}}(\mathbb{R}^n)\\&=\big\{(u_\epsilon)_\epsilon\in\mathscr{S}[\mathbb{R}^n]:\ \exists N\in\mathbb{N}:\ \forall\alpha,\beta\in\mathbb{N}^n,\ \sup_{x\in\mathbb{R}^n}|x^\alpha\partial^\beta u_\epsilon(x)|=O(\epsilon^{-N})\text{ as }\epsilon\to 0\big\}\end{aligned}$$

are called $\mathscr{S}$-regular. The elements of

$$\begin{aligned}&\mathcal{N}_{\mathscr{S}}(\mathbb{R}^n)\\&=\big\{(u_\epsilon)_\epsilon\in\mathscr{S}[\mathbb{R}^n]:\ \forall\alpha,\beta\in\mathbb{N}^n,\ \forall q\in\mathbb{N}:\ \sup_{x\in\mathbb{R}^n}|x^\alpha\partial^\beta u_\epsilon(x)|=O(\epsilon^{q})\text{ as }\epsilon\to 0\big\}\end{aligned}$$

are called $\mathscr{S}$-negligible.

The factor algebra $\mathcal{G}_{\mathscr{S}}(\mathbb{R}^n):=\mathcal{E}_{\mathscr{S}}(\mathbb{R}^n)/\mathcal{N}_{\mathscr{S}}(\mathbb{R}^n)$ is the algebra of $\mathscr{S}$-generalized functions while its subalgebra $\mathcal{G}^\infty_{\mathscr{S}}(\mathbb{R}^n):=\mathcal{E}^\infty_{\mathscr{S}}(\mathbb{R}^n)/\mathcal{N}_{\mathscr{S}}(\mathbb{R}^n)$ is called the algebra of $\mathscr{S}$-regular generalized functions. Obviously, $\mathcal{G}_c(\Omega)\subseteq\mathcal{G}_{\mathscr{S}}(\mathbb{R}^n)$ and $\mathcal{G}^\infty_c(\Omega)\subseteq\mathcal{G}^\infty_{\mathscr{S}}(\mathbb{R}^n)$. For $u\in\mathcal{G}_{\mathscr{S}}(\mathbb{R}^n)$ there is a natural definition of Fourier transform, given by $\widehat{u}:=(\widehat{u_\epsilon})_\epsilon+\mathcal{N}_{\mathscr{S}}(\mathbb{R}^n)$. The Fourier transform maps $\mathcal{G}_{\mathscr{S}}(\mathbb{R}^n)$ into $\mathcal{G}_{\mathscr{S}}(\mathbb{R}^n)$, $\mathcal{G}^\infty_{\mathscr{S}}(\mathbb{R}^n)$ into $\mathcal{G}^\infty_{\mathscr{S}}(\mathbb{R}^n)$, $\mathcal{G}_c(\Omega)$ into $\mathcal{G}_{\mathscr{S}}(\mathbb{R}^n)$ and $\mathcal{G}^\infty_c(\Omega)$ into $\mathcal{G}^\infty_{\mathscr{S}}(\mathbb{R}^n)$.

In the sequel, given $\varphi\in\mathcal{C}^\infty_c(\mathbb{R}^n)$, $\tilde{t}\in\widetilde{\Omega}_c$ and $\tilde{\tau}\in\widetilde{\mathbb{R}}_c$, $0\le\tilde{\tau}$ invertible, we denote by $\varphi_{\tilde{t},\tilde{\tau}}\in\mathcal{G}_c(\mathbb{R}^n)$ the generalized function

$$\varphi_{\tilde{t},\tilde{\tau}}(x)=\varphi\big(\frac{x-\tilde{t}}{\tilde{\tau}}\big).$$

Further, we let

$$T_\Omega(\varphi)=\{\varphi_{\tilde{t},\tilde{\tau}}:\ \tilde{\tau}\in\widetilde{\mathbb{R}}_c,\ 0\le\tilde{\tau}\text{ invertible},\ \tilde{t}\in\widetilde{\Omega}_c,\ \mathrm{supp}(\varphi_{\tilde{t},\tilde{\tau}})\subset\Omega\}$$

Proposition 2.11. *Let $u\in\mathcal{G}(\Omega)$. If there is $\varphi\in\mathcal{C}^\infty_c(\mathbb{R}^n)$, $\varphi\ge 0$, $\int\varphi(x)dx=1$ such that*

$$\int u(x)v(x)dx=0\quad\text{in }\widetilde{\mathbb{C}}$$

for all $v\in T_\Omega(\varphi)$ then $u=0$ in $\mathcal{G}(\Omega)$.

Proof. We may assume that u is real valued. If $u\ne 0$ then there exist a representative $(u_\epsilon)_\epsilon$ of u, a natural number q and a sequence $\epsilon_k\to 0$ such that

$$|u_{\epsilon_k}(t_{\epsilon_k})|\ge\epsilon_k^q$$

for all $k\in\mathbb{N}$. On the other hand, there is $N\in\mathbb{N}$ such that

$$\sup_{x\in K}|\nabla u_\epsilon(x)|\le\epsilon^{-N}$$

for sufficiently small $\epsilon\in(0,1]$, where K is a compact subset of Ω containing $(t_\epsilon)_\epsilon$ in its interior. Then

$$\begin{aligned}|u_{\epsilon_k}(x)|&=|u_{\epsilon_k}(t_{\epsilon_k})+(x-t_{\epsilon_k})\cdot\int_0^1\nabla u_{\epsilon_k}(t_{\epsilon_k}+\sigma(x-t_{\epsilon_k}))d\sigma|\\&\ge\epsilon_k^q-|x-t_{\epsilon_k}|\epsilon_k^{-N}\ge\frac{1}{2}\epsilon_k^q\end{aligned}$$

provided $|x-t_{\epsilon_k}|\le\frac{1}{2}\epsilon_k^{N+q}$. Noting that

$$\varphi\big(\frac{x-t_\epsilon}{\epsilon^{N+q+1}}\big)=0$$

when $|x-t_\epsilon| > \frac{1}{2}\epsilon^{N+q}$ eventually and that $u_{\epsilon_k}(x)$ does not change sign for $|x-t_{\epsilon_k}| \leq \frac{1}{2}\epsilon^{N+q}$, we see that

$$\Big|\int u_{\epsilon_k}(x)\varphi\Big(\frac{x-t_{\epsilon_k}}{\epsilon_k^{N+q+1}}\Big)dx\Big| \geq \frac{1}{2}\epsilon_k^{q+n(N+q+1)}.$$

Thus, with $\tilde{t}$ as above and $\tau_\epsilon = \epsilon^{N+q+1}$, we have that

$$\int u(x)\varphi_{\tilde{t},\tilde{\tau}}(x)dx \neq 0 \quad \text{in } \widetilde{\mathbb{C}}$$

contradicting the hypothesis. □

In the sequel, we denote by $L(\mathcal{G}_c(\Omega),\widetilde{\mathbb{C}})$ the space of all $\widetilde{\mathbb{C}}$-linear maps from $\mathcal{G}_c(\Omega)$ into $\widetilde{\mathbb{C}}$. It is clear that every $u \in \mathcal{G}(\Omega)$ defines an element of $L(\mathcal{G}_c(\Omega),\widetilde{\mathbb{C}})$, setting $\jmath(u)(v) = \int_\Omega u(x)v(x)dx$ for $v \in \mathcal{G}_c(\Omega)$. As an immediate consequence of Proposition 2.11 we have that the map $\jmath : \mathcal{G}(\Omega) \to L(\mathcal{G}_c(\Omega),\widetilde{\mathbb{C}}) : u \to \jmath(u)$ is injective. Our interest in $L(\mathcal{G}_c(\Omega),\widetilde{\mathbb{C}})$ is motivated by some specific properties. We begin by defining the restriction of $T \in L(\mathcal{G}_c(\Omega),\widetilde{\mathbb{C}})$ to an open subset Ω' of Ω, as the $\widetilde{\mathbb{C}}$-linear map

$$T_{|_{\Omega'}} : \mathcal{G}_c(\Omega') \to \widetilde{\mathbb{C}} : u \to T((u_\epsilon)_\epsilon + \mathcal{N}_c(\Omega)).$$

By adapting the classical proof concerning the sheaf properties of $\mathcal{D}'(\Omega)$, we obtain the following result.

Proposition 2.12. *$L(\mathcal{G}_c(\Omega),\widetilde{\mathbb{C}})$ is a sheaf.*

Thus it makes sense to define the support of $T \in L(\mathcal{G}_c(\Omega),\widetilde{\mathbb{C}})$, $\operatorname{supp} T$, as the complement of the largest open set Ω' such that $T_{|_{\Omega'}} = 0$.

Proposition 2.13. *For all $u \in \mathcal{G}(\Omega)$, $\operatorname{supp} u = \operatorname{supp} \jmath(u)$.*

Proof. The inclusion $\Omega \setminus \operatorname{supp} u \subseteq \Omega \setminus \operatorname{supp} \jmath(u)$ is immediate. Now let $x_0 \in \Omega \setminus \operatorname{supp} \jmath(u)$. There exists an open neighborhood V of x_0 such that for all $v \in \mathcal{G}_c(V)$, $\jmath(u)(v) = 0$. Therefore, from Proposition 2.11, $u_{|_V} = 0$ in $\mathcal{G}(V)$ and $x_0 \in \Omega \setminus \operatorname{supp} u$. □

We conclude this section with a discussion of operators defined by integrals. In the sequel, π_1 and π_2 are the usual projections of $\Omega \times \Omega$ on Ω.

Proposition 2.14. *Let us consider the expression*

$$Ku(x) = \int_\Omega k(x,y)u(y)dy. \tag{2.2}$$

i) *If $k \in \mathcal{G}(\Omega \times \Omega)$ then (2.2) defines a linear map $K : \mathcal{G}_c(\Omega) \to \mathcal{G}(\Omega)$: $u \to Ku$, where Ku is the generalized function with representative $\big(\int_\Omega k_\epsilon(x,y)u_\epsilon(y)dy\big)_\epsilon$;*
ii) *if $k \in \mathcal{G}^\infty(\Omega \times \Omega)$ then K maps $\mathcal{G}_c(\Omega)$ into $\mathcal{G}^\infty(\Omega)$;*
iii) *if $k \in \mathcal{G}_c(\Omega \times \Omega)$ then K maps $\mathcal{G}(\Omega)$ into $\mathcal{G}_c(\Omega)$;*
iv) *if $k \in \mathcal{G}_c^\infty(\Omega \times \Omega)$ then K maps $\mathcal{G}(\Omega)$ into $\mathcal{G}_c^\infty(\Omega)$;*
v) *if $k \in \mathcal{G}(\Omega \times \Omega)$ and $\pi_1, \pi_2 : \operatorname{supp} k \to \Omega$ are proper then $\operatorname{supp}(Ku) \Subset \Omega$ for all $u \in \mathcal{G}_c(\Omega)$ and K can be uniquely extended to a linear map from $\mathcal{G}(\Omega)$ into $\mathcal{G}(\Omega)$ such that for all $u \in \mathcal{G}(\Omega)$ and $v \in \mathcal{G}_c(\Omega)$*

$$\int_\Omega Ku(x)v(x)\,dx = \int_\Omega u(y)\ {}^tKv(y)\,dy \tag{2.3}$$

where ${}^tKv(y) = \int_\Omega k(x,y)v(x)\,dx$;

vi) *if* $k \in \mathcal{G}^\infty(\Omega \times \Omega)$ *and* $\pi_1, \pi_2 : \operatorname{supp} k \to \Omega$ *are proper then the extension defined above maps* $\mathcal{G}(\Omega)$ *into* $\mathcal{G}^\infty(\Omega)$.

The conditions on π_1 and π_2 of $v)$ and $vi)$ say that $\operatorname{supp} k$ is a proper subset of $\Omega \times \Omega$.

Proof. We give only some details of the proof of the fifth statement. The inclusion

$$\operatorname{supp}(Ku) \subseteq \pi_1(\pi_2^{-1}(\operatorname{supp} u) \cap \operatorname{supp} k), \quad u \in \mathcal{G}_c(\Omega), \tag{2.4}$$

leads to $\operatorname{supp}(Ku) \Subset \Omega$, under the assumption that $\pi_1, \pi_2 : \operatorname{supp} k \to \Omega$ are proper maps. Let $V_1 \subset V_2 \subset \dots$ be an exhausting sequence of relatively compact open sets and $F_j = \pi_2(\pi_1^{-1}(\overline{V_j}) \cap \operatorname{supp} k)$. From (2.4) it follows that

$$\operatorname{supp} u \subseteq \Omega \setminus F_j \quad \Rightarrow \quad \operatorname{supp}(Ku) \subseteq \Omega \setminus \overline{V_j}, \quad u \in \mathcal{G}_c(\Omega). \tag{2.5}$$

Let $u \in \mathcal{G}(\Omega)$. We define $K_j u \in \mathcal{G}(V_j)$ by $K(\psi_j u)_{|V_j}$ where $\psi_j \in \mathcal{C}^\infty_c(\Omega)$, $\psi_j \equiv 1$ in an open neighborhood of F_j. By the sheaf property of $\mathcal{G}(\Omega)$, there exists a generalized function Ku such that $Ku_{|V_j} = K_j u$, provided the family $\{K_j u\}_{j \in \mathbb{N}}$ is coherent. But from (2.5) we have that

$$(K_j u - K_i u)_{|V_i} = K((\psi_j - \psi_i)u)_{|V_i} = 0$$

for $i < j$, noting that $\psi_j - \psi_i \equiv 0$ on F_i. In this way we obtain a linear extension of the original map $K : \mathcal{G}_c(\Omega) \to \mathcal{G}_c(\Omega)$, which satisfies (2.3). In fact for $u \in \mathcal{G}(\Omega)$, $v \in \mathcal{G}_c(\Omega)$ and $\operatorname{supp} v \subseteq V_j$ we have

$$\begin{aligned}
\int_\Omega Ku(x)v(x)\,dx &= \int_\Omega Ku_{|V_j}(x)v(x)\,dx = \int_\Omega K(\psi_j u)(x)v(x)\,dx \\
&= \int_\Omega \psi_j u(y) \int_\Omega k(x,y)v(x)dx\,dy \\
&= \int_\Omega u(y) \int_\Omega k(x,y)v(x)dx\,dy = \int_\Omega u(y)\ {}^tKv(y)dy.
\end{aligned}$$

Finally, let us assume that there exists another linear extension K' of the operator K defined on $\mathcal{G}_c(\Omega)$ such that for all $u \in \mathcal{G}(\Omega)$ and $v \in \mathcal{G}_c(\Omega)$

$$\int_\Omega K'u(x)v(x)dx = \int_\Omega u(y)\ {}^tKv(y)dy. \tag{2.6}$$

Combining (2.3) with (2.6) we have that $\int_\Omega (K - K')u(x)v(x)dx = 0$ for all $v \in \mathcal{G}_c(\Omega)$. Thus, from Proposition 2.11, $Ku = K'u$ in $\mathcal{G}(\Omega)$. $\square$

Remark 2.15. The generalized function $k \in \mathcal{G}(\Omega \times \Omega)$ is uniquely determined by the operator $K : \mathcal{G}_c(\Omega) \to \mathcal{G}(\Omega)$. In fact, if K is identically equal to zero, $\int_{\Omega\times\Omega} k(x,y)v(x)u(y)\,dxdy = 0$ for all $u, v \in \mathcal{G}_c(\Omega)$, and so, as a consequence of Proposition 2.11, $k = 0$ in $\mathcal{G}(\Omega \times \Omega)$.

3. Generalized oscillatory integrals

In this section we summarize the meaning and the most important properties of integrals of the type

$$\int_{K\times\mathbb{R}^p} e^{i\phi(y,\xi)} a_\epsilon(y,\xi)\,dy\,d\xi,$$

where $K \Subset \Omega$, Ω an open subset of $\mathbb{R}^n$. The function $\phi(y,\xi)$ is assumed to be a phase function, i.e., it is smooth on $\Omega \times \mathbb{R}^p \setminus 0$, real valued, positively homogeneous of degree 1 in ξ and $\nabla\phi(y,\xi) \neq 0$ for all $y \in \Omega$, $\xi \in \mathbb{R}^p \setminus 0$. In the sequel we shall use the square bracket notation $\mathcal{S}^m_{\rho,\delta}[\Omega \times \mathbb{R}^p]$ for the space of nets $S^m_{\rho,\delta}(\Omega \times \mathbb{R}^p)^{(0,1]}$ where $S^m_{\rho,\delta}(\Omega \times \mathbb{R}^p)$, $m \in \mathbb{R}$, $\rho, \delta \in [0,1]$, is the usual space of Hörmander symbols. For the classical theory, we refer to [2, 10, 18, 28, 40].

Definition 3.1. An element of $\mathcal{S}^m_{\rho,\delta,\mathrm{M}}(\Omega \times \mathbb{R}^p)$ is a net $(a_\epsilon)_\epsilon \in \mathcal{S}^m_{\rho,\delta}[\Omega \times \mathbb{R}^p]$ such that

$$\forall\alpha \in \mathbb{R}^p,\ \forall\beta \in \mathbb{N}^n,\ \forall K \Subset \Omega,\ \exists N \in \mathbb{N},\ \exists\eta \in (0,1],\ \exists c > 0:$$
$$\forall y \in K,\ \forall\xi \in \mathbb{R}^p,\ \forall\epsilon \in (0,\eta],\ |\partial^\alpha_\xi \partial^\beta_y a_\epsilon(y,\xi)| \le c\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}\epsilon^{-N},$$

where $\langle\xi\rangle = \sqrt{1+|\xi|^2}$.

The subscript M underlines the moderate growth property, i.e., the bound of type ϵ^{-N} as $\epsilon \to 0$.

Definition 3.2. An element of $\mathcal{N}^m_{\rho,\delta}(\Omega \times \mathbb{R}^p)$ is a net $(a_\epsilon)_\epsilon \in \mathcal{S}^m_{\rho,\delta}[\Omega \times \mathbb{R}^p]$ satisfying the following requirement:

$$\forall\alpha \in \mathbb{N}^p,\ \forall\beta \in \mathbb{N}^n,\ \forall K \Subset \Omega,\ \forall q \in \mathbb{N},\ \exists\eta \in (0,1],\ \exists c > 0:$$
$$\forall y \in K,\ \forall\xi \in \mathbb{R}^p,\ \forall\epsilon \in (0,\eta],\ |\partial^\alpha_\xi \partial^\beta_y a_\epsilon(y,\xi)| \le c\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}\epsilon^{q}.$$

Nets of this type are called negligible.

Definition 3.3. A (generalized) symbol of order m and type (ρ,δ) is an element of the factor space $\widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega \times \mathbb{R}^p) := \mathcal{S}^m_{\rho,\delta,\mathrm{M}}(\Omega \times \mathbb{R}^p)/\mathcal{N}^m_{\rho,\delta}(\Omega \times \mathbb{R}^p)$.

In the following we denote an arbitrary representative of $a \in \widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega \times \mathbb{R}^p)$ by $(a_\epsilon)_\epsilon$.

Definition 3.4. An element $a \in \widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega \times \mathbb{R}^p)$ is called regular if it has a representative $(a_\epsilon)_\epsilon$ with the following property:

$$\begin{gathered}\forall K \Subset \Omega,\ \exists N \in \mathbb{N}:\ \forall\alpha \in \mathbb{N}^p,\ \forall\beta \in \mathbb{N}^n,\ \exists\eta \in (0,1],\ \exists c > 0: \\ \forall y \in K,\ \forall\xi \in \mathbb{R}^p,\ \forall\epsilon \in (0,\eta],\ |\partial^\alpha_\xi \partial^\beta_y a_\epsilon(y,\xi)| \le c\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}\epsilon^{-N}.\end{gathered} \tag{3.1}$$

We denote by $\widetilde{\mathcal{S}}^m_{\rho,\delta,\mathrm{rg}}(\Omega \times \mathbb{R}^p)$ the subspace of regular elements of $\widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega \times \mathbb{R}^p)$.

If the property (3.1) is true for one representative of a, it holds for every representative. Consequently, if $\mathcal{S}^m_{\rho,\delta,\mathrm{rg}}(\Omega \times \mathbb{R}^p)$ is the space defined by (3.1), we can introduce $\widetilde{\mathcal{S}}^m_{\rho,\delta,\mathrm{rg}}(\Omega \times \mathbb{R}^p)$ as the factor space $\mathcal{S}^m_{\rho,\delta,\mathrm{rg}}(\Omega \times \mathbb{R}^p)/\mathcal{N}^m_{\rho,\delta}(\Omega \times \mathbb{R}^p)$. It is easy to prove that $(a_\epsilon)_\epsilon \in \mathcal{S}^m_{\rho,\delta,\mathrm{M}}(\Omega \times \mathbb{R}^p)$ implies $(\partial^\alpha_\xi \partial^\beta_x a_\epsilon)_\epsilon \in \mathcal{S}^{m-\rho|\alpha|+\delta|\beta|}_{\rho,\delta,\mathrm{M}}(\Omega \times \mathbb{R}^p)$ and if $(a_\epsilon)_\epsilon \in \mathcal{S}^{m_1}_{\rho,\delta,\mathrm{M}}(\Omega \times \mathbb{R}^p)$, $(b_\epsilon)_\epsilon \in \mathcal{S}^{m_2}_{\rho,\delta,\mathrm{M}}(\Omega \times \mathbb{R}^p)$ then $(a_\epsilon + b_\epsilon)_\epsilon \in \mathcal{S}^{\max(m_1,m_2)}_{\rho,\delta,\mathrm{M}}(\Omega \times \mathbb{R}^p)$ and $(a_\epsilon b_\epsilon)_\epsilon \in \mathcal{S}^{m_1+m_2}_{\rho,\delta,\mathrm{M}}(\Omega \times \mathbb{R}^p)$. Since the results concerning derivatives and sums hold with $\mathcal{S}_{\rho,\delta,\mathrm{rg}}$ and $\mathcal{N}_{\rho,\delta}$ in place of $\mathcal{S}_{\rho,\delta,\mathrm{M}}$, we can define derivatives and sums on the corresponding factor spaces. Moreover, $(a_\epsilon)_\epsilon \in \mathcal{S}^{m_1}_{\rho,\delta,\mathrm{M}}(\Omega \times \mathbb{R}^p)$ and $(b_\epsilon)_\epsilon \in \mathcal{N}^{m_2}_{\rho,\delta}(\Omega \times \mathbb{R}^p)$ imply $(a_\epsilon b_\epsilon)_\epsilon \in \mathcal{N}^{m_1+m_2}_{\rho,\delta}(\Omega \times \mathbb{R}^p)$, thus we obtain that the product is a well-defined map from the space $\widetilde{\mathcal{S}}^{m_1}_{\rho,\delta}(\Omega \times \mathbb{R}^p) \times \widetilde{\mathcal{S}}^{m_2}_{\rho,\delta}(\Omega \times \mathbb{R}^p)$ into $\widetilde{\mathcal{S}}^{m_1+m_2}_{\rho,\delta}(\Omega \times \mathbb{R}^p)$. Similarly, it is well-defined as a map from $\widetilde{\mathcal{S}}^{m_1}_{\rho,\delta,\mathrm{rg}}(\Omega \times \mathbb{R}^p) \times \widetilde{\mathcal{S}}^{m_2}_{\rho,\delta,\mathrm{rg}}(\Omega \times \mathbb{R}^p)$ into $\widetilde{\mathcal{S}}^{m_1+m_2}_{\rho,\delta,\mathrm{rg}}(\Omega \times \mathbb{R}^p)$. The classical space $S^m_{\rho,\delta}(\Omega \times \mathbb{R}^p)$ is contained in $\widetilde{\mathcal{S}}^m_{\rho,\delta,\mathrm{rg}}(\Omega \times \mathbb{R}^p)$.

Let us now study the dependence of $\widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega\times\mathbb{R}^p)$ on the open set $\Omega\subseteq\mathbb{R}^n$. We can define the restriction of $a\in\widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega\times\mathbb{R}^p)$ to an open subset Ω' of Ω by setting

$$a_{|_{\Omega'}}:=(a_\epsilon{}_{|_{\Omega'}})+\mathcal{N}^m_{\rho,\delta}(\Omega'\times\mathbb{R}^p).$$

Following the same arguments adopted in the proof of Theorem 1.2.4 in [16], we obtain that $\widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega\times\mathbb{R}^p)$ is a sheaf with respect to Ω. This fact allows us to define $\operatorname{supp}_y a$ as the complement of the largest open set $\Omega'\subseteq\Omega$ such that $a_{|_{\Omega'}}=0$.

We assume from now on that $\rho>0$ and $\delta<1$ and return to the meaning of the integral

$$\int_{K\times\mathbb{R}^p} e^{i\phi(y,\xi)}a_\epsilon(y,\xi)\,dy\,d\xi. \tag{3.2}$$

Obviously, if $(a_\epsilon)_\epsilon\in\mathcal{S}^m_{\rho,\delta,\mathrm{M}}(\Omega\times\mathbb{R}^p)$ then (3.2) makes sense as an oscillatory integral for every $\epsilon\in(0,1]$. Since our aim is to estimate its asymptotic behavior with respect to ϵ, we state a lemma obtained as a simple adaptation of the reasoning presented in [10, p.122-123], [18, p.88-89], [40, p.4-5]. We recall that given the phase function ϕ, there exists an operator

$$L=\sum_{i=1}^p a_i(y,\xi)\frac{\partial}{\partial\xi_i}+\sum_{k=1}^n b_k(y,\xi)\frac{\partial}{\partial y_k}+c(y,\xi)$$

such that $a_i(y,\xi)\in S^0(\Omega\times\mathbb{R}^p)$, $b_k(y,\xi)\in S^{-1}(\Omega\times\mathbb{R}^p)$, $c(y,\xi)\in S^{-1}(\Omega\times\mathbb{R}^p)$, and such that ${}^tLe^{i\phi}=e^{i\phi}$, where ${}^tL$ is the formal adjoint.

Lemma 3.5. *Let $s=\min\{\rho,1-\delta\}$ and $j\in\mathbb{N}$. Then the following statements hold:*

i) *if $(a_\epsilon)_\epsilon\in\mathcal{S}^m_{\rho,\delta,\mathrm{M}}(\Omega\times\mathbb{R}^p)$ then $(L^ja_\epsilon)_\epsilon\in\mathcal{S}^{m-js}_{\rho,\delta,\mathrm{M}}(\Omega\times\mathbb{R}^p)$;*

ii) *i) is valid with $\mathcal{S}_{\rho,\delta,\mathrm{rg}}$ in place of $\mathcal{S}_{\rho,\delta,\mathrm{M}}$;*

iii) *i) is valid with $\mathcal{N}_{\rho,\delta}$ in place of $\mathcal{S}_{\rho,\delta,\mathrm{M}}$.*

For completeness we recall that for $m-js<-n$ and $\chi\in\mathcal{C}^\infty_c(\mathbb{R}^p)$ identically equal to 1 in a neighborhood of the origin, the oscillatory integral $I_{\phi,K}(a_\epsilon)$, at fixed ϵ, can be defined by either of the two expressions on the right hand-side of (3.3):

$$\begin{aligned} I_{\phi,K}(a_\epsilon) &:= \int_{K\times\mathbb{R}^p} e^{i\phi(y,\xi)}a_\epsilon(y,\xi)\,dy\,d\xi\\ &= \int_{K\times\mathbb{R}^p} e^{i\phi(y,\xi)}L^ja_\epsilon(y,\xi)\,dy\,d\xi \\ &= \lim_{h\to0^+}\int_{K\times\mathbb{R}^p} e^{i\phi(y,\xi)}a_\epsilon(y,\xi)\chi(h\xi)\,dy\,d\xi, \end{aligned} \tag{3.3}$$

where the equalities hold for all $\epsilon\in(0,1]$.

Proposition 3.6. *Let K be a compact set contained in Ω. Let ϕ be a phase function on $\Omega\times\mathbb{R}^p$ and a an element of $\widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega\times\mathbb{R}^p)$. The oscillatory integral*

$$I_{\phi,K}(a):=\int_{K\times\mathbb{R}^p} e^{i\phi(y,\xi)}a(y,\xi)\,dy\,d\xi:=(I_{\phi,K}(a_\epsilon))_\epsilon+\mathcal{N}$$

is a well-defined element of $\widetilde{\mathbb{C}}$.

Proof. From Lemma 3.5, if $(a_\epsilon)_\epsilon\in\mathcal{S}^m_{\rho,\delta,\mathrm{M}}(\Omega\times\mathbb{R}^p)$ then $(L^ja_\epsilon)_\epsilon\in\mathcal{S}^{m-js}_{\rho,\delta,\mathrm{M}}(\Omega\times\mathbb{R}^p)$ for every $j\in\mathbb{N}$. Taking $m-js<-n$, it is easy to see that $(I_{\phi,K}(a_\epsilon))_\epsilon\in\mathcal{E}_M$. Analogously, if $(a_\epsilon)_\epsilon$ is negligible, we have that $(I_{\phi,K}(a_\epsilon))_\epsilon\in\mathcal{N}$. $\square$

Definition 3.7. Let $a \in \widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega \times \mathbb{R}^p)$ with $\mathrm{supp}_y\, a \Subset \Omega$. We define the (generalized) oscillatory integral

$$I_\phi(a) := \int_{\Omega\times\mathbb{R}^p} e^{i\phi(y,\xi)} a(y,\xi)\, dy\, d\xi := \int_{K\times\mathbb{R}^p} e^{i\phi(y,\xi)} a(y,\xi)\, dy\, d\xi,$$

where K is any compact subset of Ω containing $\mathrm{supp}_y\, a$ in its interior.

It remains to show that this definition does not depend on the choice of K. Let $K_1, K_2 \Subset \Omega$ with $\mathrm{supp}_y\, a \subseteq \mathrm{int}\, K_1 \cap \mathrm{int}\, K_2$ and put $K_3 = K_1 \cup K_2$. But for $i = 1, 2$ and j large enough

$$\begin{aligned} &\Big| \int_{K_3\times\mathbb{R}^p} e^{i\phi(y,\xi)} a_\epsilon(y,\xi)\, dy\, d\xi - \int_{K_i\times\mathbb{R}^p} e^{i\phi(y,\xi)} a_\epsilon(y,\xi)\, dy\, d\xi \Big| \\ &\leq \int_{K_3\setminus \mathrm{int}\, K_i\times\mathbb{R}^p} |L^j a_\epsilon(y,\xi)| dy\, d\xi = O(\epsilon^q) \end{aligned}$$

for arbitrary $q \in \mathbb{N}$ since $K_3 \setminus \mathrm{int}\, K_i$ is a compact subset of $\Omega \setminus \mathrm{supp}_y\, a$, as desired.

It is clear that for each compact set K containing $\mathrm{supp}_y\, a$ in its interior, we can find representatives $(a_\epsilon)_\epsilon$ with $\mathrm{supp}_y\, a_\epsilon \subset K$ for all ϵ. For such a representative of $I_\phi(a)$, its components are defined by the classical oscillatory integral $\int_{\Omega\times\mathbb{R}^p} e^{i\phi(y,\xi)} a_\epsilon(y,\xi)\, dy\, d\!\!\bar{}\,\xi$.

Remark 3.8. A particular example of a generalized oscillatory integral on $\Omega \times \mathbb{R}^p$ is given by

$$I_\phi(au) := \int_{\Omega\times\mathbb{R}^p} e^{i\phi(y,\xi)} a(y,\xi) u(y)\, dy\, d\xi,$$

where $a \in \widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega \times \mathbb{R}^p)$ and $u \in \mathcal{G}_c(\Omega)$. We observe that the map $I_\phi(a) : \mathcal{G}_c(\Omega) \to \widetilde{\mathbb{C}} : u \to I_\phi(au)$ is well-defined and belongs to $L(\mathcal{G}_c(\Omega), \widetilde{\mathbb{C}})$.

We consider now phase functions and symbols depending on an additional parameter. We want to study oscillatory integrals of the form

$$I_{\phi,K}(a)(x) := \int_{K\times\mathbb{R}^p} e^{i\phi(x,y,\xi)} a(x,y,\xi)\, dy\, d\xi,$$

where $x \in \Omega'$, an open subset of $\mathbb{R}^{n'}$. Obviously, if for any fixed $x \in \Omega'$, $\phi(x,y,\xi)$ is a phase function with respect to the variables (y,ξ) and $a(x,y,\xi)$ belongs to $\widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega \times \mathbb{R}^p)$, the oscillatory integral $I_{\phi,K}(a)(x)$ defines a map from Ω' to $\widetilde{\mathbb{C}}$. The smooth dependence of this map on the parameter x is investigated in the following Remark 3.9 and in Proposition 3.10.

Remark 3.9. Let $\phi(x,y,\xi)$ be a real valued continuous function on $\Omega' \times \Omega \times \mathbb{R}^p$, smooth on $\Omega' \times \Omega \times \mathbb{R}^p \setminus \{0\}$ such that for all $x \in \Omega'$, $\phi(x,y,\xi)$ is a phase function with respect to (y,ξ). As in Lemma 3.5, we have that for all $j \in \mathbb{N}$, $(a_\epsilon)_\epsilon \in \mathcal{S}^m_{\rho,\delta,\mathrm{M}}(\Omega' \times \Omega \times \mathbb{R}^p)$ implies $(L^j_x a_\epsilon(x,y,\xi))_\epsilon \in \mathcal{S}^{m-js}_{\rho,\delta,\mathrm{M}}(\Omega' \times \Omega \times \mathbb{R}^p)$. The same result holds with $\mathcal{S}_{\rho,\delta,\mathrm{rg}}$ in place of $\mathcal{S}_{\rho,\delta,\mathrm{M}}$ and $\mathcal{N}_{\rho,\delta}$ in place of $\mathcal{S}_{\rho,\delta,\mathrm{M}}$ (this follows easily along the lines of [10, p.124-125] and [18, p.90]).

Proposition 3.10. *Let $\phi(x,y,\xi)$ be as in Remark 3.9.*

i) *If $a(x,y,\xi) \in \widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega' \times \Omega \times \mathbb{R}^p)$ then for all $K \Subset \Omega$*

$$w_K(x) := \int_{K\times\mathbb{R}^p} e^{i\phi(x,y,\xi)} a(x,y,\xi)\, dy\, d\xi$$

belongs to $\mathcal{G}(\Omega')$.

ii) *If* $a(x,y,\xi) \in \widetilde{\mathcal{S}}^m_{\rho,\delta,\mathrm{rg}}(\Omega' \times \Omega \times \mathbb{R}^p)$ *then* $w_K \in \mathcal{G}^\infty(\Omega')$ *for all* $K \Subset \Omega$.

iii) *If in addition* ϕ *is a phase function in* (x,y,ξ) *then for all* $K' \Subset \Omega'$

$$\int_{K'} w_K(x)\,dx = \int_{K'\times K\times\mathbb{R}^p} e^{i\phi(x,y,\xi)} a(x,y,\xi)\,dx\,dy\,d\xi.$$

Proof. An arbitrary representative of w_K is given by the oscillatory integral

$$(w_{K,\epsilon}(x))_\epsilon := \Big(\int_{K\times\mathbb{R}^p} e^{i\phi(x,y,\xi)} a_\epsilon(x,y,\xi)\,dyd\xi\Big)_\epsilon.$$

From Remark 3.9 it follows that $\big(\int_{K\times\mathbb{R}^p} e^{i\phi(x,y,\xi)} a_\epsilon(x,y,\xi)\,dyd\xi\big)_\epsilon \in \mathcal{E}[\Omega']$. At this point by computing the x-derivatives of the expression $e^{i\phi(x,y,\xi)} L^j_x a_\epsilon(x,y,\xi)$ for $\xi \neq 0$, we conclude that

$$\begin{gathered} \forall\alpha \in \mathbb{N}^{n'},\ \forall K' \Subset \Omega',\ \exists N \in \mathbb{N},\ \exists \eta \in (0,1]:\ \forall x \in K',\ \forall y \in K, \\ \forall \xi \in \mathbb{R}^p \setminus \{0\},\ \forall \epsilon \in (0,\eta],\ |\partial^\alpha_x(e^{i\phi(x,y,\xi)} L^j_x a_\epsilon(x,y,\xi))| \le \langle\xi\rangle^{m-js+|\alpha|}\epsilon^{-N}. \end{gathered} \tag{3.4}$$

Now if $m - js + |\alpha| < -n$ then we obtain for $x \in K'$ and $\epsilon \in (0,\eta]$,

$$|\partial^\alpha_x w_{K,\epsilon}(x)| \le \epsilon^{-N}.$$

Therefore $(w_{K,\epsilon})_\epsilon \in \mathcal{E}_M(\Omega')$. Obviously if $(a_\epsilon)_\epsilon \in \mathcal{N}^m_{\rho,\delta}(\Omega' \times \Omega \times \mathbb{R}^p)$ then $(w_{K,\epsilon})_\epsilon \in \mathcal{N}(\Omega')$. If $a \in \widetilde{\mathcal{S}}^m_{\rho,\delta,\mathrm{rg}}(\Omega' \times \Omega \times \mathbb{R}^p)$ the exponent N in (3.4) does not depend on the derivatives and then $(w_{K,\epsilon})_\epsilon \in \mathcal{E}^\infty_M(\Omega')$. This result completes the proof of the first two assertions. The proof of the third point is a consequence of the analogous statement in [10, (23.17.6)], applied to representatives. □

Remark 3.11. Combining Proposition 3.10 with Definition 3.7, we obtain the following results:

i) if ϕ is a phase function with respect to (y,ξ) and there exists a compact set K of Ω such that for all $x \in \Omega'$, $\mathrm{supp}_y\, a(x,\cdot,\cdot) \subseteq K$ then the oscillatory integral $\int_{\Omega\times\mathbb{R}^p} e^{i\phi(x,y,\xi)} a(x,y,\xi)\,dy\,đ\xi$ defines a generalized function belonging to $\mathcal{G}(\Omega')$;

ii) if ϕ is a phase function with respect to (y,ξ) and (x,ξ) and $\mathrm{supp}_{x,y}\, a \Subset \Omega' \times \Omega$ then the two oscillatory integrals $\int_{\Omega\times\mathbb{R}^p} e^{i\phi(x,y,\xi)} a(x,y,\xi)\,dy\,đ\xi$ and $\int_{\Omega'\times\mathbb{R}^p} e^{i\phi(x,y,\xi)} a(x,y,\xi)\,dx\,đ\xi$ belong to $\mathcal{G}(\Omega')$ and $\mathcal{G}(\Omega)$ respectively. Moreover

$$\begin{aligned} \int_{\Omega'\times\Omega\times\mathbb{R}^p} e^{i\phi(x,y,\xi)} a(x,y,\xi)\,dx\,dy\,đ\xi &= \int_{\Omega'}\int_{\Omega\times\mathbb{R}^p} e^{i\phi(x,y,\xi)} a(x,y,\xi)\,dy\,đ\xi\,dx \\ &= \int_{\Omega}\int_{\Omega'\times\mathbb{R}^p} e^{i\phi(x,y,\xi)} a(x,y,\xi)\,dx\,đ\xi\,dy. \end{aligned}$$

Remark 3.12. We recall that for each phase function $\phi(x,\xi)$

$$C_\phi := \{(x,\xi) \in \Omega \times \mathbb{R}^p \setminus \{0\} : \nabla_\xi \phi(x,\xi) = 0\}$$

is a cone-shaped subset of $\Omega \times \mathbb{R}^p \setminus \{0\}$. Let $\pi : \Omega \times \mathbb{R}^p \setminus \{0\} \to \Omega$ be the projection onto Ω and put $S_\phi := \pi C_\phi$, $R_\phi := \Omega \setminus S_\phi$. Interpreting $x \in \Omega$ as a parameter we have from Proposition 3.10 that

$$w(x) := \int_{\mathbb{R}^p} e^{i\phi(x,\xi)} a(x,\xi) d\xi = \Big(\int_{\mathbb{R}^p} e^{i\phi(x,\xi)} a_\epsilon(x,\xi) d\xi\Big)_\epsilon + \mathcal{N}$$

makes sense as an oscillatory integral for $x \in R_\phi$. More precisely, we have that

i) if $a \in \widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega \times \mathbb{R}^p)$ then $w \in \mathcal{G}(R_\phi)$;
ii) if $a \in \widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega \times \mathbb{R}^p)$ and $\mathrm{supp}_x\, a \subseteq \Omega \setminus \Omega'$, where Ω' is an open neighborhood of S_ϕ, then w can be extended to a generalized function on Ω with support contained in $\Omega \setminus \Omega'$;
iii) i) and ii) hold with $\mathcal{S}_{\rho,\delta,\mathrm{rg}}$ in place of $\mathcal{S}_{\rho,\delta}$ and $\mathcal{G}^\infty$ in place of $\mathcal{G}$;
iv) if $a \in \widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega \times \mathbb{R}^p)$ then for all $u \in \mathcal{G}_c(R_\phi)$
$$\int_{\Omega\times\mathbb{R}^p} e^{i\phi(x,\xi)} a(x,\xi)u(x)\,dx\,d\xi = \int_\Omega w(x)u(x)\,dx. \tag{3.5}$$
v) under the hypothesis of the second statement, (3.5) holds for all $u \in \mathcal{G}_c(\Omega)$.

We just give some details concerning the proof of the assertion $ii)$ and $v)$. Let $\{\Omega'_j\}_{j\in\mathbb{N}\setminus\{0\}}$ be an open covering of Ω' such that Ω'_j is relatively compact and $\Omega'_j \subseteq \overline{\Omega'_j} \subseteq \Omega'_{j+1}$ for all j. Choosing cut-off functions $\{\psi_j\}_{j\in\mathbb{N}\setminus\{0\}}$ such that $\psi_j \in \mathcal{C}^\infty_c(\Omega')$ and $\psi_j \equiv 1$ in a neighborhood of $\overline{\Omega'_j}$, we observe that $((1-\psi_j(x))a_\epsilon(x,\xi))_\epsilon$ is a representative of a identically equal to 0 on Ω'_j for all $\epsilon \in (0,1]$. At this point we see that
$$w_0(x) := \Big(\Big(\int_{\mathbb{R}^n} e^{i\phi(x,\xi)} a_\epsilon(x,\xi)\,d\xi\Big)\big|_{R_\phi}\Big)_\epsilon + \mathcal{N}(R_\phi),$$
$$w_j(x) := \Big(\Big(\int_{\mathbb{R}^n} e^{i\phi(x,\xi)}(1-\psi_j(x))a_\epsilon(x,\xi)\,d\xi\Big)\big|_{\Omega'_j}\Big)_\epsilon + \mathcal{N}(\Omega'_j), \quad j \geq 1$$
is a coherent family of generalized functions which defines $w \in \mathcal{G}(\Omega)$ such that w_0 and w_j are its restrictions to R_ϕ and Ω'_j respectively. Consequently, $\mathrm{supp}\, w \subseteq R_\phi \setminus \Omega' \equiv \Omega\setminus\Omega'$. Now for any $u \in \mathcal{G}_c(\Omega)$, $\mathrm{supp}_x(au) \cup \mathrm{supp}(wu) \subseteq (R_\phi\setminus\Omega') \cap \mathrm{supp}\, u \Subset R_\phi$. Taking $\psi \in \mathcal{C}^\infty_c(R_\phi)$ identically 1 in a neighborhood of $(R_\phi \setminus \Omega') \cap \mathrm{supp}\, u$ we have that $wu = wu\psi$ in $\mathcal{G}(\Omega)$ and $au = au\psi$ in $\widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega \times \mathbb{R}^p)$. Since $\psi u \in \mathcal{G}_c(R_\phi)$, $iv)$ gives
$$\begin{aligned}\int_{\Omega\times\mathbb{R}^n} e^{i\phi(x,\xi)} a(x,\xi)u(x)\,dx\,d\xi &= \int_{\Omega\times\mathbb{R}^n} e^{i\phi(x,\xi)} a(x,\xi)u(x)\psi(x)\,dx\,d\xi \\ &= \int_\Omega w(x)u(x)\psi(x)\,dx = \int_\Omega w(x)u(x)\,dx.\end{aligned}$$

4. Pseudodifferential operators with generalized amplitudes

As mentioned in the Introduction and as will be seen shortly, we will need different asymptotic scales. This requires an extension of Definition 3.1 and 3.2 which we now state.

Definition 4.1. Let m, μ, ρ, δ be real numbers, $\rho, \delta \in [0,1]$. Let ω be a real valued function on the interval $(0,1]$, $\omega > 0$, such that for some r in $\mathbb{R}$, for some $C > 0$ and for all $\epsilon \in (0,1]$, $\omega(\epsilon) \geq C\epsilon^r$. We denote by $\mathcal{S}^{m,\mu}_{\rho,\delta,\omega}(\Omega' \times \mathbb{R}^p)$, Ω' an open subset of $\mathbb{R}^{n'}$, the set of all $(a_\epsilon)_\epsilon \in \mathcal{S}^m_{\rho,\delta}[\Omega' \times \mathbb{R}^p]$ such that the following statement holds:
$$\begin{gathered}\forall K \Subset \Omega',\ \exists N \in \mathbb{N} : \forall\alpha \in \mathbb{N}^p,\ \forall\beta \in \mathbb{N}^{n'},\ \exists\eta \in (0,1],\ \exists c > 0 : \forall x \in K,\ \forall\xi \in \mathbb{R}^p, \\ \forall\epsilon \in (0,\eta],\ |\partial^\alpha_\xi \partial^\beta_x a_\epsilon(x,\xi)| \leq c\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}\epsilon^{-N}\omega(\epsilon)^{-(|\beta|-\mu)_+}.\end{gathered} \tag{4.1}$$

The exponent $-(|\beta|-\mu)_+ = -\max\{0,|\beta|-\mu\}$ reflects differentiability up to order μ in the case when a_ϵ is obtained from a non-smooth, classical symbol by convolution with a mollifier with scale $\omega(\epsilon)$.

Definition 4.2. An element of $\mathcal{N}^{m,\mu}_{\rho,\delta,\omega}(\Omega'\times\mathbb{R}^p)$ is a net in $\mathcal{S}^m_{\rho,\delta}[\Omega'\times\mathbb{R}^p]$ fulfilling the following condition:

$$\begin{aligned}&\forall K\Subset\Omega',\ \forall\alpha\in\mathbb{N}^p,\ \forall\beta\in\mathbb{N}^{n'},\ \forall q\in\mathbb{N},\ \exists\eta\in(0,1],\ \exists c>0:\ \forall x\in K,\\ &\forall\xi\in\mathbb{R}^p,\ \forall\epsilon\in(0,\eta],\ |\partial^\alpha_\xi\partial^\beta_x a_\epsilon(x,\xi)|\le c\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}\epsilon^q\omega(\epsilon)^{-(|\beta|-\mu)_+}.\end{aligned}\tag{4.2}$$

Nets with this property are called negligible.

The factor space $\mathcal{S}^{m,\mu}_{\rho,\delta,\omega}(\Omega'\times\mathbb{R}^p)/\mathcal{N}^{m,\mu}_{\rho,\delta,\omega}(\Omega'\times\mathbb{R}^p)$ will be denoted by $\widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega'\times\mathbb{R}^p)$. Related to Definitions 3.1 and 3.2, we note that $\mathcal{S}^{m,\mu}_{\rho,\delta,\omega}(\Omega'\times\mathbb{R}^p)\subseteq\mathcal{S}^m_{\rho,\delta,\mathrm{M}}(\Omega'\times\mathbb{R}^p)$, $\mathcal{N}^{m,\mu}_{\rho,\delta,\omega}(\Omega'\times\mathbb{R}^p)\subseteq\mathcal{N}^m_{\rho,\delta}(\Omega'\times\mathbb{R}^p)$ and $\mathcal{S}^{m,\mu}_{\rho,\delta,\omega}(\Omega'\times\mathbb{R}^p)\cap\mathcal{N}^m_{\rho,\delta}(\Omega'\times\mathbb{R}^p)\subseteq\mathcal{N}^{m,\mu}_{\rho,\delta,\omega}(\Omega'\times\mathbb{R}^p)$, provided $(\omega(\epsilon))_\epsilon$ belongs to $\mathcal{E}_M$. Therefore, $\widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega'\times\mathbb{R}^p)\subseteq\widetilde{\mathcal{S}}^m_{\rho,\delta}(\Omega'\times\mathbb{R}^p)$. One easily proves that the following mapping properties hold:

$$\begin{aligned}\partial^\alpha_\xi\partial^\beta_x:\widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega'\times\mathbb{R}^p)&\to\widetilde{\mathcal{S}}^{m-\rho|\alpha|+\delta|\beta|,\mu-|\beta|}_{\rho,\delta,\omega}(\Omega'\times\mathbb{R}^p),\\ +:\widetilde{\mathcal{S}}^{m_1,\mu}_{\rho,\delta,\omega}(\Omega'\times\mathbb{R}^p)\times\widetilde{\mathcal{S}}^{m_2,\mu}_{\rho,\delta,\omega}(\Omega'\times\mathbb{R}^p)&\to\widetilde{\mathcal{S}}^{\max(m_1,m_2),\mu}_{\rho,\delta,\omega}(\Omega'\times\mathbb{R}^p).\end{aligned}$$

As mentioned in the Introduction, an important notion is that of a *slow scale net*. Recall that a net $(r_\epsilon)\in\mathbb{C}^{(0,1]}$ is a slow scale net if for every $q\ge0$ there exist $c_q>0$ such that for all $\epsilon\in(0,1]$

$$|r_\epsilon|^q\le c_q\epsilon^{-1}.\tag{4.3}$$

Remark 4.3. If in addition to the usual assumptions on $(\omega(\epsilon))_\epsilon$, we assume that $(\omega^{-1}(\epsilon))_\epsilon$ is a slow scale net, we can uniformly bound the contributions of the derivatives in (4.1) by a single negative power of ϵ, obtaining for a suitable constant c, for $x\in K$ and for all ϵ small enough

$$|\partial^\alpha_\xi\partial^\beta_x a_\epsilon(x,\xi)|\le c\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}\epsilon^{-N-1}.$$

This means that if $(\omega^{-1}(\epsilon))_\epsilon$ is a slow scale net then $\mathcal{S}^{m,\mu}_{\rho,\delta,\omega}(\Omega'\times\mathbb{R}^p)\subseteq\mathcal{S}^m_{\rho,\delta,\mathrm{rg}}(\Omega'\times\mathbb{R}^p)$. If in addition $(\omega(\epsilon))_\epsilon\in\mathcal{E}_M$, then also $\widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega'\times\mathbb{R}^p)\subseteq\widetilde{\mathcal{S}}^m_{\rho,\delta,\mathrm{rg}}(\Omega'\times\mathbb{R}^p)$. Further, if $(\omega^{-1}(\epsilon))_\epsilon$ is a slow scale net and $\sup_{\epsilon\in(0,1]}\omega(\epsilon)<\infty$ then

$$\widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega'\times\mathbb{R}^p)=\widetilde{\mathcal{S}}^m_{\rho,\delta,\mathrm{rg}}(\Omega'\times\mathbb{R}^p)$$

since $\mathcal{S}^{m,\mu}_{\rho,\delta,\omega}(\Omega'\times\mathbb{R}^p)=\mathcal{S}^m_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^p)$ and $\mathcal{N}^{m,\mu}_{\rho,\delta,\omega}(\Omega'\times\mathbb{R}^p)=\mathcal{N}^m_{\rho,\delta}(\Omega'\times\mathbb{R}^p)$.

Definition 4.4. We denote by $\mathcal{S}^{-\infty}_{\mathrm{rg}}(\Omega'\times\mathbb{R}^p)$ the set of all $(a_\epsilon)_\epsilon\in\mathcal{S}^{-\infty}[\Omega'\times\mathbb{R}^p]$ such that

$$\begin{aligned}&\forall K\Subset\Omega',\ \exists N\in\mathbb{N}:\ \forall m\in\mathbb{R},\ \forall\alpha\in\mathbb{N}^p,\ \forall\beta\in\mathbb{N}^{n'},\ \exists\eta\in(0,1],\ \exists c>0\ :\\ &\forall x\in K,\ \forall\xi\in\mathbb{R}^p,\ \forall\epsilon\in(0,\eta],\ |\partial^\alpha_\xi\partial^\beta_x a_\epsilon(x,\xi)|\le c\langle\xi\rangle^{m-|\alpha|}\epsilon^{-N}.\end{aligned}\tag{4.4}$$

We denote by $\mathcal{N}^{-\infty}(\Omega'\times\mathbb{R}^p)$ the set of all $(a_\epsilon)_\epsilon\in\mathcal{S}^{-\infty}[\Omega'\times\mathbb{R}^p]$ such that

$$\begin{aligned}&\forall K\Subset\Omega',\ \forall m\in\mathbb{R},\ \forall q\in\mathbb{N},\ \forall\alpha\in\mathbb{N}^p,\ \forall\beta\in\mathbb{N}^{n'},\ \exists\eta\in(0,1],\ \exists c>0\ :\\ &\forall x\in K,\ \forall\xi\in\mathbb{R}^p,\ \forall\epsilon\in(0,\eta],\ |\partial^\alpha_\xi\partial^\beta_x a_\epsilon(x,\xi)|\le c\langle\xi\rangle^{m-|\alpha|}\epsilon^q.\end{aligned}\tag{4.5}$$

The factor space $\mathcal{S}^{-\infty}_{\mathrm{rg}}(\Omega'\times\mathbb{R}^p)/\mathcal{N}^{-\infty}(\Omega'\times\mathbb{R}^p)$ will be denoted by $\widetilde{\mathcal{S}}^{-\infty}_{\mathrm{rg}}(\Omega'\times\mathbb{R}^p)$.

Definition 4.5. Let Ω be an open subset of $\mathbb{R}^n$. The elements of the factor spaces $\widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n)$ and $\widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\Omega\times\mathbb{R}^n)$ will be called symbols and amplitudes of order m and type (ρ,δ,μ,ω), respectively. The elements of the factor spaces $\widetilde{\mathcal{S}}^{-\infty}_{\rm rg}(\Omega\times\mathbb{R}^n)$ and $\widetilde{\mathcal{S}}^{-\infty}_{\rm rg}(\Omega\times\Omega\times\mathbb{R}^n)$ will be called smoothing symbols and smoothing amplitudes, respectively.

Example 4.6. Let $(\omega(\epsilon))_\epsilon$ be a net as in Definition 4.1 tending to 0 as ϵ goes to 0. Given $\mu\in\mathbb{R}\setminus\mathbb{N}$, we denote by $\mathcal{G}^{\mu}_{*,loc,\omega}(\Omega)$ the space of all generalized functions $a\in\mathcal{G}(\Omega)$ having a representative $(a_\epsilon)_\epsilon$ satisfying the condition:

$$\forall K\Subset\Omega,\ \forall\alpha\in\mathbb{N}^n,\quad \|\partial^\alpha a_\epsilon\|_{L^\infty(K)}=\begin{cases}O(1), & 0\le|\alpha|\le\mu,\\ O(\omega(\epsilon)^{\mu-|\alpha|}), & |\alpha|>\mu\end{cases}\quad(\epsilon\to 0).$$

This notion is a modified version (with scales) of the generalized Zygmund regularity introduced in [22]. It is clear that for any $b\in\widetilde{\mathcal{S}}^{m}_{\rho,\delta,\rm rg}(\Omega\times\mathbb{R}^n)$ and $a\in\mathcal{G}^{\mu}_{*,loc,\omega}(\Omega)$, the product $a(x)b(x,\xi)$ defines a symbol in $\widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n)$. It follows from the results in [22] that if f is a function belonging to the Zygmund class $\mathcal{C}^{\mu}_{*}(\mathbb{R}^n)$ and $\varphi\in\mathscr{S}(\mathbb{R}^n)$ is a radial mollifier with $\int_{\mathbb{R}^n}\varphi(x)dx=1$ and $\int_{\mathbb{R}^n}x^\alpha\varphi(x)dx=0$ for all $\alpha\neq 0$ then $f*\varphi_{\omega(\epsilon)}$ satisfies the generalized Zygmund property defining $\mathcal{G}^{\mu}_{*,loc,\omega}(\Omega)$. Thus, for any b as above, $((f*\varphi_{\omega(\epsilon)})_{|\Omega}(x)b(x,\xi))_\epsilon$ may serve as a representative for a symbol in $\widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n)$.

In the sequel we assume $\rho>0$, $\delta<1$ and let $đ\xi=(2\pi)^{-n}d\xi$.

Proposition 4.7. *Let $a\in\widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\Omega\times\mathbb{R}^n)$. The oscillatory integral*

$$Au:=\int_{\Omega\times\mathbb{R}^n}e^{i(x-y)\xi}a(x,y,\xi)u(y)\,dy\,đ\xi=\big(A_\epsilon u_\epsilon(x)\big)_\epsilon+\mathcal{N}(\Omega)\tag{4.6}$$

where

$$A_\epsilon u_\epsilon(x)=\int_{\Omega\times\mathbb{R}^n}e^{i(x-y)\xi}a_\epsilon(x,y,\xi)u_\epsilon(y)\,dy\,đ\xi$$

defines a linear map from $\mathcal{G}_c(\Omega)$ into $\mathcal{G}(\Omega)$. If $(\omega^{-1}(\epsilon))_\epsilon$ is a slow scale net then the oscillatory integral (4.6) defines a linear map from $\mathcal{G}^\infty_c(\Omega)$ to $\mathcal{G}^\infty(\Omega)$. Finally, if $a\in\widetilde{\mathcal{S}}^{-\infty}_{\rm rg}(\Omega\times\Omega\times\mathbb{R}^n)$ then (4.6) defines a linear map from $\mathcal{G}_c(\Omega)$ to $\mathcal{G}^\infty(\Omega)$.

Proof. In (4.6), $\phi(x,y,\xi)=(x-y)\xi$ satisfies the assumptions of Remark 3.9. It is immediate to prove that the map $\widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\Omega\times\mathbb{R}^n)\times\mathcal{G}_c(\Omega)\to\widetilde{\mathcal{S}}^{m}_{\rho,\delta}(\Omega\times\Omega\times\mathbb{R}^n):(a,u)\to a(x,y,\xi)u(y):=(a_\epsilon(x,y,\xi)u_\epsilon(y))_\epsilon+\mathcal{N}^m_{\rho,\delta}(\Omega\times\Omega\times\mathbb{R}^n)$ is well-defined. From Proposition 3.10 and Remark 3.11, assertion i), we obtain that A is a linear map from $\mathcal{G}_c(\Omega)$ into $\mathcal{G}(\Omega)$. If $(\omega^{-1}(\epsilon))_\epsilon$ is a slow scale net and $u\in\mathcal{G}^\infty_c(\Omega)$ then $au\in\widetilde{\mathcal{S}}^{m}_{\rho,\delta,\rm rg}(\Omega\times\Omega\times\mathbb{R}^n)$ and as a consequence of Proposition 3.10, assertion ii), A maps $\mathcal{G}^\infty_c(\Omega)$ into $\mathcal{G}^\infty(\Omega)$. Finally, assuming that $a\in\widetilde{\mathcal{S}}^{-\infty}_{\rm rg}(\Omega\times\Omega\times\mathbb{R}^n)$, the integral

$$A_\epsilon u_\epsilon(x)=\int_{\Omega\times\mathbb{R}^n}e^{i(x-y)\xi}a_\epsilon(x,y,\xi)u_\epsilon(y)\,dy\,đ\xi$$

is absolutely convergent. Differentiating we obtain that

$$\begin{aligned}
&(a_\epsilon)_\epsilon\in\mathcal{S}^{-\infty}_{\rm rg}(\Omega\times\Omega\times\mathbb{R}^n),\quad (u_\epsilon)_\epsilon\in\mathcal{E}_{c,M}(\Omega)\quad\Rightarrow\quad(A_\epsilon u_\epsilon)_\epsilon\in\mathcal{E}^\infty_M(\Omega),\\
&(a_\epsilon)_\epsilon\in\mathcal{N}^{-\infty}(\Omega\times\Omega\times\mathbb{R}^n),\quad (u_\epsilon)_\epsilon\in\mathcal{E}_{c,M}(\Omega)\quad\Rightarrow\quad(A_\epsilon u_\epsilon)_\epsilon\in\mathcal{N}(\Omega),\\
&(a_\epsilon)_\epsilon\in\mathcal{S}^{-\infty}_{\rm rg}(\Omega\times\Omega\times\mathbb{R}^n),\quad (u_\epsilon)_\epsilon\in\mathcal{N}_c(\Omega)\quad\Rightarrow\quad(A_\epsilon u_\epsilon)_\epsilon\in\mathcal{N}(\Omega).
\end{aligned}$$

This completes the proof. $\square$

Definition 4.8. Let $a \in \widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega \times \Omega \times \mathbb{R}^n)$. The linear map defined by

$$A : \mathcal{G}_c(\Omega) \to \mathcal{G}(\Omega) : u \to Au(x) := \int_{\Omega\times\mathbb{R}^n} e^{i(x-y)\xi} a(x,y,\xi)u(y)\, dy\, đ\xi$$

will be called a (generalized) pseudodifferential operator with amplitude a.

The formal transpose of A is the pseudodifferential operator ${}^tA : \mathcal{G}_c(\Omega) \to \mathcal{G}(\Omega)$ defined by

$$u \to \int_{\Omega\times\mathbb{R}^n} e^{i(x-y)\xi} a(x,y,\xi)u(x)\, dx\, đ\xi = \int_{\Omega\times\mathbb{R}^n} e^{i(x-y)\xi} a(y,x,-\xi)u(y)\, dy\, đ\xi. \tag{4.7}$$

The first integral in (4.7) is an oscillatory integral in x and ξ depending on the parameter $y \in \Omega$. Renaming variables, we see that ${}^tA$ can be written in the usual pseudodifferential form and thus satisfies the mapping properties of Proposition 4.7 as well.

Definition 4.9. Let A be a pseudodifferential operator. The map $k_A \in L(\mathcal{G}_c(\Omega \times \Omega), \widetilde{\mathbb{C}})$ defined by

$$k_A(u) = \int_\Omega A(u(x,\cdot))(x)dx. \tag{4.8}$$

is called the kernel of A.

We have to prove that the integral in (4.8) makes sense. Let $a \in \widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\Omega\times\mathbb{R}^n)$ be an amplitude defining the operator A. Let $u \in \mathcal{G}_c(\Omega \times \Omega)$. From Definition 4.8,

$$A(u(x,\cdot))(x) = \int_{\Omega\times\mathbb{R}^n} e^{i(x-y)\xi} a(x,y,\xi)u(x,y)\, dy\, đ\xi,$$

where $a(x,y,\xi)u(x,y) \in \widetilde{\mathcal{S}}^{m}_{\rho,\delta}(\Omega\times\Omega\times\mathbb{R}^n)$. From Proposition 3.10 and Remark 3.11 this oscillatory integral defines a generalized function in $\mathcal{G}(\Omega)$ and $A(u(x,\cdot))(x) \in \mathcal{G}_c(\Omega)$. Consequently, $\int_\Omega A(u(x,\cdot))(x)dx$ is an element of $L(\mathcal{G}_c(\Omega\times\Omega), \widetilde{\mathbb{C}})$ and from Remark 3.11, assertion *ii)*

$$\begin{aligned} k_A(u) &= \int_\Omega \int_{\Omega\times\mathbb{R}^n} e^{i(x-y)\xi} a(x,y,\xi)u(x,y)\, dy\, đ\xi\, dx \\ &= \int_{\Omega\times\Omega\times\mathbb{R}^n} e^{i(x-y)\xi} a(x,y,\xi)u(x,y)\, dx\, dy\, đ\xi. \end{aligned}$$

Proposition 4.10. *Let $a \in \widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega \times \Omega \times \mathbb{R}^n)$ and A be the corresponding pseudodifferential operator.*

i) *For all $u \in \mathcal{G}_c(\Omega)$ and $v \in \mathcal{G}_c(\Omega)$*

$$k_A(v \otimes u) = \int_\Omega Au(x)v(x)dx = \int_\Omega u(y)\, {}^tAv(y)dy, \tag{4.9}$$

where $v \otimes u := (v_\epsilon(x)u_\epsilon(y))_\epsilon + \mathcal{N}_c(\Omega \times \Omega)$;

ii) *$k_A \in \mathcal{G}(\Omega \times \Omega \setminus \Delta)$, where Δ is the diagonal of $\Omega \times \Omega$. Moreover, for open subsets W and W' of Ω with $W \times W' \subseteq \Omega \times \Omega \setminus \Delta$, and for all $u \in \mathcal{G}_c(W')$*

$$Au_{|W}(x) = \int_\Omega k_A(x,y)u(y)dy; \tag{4.10}$$

iii) *if* $\mathrm{supp}_{x,y}\, a \subseteq \Omega \times \Omega \setminus \Omega'$, *where* Ω' *is an open neighborhood of* Δ, *then* $k_A \in \mathcal{G}(\Omega \times \Omega)$;

iv) *if* $(\omega^{-1}(\epsilon))_\epsilon$ *is a slow scale net then ii) and iii) are valid with* $\mathcal{G}^\infty(\Omega\times\Omega\setminus\Delta)$ *and* $\mathcal{G}^\infty(\Omega\times\Omega)$ *in place of* $\mathcal{G}(\Omega\times\Omega\setminus\Delta)$ *and* $\mathcal{G}(\Omega\times\Omega)$ *respectively;*

v) *if* $a \in \widetilde{\mathcal{S}}^{-\infty}_{\mathrm{rg}}(\Omega\times\Omega\times\mathbb{R}^n)$ *then* $k_A \in \mathcal{G}^\infty(\Omega\times\Omega)$.

Proof. For the first point of the assertion it is sufficient, as in the classical theory of pseudodifferential operators, to write down the three oscillatory integrals in (4.9) and to change order in integration. We observe that for $\phi(x,y,\xi) = (x-y)\xi$, $C_\phi \equiv \Delta \times \mathbb{R}^n \setminus \{0\}$ and $R_\phi \equiv \Omega\times\Omega\setminus\Delta$. Recalling the first statement of Remark 3.12, the oscillatory integral $\int_{\mathbb{R}^n} e^{i(x-y)\xi} a(x,y,\xi) d\!\!{}^-\xi$ defines a generalized function in $\mathcal{G}(\Omega\times\Omega\setminus\Delta)$. Now for all $u \in \mathcal{G}_c(\Omega\times\Omega\setminus\Delta)$

$$k_A(u) = \int_{\Omega\times\Omega} u(x,y) \int_{\mathbb{R}^n} e^{i(x-y)\xi} a(x,y,\xi)\, d\!\!{}^-\xi\, dx\, dy \tag{4.11}$$

and, since as a consequence of Proposition 2.11, $\mathcal{G}(\Omega\times\Omega\setminus\Delta)$ is included in $L(\mathcal{G}_c(\Omega\times\Omega\setminus\Delta), \widetilde{\mathbb{C}})$, (4.11) shows that $k_A \in \mathcal{G}(\Omega\times\Omega\setminus\Delta)$. In particular if $u \in \mathcal{G}_c(W')$ and $W\times W' \subseteq \Omega\times\Omega\setminus\Delta$, (4.10) follows from (4.11) and the inclusion $\mathcal{G}(W) \subseteq L(\mathcal{G}_c(W), \tilde{\mathbb{C}})$.

Under the hypothesis that$(\omega^{-1}(\epsilon))_\epsilon$ is a slow scale net, $a \in \widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\Omega\times\mathbb{R}^n)$ can also be considered as an element of $\widetilde{\mathcal{S}}^{m}_{\rho,\delta,\mathrm{rg}}(\Omega\times\Omega\times\mathbb{R}^n)$. Thus the assertions *iii)* and *iv)* are obtained from the analogous statements *ii)* and *iii)* in Remark 3.12.

Finally, if $a \in \widetilde{\mathcal{S}}^{-\infty}_{\mathrm{rg}}(\Omega\times\Omega\times\mathbb{R}^n)$, $\int_{\mathbb{R}^n} e^{i(x-y)\xi} a(x,y,\xi)\, d\!\!{}^-\xi \in \mathcal{G}^\infty(\Omega\times\Omega)$ and (4.11) holds for all $u \in \mathcal{G}_c(\Omega\times\Omega)$. Hence $k_A(x,y) = \int_{\mathbb{R}^n} e^{i(x-y)\xi} a(x,y,\xi)\, d\!\!{}^-\xi \in \mathcal{G}^\infty(\Omega\times\Omega)$. $\square$

We see from (4.9) and Proposition 2.11 that two pseudodifferential operators having the same kernel coincide. The definition of the kernel k_A is very useful in proving the following result of pseudolocality.

Proposition 4.11. *Let* $a \in \widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\Omega\times\mathbb{R}^n)$ *and* $(\omega^{-1}(\epsilon))_\epsilon$ *a slow scale net; let* A *be the corresponding pseudodifferential operator. Then for all* $u \in \mathcal{G}_c(\Omega)$

$$\mathrm{sing\,supp_g}\, Au \subseteq \mathrm{sing\,supp_g}\, u.$$

Proof. For $u \in \mathcal{G}_c(\Omega)$, we consider an arbitrary open neighborhood V of $\mathrm{sing\,supp}_g\, u$ contained in Ω and a function $\psi \in \mathcal{C}^\infty_c(V)$ identically equal to 1 in a neighborhood of $\mathrm{sing\,supp_g}\, u$. Then we write $u = \psi u + (1-\psi)u$ where $\psi u \in \mathcal{G}_c(\Omega)$ and $(1-\psi)u \in \mathcal{G}^\infty_c(\Omega)$. From Proposition 4.7, $A((1-\psi)u) \in \mathcal{G}^\infty(\Omega)$ and then our assertion becomes

$$\mathrm{sing\,supp_g}\, A(\psi u) \subseteq \mathrm{sing\,supp_g}\, u. \tag{4.12}$$

To prove (4.12), we show that for all $u \in \mathcal{G}_c(\Omega)$

$$\mathrm{sing\,supp_g}\, Au \subseteq \mathrm{supp}\, u. \tag{4.13}$$

Let $K \equiv \mathrm{supp}\ u$ and $x_0 \in \Omega\setminus K$ so that $x_0 \times K \subseteq \Omega\times\Omega\setminus\Delta$. Since $\Omega\times\Omega\setminus\Delta$ is open, there exist an open neighborhoods W and W' of x_0 and K, respectively, such that $W\times W' \subseteq \Omega\times\Omega\setminus\Delta$. We want to demonstrate that $x_0 \in \Omega\setminus\mathrm{supp}_g\, Au$, i.e. $Au_{|_W} \in \mathcal{G}^\infty(W)$. It is sufficient to recall Proposition 4.10, point *iv)*, and the equality (4.10) where $k_A \in \mathcal{G}^\infty(W\times W')$. Writing ψu in place of u in (4.13), we conclude that

$$\mathrm{sing\,supp_g}\, A(\psi u) \subseteq \mathrm{supp}\, \psi u \subseteq V.$$

Since V is arbitrary, the proof is complete. $\square$

Let us now consider a linear operator $A : \mathcal{G}_c(\Omega) \to \mathcal{G}(\Omega)$ of the form

$$Au(x) = \int_\Omega k(x,y)u(y)dy, \tag{4.14}$$

where $k \in \mathcal{G}^\infty(\Omega \times \Omega)$. As noted in Remark 2.15, k is uniquely determined by (4.14) as an element of $\mathcal{G}(\Omega \times \Omega)$. For this reason, we may call it *the* kernel of A, adopt the notation k_A, and we may call A an operator with regular generalized kernel. Obviously, every operator with regular generalized kernel is regularizing, i.e. it maps $\mathcal{G}_c(\Omega)$ into $\mathcal{G}^\infty(\Omega)$.

Proposition 4.12. *A is an operator with regular generalized kernel if and only if it is a pseudodifferential operator with smoothing amplitude in $\widetilde{\mathcal{S}}^{-\infty}_{\rm rg}(\Omega \times \Omega \times \mathbb{R}^n)$.*

Proof. Every pseudodifferential operator with smoothing amplitude has a regular generalized kernel by Proposition 4.10. To prove the converse, let $k_A \in \mathcal{G}^\infty(\Omega \times \Omega)$. Then for all $u \in \mathcal{G}_c(\Omega)$, Au has as a representative

$$A_\epsilon u_\epsilon = \int_\Omega k_{A,\epsilon}(x,y)u_\epsilon(y)dy = \int_{\Omega\times\mathbb{R}^n} e^{i(x-y)\xi}e^{-i(x-y)\xi}k_{A,\epsilon}(x,y)\chi(\xi)u_\epsilon(y)\, dy đ\xi,$$

where $\chi(\xi) \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ with $\int \chi(\xi)đ\xi = 1$. Now if we define

$$a_\epsilon(x,y,\xi) := e^{-i(x-y)\xi}k_{A,\epsilon}(x,y)\chi(\xi)$$

then each a_ϵ belongs to $S^{-\infty}(\Omega \times \Omega \times \mathbb{R}^n)$. Further, $(k_{A,\epsilon})_\epsilon \in \mathcal{E}^\infty_M(\Omega \times \Omega)$ implies $(a_\epsilon)_\epsilon \in \mathcal{S}^{-\infty}_{\rm rg}(\Omega\times\Omega\times\mathbb{R}^n)$ and $(k_{A,\epsilon})_\epsilon \in \mathcal{N}(\Omega\times\Omega)$ implies $(a_\epsilon)_\epsilon \in \mathcal{N}^{-\infty}(\Omega\times\Omega\times\mathbb{R}^n)$. In conclusion,

$$Au(x) = \int_{\Omega\times\mathbb{R}^n} e^{i(x-y)\xi}a(x,y,\xi)u(y)\, dy đ\xi$$

for $a := (a_\epsilon)_\epsilon + \mathcal{N}^{-\infty}(\Omega \times \Omega \times \mathbb{R}^n)$. $\square$

We introduce properly supported pseudodifferential operators using their kernels in $L(\mathcal{G}_c(\Omega \times \Omega), \widetilde{\mathbb{C}})$.

Definition 4.13. A pseudodifferential operator A is properly supported if and only if $\operatorname{supp} k_A$ is a proper set. An amplitude $a \in \widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega \times \Omega \times \mathbb{R}^n)$ is called properly supported if and only if $\operatorname{supp}_{x,y} a$ is a proper set.

We note that if A is properly supported then ${}^tA$ is properly supported.

Proposition 4.14. *Let A be a pseudodifferential operator with amplitude $a \in \widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega \times \Omega \times \mathbb{R}^n)$. If $(\omega(\epsilon))_\epsilon$ is bounded then A is properly supported if and only if it can be written with a properly supported amplitude in $\widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega \times \Omega \times \mathbb{R}^n)$. If $(\omega^{-1}(\epsilon))_\epsilon$ is a slow scale net then A is properly supported if and only if it can be written with a properly supported amplitude in $\widetilde{\mathcal{S}}^{m}_{\rho,\delta,\rm rg}(\Omega \times \Omega \times \mathbb{R}^n)$.*

Proof. Let us consider the first case when $(\omega(\epsilon))_\epsilon$ is bounded. If A is properly supported then choosing a proper function $\chi \in \mathcal{C}^\infty(\Omega \times \Omega)$ identically equal to 1 in a neighborhood of $\operatorname{supp} k_A$ we have that $\chi a := (\chi a_\epsilon)_\epsilon + \mathcal{N}^{m,\mu}_{\rho,\delta,\omega}(\Omega \times \mathbb{R}^n)$ belongs to $\widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\Omega\times\mathbb{R}^n)$. This uses the boundedness of $\omega(\epsilon)$. Clearly, χa is properly supported. Moreover, since for all $u\in\mathcal{G}_c(\Omega\times\Omega)$

$$k_A((1-\chi)u)=\int_{\Omega\times\mathbb{R}^n} e^{i(x-y)\xi}a(x,y,\xi)(1-\chi(x,y))u(x,y)\,dx\,dy\,đ\xi=0\quad\text{in }\widetilde{\mathbb{C}},$$

the operators with amplitudes a and χa have the same kernel and hence they coincide.

To prove the converse, assume that $a\in\widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\Omega\times\mathbb{R}^n)$ is a properly supported amplitude. Since $\operatorname{supp}k_A\subseteq\operatorname{supp}_{x,y}a$ we see that A is a properly supported pseudodifferential operator.

If $(\omega^{-1}(\epsilon))_\epsilon$ is a slow scale net we can repeat the same arguments, substituting $\widetilde{\mathcal{S}}^{m}_{\rho,\delta,\mathrm{rg}}(\Omega\times\Omega\times\mathbb{R}^n)$ for $\widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\Omega\times\mathbb{R}^n)$. □

Remark 4.15. Proposition 4.14 implies that any properly supported pseudodifferential operator A can be written with a properly supported amplitude a such that there exist a representative $(a_\epsilon)_\epsilon$ of a and a proper closed subset of $\Omega\times\Omega$ containing $\operatorname{supp}_{x,y}a_\epsilon$ for all ϵ.

Proposition 4.16. *If A is a properly supported pseudodifferential operator then for all $K\Subset\Omega$ there exist $K',K''\Subset\Omega$ such that for all $u\in\mathcal{G}_c(\Omega)$ the following statements hold:*

i) $\operatorname{supp}u\subset K$ *implies* $\operatorname{supp}Au\subset K'$,
ii) $\operatorname{supp}u\subset\Omega\setminus K''$ *implies* $\operatorname{supp}Au\subset\Omega\setminus K$.

Proof. From (4.9) we obtain for all $u\in\mathcal{G}_c(\Omega)$

$$\operatorname{supp}Au\subseteq\pi_1(\pi_2^{-1}(\operatorname{supp}u)\cap\operatorname{supp}k_A).\tag{4.15}$$

The first assertion follows from (4.15) putting $K'=\pi_1(\pi_2^{-1}(K)\cap\operatorname{supp}k_A)$. Defining K'' as $\pi_2(\pi_1^{-1}(K)\cap\operatorname{supp}k_A)$, (4.15) leads to assertion *ii*). □

Proposition 4.16 is the well-known topological characterization of a properly supported linear operator.

Proposition 4.17. *If A is a properly supported pseudodifferential operator with amplitude $a\in\widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\Omega\times\mathbb{R}^n)$ then*

i) *A maps $\mathcal{G}_c(\Omega)$ into $\mathcal{G}_c(\Omega)$,*
ii) *A can be uniquely extended to a linear map from $\mathcal{G}(\Omega)$ into $\mathcal{G}(\Omega)$ such that for all $u\in\mathcal{G}(\Omega)$ and $v\in\mathcal{G}_c(\Omega)$,*

$$\int_\Omega Au(x)v(x)dx=\int_\Omega u(y)\ {}^tAv(y)dy.\tag{4.16}$$

In the particular case when $(\omega^{-1}(\epsilon))_\epsilon$ is a slow scale net

iii) *A maps $\mathcal{G}^\infty_c(\Omega)$ into $\mathcal{G}^\infty_c(\Omega)$,*
iv) *the extension defined above maps $\mathcal{G}^\infty(\Omega)$ into $\mathcal{G}^\infty(\Omega)$.*

The same results hold with ${}^tA$ in place of A.

Proof. The first assertion is clear from *i*) in Proposition 4.16. To prove the second, we use the well-known sheaf-theoretic argument. We only need to define A locally. Let $V_1\subset V_2\subset\dots$ be an exhausting sequence of relatively compact open sets, let $K_j=\overline{V_j}$ and K''_j as in *ii*) of Proposition 4.16, where we assume that $\{K''_j\}_{j\in\mathbb{N}}$ is increasing. Given $u\in\mathcal{G}(\Omega)$, we define $A_ju\in\mathcal{G}(V_j)$ by $(A(\psi_ju))_{|_{V_j}}$ where $\psi_j\in$

$\mathcal{C}_c^\infty(\Omega)$, $\psi_j \equiv 1$ in an open neighborhood of K_j''. As in the proof of Proposition 2.14, it is clear that the family $\{A_j u\}_{j\in\mathbb{N}}$ is coherent. In this way we obtain a linear extension of the original pseudodifferential operator on $\mathcal{G}_c(\Omega)$, which satisfies (4.16). In fact, choosing $u \in \mathcal{G}(\Omega)$ and $v \in \mathcal{G}_c(\Omega)$, we have that $\operatorname{supp} v \subseteq V_j$ for some j. Since the function ψ_j is identically 1 in an open neighborhood of $\pi_2(\pi_1^{-1}(\overline{V_j}) \cap \operatorname{supp} k_A)$ and $\operatorname{supp}\ {}^tAv \subseteq \pi_2(\pi_1^{-1}(\overline{V_j}) \cap \operatorname{supp} k_A)$, we conclude that

$$\int_\Omega Au(x)v(x)dx = \int_\Omega A(\psi_j u)v(x)dx = \int_\Omega \psi_j u(y)\ {}^tAv(y)dy = \int_\Omega u(y)\ {}^tAv(y)dy.$$

The uniqueness is proved as in (2.6).

Assume now that $(\omega^{-1}(\epsilon))_\epsilon$ is a slow scale net. From Proposition 4.7 we already know that A maps $\mathcal{G}_c^\infty(\Omega)$ into $\mathcal{G}^\infty(\Omega)$. This mapping property combined with assertion $i)$ implies that $A : \mathcal{G}_c^\infty(\Omega) \to \mathcal{G}_c^\infty(\Omega)$. Finally using the sheaf property of $\mathcal{G}^\infty(\Omega)$, the extension to $\mathcal{G}(\Omega)$ defined above maps $\mathcal{G}^\infty(\Omega)$ into $\mathcal{G}^\infty(\Omega)$. □

It is clear that if A is properly supported and $(\omega^{-1}(\epsilon))_\epsilon$ is a slow scale net then the pseudolocality property $\operatorname{sing\,supp}_g Au \subseteq \operatorname{sing\,supp}_g u$ holds for every $u \in \mathcal{G}(\Omega)$. In fact, it suffices to recall that the restrictions of Au to the open subsets V_j are expressed by $(A(\psi_j u))_{|V_j}$, where $\psi_j u$ has compact support.

Proposition 4.18. *Let $a \in \widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\Omega\times\mathbb{R}^n)$ with $(\omega^{-1}(\epsilon))_\epsilon$ a slow scale net. The corresponding pseudodifferential operator A can be written as the sum $A_0 + A_1$ where A_0 is a properly supported pseudodifferential operator and A_1 has regular generalized kernel.*

Proof. Take a proper function $\chi \in \mathcal{C}^\infty(\Omega\times\Omega)$ identically equal to 1 in a neighborhood of the diagonal $\Delta \subset \Omega\times\Omega$. Given $u \in \mathcal{G}_c(\Omega)$ we can write $Au = A_0u + A_1u$ where A_0 is the properly supported pseudodifferential operator with amplitude $a_0 = \chi a \in \widetilde{\mathcal{S}}^m_{\rho,\delta,\mathrm{rg}}(\Omega\times\Omega\times\mathbb{R}^n)$ and A_1 is the pseudodifferential operator with amplitude $a_1 = a(1-\chi) \in \widetilde{\mathcal{S}}^m_{\rho,\delta,\mathrm{rg}}(\Omega\times\Omega\times\mathbb{R}^n)$. Since $\operatorname{supp}_{x,y} a_1$ is included in the complement of an open neighborhood of Δ and $(\omega^{-1}(\epsilon))_\epsilon$ is a slow scale net, the arguments in the proof of Proposition 4.10 show that the kernel of A_1 belongs to $\mathcal{G}^\infty(\Omega\times\Omega)$. □

5. Formal series and generalized symbols

In this section we develop a pseudodifferential calculus: formal series, symbols, transposition, and composition. Formal series and symbols play a basic role in the classical theory of pseudodifferential operators. The aim of this section is to generalize these concepts to our context. As we shall see in detail in Theorem 5.3, we will have to consider the subspaces

$$\underline{\widetilde{\mathcal{S}}}^{m,\mu}_{\rho,\delta,\omega}(\Omega'\times\mathbb{R}^p) := \underline{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega'\times\mathbb{R}^p)/\underline{\mathcal{N}}^{m,\mu}_{\rho,\delta,\omega}(\Omega'\times\mathbb{R}^p)$$

of $\widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega'\times\mathbb{R}^p)$,

$$\underline{\widetilde{\mathcal{S}}}^{m}_{\rho,\delta,\mathrm{rg}}(\Omega'\times\mathbb{R}^p) := \underline{\mathcal{S}}^{m}_{\rho,\delta,\mathrm{rg}}(\Omega'\times\mathbb{R}^p)/\underline{\mathcal{N}}^{m}_{\rho,\delta}(\Omega'\times\mathbb{R}^p)$$

of $\widetilde{\mathcal{S}}^{m}_{\rho,\delta,\mathrm{rg}}(\Omega'\times\mathbb{R}^p)$ and

$$\underline{\widetilde{\mathcal{S}}}^{-\infty}_{\mathrm{rg}}(\Omega'\times\mathbb{R}^p) := \underline{\mathcal{S}}^{-\infty}_{\mathrm{rg}}(\Omega'\times\mathbb{R}^p)/\underline{\mathcal{N}}^{-\infty}(\Omega'\times\mathbb{R}^p)$$

of $\widetilde{\mathcal{S}}^{-\infty}_{\rm rg}(\Omega' \times \mathbb{R}^p)$, obtained by fixing $\eta = 1$ in (4.1), (4.2), in the definitions of $\mathcal{S}^m_{\rho,\delta,\rm rg}(\Omega' \times \mathbb{R}^p)$ and $\mathcal{N}^m_{\rho,\delta}(\Omega' \times \mathbb{R}^p)$ and in (4.4), (4.5), respectively. This is needed in order to guarantee that the infinite number of terms in the formal series will be defined for ϵ in a common interval.

What concerns smoothing symbols, we need a refined version of Definition 4.4. We denote by $\underline{\mathcal{S}}^{-\infty,\mu}_{\omega}(\Omega \times \mathbb{R}^n)$ the set of all $(a_\epsilon)_\epsilon \in \mathcal{E}[\Omega \times \mathbb{R}^n]$ such that

$$\forall K \Subset \Omega,\ \exists N \in \mathbb{N}:\ \forall m \in \mathbb{R},\ \forall \alpha, \beta \in \mathbb{N}^n,\ \exists c > 0:\ \forall x \in K,\ \forall \xi \in \mathbb{R}^n,\ \forall \epsilon \in (0,1],$$
$$|\partial^\alpha_\xi \partial^\beta_x a_\epsilon(x,\xi)| \le c\langle\xi\rangle^{m-|\alpha|}\epsilon^{-N}\omega(\epsilon)^{-(|\beta|-\mu)_+}$$

and by $\underline{\mathcal{N}}^{-\infty,\mu}_{\omega}(\Omega \times \mathbb{R}^n)$ the set of all $(a_\epsilon)_\epsilon \in \mathcal{E}[\Omega \times \mathbb{R}^n]$ such that

$$\forall K \Subset \Omega,\ \forall q \in \mathbb{N},\ \forall m \in \mathbb{R},\ \forall \alpha, \beta \in \mathbb{N}^n,\ \exists c > 0:\ \forall x \in K,\ \forall \xi \in \mathbb{R}^n,\ \forall \epsilon \in (0,1],$$
$$|\partial^\alpha_\xi \partial^\beta_x a_\epsilon(x,\xi)| \le c\langle\xi\rangle^{m-|\alpha|}\epsilon^{q}\omega(\epsilon)^{-(|\beta|-\mu)_+}.$$

We introduce the notation $\widetilde{\underline{\mathcal{S}}}^{-\infty,\mu}_{\omega}(\Omega \times \mathbb{R}^n)$ for $\underline{\mathcal{S}}^{-\infty,\mu}_{\omega}(\Omega \times \mathbb{R}^n)/\underline{\mathcal{N}}^{-\infty,\mu}_{\omega}(\Omega \times \mathbb{R}^n)$. Obviously, if $(\omega(\epsilon))_\epsilon$ is bounded and $(\omega^{-1}(\epsilon))_\epsilon$ is a slow scale net then $\widetilde{\underline{\mathcal{S}}}^{-\infty,\mu}_{\omega}(\Omega \times \mathbb{R}^n) = \widetilde{\underline{\mathcal{S}}}^{-\infty}_{\rm rg}(\Omega\times\mathbb{R}^n)$. Finally, let $(a_\epsilon)_\epsilon \in \mathcal{S}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n)$ and K be a compact subset of Ω. We say that $(a_\epsilon)_\epsilon$ is of *growth type* $N_K \in \mathbb{N}$ on K if and only if

$$\forall \alpha, \beta \in \mathbb{N}^n,\ \exists c > 0:\ \forall x \in K,\ \forall \xi \in \mathbb{R}^n,\ \forall \epsilon \in (0,1],$$
$$|\partial^\alpha_\xi \partial^\beta_x a_\epsilon(x,\xi)| \le c\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}\epsilon^{-N_K}\omega(\epsilon)^{-(|\beta|-\mu)_+}.$$

Definition 5.1. Let $\{m_j\}_{j\in\mathbb{N}}$ and $\{\mu_j\}$ be sequences of real numbers with $m_j \searrow -\infty$, $m_0 = m$ and $\mu_0 = \mu$. Let $\{(a_{j,\epsilon})_\epsilon\}_{j\in\mathbb{N}}$ be a sequence of elements $(a_{j,\epsilon})_\epsilon \in \underline{\mathcal{S}}^{m_j,\mu_j}_{\rho,\delta,\omega}(\Omega \times \mathbb{R}^n)$, satisfying the following condition:

$$\forall K \Subset \Omega,\ \exists N_K \in \mathbb{N}:\ \forall j \in \mathbb{N} \quad (a_{j,\epsilon})_\epsilon \text{ is of growth type } N_K \text{ on } K. \tag{5.1}$$

We say that the formal series $\sum_{j=0}^\infty (a_{j,\epsilon})_\epsilon$ is the asymptotic expansion of $(a_\epsilon)_\epsilon \in \mathcal{E}[\Omega \times \mathbb{R}^n]$, $(a_\epsilon)_\epsilon \sim \sum_j (a_{j,\epsilon})_\epsilon$ for short, if and only if for all $K \Subset \Omega$ there exists $M_K \in \mathbb{N}$ such that for all $r \ge 1$, $(a_\epsilon - \sum_{j=0}^{r-1} a_{j,\epsilon})_\epsilon$ is an element of $\underline{\mathcal{S}}^{m_r,\mu}_{\rho,\delta,\omega}(\Omega \times \mathbb{R}^n)$ of growth type M_K on K.

Remark 5.2. If the elements $(a_\epsilon)_\epsilon$ and $(a'_\epsilon)_\epsilon$ in $\in \mathcal{E}[\Omega \times \mathbb{R}^n]$ have the same asymptotic expansion $\sum_j (a_{j,\epsilon})_\epsilon$ then $(a_\epsilon - a'_\epsilon)_\epsilon \in \underline{\mathcal{S}}^{-\infty,\mu}_{\omega}(\Omega \times \mathbb{R}^n)$. Indeed, we can write

$$a_\epsilon - a'_\epsilon = a_\epsilon - \sum_{j=0}^r a_{j,\epsilon} + \sum_{j=0}^r a_{j,\epsilon} - a'_\epsilon.$$

From the definition of an asymptotic expansion, we have that for all $K \Subset \Omega$ there exists a natural number M_K such that $\big(a_\epsilon - \sum_{j=0}^r a_{j,\epsilon}\big)_\epsilon$ and $\big(\sum_{j=0}^r a_{j,\epsilon} - a'_\epsilon\big)_\epsilon$ are elements of $\underline{\mathcal{S}}^{m_r,\mu}_{\rho,\delta,\omega}(\Omega \times \mathbb{R}^n)$ of growth type M_K on K. Since M_K does not depend on r and the sequence of $\{m_r\}_{r\in\mathbb{N}\setminus\{0\}}$ tends to $-\infty$, we conclude that $(a_\epsilon - a'_\epsilon)_\epsilon \in \underline{\mathcal{S}}^{-\infty,\mu}_{\omega}(\Omega \times \mathbb{R}^n)$, as desired.

Theorem 5.3. *Let $\{m_j\}_j$, $\{\mu_j\}_j$ and $(a_{j,\epsilon})_\epsilon \in \underline{\mathcal{S}}^{m_j,\mu_j}_{\rho,\delta,\omega}(\Omega \times \mathbb{R}^n)$ for all $j \in \mathbb{N}$ as in Definition 5.1. If in addition one of the following hypotheses*

$$\mu_j \ge \mu \text{ for all } j \in \mathbb{N} \text{ and } \sup_{\epsilon\in(0,1]} \omega(\epsilon) < \infty \tag{5.2}$$

or

$$\mu_j \le \mu \text{ for all } j \in \mathbb{N} \text{ and } (\omega^{-1}(\epsilon))_\epsilon \text{ is a slow scale net} \tag{5.3}$$

holds then there exists $(a_\epsilon)_\epsilon \in \underline{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n)$ *such that* $(a_\epsilon)_\epsilon \sim \sum_j (a_{j,\epsilon})_\epsilon$.

Proof. The proof follows the classical line of arguments, but we will have to keep track of the ϵ-dependence carefully. We consider a sequence of relatively compact open sets $\{V_l\}$ contained in Ω, such that for all $l \in \mathbb{N}$, $V_l \subset K_l = \overline{V_l} \subset V_{l+1}$ and $\bigcup_{l\in\mathbb{N}} V_l = \Omega$. Let $\psi \in \mathcal{C}^\infty(\mathbb{R}^n)$, $0 \le \psi(\xi) \le 1$, such that $\psi(\xi) = 0$ for $|\xi| \le 1$ and $\psi(\xi) = 1$ for $|\xi| \ge 2$. We introduce

$$b_{j,\epsilon}(x,\xi) = \psi(\lambda_j\xi)a_{j,\epsilon}(x,\xi),$$

where λ_j will be positive constants, independent of ϵ, with $\lambda_{j+1} < \lambda_j < 1$, $\lambda_j \to 0$. We can define

$$a_\epsilon(x,\xi) = \sum_{j\in\mathbb{N}} b_{j,\epsilon}(x,\xi). \tag{5.4}$$

This sum is locally finite. We observe that $\partial^\alpha(\psi(\lambda_j\xi)) = \partial^\alpha\psi(\lambda_j\xi)\lambda_j^{|\alpha|}$, that $\mathrm{supp}(\partial^\alpha\psi(\lambda_j\xi)) \subseteq \{\xi : 1/\lambda_j \le |\xi| \le 2/\lambda_j\}$, and that $1/\lambda_j \le |\xi| \le 2/\lambda_j$ implies $\lambda_j \le 2/|\xi| \le 4/(1+|\xi|)$.

Case 1: We assume now hypothesis (5.2). Fixing $K \Subset \Omega$ and $\alpha,\beta \in \mathbb{N}^n$, we obtain for $j \in \mathbb{N}$, $\epsilon \in (0,1]$, $x \in K$, $\xi \in \mathbb{R}^n$,

$$\begin{aligned}
&|\partial_\xi^\alpha\partial_x^\beta b_{j,\epsilon}(x,\xi)|\\
&\le \sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}\lambda_j^{|\alpha-\gamma|}|\partial^{\alpha-\gamma}\psi(\lambda_j\xi)|c_{j,\gamma,\beta,K}\langle\xi\rangle^{m_j-\rho|\gamma|+\delta|\beta|}\epsilon^{-N_K}\omega(\epsilon)^{-(|\beta|-\mu_j)_+}\\
&\le \sum_{\gamma\le\alpha} c_{j,\gamma,\beta,K}4^{|\alpha-\gamma|}\langle\xi\rangle^{-|\alpha-\gamma|}\langle\xi\rangle^{m_j-\rho|\gamma|+\delta|\beta|}\epsilon^{-N_K}\omega(\epsilon)^{-(|\beta|-\mu_j)_+}\\
&\le C_{j,\alpha,\beta,K}\langle\xi\rangle^{m_j-\rho|\alpha|+\delta|\beta|}\epsilon^{-N_K}\omega(\epsilon)^{-(|\beta|-\mu)_+},
\end{aligned}\tag{5.5}$$

where in the last computations we use the inequality $(|\beta|-\mu_j)_+ \le (|\beta|-\mu)_+$ for $\mu_j \ge \mu$ and the boundeness of ω. At this point we choose λ_j such that for $|\alpha+\beta| \le j$, $l \le j$

$$C_{j,\alpha,\beta,K_l}\lambda_j \le 2^{-j}. \tag{5.6}$$

Our aim is to prove that $a_\epsilon(x,\xi)$ defined in (5.4) belongs to $\underline{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n)$. We already know that $(a_\epsilon)_\epsilon \in \mathcal{E}[\Omega\times\mathbb{R}^n]$. We observe that

$$\begin{gathered}
\forall K \Subset \Omega,\ \exists l \in \mathbb{N}:\ K \subset V_l \subset K_l,\\
\forall \alpha_0,\beta_0 \in \mathbb{N}^n,\ \exists j_0 \in \mathbb{N},\ j_0 \ge l:\ |\alpha_0+\beta_0| \le j_0, \quad m_{j_0}+1 \le m.
\end{gathered}\tag{5.7}$$

Now, $(a_\epsilon)_\epsilon$ as the sum of the following two terms:

$$a_\epsilon(x,\xi) = \sum_{j=0}^{j_0-1} b_{j,\epsilon}(x,\xi) + \sum_{j=j_0}^{+\infty} b_{j,\epsilon}(x,\xi) = f_\epsilon(x,\xi) + s_\epsilon(x,\xi).$$

First we study $f_\epsilon(x,\xi)$. For $x \in K$, using hypothesis (5.2), we have that

$$\begin{aligned}
&|\partial_\xi^{\alpha_0}\partial_x^{\beta_0} f_\epsilon(x,\xi)|\\
&\le \sum_{j=0}^{j_0-1} c_{j,\alpha_0,\beta_0,K}\langle\xi\rangle^{m_j-\rho|\alpha_0|+\delta|\beta_0|}\epsilon^{-N_K}\omega(\epsilon)^{-(|\beta_0|-\mu_j)_+}\\
&\le c'_{\alpha_0,\beta_0,K}\langle\xi\rangle^{m-\rho|\alpha_0|+\delta|\beta_0|}\epsilon^{-N_K}\omega(\epsilon)^{-(|\beta_0|-\mu)_+}.
\end{aligned}\tag{5.8}$$

We now turn to $s_\epsilon(x,\xi)$. From (5.5) and (5.6), we get for $x\in K$ and $\epsilon\in(0,1]$,

$$
\begin{aligned}
&|\partial_\xi^{\alpha_0}\partial_x^{\beta_0} s_\epsilon(x,\xi)|\\
&\le \sum_{j=j_0}^{+\infty} C_{j,\alpha_0,\beta_0,K_l}\langle\xi\rangle^{m_j-\rho|\alpha_0|+\delta|\beta_0|}\epsilon^{-N_{K_l}}\omega(\epsilon)^{-(|\beta_0|-\mu)_+}\\
&\le \sum_{j=j_0}^{+\infty} 2^{-j}\lambda_j^{-1}\langle\xi\rangle^{-1}\langle\xi\rangle^{m_j+1-\rho|\alpha_0|+\delta|\beta_0|}\epsilon^{-N_{K_l}}\omega(\epsilon)^{-(|\beta_0|-\mu)_+}.
\end{aligned}
$$

Since $\psi(\xi)$ is identically equal to 0 for $|\xi|\le 1$, we can assume in our estimates that $\langle\xi\rangle^{-1}\le\lambda_j$, and therefore from (5.7), we conclude that

$$
|\partial_\xi^{\alpha_0}\partial_x^{\beta_0} s_\epsilon(x,\xi)|\le \langle\xi\rangle^{m-\rho|\alpha_0|+\delta|\beta_0|}\epsilon^{-N_{K_l}}\omega(\epsilon)^{-(|\beta_0|-\mu)_+}. \tag{5.9}
$$

In conclusion, we obtain that $(a_\epsilon)_\epsilon\in\mathcal{S}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n)$. In order to prove that $(a_\epsilon)_\epsilon\sim\sum_j(a_{j,\epsilon})_\epsilon$ we fix $r\ge 1$ and we write

$$
\begin{aligned}
a_\epsilon(x,\xi)-\sum_{j=0}^{r-1}a_{j,\epsilon}(x,\xi)&=\sum_{j=0}^{r-1}(\psi(\lambda_j\xi)-1)a_{j,\epsilon}(x,\xi)+\sum_{j=r}^{+\infty}\psi(\lambda_j\xi)a_{j,\epsilon}(x,\xi)\\
&=g_\epsilon(x,\xi)+t_\epsilon(x,\xi).
\end{aligned}
$$

Recall that $\psi\in\mathcal{C}^\infty(\mathbb{R}^n)$ was chosen such that $\psi-1\in\mathcal{C}^\infty_c(\mathbb{R}^n)$ and $\mathrm{supp}(\psi-1)\subseteq\{\xi:\ |\xi|\le 2\}$. Thus, for $0\le j\le r-1$,

$$
\mathrm{supp}(\psi(\lambda_j\xi)-1)\subseteq\{\xi:\ |\lambda_j\xi|\le 2\}\subseteq\{\xi:\ |\xi|\le 2\lambda_{r-1}^{-1}\}.
$$

As a consequence, for fixed $K\Subset\Omega$ and for all $\epsilon\in(0,1]$,

$$
|\partial_\xi^\alpha\partial_x^\beta g_\epsilon(x,\xi)|\le c_{\alpha,\beta,K}\langle\xi\rangle^{m_r-\rho|\alpha|+\delta|\beta|}\epsilon^{-N_K}\omega(\epsilon)^{-(|\beta|-\mu)_+}.
$$

In this way $(g_\epsilon)_\epsilon$ is an element of $\underline{\mathcal{S}}^{m_r,\mu}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n)$ of growth type N_K on the compact set K. Moreover, repeating the same arguments used in the construction of $(a_\epsilon)_\epsilon$ we have that $(t_\epsilon)_\epsilon$ belongs to $\underline{\mathcal{S}}^{m_r,\mu}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n)$ and it is of growth type N_{K_l} on K, where $K\subset V_l\subset K_l$. Summarizing, we have that for all $r\ge 1$, $(a_\epsilon)_\epsilon-\sum_{j=0}^{r-1}(a_{j,\epsilon})_\epsilon\in\underline{\mathcal{S}}^{m_r,\mu}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n)$ and it is of growth type $\max(N_K,N_{K_l})$.

Case 2: The proof can be easily adapted to cover hypothesis (5.3). The crucial point is to observe that if $\mu_j\le\mu$ then $(|\beta|-\mu_j)_+\ge(|\beta|-\mu)_+$, and that we get the inequality

$$
\omega(\epsilon)^{-(|\beta|-\mu_j)_+}=\omega(\epsilon)^{-(|\beta|-\mu)_+}\omega(\epsilon)^{(|\beta|-\mu)_+-(|\beta|-\mu_j)_+}\le c_{j,\beta}\,\omega(\epsilon)^{-(|\beta|-\mu)_+}\epsilon^{-1}, \tag{5.10}
$$

which follows from the definition of a slow scale net. Using (5.10), we may transform (5.5), (5.8), (5.9), respectively, into

$$
|\partial_x^\alpha\partial_x^\beta b_{j,\epsilon}(x,\xi)|\le C_{j,\alpha,\beta,K}\langle\xi\rangle^{m_j+1-\rho|\alpha|+\delta|\beta|}\langle\xi\rangle^{-1}\epsilon^{-N_K-1}\omega(\epsilon)^{-(|\beta|-\mu)_+}, \tag{5.11}
$$

$$
|\partial_\xi^{\alpha_0}\partial_x^{\beta_0} f_\epsilon(x,\xi)|\le c'_{\alpha_0,\beta_0,K}\langle\xi\rangle^{m-\rho|\alpha_0|+\delta|\beta_0|}\epsilon^{-N_K-1}\omega(\epsilon)^{-(|\beta_0|-\mu)_+}, \tag{5.12}
$$

$$
|\partial_\xi^{\alpha_0}\partial_x^{\beta_0} s_\epsilon(x,\xi)|\le c''_{\alpha_0,\beta_0,K}\langle\xi\rangle^{m-\rho|\alpha_0|+\delta|\beta_0|}\epsilon^{-N_{K_l}-1}\omega(\epsilon)^{-(|\beta_0|-\mu)_+}, \tag{5.13}
$$

where $x\in K$, $\xi\in\mathbb{R}^n$ and $\epsilon\in(0,1]$.

In this way $(g_\epsilon)_\epsilon \in \underline{\mathcal{S}}^{m_r,\mu}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n)$ is of growth type N_K+1 on K and $(t_\epsilon)_\epsilon \in \underline{\mathcal{S}}^{m_r,\mu}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n)$ is of growth type $N_{K_l}+1$ on K. Finally, $(a_\epsilon)_\epsilon-\sum_{j=0}^{r-1}(a_{j,\epsilon})_\epsilon$ belongs to $\underline{\mathcal{S}}^{m_r,\mu}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n)$ and it is of growth type $\max(N_K,N_{K_l})+1$. □

The following theorem studies the special case of formal series of negligible elements.

Theorem 5.4. *Let $\{m_j\}_j$, $\{\mu_j\}_j$ be sequences of real numbers as in Definition 5.1 and let $(a_{j,\epsilon})_\epsilon \in \underline{\mathcal{N}}^{m_j,\mu_j}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n)$ for all $j\in\mathbb{N}$. If the hypothesis* (5.2) *holds or if $\mu_j\le\mu$ for all $j\in\mathbb{N}$ then there exists $(a_\epsilon)_\epsilon\in\underline{\mathcal{N}}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n)$ such that for all $r\ge1$*

$$\Big(a_\epsilon-\sum_{j=0}^{r-1}a_{j,\epsilon}\Big)_\epsilon\in\underline{\mathcal{N}}^{m_r,\mu}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n). \tag{5.14}$$

If $(a'_\epsilon)_\epsilon\in\mathcal{E}[\Omega\times\mathbb{R}^n]$ satisfies (5.14) *then $(a_\epsilon-a'_\epsilon)_\epsilon\in\underline{\mathcal{N}}^{-\infty,\mu}_{\omega}(\Omega\times\mathbb{R}^n)$.*

Proof. Repeating the arguments and constructions from the proof of Theorem 5.3, we conclude that

$$\begin{gathered}\forall K\Subset\Omega,\forall\alpha,\beta\in\mathbb{N}^n,\ \forall q\in\mathbb{N},\ \forall j\in\mathbb{N},\ \exists C_{j,\alpha,\beta,K,q}>0:\ \forall x\in K,\ \forall\xi\in\mathbb{R}^n,\\ \forall\epsilon\in(0,1],\ |\partial^\alpha_\xi\partial^\beta_x b_{j,\epsilon}(x,\xi)|\le C_{j,\alpha,\beta,K,q}\langle\xi\rangle^{m_j-\rho|\alpha|+\delta|\beta|}\epsilon^q\omega(\epsilon)^{-(|\beta|-\mu)_+}.\end{gathered}$$

holds under either of the two hypotheses stated in Theorem 5.4. At this point we choose λ_j such that for $|\alpha+\beta|\le j$, $l\le j$, $q\le j$,

$$C_{j,\alpha,\beta,K_l,q}\lambda_j\le 2^{-j}. \tag{5.15}$$

We observe that (5.7) still holds, where, given $q_0\in\mathbb{N}$, we may take $j_0\ge\max(l,q_0)$. As a consequence, writing as before $a_\epsilon(x,\xi)=f_\epsilon(x,\xi)+s_\epsilon(x,\xi)$, we have

$$|\partial^{\alpha_0}_\xi\partial^{\beta_0}_x f_\epsilon(x,\xi)|\le c_{\alpha_0,\beta_0,q_0,K}\langle\xi\rangle^{m-\rho|\alpha_0|+\delta|\beta_0|}\epsilon^{q_0}\omega(\epsilon)^{-(|\beta_0|-\mu)_+}$$

and from (5.15)

$$\begin{aligned}|\partial^{\alpha_0}_\xi\partial^{\beta_0}_x s_\epsilon(x,\xi)|&\le\sum_{j=j_0}^{+\infty}C_{j,\alpha_0,\beta_0,K_l,q_0}\langle\xi\rangle^{m_j-\rho|\alpha_0|+\delta|\beta_0|}\epsilon^{q_0}\omega(\epsilon)^{-(|\beta_0|-\mu)_+}\\&\le\sum_{j=j_0}^{+\infty}2^{-j}\lambda_j^{-1}\langle\xi\rangle^{-1}\langle\xi\rangle^{m_j+1-\rho|\alpha_0|+\delta|\beta_0|}\epsilon^{q_0}\omega(\epsilon)^{-(|\beta_0|-\mu)_+}\\&\le\langle\xi\rangle^{m-\rho|\alpha_0|+\delta|\beta_0|}\epsilon^{q_0}\omega(\epsilon)^{-(|\beta_0|-\mu)_+}.\end{aligned}$$

These results lead us to the conclusion that $(a_\epsilon)_\epsilon\in\underline{\mathcal{N}}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n)$. We omit the rest of the proof since it is a simple adaptation of the proof of Theorem 5.3, where in place of the growth type N_K we consider an arbitrary exponent $q\in\mathbb{N}$. □

Definition 5.5. Let $\{m_j\}_{j\in\mathbb{N}}$ and $\{\mu_j\}$ be sequences of real numbers with $m_j\searrow-\infty$, $m_0=m$ and $\mu_0=\mu$. Let $\{a_j\}_{j\in\mathbb{N}}$ be a sequence of symbols $a_j\in\underline{\widetilde{\mathcal{S}}}^{m_j,\mu_j}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n)$ whose representatives $(a_{j,\epsilon})_\epsilon$ satisfy (5.1).

We say that the formal series $\sum_j a_j$ is the asymptotic expansion of $a\in\underline{\widetilde{\mathcal{S}}}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n)$, $a\sim\sum_j a_j$ for short, if and only if there exist a representative $(a_\epsilon)_\epsilon$ of a and, for every j, representatives $(a_{j,\epsilon})_\epsilon$ of a_j, such that $(a_\epsilon)_\epsilon\sim\sum_j(a_{j,\epsilon})_\epsilon$.

Note that if some representative of a_j satisfies (5.1) then every representative does. Theorems 5.1 and 5.2 lead us to the following characterization of $a\sim\sum_j a_j$.

Proposition 5.6. *Under the hypotheses of Theorem 5.4, $a \sim \sum_j a_j$ if and only if for any choice of representatives $(a_{j,\epsilon})_\epsilon$ of a_j there exists a representative $(a_\epsilon)_\epsilon$ of a such that $(a_\epsilon)_\epsilon \sim \sum_j (a_{j,\epsilon})_\epsilon$.*

Proof. We assume that there exist $(a_\epsilon)_\epsilon$ and $(a_{j,\epsilon})_\epsilon$ such that $(a_\epsilon)_\epsilon \sim \sum_j (a_{j,\epsilon})_\epsilon$. Let $(a'_{j,\epsilon})_\epsilon$ be another choice of representatives of a_j. It is clear that $\sum_j (a_{j,\epsilon} - a'_{j,\epsilon})_\epsilon$ fulfills the requirements of Theorem 5.4 and therefore there exists $(a''_\epsilon)_\epsilon \in \underline{\mathcal{N}}^{m,\mu}_{\rho,\delta,\omega}(\Omega \times \mathbb{R}^n)$ such that for all $r \geq 1$, $(a''_\epsilon - \sum_{j=0}^{r-1}(a_{j,\epsilon} - a'_{j,\epsilon}))_\epsilon \in \underline{\mathcal{N}}^{m_r,\mu}_{\rho,\delta,\omega}(\Omega \times \mathbb{R}^n)$. As a consequence, $(a_\epsilon - a''_\epsilon)_\epsilon \in \underline{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega \times \mathbb{R}^n)$ is another representative of a and $(a_\epsilon - a''_\epsilon)_\epsilon \sim \sum_j (a'_{j,\epsilon})_\epsilon$. In fact, for all $r \geq 1$ we have that $(a_\epsilon - a''_\epsilon - \sum_{j=0}^{r-1} a'_{j,\epsilon})_\epsilon$ can be written as the difference of $(a_\epsilon - \sum_{j=0}^{r-1} a_{j,\epsilon})_\epsilon \in \underline{\mathcal{S}}^{m_r,\mu}_{\rho,\delta,\omega}(\Omega \times \mathbb{R}^n)$ and $(a''_\epsilon - \sum_{j=0}^{r-1}(a_{j,\epsilon} - a'_{j,\epsilon}))_\epsilon \in \underline{\mathcal{N}}^{m_r,\mu}_{\rho,\delta,\omega}(\Omega \times \mathbb{R}^n)$, where the growth order on every compact set is independent of r. □

Theorem 5.7. *Let $\{m_j\}_j$, $\{\mu_j\}_j$ and $a_j \in \widetilde{\underline{\mathcal{S}}}^{m_j,\mu_j}_{\rho,\delta,\omega}(\Omega \times \mathbb{R}^n)$ for all $j \in \mathbb{N}$ as in Definition 5.5. If in addition the hypothesis* (5.2) *or* (5.3) *holds then there exists $a \in \widetilde{\underline{\mathcal{S}}}^{m,\mu}_{\rho,\delta,\omega}(\Omega \times \mathbb{R}^n)$ such that $a \sim \sum_j a_j$. Moreover, if $b \in \widetilde{\underline{\mathcal{S}}}^{m,\mu}_{\rho,\delta,\omega}(\Omega \times \mathbb{R}^n)$ has asymptotic expansion $\sum_j a_j$ then there exists a representative $(a_\epsilon)_\epsilon$ of a and a representative $(b_\epsilon)_\epsilon$ of b such that $(a_\epsilon - b_\epsilon)_\epsilon \in \underline{\mathcal{S}}^{-\infty,\mu}_{\omega}(\Omega \times \mathbb{R}^n)$.*

Proof. The existence of a is a direct consequence of Theorem 5.3. In particular, there is a choice of representatives $(a_{j,\epsilon})_\epsilon$ of a_j and a representative $(a_\epsilon)_\epsilon$ of a such that $(a_\epsilon)_\epsilon \sim \sum_j (a_{j,\epsilon})_\epsilon$. Now if $b \sim \sum_j a_j$, the previous proposition guarantees the existence of a representative $(b_\epsilon)_\epsilon$ such that $(b_\epsilon)_\epsilon \sim \sum_j (a_{j,\epsilon})_\epsilon$. Therefore, from Remark 5.2, $(a_\epsilon - b_\epsilon)_\epsilon \in \underline{\mathcal{S}}^{-\infty,\mu}_{\omega}(\Omega \times \mathbb{R}^n)$. □

Combining Theorem 5.4 with Remark 5.2 we have that if $a \in \widetilde{\underline{\mathcal{S}}}^{m,\mu}_{\rho,\delta,\omega}(\Omega \times \mathbb{R}^n)$ has asymptotic expansion $\sum_j a_j$ where each term $a_j = 0$ in $\widetilde{\underline{\mathcal{S}}}^{m_j,\mu_j}_{\rho,\delta,\omega}(\Omega \times \mathbb{R}^n)$ then it has a representative of the form $a_\epsilon = a'_\epsilon + a''_\epsilon$, where $(a'_\epsilon)_\epsilon \in \underline{\mathcal{N}}^{m,\mu}_{\rho,\delta,\omega}(\Omega \times \mathbb{R}^n)$ and $(a''_\epsilon)_\epsilon \in \underline{\mathcal{S}}^{-\infty,\mu}_{\omega}(\Omega \times \mathbb{R}^n)$.

In the sequel we always assume $0 \leq \delta < \rho \leq 1$.

Theorem 5.8. *Let A be a properly supported pseudodifferential operator with amplitude $a \in \widetilde{\underline{\mathcal{S}}}^{m,\mu}_{\rho,\delta,\omega}(\Omega \times \Omega \times \mathbb{R}^n)$ where $(\omega^{-1}(\epsilon))_\epsilon$ is a slow scale net. Then there exists $\sigma \in \widetilde{\underline{\mathcal{S}}}^{m}_{\rho,\delta,\mathrm{rg}}(\Omega \times \mathbb{R}^n)$ such that for all $u \in \mathcal{G}_{\mathscr{S}}(\mathbb{R}^n)$*

$$A(u_{|\Omega})(x) = \int_{\mathbb{R}^n} e^{ix\xi} \sigma(x,\xi) \widehat{u}(\xi)\, đ\xi. \tag{5.16}$$

Moreover, $\sigma \sim \sum_j \frac{1}{\gamma!} \partial^\gamma_\xi D^\gamma_y a(x,y,\xi)_{|x=y}$ where $D^\gamma = (-i)^{|\gamma|}\partial^\gamma$ and the asymptotic expansion is understood in the sense of $\widetilde{\underline{\mathcal{S}}}^{m}_{\rho,\delta,\mathrm{rg}}(\Omega \times \mathbb{R}^n)$.

Proof. As shown in Proposition 4.14, given a proper function $\chi \in \mathcal{C}^\infty(\Omega \times \Omega)$ identically equal to 1 in a neighborhood of $\operatorname{supp} k_A \cup \Delta$, the pseudodifferential operator A can be written with the properly supported amplitude $\chi a := (\chi a_\epsilon)_\epsilon + \underline{\mathcal{N}}^{m}_{\rho,\delta}(\Omega \times$

$\Omega\times\mathbb{R}^n)$ and can be viewed as an element of $\underline{\widetilde{\mathcal{S}}}^{\,m}_{\rho,\delta,\mathrm{rg}}(\Omega\times\Omega\times\mathbb{R}^n)$. Now we define

$$\sigma_\epsilon(x,\xi)=\int_{\mathbb{R}^n}\widehat{b_\epsilon}(x,\eta,\xi+\eta)\,d\eta\,,$$
$$\widehat{b_\epsilon}(x,\eta,\xi)=\int_\Omega e^{i(x-y)\eta}\chi(x,y)a_\epsilon(x,y,\xi)\,đy.$$

The net $(\widehat{b_\epsilon})_\epsilon$ belongs to $\mathcal{E}[\Omega\times\Omega\times\mathbb{R}^n]$. Using integration by parts and the assumptions on $(\omega^{-1}(\epsilon))_\epsilon$, we obtain for $x\in K\Subset\Omega$ and $\epsilon\in(0,1]$,

$$\begin{aligned}
&|(-i\eta)^\gamma\partial^\alpha_\xi\partial^\beta_x\widehat{b_\epsilon}(x,\eta,\xi)|\\
&=\Big|\sum_{\beta'\le\beta}\binom{\beta}{\beta'}\int_\Omega e^{i(x-y)\eta}\partial^\alpha_\xi\partial^{\beta-\beta'}_x\partial^{\gamma+\beta'}_y(\chi(x,y)a_\epsilon(x,y,\xi))đy\Big|\\
&\le c_{\alpha,\beta,\gamma}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta+\gamma|}\epsilon^{-N_K-1}.
\end{aligned}$$

Consequently, we have for any $M\in\mathbb{N}$ that

$$|\partial^\alpha_\xi\partial^\beta_x\widehat{b_\epsilon}(x,\eta,\xi)|\le c_{\alpha,\beta,M}\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|+\delta M}\langle\eta\rangle^{-M}\epsilon^{-N_K-1}. \tag{5.17}$$

From (5.17) we obtain for $x\in K$ and $\epsilon\in(0,1]$

$$\begin{aligned}
|\partial^\alpha_\xi\partial^\beta_x\sigma_\epsilon(x,\xi)|&\le c_{\alpha,\beta,M}\epsilon^{-N_K-1}\int_{\mathbb{R}^n}\langle\xi+\eta\rangle^{m-\rho|\alpha|+\delta|\beta|+\delta M}\langle\eta\rangle^{-M}d\eta\\
&\le c'_{\alpha,\beta,M}\epsilon^{-N_K-1}\langle\xi\rangle^{p+\delta M},
\end{aligned}$$

where $p=\max(m-\rho|\alpha|+\delta|\beta|,0)$ and M is large enough. Next we estimate

$$\sigma_\epsilon(x,\xi)-\sum_{|\gamma|=0}^{h-1}\frac{1}{\gamma!}\partial^\gamma_\xi D^\gamma_y a_\epsilon(x,y,\xi)|_{x=y}$$

for $h\ge1$. Recalling that $\partial^\gamma_\xi D^\gamma_y a_\epsilon(x,y,\xi)|_{x=y}=\partial^\gamma_\xi D^\gamma_y(\chi(x,y)a_\epsilon(x,y,\xi))|_{x=y}$, a power series expansion of $\widehat{b_\epsilon}(x,\eta,\xi+\eta)$ in the last argument about ξ and the same reasoning as in [40, p.24-25] leads to the following estimates:

$$|\sigma_\epsilon(x,\xi)-\sum_{|\gamma|<h}\frac{1}{\gamma!}\partial^\gamma_\xi D^\gamma_y a_\epsilon(x,y,\xi)|_{x=y}|\le C_h\epsilon^{-N_K-1}\langle\xi\rangle^{m-(\rho-\delta)h+n},$$
$$|\partial^\alpha_\xi\partial^\beta_x\big(\sigma_\epsilon(x,\xi)-\sum_{|\gamma|<h}\frac{1}{\gamma!}\partial^\gamma_\xi D^\gamma_y a_\epsilon(x,y,\xi)|_{x=y}\big)|\le C_h\epsilon^{-N_K-1}\langle\xi\rangle^{m-(\rho-\delta)h+n-\rho|\alpha|+\delta|\beta|}.$$

We now write

$$\begin{aligned}
&\sigma_\epsilon(x,\xi)-\sum_{|\gamma|<h}\frac{1}{\gamma!}\partial^\gamma_\xi D^\gamma_y a_\epsilon(x,y,\xi)|_{x=y}\\
&=\sigma_\epsilon(x,\xi)-\sum_{|\gamma|<h'}\frac{1}{\gamma!}\partial^\gamma_\xi D^\gamma_y a_\epsilon(x,y,\xi)|_{x=y}+\sum_{h\le|\gamma|<h'}\frac{1}{\gamma!}\partial^\gamma_\xi D^\gamma_y a_\epsilon(x,y,\xi)|_{x=y}.
\end{aligned}\tag{5.18}$$

where $h'=h+n/(\rho-\delta)$.

¿From the previous computations, $(\sigma_\epsilon-\sum_{|\alpha|<h'}\frac{1}{\gamma!}\partial^\gamma_\xi D^\gamma_y a_\epsilon(x,y,\xi)|_{x=y})_\epsilon$ is an element of $\underline{\mathcal{S}}^{m-(\rho-\delta)h}_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ and the last sum in (5.18) belongs to $\underline{\mathcal{S}}^{m-(\rho-\delta)h}_{\rho,\delta,\mathrm{rg}}(\Omega\times$

$\mathbb{R}^n$). This result shows that $(\sigma_\epsilon(x,\xi))_\epsilon \in \underline{\mathcal{S}}^m_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ and

$$(\sigma_\epsilon)_\epsilon \sim \sum_\gamma \frac{1}{\gamma!}(\partial^\gamma_\xi D^\gamma_y a_\epsilon(x,y,\xi)_{|_{x=y}})_\epsilon.$$

It remains to prove that (5.16) holds with $\sigma := (\sigma_\epsilon)_\epsilon + \underline{\mathcal{N}}^m_{\rho,\delta}(\Omega\times\mathbb{R}^n)$. It is sufficient to show that the generalized functions involved in (5.16) coincide locally. In fact, with the notations introduced in the proof of Proposition 4.17, we have for $u \in \mathcal{G}_{\mathscr{S}}(\mathbb{R}^n)$,

$$\begin{aligned}
(A(u_{|\Omega}))_{|V_j} &= A_j(u_{|\Omega})\\
&= \Big(\int_{\Omega\times\mathbb{R}^n} e^{i(x-y)\xi}\chi(x,y)a(x,y,\xi)\psi_j(y)u(y)\,dy\,đ\xi\Big)\Big|_{V_j}\\
&= \Big(\int_{\mathbb{R}^n} e^{ix\eta}\int_{\Omega\times\mathbb{R}^n} e^{i(x-y)\xi}e^{i(y-x)\eta}\chi(x,y)a(x,y,\xi)\psi_j(y)\,dy\,đ\xi\ \widehat{u}(\eta)đ\eta\Big)\Big|_{V_j},
\end{aligned}$$

where $\psi_j \in \mathcal{C}^\infty_c(\Omega)$ and is identically 1 in a neighborhood of $\pi_2(\pi_1^{-1}(\overline{V_j})\cap\operatorname{supp}\chi)$. Now since $\chi(x,y)a(x,y,\xi)(\psi_j(y)-1)\equiv 0$ on V_j, we can conclude that

$$(A(u_{|\Omega}))_{|V_j} = \Big(\int_{\mathbb{R}^n} e^{ix\eta}\sigma(x,\eta)\widehat{u}(\eta)\,đ\eta\Big)\Big|_{V_j}.$$

□

Remark 5.9. From the above computations it follows that $(a_\epsilon)_\epsilon \in \underline{\mathcal{N}}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\Omega\times\mathbb{R}^n)$ implies $(\sigma_\epsilon)_\epsilon \in \underline{\mathcal{N}}^m_{\rho,\delta}(\Omega\times\mathbb{R}^n)$. Therefore, the integral

$$\begin{aligned}
\sigma(x,\xi) &= \int_{\Omega\times\mathbb{R}^n} e^{i(x-y)\eta}\chi(x,y)a(x,y,\xi+\eta)\,đy\,d\eta\\
&:= \Big(\int_{\mathbb{R}^n}\widehat{b_\epsilon}(x,\eta,\xi+\eta)\,d\eta\Big)_\epsilon + \underline{\mathcal{N}}^m_{\rho,\delta}(\Omega\times\mathbb{R}^n)
\end{aligned} \tag{5.19}$$

yields a well-defined element of $\widetilde{\underline{\mathcal{S}}}^m_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$. In this way (5.19) gives a map, depending on χ, from the set of the amplitudes in $\widetilde{\underline{\mathcal{S}}}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\Omega\times\mathbb{R}^n)$ which define A to the space of symbols $\widetilde{\underline{\mathcal{S}}}^m_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$.

We note that since $\chi(x,y)a(x,y,\xi)$ is a properly supported amplitude, the existence of σ in $\widetilde{\underline{\mathcal{S}}}^m_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ can be deduced, using the sheaf properties of $\widetilde{\underline{\mathcal{S}}}^m_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ with respect to Ω, by means of the local definition

$$\sigma_{|V_j}(x,\xi) = \Big(\int_{W_j} e^{i(x-y)\eta}\chi(x,y)a(x,y,\xi+\eta)\,đy\,d\eta\Big)_{|V_j} \in \widetilde{\underline{\mathcal{S}}}^m_{\rho,\delta,\mathrm{rg}}(V_j\times\mathbb{R}^n)$$

where $W_j = \pi_2(\pi_1^{-1}(\overline{V_j})\cap\operatorname{supp}\chi)\times\mathbb{R}^n$. Since the proper set $\operatorname{supp}\chi$ contains $\operatorname{supp}_{x,y} a_\epsilon$ for each ϵ, we arrive at the global expression (5.19) for σ.

Formula (5.19) gives only a map from amplitudes to symbols, but the spaces of generalized symbols contain different symbols for the same operator. This is due to the fact that the difference of two symbols may be *not* negligible in the sense of the symbol space, though the action of the corresponding operators may be the same. A simple example of this effect is given by the net of zero-order symbols $(\zeta_\epsilon(x,\xi))_\epsilon = (\varphi(\eta_\epsilon\xi)-1)_\epsilon$ where φ is some function in $\mathscr{S}(\mathbb{R}^n)$ with $\varphi(0)=1$ and $(\eta_\epsilon)_\epsilon$ belongs to $\mathcal{N}$, i.e., is a negligible net of real numbers. The net $(\tau_\epsilon)_\epsilon$ represents an element of $\widetilde{\underline{\mathcal{S}}}^m_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ which is not the zero element there, though the corresponding operator defined through (5.16) is the zero operator. This lack of uniqueness is remedied by the following observation.

Remark 5.10. The family of generalized functions $A(e^{i\cdot\xi})(x)$ in $\mathcal{G}(\Omega)$, parametrized by ξ, defines a generalized function in $\mathcal{G}(\Omega\times\mathbb{R}^n)$. Taking any proper function χ as above,

$$A(e^{i\cdot\xi})(x)=\Big(\int_{\Omega\times\mathbb{R}^n}e^{i(x-y)\eta}\chi(x,y)a_\epsilon(x,y,\eta)e^{iy\xi}\,dy\,đ\eta\Big)_\epsilon+\mathcal{N}(\Omega\times\mathbb{R}^n),$$

and after a suitable change of coordinates

$$\begin{aligned}e^{-ix\xi}A(e^{i\cdot\xi})(x)&=\Big(\int_{\Omega\times\mathbb{R}^n}e^{i(x-y)\eta}\chi(x,y)a_\epsilon(x,y,\xi+\eta)\,dy\,đ\eta\Big)_\epsilon+\mathcal{N}(\Omega\times\mathbb{R}^n)\\&=\Big(\sigma_\epsilon(x,\xi)\Big)_\epsilon+\mathcal{N}(\Omega\times\mathbb{R}^n).\end{aligned}\tag{5.20}$$

Thus (5.20) defines a map from the space of properly supported pseudodifferential operators with amplitude of type (ρ,δ,μ,ω) to the algebra $\mathcal{G}(\Omega\times\mathbb{R}^n)$. It follows that the symbol, viewed as an element of $\mathcal{G}(\Omega\times\mathbb{R}^n)$, depends only on the operator A and not on the choice of amplitude a or the proper function χ. This justifies to refer to it as *the* symbol of the operator A. Note that the symbol ζ described above vanishes as an element of $\mathcal{G}(\Omega\times\mathbb{R}^n)$. More precisely, one can actually show that the symbol is already unique in the space $\widetilde{\mathcal{G}}_\tau(\Omega\times\mathbb{R}^n)$ of generalized functions which are tempered in the second variable, for which we refer to [16, Def. 1.2.52].

Remark 5.11. The formal series $\sum_\gamma\frac{1}{\gamma!}\partial_\xi^\gamma D_y^\gamma a(x,y,\xi)_{|x=y}$ satisfies the requirements of Definition 5.5 with $m_j=m-(\rho-\delta)j$, $\mu_j=\mu-j$ and

$$a_j=\sum_{|\gamma|=j}\frac{1}{\gamma!}\partial_\xi^\gamma D_y^\gamma a(x,y,\xi)_{|x=y}.$$

Theorem 5.7 says that there exists a symbol σ_0 belonging, more specifically, to the space $\underline{\widetilde{\mathcal{S}}}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n)$ and there are representatives $(\sigma_{0,\epsilon})_\epsilon$ and $(\sigma_\epsilon)_\epsilon$ of σ_0 and σ, respectively, such that $(\sigma_\epsilon-\sigma_{0,\epsilon})_\epsilon\in\underline{\mathcal{S}}^{-\infty}_{\mathrm{rg}}(\Omega\times\mathbb{R}^n)$. In place of the equation (5.16), we can only assert that the equation $Au(x)=\int_{\mathbb{R}^n}e^{ix\xi}\sigma_0(x,\xi)\widehat{u}(\xi)\,đ\xi$ is valid modulo an operator with regular generalized kernel on $\mathcal{G}_c(\Omega)$.

Theorem 5.12. *Let A be a properly supported pseudodifferential operator with amplitude $a\in\underline{\widetilde{\mathcal{S}}}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\Omega\times\mathbb{R}^n)$ where $(\omega^{-1}(\epsilon))_\epsilon$ is a slow scale net. Let σ be given by* (5.16). *Then there exists $\sigma'\in\underline{\widetilde{\mathcal{S}}}^{m}_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ such that for all $u\in\mathcal{G}_{\mathscr{S}}(\mathbb{R}^n)$*

$${}^tA(u_{|\Omega})(x)=\int_{\Omega\times\mathbb{R}^n}e^{ix\xi}\sigma'(x,\xi)\widehat{u}(\xi)\,đ\xi$$

and $\sigma'\sim\sum_\gamma\frac{(-1)^{|\gamma|}}{\gamma!}\partial_\xi^\gamma D_x^\gamma\sigma(x,-\xi)$.

Proof. From (4.16) we have for all $u,v\in\mathcal{G}_c(\Omega)$ that

$$\begin{aligned}\int_\Omega v(x)\,{}^tAu(x)dx&=\int_\Omega\int_{\Omega\times\mathbb{R}^n}e^{i(y-x)\xi}\sigma(y,\xi)u(y)\,dy\,đ\xi\,v(x)\,dx\\&=\int_\Omega\int_{\Omega\times\mathbb{R}^n}e^{i(x-y)\xi}\sigma(y,-\xi)u(y)\,dy\,đ\xi\,v(x)\,dx.\end{aligned}$$

As a consequence of the injectivity of the inclusion of $\mathcal{G}(\Omega)$ in $L(\mathcal{G}_c(\Omega), \widetilde{\mathbb{C}})$ we obtain that

$$ {}^tAu(x) = \int_{\Omega\times\mathbb{R}^n} e^{i(x-y)\xi}\sigma(y,-\xi)u(y)\,dy\,đ\xi, \quad u \in \mathcal{G}_c(\Omega). $$

Thus ${}^tA$ is a properly supported pseudodifferential operator with amplitude $\sigma(y,-\xi) \in \underline{\widetilde{\mathcal{S}}}^m_{\rho,\delta,\mathrm{rg}}(\Omega\times\Omega\times\mathbb{R}^n)$ and satisfies the assumptions of Theorem 5.8. A direct application of that theorem guarantees the existence of $\sigma' \in \underline{\widetilde{\mathcal{S}}}^m_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ as required. $\square$

Remark 5.13. Theorem 5.12 combined with Proposition 4.17 shows that for all $u \in \mathcal{G}_c(\Omega)$

$$ Au(x) = {}^t({}^tAu)(x) = \int_{\Omega\times\mathbb{R}^n} e^{i(x-y)\xi}\sigma'(y,-\xi)u(y)\,dy\,đ\xi. $$

Defining the dual symbol $\tilde{\sigma}(x,\xi) := \sigma'(x,-\xi) \in \underline{\widetilde{\mathcal{S}}}^m_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$, we have the asymptotic expansion $\tilde{\sigma} \sim \sum_\gamma \frac{(-1)^{|\gamma|}}{\gamma!}\partial_\xi^\gamma D_x^\gamma\sigma(x,\xi)$ and

$$ Au(x) = \int_{\Omega\times\mathbb{R}^n} e^{i(x-y)\xi}\tilde{\sigma}(y,\xi)u(y)\,dy\,đ\xi. $$

Since $Au \in \mathcal{G}_c(\Omega)$ and we can choose $(\int_{\Omega\times\mathbb{R}^n} e^{i(x-y)\xi}\tilde{\sigma}_\epsilon(y,\xi)u_\epsilon(y)\,dy\,đ\xi)_\epsilon$ in $\mathcal{E}_{c,M}(\Omega)$ as a (compactly supported) representative, we see that

$$ \widehat{A_\epsilon u_\epsilon}(\xi) = \int_{\mathbb{R}^n} e^{-iy\xi}\tilde{\sigma}_\epsilon(y,\xi)u_\epsilon(y)\,dy. $$

Therefore, $\widehat{Au}(\xi) = \int_{\mathbb{R}^n} e^{-iy\xi}\tilde{\sigma}(y,\xi)u(y)\,dy$.

Remark 5.14. Let A be pseudodifferential operator with amplitude $a \in \underline{\widetilde{\mathcal{S}}}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\Omega\times\mathbb{R}^n)$ where $(\omega^{-1}(\epsilon))_\epsilon$ is a slow scale net. As a consequence of Theorem 5.8, Theorem 5.12 and Proposition 4.18 we have

$$ \begin{aligned} Au(x) &= \int_{\mathbb{R}^n} e^{ix\xi}\sigma(x,\xi)\widehat{u}(\xi)\,đ\xi + Ru(x), \\ {}^tAu(x) &= \int_{\mathbb{R}^n} e^{ix\xi}\sigma'(x,\xi)\widehat{u}(\xi)\,đ\xi + Su(x), \end{aligned} \tag{5.21} $$

for all $u \in \mathcal{G}_c(\Omega)$, where σ, σ' belong to $\underline{\widetilde{\mathcal{S}}}^m_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ and R and S are operators with regular generalized kernels.

Theorem 5.15. *Let A and B be two properly supported pseudodifferential operators with amplitude $a \in \underline{\widetilde{\mathcal{S}}}^{m_1,\mu}_{\rho,\delta,\omega}(\Omega\times\Omega\times\mathbb{R}^n)$ and $b \in \underline{\widetilde{\mathcal{S}}}^{m_2,\mu}_{\rho,\delta,\omega}(\Omega\times\Omega\times\mathbb{R}^n)$, respectively. Assume that $(\omega^{-1}(\epsilon))_\epsilon$ is a slow scale net. Then, given $\sigma_1 \in \underline{\widetilde{\mathcal{S}}}^{m_1}_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ and $\sigma_2 \in \underline{\widetilde{\mathcal{S}}}^{m_2}_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ satisfying (5.16) for A and B, respectively, there exists $\sigma \in \underline{\widetilde{\mathcal{S}}}^{m_1+m_2}_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ such that the properly supported pseudodifferential operator AB can be written in the form*

$$ AB(u_{|\Omega})(x) = \int_{\mathbb{R}^n} e^{ix\xi}\sigma(x,\xi)\widehat{u}(\xi)\,đ\xi, \quad u \in \mathcal{G}_{\mathscr{S}}(\mathbb{R}^n) \tag{5.22} $$

and $\sigma \sim \sum_\gamma \frac{1}{\gamma!}\partial_\xi^\gamma\sigma_1 D_x^\gamma\sigma_2$.

Proof. Using Theorem 5.8, we can write for all $u \in \mathcal{G}_c(\Omega)$
$$ABu(x) = \int_{\Omega\times\mathbb{R}^n} e^{i(x-y)\xi}\sigma_1(x,\xi)Bu(y)\,dy\,đ\xi.$$
Since $Bu \in \mathcal{G}_c(\Omega)$, from Remark 5.13,
$$\begin{aligned} ABu(x) &= \int_{\mathbb{R}^n} e^{i(x-y)\xi}\sigma_1(x,\xi)Bu(y)\,dy\,đ\xi = \int_{\mathbb{R}^n} e^{ix\xi}\sigma_1(x,\xi)\widehat{Bu}(\xi)\,đ\xi \\ &= \int_{\mathbb{R}^n} e^{ix\xi}\sigma_1(x,\xi)\int_{\Omega} e^{-iy\xi}\tilde{\sigma_2}(y,\xi)u(y)\,dy\,đ\xi \\ &= \int_{\Omega\times\mathbb{R}^n} e^{i(x-y)\xi}\sigma_1(x,\xi)\tilde{\sigma_2}(y,\xi)u(y)\,dy\,đ\xi. \end{aligned} \tag{5.23}$$
This equality shows that AB is a pseudodifferential operator and its amplitude $\sigma_1(x,\xi)\tilde{\sigma_2}(y,\xi)$ belongs to $\widetilde{\underline{\mathcal{S}}}^{m_1+m_2}_{\rho,\delta,\mathrm{rg}}(\Omega\times\Omega\times\mathbb{R}^n)$. By considering the kernel of AB, we can prove that the composition of two properly supported pseudodifferential operators is a properly supported pseudodifferential operator as in the classical case. Therefore, applying again Theorem 5.8, there exists $\sigma \in \widetilde{\underline{\mathcal{S}}}^{m_1+m_2}_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ such that (5.22) holds and
$$\sigma \sim \sum_\gamma \frac{1}{\gamma!}\partial_\xi^\gamma D_y^\gamma(\sigma_1(x,\xi)\tilde{\sigma_2}(y,\xi))_{|x=y}. \tag{5.24}$$
As in [40, p.27-28], (5.24) leads to $\sigma \sim \sum_\gamma \frac{1}{\gamma!}\partial_\xi^\gamma\sigma_1 D_x^\gamma\sigma_2$. □

To conclude this section we want to consider the composition of two pseudodifferential operators when only one operator is properly supported. This requires some preliminary results.

Lemma 5.16. *Let A be a properly supported pseudodifferential operator with amplitude a belonging to $\widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\Omega\times\mathbb{R}^n)$ where $(\omega^{-1}(\epsilon))_\epsilon$ is a slow scale net and let R be an operator with regular generalized kernel $k_R \in \mathcal{G}^\infty(\Omega\times\Omega)$. Then*
$$\begin{gathered} A(k_R(\cdot,y))(x) := (A_\epsilon(k_{R,\epsilon}(\cdot,y))(x))_\epsilon + \mathcal{N}(\Omega\times\Omega), \\ {}^tA(k_R(x,\cdot))(y) := ({}^tA_\epsilon(k_{R,\epsilon}(x,\cdot))(y))_\epsilon + \mathcal{N}(\Omega\times\Omega) \end{gathered}$$
are well-defined elements of $\mathcal{G}^\infty(\Omega\times\Omega)$.

Proof. We prove the lemma for $A(k_R(\cdot,y))(x)$. The proof for ${}^tA(k_R(x,\cdot))(y)$ is analogous. We begin by observing that for all fixed $y \in \Omega$, $k_R(\cdot,y) := (k_{R,\epsilon}(\cdot,y))_\epsilon + \mathcal{N}(\Omega)$ is a generalized function in $\mathcal{G}^\infty(\Omega)$. Since A is properly supported and $(\omega^{-1}(\epsilon))_\epsilon$ a slow scale net, Proposition 4.17 says that $A(k_R(\cdot,y))$ belongs to $\mathcal{G}^\infty(\Omega)$. In detail,
$$\begin{aligned} A(k_R(\cdot,y))_{|V_j}(x) &= A_j(k_R(\cdot,y))(x) = A(\psi_j(\cdot)k_R(\cdot,y))_{|V_j}(x) \\ &= \Big(\int_{\Omega\times\mathbb{R}^n} e^{i(x-z)\xi}a(x,z,\xi)\psi_j(z)k_R(z,y)\,dz\,đ\xi\Big)_{|V_J}, \end{aligned} \tag{5.25}$$
where $\{V_j\}_j$ is an exhausting sequence of relatively compact sets, $K_j = \overline{V_j}$, $K_j'' = \pi_2(\pi_1^{-1}(K_j)\cap \operatorname{supp} k_A)$ and $\psi_j \in \mathcal{C}_c^\infty(\Omega)$ with $\psi_j \equiv 1$ in an open neighborhood of K_j''. From Proposition 3.10 and the assumption on $(\omega^{-1}(\epsilon))_\epsilon$ the oscillatory integral in (5.25), depending on the parameters $(x,y)\in\Omega\times\Omega$ defines a generalized function in $\mathcal{G}^\infty(\Omega\times\Omega)$. Hence $A(k_R(\cdot,y))(x) \in \mathcal{G}^\infty(\Omega\times\Omega)$. □

Proposition 5.17. *Let A be a properly supported pseudodifferential operator with amplitude $a \in \widetilde{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega\times\Omega\times\mathbb{R}^n)$ where $(\omega^{-1}(\epsilon))_\epsilon$ is a slow scale net and let R be an operator with regular generalized kernel. Then AR and RA are operators with regular generalized kernel.*

Proof. Let us consider AR. From Proposition 4.17 we have for all $u, v \in \mathcal{G}_c(\Omega)$ that

$$\begin{aligned}\int_\Omega ARu(x)v(x)\,dx &= \int_\Omega Ru(x)\ {}^tAv(x)\,dx\\ &= \int_\Omega\int_\Omega k_R(x,y)u(y)\,dy\ {}^tAv(x)\,dx\\ &= \int_\Omega\int_\Omega k_R(x,y)\ {}^tAv(x)\,dx\,u(y)\,dy,\end{aligned}\tag{5.26}$$

where ${}^tAv \in \mathcal{G}_c(\Omega)$ and $k_R \in \mathcal{G}^\infty(\Omega\times\Omega)$. Since ${}^tA$ is properly supported and $k_R(\cdot,y) \in \mathcal{G}^\infty(\Omega)$ for every fixed $y \in \Omega$, we have

$$\int_\Omega k_R(x,y)\ {}^tAv(x)\,dx = \int_\Omega A(k_R(\cdot,y))(x)v(x)\,dx,\tag{5.27}$$

where from the previous lemma $A(k_R(\cdot,y))(x) \in \mathcal{G}^\infty(\Omega\times\Omega)$. Combining (5.26) with (5.27) we obtain that for all $v \in \mathcal{G}_c(\Omega)$

$$\int_\Omega\Big(ARu(x) - \int_\Omega A(k_R(\cdot,y))(x)u(y)\,dy\Big)v(x)\,dx = 0.$$

Finally, Proposition 2.11 shows that $ARu(x) = \int_\Omega A(k_R(\cdot,y))(x)u(y)dy$. In an analogous way one sees that RA has regular generalized kernel ${}^tA(k_R(x,\cdot))(y)$. $\square$

Proposition 5.18. *Let A be a properly supported pseudodifferential operator with amplitude in $\underline{\widetilde{\mathcal{S}}}^{m_1,\mu}_{\rho,\delta,\omega}(\Omega\times\Omega\times\mathbb{R}^n)$ and let B be a pseudodifferential operator with amplitude in $\underline{\widetilde{\mathcal{S}}}^{m_2,\mu}_{\rho,\delta,\omega}(\Omega\times\Omega\times\mathbb{R}^n)$. Assuming that $(\omega^{-1}(\epsilon))_\epsilon$ is a slow scale net, there exist σ and τ in $\underline{\widetilde{\mathcal{S}}}^{m_1+m_2}_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ such that for all $u \in \mathcal{G}_c(\Omega)$*

$$\begin{aligned}ABu(x) &= \int_{\mathbb{R}^n} e^{ix\xi}\sigma(x,\xi)\widehat{u}(\xi)\,đ\xi + Ru(x),\\ BAu(x) &= \int_{\mathbb{R}^n} e^{ix\xi}\tau(x,\xi)\widehat{u}(\xi)\,đ\xi + Su(x),\end{aligned}$$

where R and S have regular generalized kernel.

Proof. From Proposition 4.18 we have that $A = A_0 + A_1$, where A_0 is properly supported and A_1 has regular generalized kernel. At this point, an application of Theorem 5.15 and Proposition 5.17 lead us to our assertion. $\square$

6. Hypoellipticity and regularity results

This section is devoted to regularity theory for equations with $\mathcal{G}^\infty$-right hand side. We give a general definition of hypoelliptic symbols and construct parametrices for these symbols. The $\mathcal{G}^\infty$-regularity result for pseudodifferential equations then follows along the lines of the classical arguments.

A *strongly positive slow scale net* is a slow scale net $(r_\epsilon)_\epsilon \in \mathbb{R}^{(0,1]}$ such that $r_\epsilon > 0$ for all $\epsilon \in (0,1]$ and $\inf_\epsilon r_\epsilon \neq 0$.

Definition 6.1. Let m, l, μ, ρ, δ be real numbers with $l \leq m$ and $0 \leq \delta < \rho \leq 1$. We say that $(a_\epsilon)_\epsilon \in \underline{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega \times \mathbb{R}^n)$ is an element of $H\underline{\mathcal{S}}^{m,l,\mu}_{\rho,\delta,\omega}(\Omega \times \mathbb{R}^n)$ if and only if for all $K \Subset \Omega$ there exists a strongly positive slow scale net $(r_{K,\epsilon})_\epsilon$, a net $(\omega_{1,K,\epsilon})_\epsilon$, $\omega_{1,K,\epsilon} \geq C_K \epsilon^{s_K}$ on the interval $(0,1]$ for certain constants $C_K > 0$, $s_K \in \mathbb{R}$, and slow scale nets $(\omega_{2,K,\alpha,\beta,\epsilon})_\epsilon$, such that for all $x \in K$, for $|\xi| \geq r_{K,\epsilon}$, for all $\epsilon \in (0,1]$,

$$|a_\epsilon(x,\xi)| \geq \omega_{1,K,\epsilon}\langle\xi\rangle^l \tag{6.1}$$

and

$$|\partial^\alpha_\xi \partial^\beta_x a_\epsilon(x,\xi)| \leq \omega_{2,K,\alpha,\beta,\epsilon}|a_\epsilon(x,\xi)|\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|}. \tag{6.2}$$

for all $(\alpha,\beta) \neq (0,0)$. When ω is independent of ϵ we use the notation $H\underline{\mathcal{S}}^{m,l}_{\rho,\delta,\mathrm{rg}}(\Omega \times \mathbb{R}^n)$.

Before proceeding, we indicate a simple example of an element of $H\underline{\mathcal{S}}^{m,l,\mu}_{\rho,\delta,\omega}(\Omega \times \mathbb{R}^n)$.

Example 6.2. Let $(\omega^{-1}(\epsilon))_\epsilon$ be a slow scale net with $\sup_\epsilon \omega(\epsilon) < \infty$. Given $\mu \in \mathbb{R} \setminus \mathbb{N}$, let $(a_\epsilon)_\epsilon$ be a representative of a generalized function in $\mathcal{G}^\mu_{*,loc,\omega}(\Omega)$ (see Example 4.6) such that

$$\forall K \Subset \Omega,\ \forall \alpha \in \mathbb{N}^n,\ \exists c > 0:\ \forall x \in K,\ \forall \epsilon \in (0,1], \quad |\partial^\alpha a_\epsilon(x)| \leq c\,\omega(\epsilon)^{-(|\alpha|-\mu)_+} \tag{6.3}$$

and

$$\forall K \Subset \Omega,\ \exists(\omega_{1,K,\epsilon})_\epsilon:\ \forall \epsilon \in (0,1], \quad \inf_{x\in K}|a_\epsilon(x)| \geq \omega_{1,K,\epsilon}, \tag{6.4}$$

where $(\omega^{-1}_{1,K,\epsilon})_\epsilon$ is a slow scale net. Now, for any classical hypoelliptic symbol $b(x,\xi) \in HS^m_{\rho,\delta}(\Omega \times \mathbb{R}^n)$, the product $(a_\epsilon(x)b(x,\xi))_\epsilon$ belongs to $H\underline{\mathcal{S}}^{m,l,\mu}_{\rho,\delta,\omega}(\Omega \times \mathbb{R}^n)$. Since, as already proved in Section 4, $(a_\epsilon(x)b(x,\xi))_\epsilon \in \underline{\mathcal{S}}^{m,\mu}_{\rho,\delta,\omega}(\Omega \times \mathbb{R}^n)$, we simply have to check the estimates (6.1) and (6.2). Combining (6.3) and (6.4) with the properties of b, we obtain that for all compact sets K there exists a radius R_K such that for $x \in K$, $|\xi| \geq R_K$, $\epsilon \in (0,1]$,

$$|a_\epsilon(x)b(x,\xi)| \geq c\omega_{1,K,\epsilon}\langle\xi\rangle^l,$$

and

$$\begin{aligned}
|\partial^\alpha_\xi \partial^\beta_x (a_\epsilon(x)b(x,\xi))| &\leq \sum_{\beta' \leq \beta} \binom{\beta}{\beta'} |\partial^{\beta'}_x a_\epsilon(x)||\partial^\alpha_\xi \partial^{\beta-\beta'}_x b(x,\xi)| \\
&\leq c\omega(\epsilon)^{-(|\beta|-\mu)_+}\omega^{-1}_{1,K,\epsilon}\omega_{1,K,\epsilon}\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|} \\
&\leq c'\omega(\epsilon)^{-(|\beta|-\mu)_+}\omega^{-1}_{1,K,\epsilon}|a_\epsilon(x)b(x,\xi)|\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|},
\end{aligned}$$

where $(\omega(\epsilon)^{-(|\beta|-\mu)_+}\omega^{-1}_{1,K,\epsilon})_\epsilon$ is a slow scale net.

Returning to $(a_\epsilon)_\epsilon$ in Definition 6.1, it is evident from (6.1) that $a_\epsilon(x,\xi) \neq 0$ for $x \in K$ and $|\xi| \geq r_{K,\epsilon}$. Let us choose a locally finite open covering $(\Omega_j)_{j\in\mathbb{N}}$ of Ω such that $\Omega_j \subset \overline{\Omega}_j \Subset \Omega_{j+1}$ for all j. Let $(\psi_j)_{j\in\mathbb{N}}$ be a partition of unity subordinate to $(\Omega_j)_j$ and let $(r_{j,\epsilon})_\epsilon := (r_{\overline{\Omega}_j,\epsilon})_\epsilon$ be an increasing sequence of strongly positive slow scale nets satisfying (6.1) and (6.2) with $K = \overline{\Omega}_j$. We take a function $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $\varphi(\xi) = 0$ for $|\xi| \leq 1$ and $\varphi(\xi) = 1$ for $|\xi| \geq 2$. At this point we can define

$$p_{0,\epsilon}(x,\xi) = \sum_j a^{-1}_\epsilon(x,\xi)\varphi\big(\frac{\xi}{r_{j,\epsilon}}\big)\psi_j(x), \tag{6.5}$$

where by construction $(p_{0,\epsilon})_\epsilon \in \mathcal{E}[\Omega\times\mathbb{R}^n]$ since every $(a_\epsilon^{-1}(x,\xi)\varphi(\frac{\xi}{r_{j,\epsilon}}))_\epsilon \in \mathcal{E}[\Omega\times\mathbb{R}^n]$ and the sum is locally finite. Before proceeding with the study of $(p_{0,\epsilon})_\epsilon$ we need a technical lemma.

Lemma 6.3. *Let $(a_\epsilon)_\epsilon \in H\underline{\mathcal{S}}^{m,l,\mu}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n)$. For all $K \Subset \Omega$ and $\alpha,\beta\in\mathbb{N}^n$, there exists a slow scale net $(d_{K,\alpha,\beta,\epsilon})_\epsilon$ such that*

$$|\partial_\xi^\alpha\partial_x^\beta a_\epsilon^{-1}(x,\xi)| \le d_{K,\alpha,\beta,\epsilon}\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|}|a_\epsilon^{-1}(x,\xi)|, \quad x\in K,\ |\xi|\ge r_{K,\epsilon},\ \epsilon\in(0,1]. \tag{6.6}$$

Proof. Obviously (6.6) is true for $\alpha,\beta=0$. Differentiating $a_\epsilon^{-1}a_\epsilon(x,\xi)\equiv 1$ on $K\times\{|\xi|\ge r_{K,\epsilon}\}$ we obtain

$$\frac{\partial_\xi^\alpha\partial_x^\beta a_\epsilon^{-1}(x,\xi)}{a_\epsilon^{-1}(x,\xi)} = -\sum_{\substack{0<\alpha'\le\alpha\\ 0<\beta'\le\beta}}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}\frac{\partial_\xi^{\alpha'}\partial_x^{\beta'}a_\epsilon(x,\xi)}{a_\epsilon(x,\xi)}\frac{\partial_\xi^{\alpha-\alpha'}\partial_x^{\beta-\beta'}a_\epsilon^{-1}(x,\xi)}{a_\epsilon^{-1}(x,\xi)}.$$

Using induction, we conclude that

$$\begin{aligned}\frac{|\partial_\xi^\alpha\partial_x^\beta a_\epsilon^{-1}(x,\xi)|}{|a_\epsilon^{-1}(x,\xi)|} &\le \sum_{\substack{\alpha'\le\alpha\\ \beta'\le\beta}} d_{K,\alpha',\beta',\epsilon}\langle\xi\rangle^{-\rho|\alpha'|+\delta|\beta'|}\omega_{2,K,\alpha-\alpha',\beta-\beta',\epsilon}\langle\xi\rangle^{-\rho|\alpha-\alpha'|+\delta|\beta-\beta'|}\\ &\le d_{K,\alpha,\beta,\epsilon}\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|},\end{aligned}$$

where $(d_{K,\alpha,\beta,\epsilon})_\epsilon$ is a slow scale net since it is a finite sum of products of slow scale nets. □

Proposition 6.4. *Let $(a_\epsilon)_\epsilon \in H\underline{\mathcal{S}}^{m,l,\mu}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n)$. Then $(p_{0,\epsilon})_\epsilon$ is an element of $H\underline{\mathcal{S}}^{-l,-m}_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ and for all $K\Subset\Omega$ there exists a strongly positive slow scale net $(r'_{K,\epsilon})_\epsilon$ such that for all $x\in K$, $|\xi|\ge r'_{K,\epsilon}$ and for all $\epsilon\in(0,1]$,*

$$p_{0,\epsilon}(x,\xi)a_\epsilon(x,\xi)=1. \tag{6.7}$$

Proof. We begin by observing that for all $K\Subset\Omega$ there exists $j_0\in\mathbb{N}$ such that for all $j\ge j_0$, $\operatorname{supp}\psi_j\cap K=\emptyset$ and then

$$p_{0,\epsilon}(x,\xi)_{|_{K\times\mathbb{R}^n}} = \sum_{j=0}^{j_0}\Big(a_\epsilon^{-1}(x,\xi)\varphi(\frac{\xi}{r_{j,\epsilon}})\psi_j(x)\Big)_{|_{K\times\mathbb{R}^n}}$$

For $x\in K$ and $|\xi|\ge 2r_{j_0,\epsilon}$ we have $p_{0,\epsilon}(x,\xi)=\sum_{j=0}^{j_0}a_\epsilon^{-1}(x,\xi)\psi_j(x)=a_\epsilon^{-1}(x,\xi)$. This result proves (6.7). Using (6.1) and the definition of $(p_{0,\epsilon})_\epsilon$ we conclude

$$\begin{aligned}&|p_{0,\epsilon}(x,\xi)| = |a_\epsilon^{-1}(x,\xi)| \ge c_K\langle\xi\rangle^{-m}\epsilon^{N_K}\omega(\epsilon)^{(-\mu)_+}, \quad x\in K,\ |\xi|\ge 2r_{j_0,\epsilon},\ \epsilon\in(0,1],\\ &|p_{0,\epsilon}(x,\xi)| \le c_K\max_{0\le j\le j_0}(\omega^{-1}_{1,\overline{\Omega}_j,\epsilon})\langle\xi\rangle^{-l} \le c'_K\epsilon^{-M_K}\langle\xi\rangle^{-l}, \quad x\in K,\ \xi\in\mathbb{R}^n,\ \epsilon\in(0,1],\end{aligned}$$

where $M_K=\max_{0\le j\le j_0}(s_{\overline{\Omega}_j})_+$. Let us now consider $\partial_\xi^\alpha\partial_x^\beta p_{0,\epsilon}(x,\xi)$ for $(\alpha,\beta)\ne(0,0)$. Since $p_{0,\epsilon}$ coincides with $a_\epsilon^{-1}(x,\xi)$ on $K\times\{|\xi|\ge 2r_{j_0,\epsilon}\}$, Lemma 6.3 guarantees the estimate

$$|\partial_\xi^\alpha\partial_x^\beta p_{0,\epsilon}(x,\xi)| \le d_{K,\alpha,\beta,\epsilon}|p_{0,\epsilon}(x,\xi)|\langle\xi\rangle^{-\rho|\alpha|+\delta|\beta|} \tag{6.8}$$

on this set. In order to prove that $(p_{0,\epsilon})_\epsilon$ is an element of $H\underline{\mathcal{S}}^{-l,-m}_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$, it remains to estimate every term $\partial_\xi^\alpha\partial_x^\beta(a_\epsilon^{-1}(x,\xi)\varphi(\frac{\xi}{r_{j,\epsilon}})\psi_j(x))$, $j\le j_0$, on $K\times\{r_{j,\epsilon}\le$

$|\xi| \le 2r_{j_0,\epsilon}\}$. From (6.6) and (6.2), recalling the assumptions on the nets involved in our formulas, we obtain

$$\begin{aligned}
&|\partial_\xi^\alpha \partial_x^\beta (a_\epsilon^{-1}(x,\xi)\varphi\big(\frac{\xi}{r_{j,\epsilon}}\big)\psi_j(x))| \\
&\le \sum_{\alpha'\le\alpha,\beta'\le\beta} d_{\overline{\Omega}_j,\alpha',\beta',\epsilon} 1_{j,j_0}(|\xi|)\langle\xi\rangle^{-\rho|\alpha'|+\delta|\beta'|}|a_\epsilon^{-1}(x,\xi)| \\
&\quad\times \sup_{1\le|\xi|\le 2}|\partial^{\alpha-\alpha'}\varphi(\xi)|\sup_x|\partial^{\beta-\beta'}\psi_j(x)| \\
&\le \sum_{\alpha'\le\alpha,\beta'\le\beta} d'_{\overline{\Omega}_j,\alpha',\beta',\epsilon}\langle\xi\rangle^{-l-\rho|\alpha|+\delta|\beta|}\omega^{-1}_{1,\overline{\Omega}_j,\epsilon}\langle 2r_{j_0,\epsilon}\rangle^{\rho|\alpha-\alpha'|} \\
&\le g_{\overline{\Omega}_j,\alpha,\beta,\epsilon}\epsilon^{-M_K}\langle\xi\rangle^{-l-\rho|\alpha|+\delta|\beta|}, \quad x\in K,\ |\xi|\le 2r_{j_0,\epsilon},
\end{aligned} \tag{6.9}$$

where $1_{j,j_0}$ is the characteristic function of the interval $[r_{j,\epsilon}, 2r_{j_0,\epsilon}]$. As a consequence there exist certain slow scale nets $(g_{K,\alpha,\beta,\epsilon})_\epsilon$ such that the following estimate holds on $K\times\mathbb{R}^n$:

$$|\partial_\xi^\alpha\partial_x^\beta p_{0,\epsilon}(x,\xi)| \le g_{K,\alpha,\beta,\epsilon}\epsilon^{-M_K}\langle\xi\rangle^{-l-\rho|\alpha|+\delta|\beta|}. \tag{6.10}$$

In conclusion, combining the estimate from below with (6.8) and (6.10), we have that $(p_{0,\epsilon})_\epsilon$ belongs to $H\underline{\mathcal{S}}^{-l,-m}_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ and it is of growth type M_K+1 on the compact set K. $\square$

Proposition 6.5. *Let $(a_\epsilon)_\epsilon$ be in $H\underline{\mathcal{S}}^{m,l,\mu}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n)$. Then for all α,β in $\mathbb{N}^n$, we have $(p_{0,\epsilon}(x,\xi)\partial_\xi^\alpha\partial_x^\beta a_\epsilon(x,\xi))_\epsilon$ in $\underline{\mathcal{S}}^{-\rho|\alpha|+\delta|\beta|}_{\rho,\delta,\mathrm{rg}}$. More precisely, for every $K\Subset\Omega$, for all $x\in K$, $\xi\in\mathbb{R}^n$ and $\epsilon\in(0,1]$,*

$$|\partial_\xi^\gamma\partial_x^\sigma(p_{0,\epsilon}(x,\xi)\partial_\xi^\alpha\partial_x^\beta a_\epsilon(x,\xi))| \le s_{K,\alpha,\beta,\gamma,\sigma,\epsilon}\langle\xi\rangle^{-\rho|\alpha+\gamma|+\delta|\beta+\sigma|}, \tag{6.11}$$

where $(s_{K,\alpha,\beta,\gamma,\sigma,\epsilon})_\epsilon$ is a slow scale net.

Proof. We fix $K\Subset\Omega$. From (6.2) and (6.8) we easily see that there exists a slow scale net satisfying (6.11) on $K\times\{|\xi|\ge 2r_{j_0,\epsilon}\}$. Let us assume now $|\xi|\le 2r_{j_0,\epsilon}$. From (6.2) and the same arguments as used in (6.9) we obtain

$$\begin{aligned}
&|\partial_\xi^\gamma\partial_x^\sigma(p_{0,\epsilon}(x,\xi)\partial_\xi^\alpha\partial_x^\beta a_\epsilon(x,\xi))| \\
&\le \sum_{\gamma'\le\gamma,\sigma'\le\sigma}\binom{\gamma}{\gamma'}\binom{\sigma}{\sigma'}|\partial_\xi^{\gamma'}\partial_x^{\sigma'}p_{0,\epsilon}(x,\xi)||\partial_\xi^{\alpha+\gamma-\gamma'}\partial_x^{\beta+\sigma-\sigma'}a_\epsilon(x,\xi)| \\
&\le \sum_{\gamma'\le\gamma,\sigma'\le\sigma,j\le j_0} l_{\overline{\Omega}_j,\gamma',\sigma',\epsilon}\langle\xi\rangle^{-\rho|\gamma'|+\delta|\sigma'|}\omega_{2,\overline{\Omega}_j,\alpha+\gamma-\gamma',\beta+\sigma-\sigma',\epsilon}\langle\xi\rangle^{-\rho|\alpha+\gamma-\gamma'|+\delta|\beta+\sigma-\sigma'|} \\
&\le s_{K,\alpha,\beta,\gamma,\sigma,\epsilon}\langle\xi\rangle^{-\rho|\alpha+\gamma|+\delta|\beta+\sigma|},
\end{aligned}$$

where $(l_{\overline{\Omega}_j,\gamma',\sigma',\epsilon})_\epsilon$ and $(s_{K,\alpha,\beta,\gamma,\sigma,\epsilon})_\epsilon$ are slow scale nets. $\square$

Proposition 6.6. *Let $(a_\epsilon)_\epsilon\in H\underline{\mathcal{S}}^{m,l,\mu}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n)$. We define for $h\ge 1$*

$$p_{h,\epsilon}(x,\xi) = -\Big\{\sum_{\substack{|\gamma|+j=h\\ j<h}}\frac{(-i)^{|\gamma|}}{\gamma!}\partial_x^\gamma a_\epsilon(x,\xi)\partial_\xi^\gamma p_{j,\epsilon}(x,\xi)\Big\}p_{0,\epsilon}(x,\xi).$$

Then, each $(p_{j,\epsilon})_\epsilon\in\underline{\mathcal{S}}^{-l-(\rho-\delta)j}_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ and the requirements of Definition 5.1 are satisfied.

Proof. We argue by induction. For $h = 1$,

$$\begin{aligned}
&\partial_\xi^\alpha \partial_x^\beta p_{1,\epsilon}(x,\xi) \\
&= -i \sum_{|\gamma|=1} \sum_{\substack{\alpha' \le \alpha \\ \beta' \le \beta}} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} \partial_\xi^{\alpha'} \partial_x^{\beta'} (\partial_x^\gamma a_\epsilon p_{0,\epsilon})(x,\xi) \partial_\xi^{\alpha-\alpha'} \partial_x^{\beta-\beta'} \partial_\xi^\gamma p_{0,\epsilon}(x,\xi).
\end{aligned}$$

Using Proposition 6.5 and (6.10) we obtain that

$$\begin{aligned}
&|\partial_\xi^\alpha \partial_x^\beta p_{1,\epsilon}(x,\xi)| \\
&\le \sum_{|\gamma|=1} \sum_{\substack{\alpha' \le \alpha \\ \beta' \le \beta}} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} |\partial_\xi^{\alpha'} \partial_x^{\beta'} (\partial_x^\gamma a_\epsilon p_{0,\epsilon})(x,\xi)| |\partial_\xi^{\alpha-\alpha'+\gamma} \partial_x^{\beta-\beta'} p_{0,\epsilon}(x,\xi)| \\
&\le \sum_{|\gamma|=1} \sum_{\substack{\alpha' \le \alpha \\ \beta' \le \beta}} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} s_{K,\alpha',\beta',\gamma,\epsilon} \langle\xi\rangle^{-\rho|\alpha'|+\delta|\beta'+\gamma|} \\
&\quad \times g_{K,\alpha-\alpha'+\gamma,\beta-\beta',\epsilon} \epsilon^{-M_K} \langle\xi\rangle^{-l-\rho|\alpha-\alpha'+\gamma|+\delta|\beta-\beta'|} \\
&\le t_{1,K,\alpha,\beta,\epsilon} \epsilon^{-M_K} \langle\xi\rangle^{-l-(\rho-\delta)-\rho|\alpha|+\delta|\beta|},
\end{aligned}$$

where $(t_{1,K,\alpha,\beta,\epsilon})_\epsilon$ is a slow scale net. We assume that for all $K \Subset \Omega$ there is $M_K \in \mathbb{N}$ such that $\forall \alpha, \beta \in \mathbb{N}^n$ there exists a slow scale net $(t_{h,K,\alpha,\beta,\epsilon})_\epsilon$ so that

$$|\partial_\xi^\alpha \partial_x^\beta p_{h,\epsilon}(x,\xi)| \le t_{h,K,\alpha,\beta,\epsilon} \epsilon^{-M_K} \langle\xi\rangle^{-l-(\rho-\delta)h-\rho|\alpha|+\delta|\beta|} \tag{6.12}$$

for all $x \in K, \xi \in \mathbb{R}^n$ and $\epsilon \in (0,1]$. We want to prove that (6.12) holds for $h+1$. We have

$$\begin{aligned}
&|\partial_\xi^\alpha \partial_x^\beta p_{h+1,\epsilon}(x,\xi)| \\
&\le \sum_{\substack{|\gamma|+j=h+1 \\ j<h+1}} \frac{1}{\gamma!} \sum_{\substack{\alpha' \le \alpha \\ \beta' \le \beta}} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} |\partial_\xi^{\alpha'} \partial_x^{\beta'} (\partial_x^\gamma a_\epsilon p_{0,\epsilon})(x,\xi)| |\partial_\xi^{\alpha-\alpha'+\gamma} \partial_x^{\beta-\beta'} p_{j,\epsilon}(x,\xi)| \\
&\le \sum_{\substack{|\gamma|+j=h+1 \\ j<h+1}} \sum_{\substack{\alpha' \le \alpha \\ \beta' \le \beta}} \binom{\alpha}{\alpha'} \binom{\beta}{\beta'} s_{K,\alpha',\beta',\gamma,\epsilon} \langle\xi\rangle^{-\rho|\alpha'|+\delta|\beta'+\gamma|} \\
&\quad \times t_{j,K,\alpha-\alpha'+\gamma,\beta-\beta',\epsilon}\, \epsilon^{-M_K} \langle\xi\rangle^{-l-(\rho-\delta)j} \langle\xi\rangle^{-\rho|\alpha-\alpha'+\gamma|+\delta|\beta-\beta'|} \\
&\le t_{h+1,K,\alpha,\beta,\epsilon} \epsilon^{-M_K} \langle\xi\rangle^{-l-(\rho-\delta)(h+1)-\rho|\alpha|+\delta|\beta|}.
\end{aligned}$$

This estimate concludes the proof. $\square$

Definition 6.7. A symbol $a \in \widetilde{\underline{\mathcal{S}}}^{m,\mu}_{\rho,\delta,\omega}(\Omega \times \mathbb{R}^n)$ is called hypoelliptic if one of its representatives $(a_\epsilon)_\epsilon$ belongs to $H\underline{\mathcal{S}}^{m,l,\mu}_{\rho,\delta,\omega}(\Omega \times \mathbb{R}^n)$.

The set of hypoelliptic symbols is denoted by $H\widetilde{\underline{\mathcal{S}}}^{m,l,\mu}_{\rho,\delta,\omega}(\Omega \times \mathbb{R}^n)$. The next, central result shows that operators with symbols of this type admit a (generalized) parametrix.

Theorem 6.8. *Let $a \in H\widetilde{\underline{\mathcal{S}}}^{m,l,\mu}_{\rho,\delta,\omega}(\Omega \times \mathbb{R}^n)$ where $(\omega^{-1}(\epsilon))_\epsilon$ is a slow scale net and let A be the corresponding pseudodifferential operator. Then there exists a properly* *supported pseudodifferential operator P with symbol in $\widetilde{\underline{\mathcal{S}}}^{-l}_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ such that for all $u\in\mathcal{G}_c(\Omega)$ the equalities*

$$\begin{aligned} PAu &= u+Ru,\\ APu &= u+Su \end{aligned} \tag{6.13}$$

hold in $\mathcal{G}(\Omega)$, where R and S are operators with regular generalized kernel.

Proof. We work with a representative $(a_\epsilon)_\epsilon\in H\underline{\mathcal{S}}^{m,l,\mu}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n)$ of a. From Proposition 6.6 and Theorem 5.3, the formal series $\sum_j p_{j,\epsilon}$ defines an element $(p_\epsilon)_\epsilon\in\underline{\mathcal{S}}^{-l}_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ such that $(p_\epsilon)_\epsilon\sim\sum_j(p_{j,\epsilon})_\epsilon$. Let $\chi\in\mathcal{C}^\infty(\Omega\times\Omega)$ be a proper function identically equal to 1 in a neighborhood of the diagonal. Then the pseudodifferential operator P with amplitude $(\chi(x,y)p_\epsilon(x,\xi))_\epsilon+\underline{\mathcal{N}}^{-l}_{\rho,\delta}(\Omega\times\Omega\times\mathbb{R}^n)\in\widetilde{\underline{\mathcal{S}}}^{-l}_{\rho,\delta,\mathrm{rg}}(\Omega\times\Omega\times\mathbb{R}^n)$ is properly supported and, using Theorem 5.8, it can be written in the form

$$Pu(x)=\int_{\mathbb{R}^n}e^{ix\xi}\sigma_1(x,\xi)\widehat{u}(\xi)\,đ\xi,$$

where $u\in\mathcal{G}_c(\Omega)$ and $\sigma_1\in\widetilde{\underline{\mathcal{S}}}^{-l}_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$.

We observe that there exists a representative $(\sigma_{1,\epsilon})_\epsilon$ of σ_1 such that

$$(\sigma_{1,\epsilon}-p_\epsilon)_\epsilon\in\underline{\mathcal{S}}^{-\infty}_{\mathrm{rg}}(\Omega\times\mathbb{R}^n). \tag{6.14}$$

Analogously, by Remark 5.14, there exists $\sigma_2\in\widetilde{\underline{\mathcal{S}}}^{m}_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ such that for all $u\in\mathcal{G}_c(\Omega)$, $Au(x)=\int_{\mathbb{R}^n}e^{ix\xi}\sigma_2(x,\xi)\widehat{u}(\xi)\,đ\xi$ modulo an operator with regular generalized kernel, and in particular there is a representative $(\sigma_{2,\epsilon})_\epsilon$ of σ_2 with the property

$$(\sigma_{2,\epsilon}-a_\epsilon)_\epsilon\in\underline{\mathcal{S}}^{-\infty}_{\mathrm{rg}}(\Omega\times\mathbb{R}^n). \tag{6.15}$$

At this stage, Proposition 5.18 guarantees the existence of $\sigma\in\widetilde{\underline{\mathcal{S}}}^{m-l}_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ such that for all $u\in\mathcal{G}_c(\Omega)$

$$PAu(x)=\int_{\mathbb{R}^n}e^{ix\xi}\sigma(x,\xi)\widehat{u}(\xi)\,đ\xi+Tu(x),$$

where T is an operator with regular generalized kernel. Combining (6.14) and (6.15) with $\sigma\sim\sum_\gamma\frac{1}{\gamma!}\partial^\gamma_\xi\sigma_1D^\gamma_x\sigma_2$, we may assume that there exists a representative $(\sigma_\epsilon)_\epsilon$ of σ such that for all $h\in\mathbb{N}$, $h\ge1$,

$$\Big(\sigma_\epsilon-\sum_{|\gamma|<h}\frac{1}{\gamma!}\partial^\gamma_\xi p_\epsilon D^\gamma_x a_\epsilon\Big)_\epsilon\in\underline{\mathcal{S}}^{m-l-(\rho-\delta)h}_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n), \tag{6.16}$$

and it is of growth type M_K+N_K, where M_K and N_K are suitable growth types of $(p_\epsilon)_\epsilon\in\underline{\mathcal{S}}^{-l}_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ and $(a_\epsilon)_\epsilon\in\underline{\mathcal{S}}^{m}_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ on the compact set K, respectively. We shall show that $(\sigma_\epsilon-1)_\epsilon$ is an element of $\underline{\mathcal{S}}^{-\infty}_{\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ of growth type M_K+N_K+1. Since $(p_\epsilon)_\epsilon\sim\sum_j(p_{j,\epsilon})_\epsilon$,

$$\sigma_\epsilon-\sum_{|\gamma|<h}\frac{1}{\gamma!}\partial^\gamma_\xi p_\epsilon D^\gamma_x a_\epsilon=\sigma_\epsilon-\sum_{|\gamma|<h}\frac{1}{\gamma!}D^\gamma_x a_\epsilon\sum_{j=0}^{h-1}\partial^\gamma_\xi p_{j,\epsilon}-\sum_{|\gamma|<h}\frac{1}{\gamma!}\partial^\gamma r_{h,\epsilon}D^\gamma_x a_\epsilon, \tag{6.17}$$

where $(\partial^\gamma_\xi r_{h,\epsilon}D^\gamma_x a_\epsilon)_\epsilon$ is an element of $\underline{\mathcal{S}}^{m-l-(\rho-\delta)(h+|\gamma|)}_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ of growth type M_K+N_K. Next, (6.16) combined with (6.17) proves that the difference

$$\Big(\sigma_\epsilon-\sum_{|\gamma|<h}\frac{1}{\gamma!}D^\gamma_x a_\epsilon\sum_{j=0}^{h-1}\partial^\gamma_\xi p_{j,\epsilon}\Big)_\epsilon$$

belongs to $\underline{\mathcal{S}}^{m-l-(\rho-\delta)h}_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ and it is of the growth type M_K+N_K. Now let us write

$$\begin{aligned}&\sum_{|\gamma|<h}\frac{1}{\gamma!}D^\gamma_x a_\epsilon\sum_{j=0}^{h-1}\partial^\gamma_\xi p_{j,\epsilon}\\ &= p_{0,\epsilon}a_\epsilon+\sum_{k=1}^{h-1}\{p_{k,\epsilon}a_\epsilon+\sum_{\substack{|\gamma|+j=k\\ j<k}}\frac{1}{\gamma!}\partial^\gamma_\xi p_{j,\epsilon}D^\gamma_x a_\epsilon\}+\sum_{\substack{|\gamma|+j\ge h\\ |\gamma|<h,j<h}}\frac{1}{\gamma!}\partial^\gamma_\xi p_{j,\epsilon}D^\gamma_x a_\epsilon.\end{aligned}\tag{6.18}$$

From Proposition 6.6 and the equality $(p_{0,\epsilon}a_\epsilon)(x,\xi)=1$ for $\epsilon\in(0,1]$, $x\in K$, $|\xi|\ge r'_{K,\epsilon}$, where $(r'_{K,\epsilon})_\epsilon$ is a strongly positive slow scale net, we conclude that

$$\sum_{|\gamma|<h}\frac{1}{\gamma!}D^\gamma_x a_\epsilon\sum_{j=0}^{h-1}\partial^\gamma_\xi p_{j,\epsilon}=1+\sum_{\substack{|\gamma|+j\ge h\\ |\gamma|<h,j<h}}\frac{1}{\gamma!}\partial^\gamma_\xi p_{j,\epsilon}D^\gamma_x a_\epsilon,\quad \epsilon\in(0,1],\ x\in K,|\xi|\ge r'_{K,\epsilon},$$

where the sum on the right-hand side satisfies the estimates of an element of $\underline{\mathcal{S}}^{m-l-(\rho-\delta)(j+|\gamma|)}_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)\subseteq\underline{\mathcal{S}}^{m-l-(\rho-\delta)h}_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ of growth type M_K+N_K on $K\times\{|\xi|\ge r'_{K,\epsilon}\}$. It is important to note that, from the properties of $(p_{0,\epsilon})_\epsilon$ and $(p_{j,\epsilon})_\epsilon$, the continuity of the functions involved in (6.18) on compact sets and the assumptions on $(r'_{K,\epsilon})_\epsilon$, we can omit the condition $|\xi|\ge r'_{K,\epsilon}$ adding 1 in the growth type. Therefore,

$$\Big(\sum_{|\gamma|<h}\frac{1}{\gamma!}D^\gamma_x a_\epsilon\sum_{j=0}^{h-1}\partial^\gamma_\xi p_{j,\epsilon}-1\Big)_\epsilon$$

belongs to $\underline{\mathcal{S}}^{m-l-(\rho-\delta)h}_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ and it is of growth type M_K+N_K+1. This means that $(\sigma_\epsilon-1)_\epsilon\in\underline{\mathcal{S}}^{-\infty}_{\mathrm{rg}}(\Omega\times\mathbb{R}^n)$. In conclusion,

$$\begin{aligned}PAu(x)&=\Big(\int_{\Omega\times\mathbb{R}^n}e^{i(x-y)\xi}u_\epsilon(y)dy\,{\it đ}\xi\Big)_\epsilon+\mathcal{N}(\Omega)\\ &\quad+\Big(\int_{\Omega\times\mathbb{R}^n}e^{i(x-y)\xi}(\sigma_\epsilon(x,\xi)-1)u_\epsilon(y)dy\,{\it đ}\xi\Big)_\epsilon+\mathcal{N}(\Omega)+Tu(x)\\ &=Iu(x)+Ru(x),\end{aligned}$$

for all $u\in\mathcal{G}_c(\Omega)$, where R is an operator with regular generalized kernel and is the sum of the pseudodifferential operator with symbol $(\sigma_\epsilon-1)_\epsilon+\underline{\mathcal{N}}^{-\infty}(\Omega\times\mathbb{R}^n)$ and T.

In an analogous way we can easily prove that there is a properly supported pseudodifferential operator Q with symbol $q\in\widetilde{\underline{\mathcal{S}}}^{-l}_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ such that $AQ=I+R'$, where R' has regular generalized kernel. Since $P(AQ)=P+PR'$ and $P(AQ)=(PA)Q=Q+RQ$, Proposition 5.17 shows that $P-Q=RQ-PR'$ is an operator with regular generalized kernel. Consequently, (6.13) holds. ☐

Corollary 6.9. *Let $a\in H\widetilde{\underline{\mathcal{S}}}^{m,l,\mu}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n)$ where $(\omega^{-1}(\epsilon))_\epsilon$ is a slow scale net. We assume that the corresponding pseudodifferential operator A is properly supported. Then there is a properly supported pseudodifferential operator P with symbol in*

$\widetilde{\underline{\mathcal{S}}}^{-l}_{\rho,\delta,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$ *such that for all* $u\in\mathcal{G}(\Omega)$

$$\begin{aligned} PAu &= u + Ru,\\ APu &= u + Su, \end{aligned} \tag{6.19}$$

where R *and* S *have regular generalized kernels.*

Proof. From the previous theorem we know the existence of a parametrix P such that (6.19) holds for all $u\in\mathcal{G}_c(\Omega)$. Since $PA-I$ and $AP-I$ are properly supported, R and S are properly supported operators with regular generalized kernel. We extend the operators PA and $I+R$ from $\mathcal{G}_c(\Omega)$ into $\mathcal{G}(\Omega)$. Guided by Proposition 4.17, we have that for every $u\in\mathcal{G}(\Omega)$ the coherent sequence

$$PAu_{|_{V_i}} = PA(\psi_i u)_{|_{V_i}},$$

defines PAu. Here $V_1\subset V_2\subset\dots$ is an exhausting sequence of relatively compact sets of Ω and $\psi_i\in\mathcal{C}^\infty_c(\Omega)$ is identically 1 in a neighborhood of $\pi_1(\pi_2^{-1}(\overline{V_i})\cap(\Delta\cup\operatorname{supp}k_R))$. Since $\psi_i u\in\mathcal{G}_c(\Omega)$ we obtain from (6.13) that

$$PAu_{|_{V_i}} = I(\psi_i u)_{|_{V_i}} + R(\psi_i u)_{|_{V_i}},$$

where the sequences $I(\psi_i u)_{|_{V_i}}$ and $R(\psi_i u)_{|_{V_i}}$ define u and Ru respectively. In conclusion, $PAu = u + Ru$ for every $u\in\mathcal{G}(\Omega)$. The equality $APu = u + Su$ is proved analogously. □

Theorem 6.10. *Let* $a\in H\widetilde{\underline{\mathcal{S}}}^{m,l,\mu}_{\rho,\delta,\omega}(\Omega\times\mathbb{R}^n)$ *where* $(\omega^{-1}(\epsilon))_\epsilon$ *is a slow scale net. We assume that the corresponding pseudodifferential operator* A *is properly supported. Then for every* $u\in\mathcal{G}(\Omega)$*,* $\operatorname{sing\,supp}_{\mathrm{g}}(Au)\equiv\operatorname{sing\,supp}_{\mathrm{g}}u$*.*

Proof. The inclusion $\operatorname{sing\,supp}_{\mathrm{g}}(Au)\subseteq\operatorname{sing\,supp}_{\mathrm{g}}u$ is clear from the pseudo-locality property of A. Now, let us consider a parametrix P of A. From (6.19) we have that u can be written as $PAu - Ru$ where R has regular generalized kernel. Therefore, $\operatorname{sing\,supp}_{\mathrm{g}}u\subseteq\operatorname{sing\,supp}_{\mathrm{g}}(PAu)$. The pseudo-locality property of P allows us to conclude that $\operatorname{sing\,supp}_g u\subseteq\operatorname{sing\,supp}_g(Au)$. □

Theorem 6.10 is the main regularity result of this section. It says that a hypoelliptic symbol in the sense of Definition 6.7 leads to a $\mathcal{G}^\infty$-hypoelliptic operator. In [24] partial differential operators with generalized constant coefficients were considered. In particular, the symbol of a *(WH)-elliptic operator with slow scale radius* – as defined there – belongs to $H\widetilde{\underline{\mathcal{S}}}^{m,l}_{1,0,\mathrm{rg}}(\Omega\times\mathbb{R}^n)$. Therefore, the constant coefficient $\mathcal{G}^\infty$-regularity result [24, Thm. 5.5] is a special case of Theorem 6.10.

Acknowledgements: The authors are grateful to Günther Hörmann for many valuable discussions during the preparation of the paper. The authors also thank Luigi Rodino for useful discussions on pseudodifferential and Fourier integral operators.

References

[1] H. A. Biagioni. *A Nonlinear Theory of Generalized Functions.* Lecture Notes in Math. 1421. Springer-Verlag, Berlin, 1990.

[2] P. Boggiatto, E. Buzano, and L. Rodino. *Global Hypoellipticity and Spectral Theory.* Math. Research 92. Academie-Verlag, Berlin, 1996.

[3] P. Boggiatto and L. Rodino. Quantization and pseudo-differential operators. *Cubo Mat. Educ.*, 5(1):237–272, 2003.

[4] R. R. Coifman and Y. Meyer. Au delà des opérateurs pseudo-différentiels. *Astérisque*, 57:1 – 184, 1978.

[5] R. R. Coifman and Y. Meyer. *Calderón–Zygmund and multilinear operators.* Cambridge University Press, Cambridge, 1997.

[6] J. F. Colombeau. *New Generalized Functions and Multiplication of Distributions.* North-Holland Math. Studies 84. North-Holland, Amsterdam, 1984.

[7] J. F. Colombeau. *Elementary Introduction to New Generalized Functions.* North-Holland Math. Studies 113. North-Holland, Amsterdam, 1985.

[8] J. F. Colombeau. *Multiplication of Distributions. A Tool in Mathematics, Numerical Engineering and Theoretical Physics.* Lecture Notes in Math. 1532. Springer-Verlag, New York, 1992.

[9] A. Delcroix. Generalized integral operators and Schwartz kernel type theorems. *Jour. Math. Analysis and Appl.*, 306(2):481–501, 2005.

[10] J. Dieudonné. *Elements d'analyse*, volume VII. Gauthier-Villars, Paris, 1978.

[11] C. Garetto. Pseudo-differential operators in algebras of generalized functions and global hypoellipticity. *Acta Appl. Math.*, 80(2):123–174, 2004.

[12] C. Garetto. Topological structures in Colombeau algebras: topological $\widetilde{\mathbb{C}}$-modules and duality theory. *Acta Appl. Math.*, 88(1):81–123, 2005.

[13] C. Garetto. Topological structures in Colombeau algebras: investigation of the duals of $\mathcal{G}_c(\Omega)$, $\mathcal{G}(\Omega)$ and $\mathcal{G}_{\mathscr{S}}(\mathbb{R}^n)$. *arXiv:math. FA/0408278*, 2005. To appear in Monatsh. Math.

[14] C. Garetto and G. Hörmann. Microlocal analysis of generalized functions: pseudodifferential techniques and propagation of singularities. *Proc. Edinburgh Math. Soc.*, 48(3):603–629, 2005.

[15] M. Grosser, E. Farkas, M. Kunzinger, and R. Steinbauer. On the foundations of nonlinear generalized functions I and II. *Mem. Amer. Math. Soc.*, 153:1–93, 2001.

[16] M. Grosser, M. Kunzinger, M. Oberguggenberger, and R. Steinbauer. *Geometric theory of generalized functions.* Mathematics and its Applications 537. Kluwer Acad. Publ., Dordrecht, 2001.

[17] M. Grosser, M. Kunzinger, R. Steinbauer, and J. Vickers. A global theory of algebras of generalized functions. *Adv. Math.*, 166:50–72, 2002.

[18] L. Hörmander. Fourier integral operators I. *Acta Math.*, 127:79–183, 1971.

[19] L. Hörmander. *The Analysis of Linear Partial Differential Operators Vol. III.* Grundlehren math. Wiss. 274. Springer-Verlag, Berlin, 1985. Second printing 1994.

[20] G. Hörmann. Integration and microlocal analysis in Colombeau algebras. *J. Math. Anal. Appl.*, 239(2):332–348, 1999.

[21] G. Hörmann. First-order hyperbolic pseudodifferential equations with generalized symbols. *J. Math. Anal. Appl.*, 293(1):40–56, 2004.

[22] G. Hörmann. Hölder-Zygmund regularity in algebras of generalized functions. *Z. Anal. Anwendungen*, 23(1):139–165, 2004.

[23] G. Hörmann and M. Kunzinger. Microlocal analysis of basic operations in Colombeau algebras. *J. Math. Anal. Appl.*, 261(1):254–270, 2001.

[24] G. Hörmann and M. Oberguggenberger. Elliptic regularity and solvability for partial differential equations with Colombeau coefficients. *Electron. J. Diff. Eqns.*, 2004(14):1–30, 2004.

[25] G. Hörmann, M. Oberguggenberger, and S. Pilipovic. Microlocal hypoellipticity of linear partial differential operators with generalized functions as coefficients. *arXiv:math AP/0303248*, 2005. To appear in Trans. Amer. Math. Soc.

[26] J. Hounie. On the L^2 continuity of pseudodifferential operators. *Comm. Partial Differential Equations*, 11:765–778, 1986.

[27] H. Kumano-go. *Pseudodifferential Operators.* MIT Press, Cambridge, Mass., 1981.

[28] M. Mascarello and L. Rodino. *Partial differential equations with multiple characteristics.* Mathematical Topics 13. Akademie Verlag, Berlin, 1997.

[29] V. P. Maslov and V. A. Tsupin. δ-shaped Sobolev generalized solutions of quasi-linear equations. *Russian Math. Surveys*, 34(1):231–232, 1979.

[30] V. P. Maslov and V. A. Tsupin. Necessary conditions for the existence of infinitesimally narrow solitons in gas dynamics. *Soviet Phys. Dokl.*, 24(5):354–356, 1979.

[31] M. Nedeljkov and S. Pilipović. Hypoelliptic differential operators with generalized constant coefficients. *Proc. Edinb. Math. Soc.(2)*, 41(1):47–60, 1998.

[32] M. Nedeljkov, S. Pilipović, and D. Scarpalézos. *The Linear Theory of Colombeau Generalized Functions*. Pitman Research Notes in Mathematics 385. Longman Scientific & Technical, Harlow, 1998.
[33] M. Oberguggenberger. Pseudodifferential operators on algebras of generalized functions: elliptic regularity. Unpublished manuscript, lectures at University of Pretoria and École Normale Supérieure de Lyon, 1990.
[34] M. Oberguggenberger. *Multiplication of Distributions and Applications to Partial Differential Equations*. Pitman Research Notes in Mathematics 259. Longman Scientific & Technical, Harlow, 1992.
[35] M. Oberguggenberger and M. Kunzinger. Characterization of Colombeau generalized functions by their point values. *Math. Nachr.*, 203:147–157, 1999.
[36] S. Pilipović. *Colombeau's Generalized Functions and Pseudo-Differential Operators*. Lect. Math. Sci., Univ. Tokyo, Tokyo, 1994.
[37] D. Robert. *Autour de l'Approximation Semi-Classique*. Progress in Math. 68. Birkhäuser, Boston, 1987.
[38] L. Rodino. On the boundedness of pseudo-differential operators in the class $L^m_{\rho,1}$. *Proc. Am. Math. Soc.*, 58:211–215, 1976.
[39] D. Scarpalézos. Colombeau's generalized functions: topological structures; microlocal properties. A simplified point of view. *I. Bull. Cl. Sci. Math. Nat. Sci. Math.*, 25:89–114, 2000.
[40] M. A. Shubin. *Pseudodifferential Operators and Spectral Theory*. Springer-Verlag, Berlin, 1987. Nauka, Moscow 1978; English translation.

Claudia Garetto
Institut für Technische Mathematik, Geometrie und Bauinformatik, Universität Innsbruck, A - 6020 Innsbruck, Austria
E-mail address: claudia@mat1.uibk.ac.at

Todor Gramchev
Dipartimento di Matematica e Informatica, Università di Cagliari, I - 09124 Cagliari, Italia
E-mail address: todor@unica.it

Michael Oberguggenberger
Institut für Technische Mathematik, Geometrie und Bauinformatik, Universität Innsbruck, A - 6020 Innsbruck, Austria
E-mail address: michael@mat1.uibk.ac.at